Robotica (2009) volume 27, pp. 929–940. © Cambridge University Press 2009
doi:10.1017/S0263574709005359

Geometrical method to determine the reciprocal screws and applications to parallel manipulators

Jianguo Zhao†, Bing Li‡§*, Xiaojun Yang‡ and Hongjian Yu‡

†*Department of Electrical and Computer Engineering, Michigan State University, MI 48824, USA*
‡*Shenzhen Graduate School, Harbin Institute of Technology, Shenzhen 518055, P.R. China*
§*State Key Laboratory of Robotics and System (HIT), Harbin 150001, P.R. China*

(Received in Final Form: January 16, 2009. First published online: April 3, 2009)

SUMMARY
Screw theory has demonstrated its wide applications in robot kinematics and statics. We aim to propose an intuitive geometrical approach to obtain the reciprocal screws for a given screw system. Compared with the traditional Plücker coordinate method, the new approach is free from algebraic manipulation and can be used to obtain the reciprocal screws just by inspecting the structure of manipulator. The approach is based on three observations that describe the geometrical relation for zero pitch screw and infinite pitch screw. Based on the observations, the reciprocal screw systems of several common kinematic elements are analyzed, including usual kinematic pairs and chains. We also demonstrate usefulness of the geometrical approach by a variety of applications in mobility analysis, Jacobian formulation, and singularity analysis for parallel manipulator. This new approach can facilitate the parallel manipulator design process and provide sufficient insights for existing manipulators.

KEYWORDS: Screw theory; Geometrical method; Reciprocal screw; Line vector; Couple vector.

1. Introduction

The instantaneous motion of a rigid body can be represented by a six-component vector called twist. Similarly, all the forces acting on a rigid body can be described by another six-component vector called wrench. Although twist and wrench have different physical meanings, they have the same mathematical representation called screws. Two screws are said to be reciprocal to each other if their reciprocal product equals to zero. If one screw is twist and the other is wrench, the physical meaning for reciprocity between these two screws is that the instantaneous work for the wrench along the twist is zero.[1] Hence if a screw system of twists represents the degrees of freedom (DOF) of a rigid body, the reciprocal system of wrenches is the constraint forces acting on it. The dual proposition also holds: When a screw system of wrenches act on a rigid body, then the reciprocal system of twists is the DOF of the body. Therefore, the reciprocity property makes it possible to obtain motion information from the corresponding constraint counterpart and vice versa,[2] and it is necessary to study how to find the reciprocal screw system for a given one.

*Corresponding author. E-mail: libing.sgs@hit.edu.cn

Traditionally, the reciprocal screw system is obtained by algebraic approach. First of all, a Cartesian coordinate frame is established and the Plücker coordinates for screws in the frame are derived. Then a set of homogeneous equations is formed by the reciprocal condition. Finally, various methods are applied to obtain null space of the equations.[3,4] Different from conventional method, we try to propose a geometrical method to determine the reciprocal screw systems. Compared with the algebraic approach, one only needs to implement three simple observations developed in the paper. Simply by inspecting the joint axes and applying the observations, one can easily obtain a basis of the reciprocal screw system.

Relative motion between two rigid bodies can be achieved by mechanical joints. For example, rotation and translation can be obtained by revolute and prismatic joints respectively. In robot manipulator design, revolute and prismatic joints are widely used. In fact, most of the other joints can be conceived as the combination of several revolute and prismatic joints. For example, a universal joint can be substituted by two revolute joints whose axis intersects with each other perpendicularly. At the same time, common constraints in statics are pure force or moment.[5] For both the motion and constraint, the screw associated with rotation or a pure force is a zero pitch screw which we call a line vector, while the screw associated with translation or a pure moment is an infinite pitch screw which we call a couple vector. We summarize the physical meaning for line and couple vectors for motion and constraint in Table I. For simplicity, only the line vector and couple vector will be discussed in this paper.

A line vector is a zero pitch screw which can be written as

$$\mathbf{L} = (\mathbf{s}_l, \mathbf{r} \times \mathbf{s}_l),$$

where $\mathbf{s}_l$ is the direction of the screw axis and $\mathbf{r}$ is any point on the line vector.[6] Note that we have used the ray coordinates for screws.

A couple vector is an infinite pitch screw which has the following form:

$$\mathbf{C} = (0, \mathbf{s}_c),$$

where $\mathbf{s}_c$ is the direction of the couple vector. Note that a couple vector is a free vector which can move freely in space, and only its direction is important.

In this paper, many figures will be used to illustrate the geometrical ideas. For convenience, a line vector is represented

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Table I. Physical meaning for line and couple vector.

Screw	Twist	Wrench
Line vector	Rotation	Force
Couple vector	Translation	Moment

by a line with a solid arrow, while a couple vector is represented by a line with a hollow arrow. The screws for twists are depicted as dashed lines, while for wrenches as solid lines.

2. Basic Observations and Steps to Obtain Reciprocal System

In this section, we will first propose three observations about the reciprocal conditions for a line vector and a couple vector. Huang *et al.*[8] list these propositions as corollaries; however, they do not provide proofs for them and apply them to the derivation of reciprocal screws. After introducing the basic observations, general steps for obtaining the reciprocal screw system are outlined for both the traditional algebraic method and our geometrical method.

2.1. Basic observations

Observation 1: Two line vectors

Two line vectors are reciprocal to each other if and only if they are coplanar.

Proof: Suppose the two line vectors are $\mathbf{L}_1 = l_1\hat{\mathbf{L}}_1$ and $\mathbf{L}_2 = l_2\hat{\mathbf{L}}_2$, where l_1 and l_2 are the amplitude of the line vectors, and $\hat{\mathbf{L}}_1 = (\mathbf{s}_{l1}, \mathbf{r}_1 \times \mathbf{s}_{l1})$ and $\hat{\mathbf{L}}_2 = (\mathbf{s}_{l2}, \mathbf{r}_2 \times \mathbf{s}_{l2})$ are unit screws associated with them. Then the reciprocal product between them is

$$\mathbf{L}_1 \circ \mathbf{L}_2 = l_1 l_2[(\mathbf{s}_{l1} \bullet (\mathbf{r}_2 \times \mathbf{s}_{l2}) + \mathbf{s}_{l2} \bullet (\mathbf{r}_1 \times \mathbf{s}_{l1})] = -l_1 l_2 a \sin\alpha,$$

where α is the twist angle between the two lines and a is the length of common perpendicular.

If two line vectors are coplanar, they are either parallel to each other or intersecting at a point. If they are parallel, then $\alpha = 0$, while when they intersect, then $a = 0$. Either condition will lead to $\mathbf{L}_1 \circ \mathbf{L}_2 = 0$ which means two coplanar line vectors are reciprocal to each other.

If two line vectors are reciprocal, we have $a \sin\alpha = 0$. The solution is either $a = 0$, which means the two line vectors intersect, or $\alpha = 0$, which means they are parallel to each other. For both situations, the line vectors are coplanar. Thus we complete the proof of observation 1.

Observation 2: Two couple vectors

Two couple vectors are always reciprocal to each other.

This observation is self-evident since the first three components of a couple vector is zero. The reciprocal product between two couple vectors is always zero.

Observation 3: Line vector and couple vector

A line vector is reciprocal to a couple vector if and only if they are perpendicular to each other.

Proof: This observation is also obvious from the definition of reciprocal product.

$$\mathbf{L} \circ \mathbf{C} = \mathbf{s}_l \bullet \mathbf{s}_c.$$

From the above equation, we can see that the reciprocity condition for a line vector and a couple vector is equivalent to the perpendicular condition.

2.2. Steps to obtain reciprocal system

A screw system is a span of $k(k \leq 6)$ linearly independent screws, and is often called a k system. For this screw system, any k linearly independent screws can be a basis of the system. The reciprocal screw system for the given one is a $6-k$ system in which any one screw is reciprocal to all the screws in the given system. Similarly, any $6-k$ linearly independent screws form a basis of the reciprocal system.[6] Hence we only need to find $6-k$ convenient linearly independent screws to represent the reciprocal system.

For traditional Plücker coordinate method, the steps to obtain the reciprocal screws are outlined as follows:

(1) Establish a convenient coordinate frame;
(2) Write the Plücker coordinate for each of the screws;
(3) Use linear algebra to check the linear dependence of the screws and find one basis of the screw system;
(4) Use the reciprocal condition to establish linear equations and solve the reciprocal screws.

In contrast, the steps for proposed geometrical method are as follows:

(1) Use Grassmann geometry to determine the dependence of the screws by inspection[7] and find one basis of the screw system;
(2) Use the three observations to find the reciprocal screws and reciprocal screw system.

Simply from the steps of the two methods, we can see that the geometrical method is simpler than the Plücker coordinate method. In what follows, we will use both methods to obtain the reciprocal system and demonstrate the simplicity of the geometrical method. Due to the length of the paper, the Plücker coordinate method will be used in only a few examples to make necessary comparison.

3. Reciprocal Screw System of Common Kinematic Element

In this section, reciprocal screws of some common kinematic elements will be discussed. Tsai[5] has performed the same discussion; however, he directly puts the results without any analysis. Here, instead, we will use both the Plücker coordinate method and the geometrical method to give necessary explanation. At first, the reciprocal screws of some kinematic pairs will be analyzed. Then some common kinematic chains will be discussed. All the analyses in this section will facilitate further development of the geometrical approach in subsequent sections.

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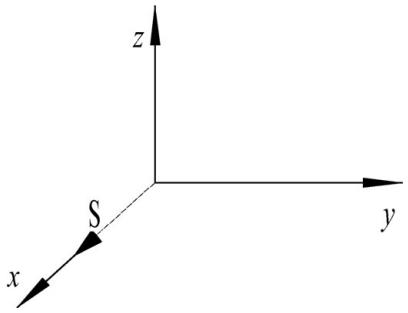


Fig. 1. Coordinate frame for a revolute pair.

3.1. Reciprocal screws of common kinematic pairs

As stated in Section 1, we only concern the kinematic pair capable of being decomposed into prismatic or revolute joints in this paper, and thus only revolute pair, prismatic pair, cylindrical pair, universal pair, spherical pair, and planar pair will be discussed.

3.1.1. Revolute pair.

• Plücker coordinate method

As stated before, the associated screw of a revolute pair is a line vector. Hence, by the convention in this paper, it should be a dashed line with a solid arrow. We establish a coordinate frame as shown in Fig. 1. Then the Plücker coordinate for it is

$$\mathbf{S} = (1 \quad 0 \quad 0; \quad 0 \quad 0 \quad 0).$$

By the reciprocal condition, we can find one basis for the reciprocal screw system as follows:

$$\begin{aligned}
\mathbf{S}_1^r &= (1 \quad 0 \quad 0; \quad 0 \quad 0 \quad 0);\\
\mathbf{S}_2^r &= (0 \quad 1 \quad 0; \quad 0 \quad 0 \quad 0);\\
\mathbf{S}_3^r &= (0 \quad 0 \quad 1; \quad 0 \quad 0 \quad 0);\\
\mathbf{S}_4^r &= (0 \quad 0 \quad 0; \quad 0 \quad 1 \quad 0);\\
\mathbf{S}_5^r &= (0 \quad 0 \quad 0; \quad 0 \quad 0 \quad 1).
\end{aligned}$$

• Geometrical method

By observation 1, all the forces along the lines which are coplanar with the revolute joint axis are reciprocal screws. The above $\mathbf{S}_1^r$, $\mathbf{S}_2^r$, and $\mathbf{S}_3^r$ are such forces. By observation 3, all the moments lying in planes perpendicular to the joint axis are reciprocal screws. The above $\mathbf{S}_4^r$ and $\mathbf{S}_5^r$ are such moments. In fact, we can choose any five linearly independent ones to form the reciprocal 5 system.

As only a line vector is considered in this example, both methods are very simple. However, we can still conclude that the geometrical method provides insights to all the reciprocal screws without algebraic manipulation. In the next few examples of this subsection, only the geometrical method will be used to obtain the reciprocal system.

3.1.2. Prismatic pair. The associated screw of a prismatic pair is a couple vector. By observation 2, any moment is a reciprocal screw. Also by observation 3, all the forces along the lines lying in the plane perpendicular to the joint axis are reciprocal screws. We can choose any five linearly independent ones to form the reciprocal 5 system.

3.1.3. Cylindrical pair. The associated screws of a cylindrical pair are a couple vector and a line vector with the same direction. By observations 1 and 3, all the forces along the lines lying in the plane perpendicular to the joint axis and intersect it are reciprocal screws. They form a planar pencil. By observations 2 and 3, all the moments lying in the plane perpendicular to the joint axis are also reciprocal screws. We can choose any four linearly independent ones, two from the line vectors and two from the couple vectors stated above, to form the basis of the reciprocal 4 system.

3.1.4. Universal pair. The associated screws of a universal pair are two intersecting line vectors. By observation 1, all the forces lying in the intersecting plane or passing through the intersecting point are reciprocal screws. By observation 3, the moments perpendicular to the intersecting plane are also reciprocal screws. We can choose any four linearly independent ones to form the reciprocal 4 system.

3.1.5. Spherical pair. The associated screws of a spherical pair are three intersecting line vectors. By observation 1, all the forces passing through the intersecting point are reciprocal screws. By observation 3, for a general situation, there exist no reciprocal screws of pure moments. All the reciprocal line vectors form a 3 system.

3.1.6. Planar pair. The associated screws of a planar pair are two couple vectors and a line vector. The plane formed by the two couple vectors is perpendicular to the line vector. By observations 1 and 3, all the forces perpendicular to the plane are reciprocal screws. By observations 2 and 3, all the moments lying in the plane perpendicular to the line vector are also reciprocal screws. All these reciprocal screws form a 3 system.

3.2. Reciprocal screws of common kinematic chain

A reciprocal screw for a kinematic chain is reciprocal to the screws associated with all pairs in the chain. Hence the reciprocal screw system of a kinematic chain is the intersection of reciprocal screw systems for every pair. Four different common kinematic chains will be discussed here.

3.2.1. Revolute-revolute-revolute. The associated screws for this kinematic chain are three line vectors. For three linearly independent line vectors, there exist four cases by Grassmann geometry.[8] The first one is that they belong to a regulus which means they are neither parallel to nor intersect with each other; the second one is that they are parallel to each other but not coplanar; the third one is that they are arbitrarily placed in a plane but are not parallel to each other; and the last one is that they intersect at a common point. Since the first situation is the most general one, we use both the Plücker coordinate method and our geometrical method to obtain the reciprocal system. For the other situations, only the geometrical method will be used.

• Plücker coordinate method

Without loss of generality, let these three line vectors be perpendicular to each other and have the same unit length

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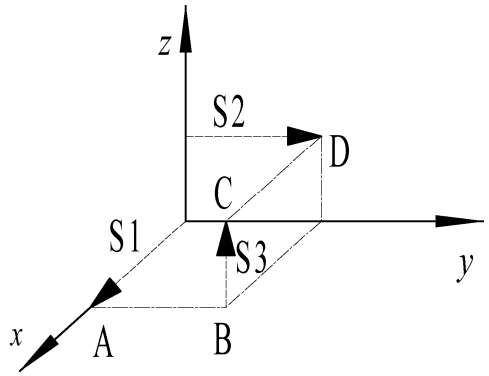


Fig. 2. Coordinate method for the general case of three revolute joints.

of common perpendicular. They are shown in Fig. 2 as $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{S}_3$. Establishing a coordinate frame as shown in the figure, the Plücker coordinates for them are

$$\mathbf{S}_1 = (1 \quad 0 \quad 0; \quad 0 \quad 0 \quad 0);$$
$$\mathbf{S}_2 = (0 \quad 1 \quad 0; \quad 1 \quad 0 \quad 0);$$
$$\mathbf{S}_3 = (0 \quad 0 \quad 1; \quad 1 \quad -1 \quad 0).$$

Suppose the coordinate of reciprocal screw is (L, M, N, P, Q, R); then by the reciprocal condition we can form the following equations:

$$\begin{cases} P = 0; \\ L + Q = 0; \\ L - M + R = 0. \end{cases}$$

Any screw satisfying the above equations will be a reciprocal screw. In order to compare with the results of the geometrical method, we choose the following three to form the basis for reciprocal system.

$$\mathbf{S}_1^r = (0 \quad 1 \quad 0; \quad 0 \quad 0 \quad 1);$$
$$\mathbf{S}_2^r = (1 \quad 0 \quad 0; \quad 0 \quad -1 \quad -1);$$
$$\mathbf{S}_3^r = (0 \quad 0 \quad 1; \quad 0 \quad 0 \quad 0).$$

- **Geometrical method**

Suppose R_1, R_2, and R_3 are the three joint axes as shown in Fig 3. In order to find the reciprocal screws, select a random point Q on R_3. Then Q and R_1, Q and R_2 will form two planes. The intersection of the two planes is a line vector coplanar with each of the three lines (F_1 shown in the figure). Thus by observation 1 it is reciprocal to them. By choosing different points on the line, we can get infinitely many different reciprocal line vectors; however, only three of them are linearly independent, and we can choose any three independent ones to form the basis of the reciprocal system.

In fact, the three basis screws obtained by the coordinate method can be interpreted by the geometrical method. For $\mathbf{S}_1^r$, it is a line vector along CD, which can be obtained by choosing C on $\mathbf{S}_3$, and CD is the intersection of the two planes formed by C and $\mathbf{S}_1$, $\mathbf{S}_2$. Similarly, for $\mathbf{S}_2^r$, it is a line vector along AB, which can be obtained by choosing B on $\mathbf{S}_3$, and AB is the intersection of the two planes formed by B and $\mathbf{S}_1$, $\mathbf{S}_2$. $\mathbf{S}_3^r$ is a line vector along the z-axis, which can be considered as choosing the infinity point on $\mathbf{S}_3$. From this comparison, we can see that the geometrical method is simpler and more intuitive than the coordinate one.

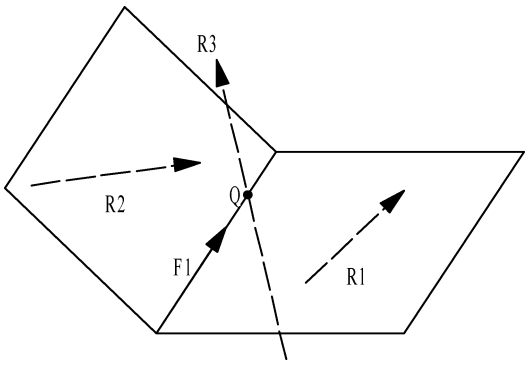


Fig. 3. Geometrical method for the general case of three revolute joints.

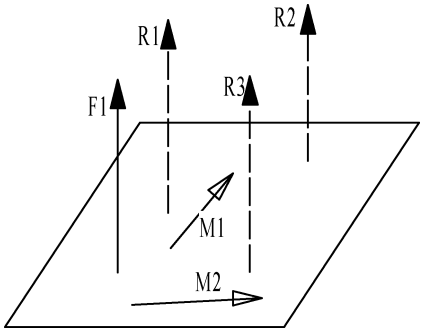


Fig. 4. Reciprocal screws for revolute-revolute-revolute kinematic chain: parallel case.

The second case of three revolute joints is shown in Fig. 4. R_1, R_2, and R_3 are the three parallel joint axes. By observation 3, any couple vector in the plane perpendicular to the given line vectors is a reciprocal screw. In addition, all the line vectors parallel to them are also reciprocal screws by observation 1. Therefore, we choose one parallel line vector F_1 and two couple vectors M_1 and M_2 in the perpendicular plane to form the basis of the reciprocal system as shown in the figure.

The coplanar case is shown in Fig. 5, where R_1, R_2, and R_3 lie on the same plane. By observations 1 and 3, any line

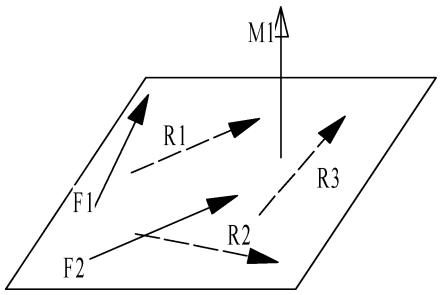
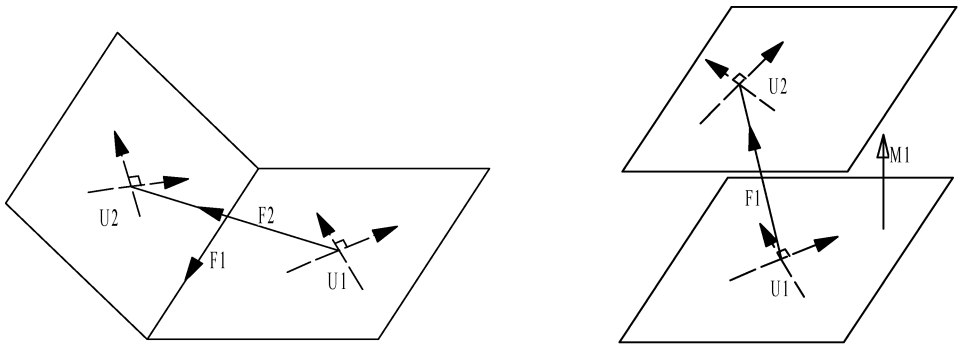


Fig. 5. Reciprocal screws for revolute-revolute-revolute kinematic chain: coplanar case.

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(a) Two planes intersect with each other; (b) Two planes are parallel to each other

Fig. 6. Reciprocal screws for universal-universal kinematic chain.

vector in the plane and any couple vector perpendicular to the plane are reciprocal screws. We choose two line vectors F_1 and F_2 in the plane and one perpendicular couple vector M_1 to form the basis of reciprocal system as shown in the figure. The last case is the same as a spherical pair discussed above and is omitted here.

3.2.2. Universal-universal. This kinematic chain is usually used in translational manipulators to constrain the relative rotational motion between the fixed base and the moving platform. Generally, the two universal joints can be placed arbitrarily, which means the two planes formed by each two joints' axes can either intersect or parallel. For the first case shown in Fig. 6(a), two linearly independent reciprocal line vectors can be found by observation 1. One is F_2 which passes through two joint centers, and the other is F_1 which is the intersection line of two planes. They form a basis for the reciprocal 2 system. Similarly, in the parallel situation shown in Fig. 6(b), by observations 1 and 2, one reciprocal line vector F_1 passing through two joint centers and one reciprocal couple vector M_1 perpendicular to the two parallel planes can be used as a basis for the reciprocal system.

3.2.3. Universal-spherical. This kinematic chain can be decomposed into five revolute joints that form a 5 system. The reciprocal screw is thus a 1 system. The line vector which passes through both the universal joint center and the spherical joint center is coplanar to all the line vectors associated with the five revolute joints. By observation 1, it is a reciprocal screw.

3.2.4. Revolute-spherical. The line vectors associated with all the pairs of this kinematic chain form a 4 system and the reciprocal system is thus a 2 system. By observation 1, the reciprocal screws are a planar pencil passing through the spherical joint center and lying in the plane formed by the spherical joint center and the revolute joint axis. They are shown in Fig. 7.

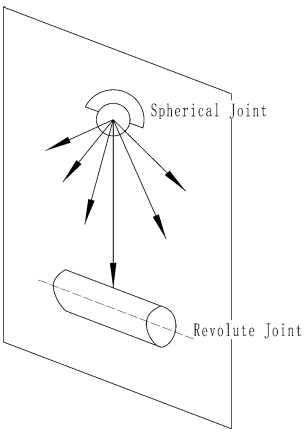


Fig. 7. Reciprocal screws for the revolute-spherical kinematic chain.

4. Applications

4.1. Mobility analysis of a rigid body

A free rigid body has six DOF. Analyzing the motion twists for a rigid body under given constraint wrenches is called the exact constraint.[10] It is useful in the workpiece localization and parts assembly.[9] Blading[10] discusses the R and C patterns for the motion and constraint of a rigid body. However, he only analyzes the pure force constraint, and he also treats the translation as rotation about an axis at infinity which is not intuitive. In this section, more general results will be obtained based on the three observations. The method is illustrated by three examples.

Example 1: Pure force constraints

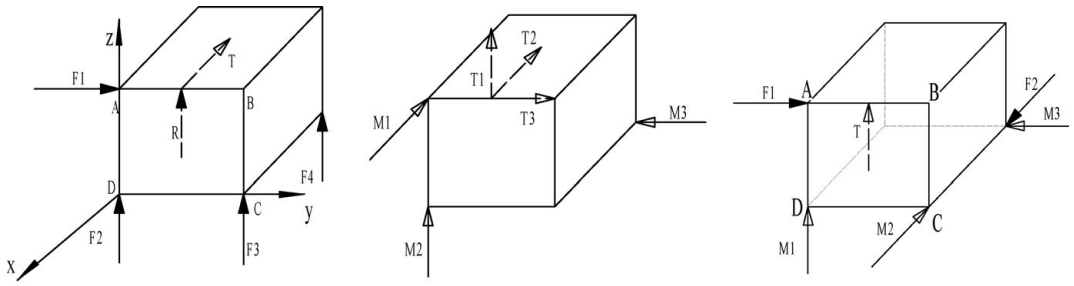
In this example, a cube is subjected to four unit pure forces as shown in Fig. 8(a). All of them exert on the cube's vertices and parallel to the edges.

- **Plücker coordinate method**

Establish a coordinate frame at vertex D as shown in the figure, with its y-axis along DC and z-axis along DA. Then the screw coordinates $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, and $\mathbf{S}_4$ corresponding to the four wrenches F_1, F_2, F_3, and F_4 are

$$\mathbf{S}_1 = (0 \quad 1 \quad 0; \quad -1 \quad 0 \quad 0);$$
$$\mathbf{S}_2 = (0 \quad 0 \quad 1; \quad 0 \quad 0 \quad 0);$$
$$\mathbf{S}_3 = (0 \quad 0 \quad 1; \quad 1 \quad 0 \quad 0);$$
$$\mathbf{S}_4 = (0 \quad 0 \quad 1; \quad 1 \quad 1 \quad 0).$$

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(a) Pure force constraints (b) Pure moment constraints (c) Hybrid constraints

Fig. 8. Mobility analysis of a rigid cube.

We can easily check that these four vectors are linearly independent; hence they form a 4 system of screws, and its reciprocal system is a 2 system. Suppose the coordinate of reciprocal screw is (L, M, N, P, Q, R); then by the reciprocal condition we have

$$L = M = R = Q = 0,$$

which means if the reciprocal screw is a line vector, it must be along the z-direction and in the y–z plane, and if it is a couple vector, it must be along the x-direction. We choose one line vector and one couple vector to form the basis of the reciprocal 2 system.

$$\mathbf{S}_1^r = (0 \quad 0 \quad 1; \quad 1 \quad 0 \quad 0);$$
$$\mathbf{S}_2^r = (0 \quad 0 \quad 0; \quad 1 \quad 0 \quad 0).$$

- **Geometrical method**

By Grassmann geometry, the four line vectors F_1, F_2, F_3, and F_4 are linearly independent. By observation 1, all the line vectors parallel to the three forces F_2, F_3, and F_4 and contained in plane ABCD can be reciprocal line vectors. The R in Fig. 8(a) is one of such line vectors. By observation 2, any couple vector parallel to x-axis can be a reciprocal couple vector since it is perpendicular to all the forces. The T in Fig. 8(a) is one of such couple vectors. R and T form a basis for the reciprocal 2 system. The physical meaning is that the cube has two DOF under the constraint of four wrenches: one rotation about R and one translation along T. In fact, the reciprocal screws obtained by the coordinate method can be interpreted by the geometrical method. $\mathbf{S}_1^r$ is one of the line vectors obtained by observation 1, while $\mathbf{S}_2^r$ is one of the couple vectors obtained by observation 2.

Example 2: Pure moment constraints

In this example, a cube is subject to three pure moments M_1, M_2, and M_3 as shown in Fig. 8(b). As couple vectors are free vectors, there are at most three linearly independent couple vectors in three-dimensional space. The three couple vectors in the figure are linearly independent since they point to different directions. Hence the reciprocal system is a 3 system.

There cannot be any line vector reciprocal to M_1, M_2, and M_3 because there exists no line vector perpendicular to all of them. By observation 2, we can have three linearly independent couple vectors reciprocal to them. We have chosen T_1, T_2, and T_3 shown in the figure as a basis for the reciprocal 3 system. It means the cube has three translational DOF along the three directions.

Example 3: Hybrid constraints

The same cube is subjected to two forces F_1 and F_2 and three moments M_1, M_2, and M_3 as shown in Fig. 8(c). All of them exert on the cube's vertices and parallel to the edges. By Grassmann geometry, the five constraints are linearly independent. Hence its reciprocal system is a 1 system.

Similar to example 2, the three linearly independent couple vectors mean that there are no reciprocal line vectors. We can have a reciprocal couple vector perpendicular to the two forces. The T in the plane ABCD and parallel to M_1 is such a couple vector. It means the cube has only one translational DOF along the couple vector.

4.2. Mobility analysis of some parallel manipulators

A typical parallel manipulator consists of a fixed base, a moving platform, and several identical kinematic chains called limbs which connect the moving platform to the fixed base. By these limbs, the moving platform is constrained to perform the prescribed motion.

Traditionally, the mobility for a manipulator is derived by the Kutzbach–Grübler formula.[5] However, the formula only tells the number of DOF, and the types of motion for each DOF cannot be specified. Furthermore, for some manipulators the formula may lead to erroneous results.[8] In this section, based on previous analysis for common kinematic elements and a rigid body, the exact DOF of parallel manipulators can be derived. Here "exact" means both the number and the motion types.

First of all, we should define some screw systems. *Limb motion screw system* is the screw system formed by every joint in each limb of the manipulator, while *limb constraint screw system* is the reciprocal system of limb motion screw system. *Platform motion screw system* is the motion screw system of the moving platform, which can be considered as the DOF for the manipulator. It is the intersection of all the limb motion screw systems. *Platform constraint screw system* is the reciprocal system of the platform motion screw system. It is the union of all the limb constraint screw systems.[11]

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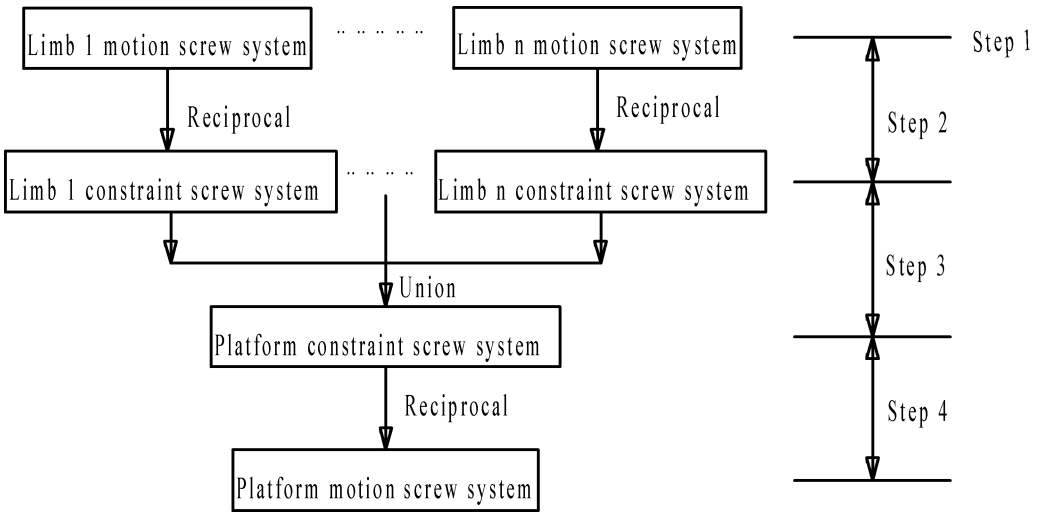


Fig. 9. Steps to obtain the exact DOF of parallel manipulators.

The procedure to obtain the exact DOF of parallel manipulator is listed as follows and outlined in Fig. 9.

(1) Find a basis for each limb motion screw system;
(2) Solve limb constraint screw system for each limb by the reciprocal condition and find a basis for the system;
(3) Find the union of all limb constraint screw systems to get the platform constraint screw system. Determine a basis for the system;
(4) Solve platform motion screw system using the reciprocal condition again and find a basis for the system. It is the desired DOF of the manipulator.

Two examples are used to illustrate the procedure.

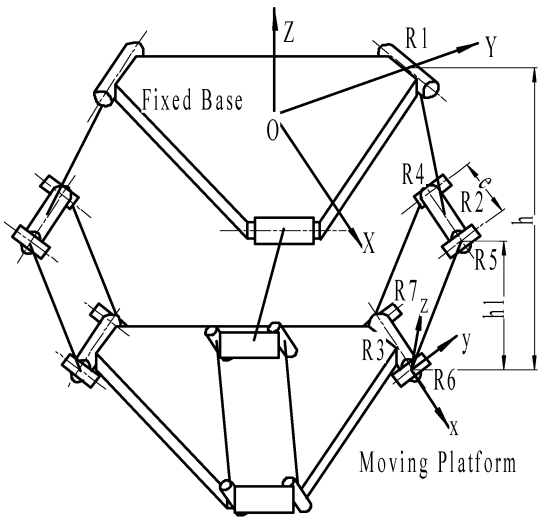
Example 4: Rotational Delta robot

The rotational Delta robot evolves from the traditional Delta robot which substitutes the spherical or universal joints in the parallelogram by revolute joints.[12] The architecture is shown in Fig. 10(a). The robot has three limbs, and every limb has seven revolute joints as labeled on one limb in the figure. Four of them, R_4, R_5, R_6, and R_7, form a parallelogram. The other three R_1, R_2, and R_3 are parallel to each other. Moreover, the above two sets of revolute joints are perpendicular to each other. Establish the moving coordinate frame in the limb as shown in Fig. 10(b): x-axis along the axis of R_3, y-axis along the axis of R_6, and z-axis is determined by the right-hand rule. If we use the Kutzbach–Grübler formula to calculate its mobility, a negative result will be obtained.[5] In this example, we will use both the coordinate method and the geometrical method to obtain the exact DOF for the manipulator.

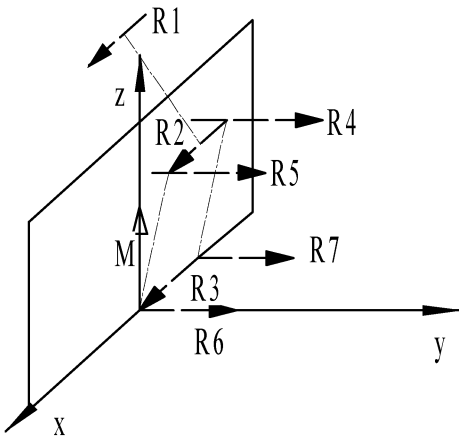
- **Plücker coordinate method**

The screw coordinates $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, $\mathbf{S}_4$, $\mathbf{S}_5$, $\mathbf{S}_6$, and $\mathbf{S}_7$ corresponding to the seven revolute joints are

$$\mathbf{S}_1 = (1 \quad 0 \quad 0; \quad 0 \quad \mathrm{h} \quad a);$$
$$\mathbf{S}_2 = (1 \quad 0 \quad 0; \quad 0 \quad h_1 \quad -b);$$



(a) The architecture



(b) Geometrical presentation for the joints in a limb

Fig. 10. Mobility analysis for the rotational Delta robot.

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$$\mathbf{S}_3 = (1 \quad 0 \quad 0; \quad 0 \quad 0 \quad 0);$$
$$\mathbf{S}_4 = (0 \quad 1 \quad 0; \quad -h_1 \quad 0 \quad -(c+e));$$
$$\mathbf{S}_5 = (0 \quad 1 \quad 0; \quad -h_1 \quad 0 \quad -c);$$
$$\mathbf{S}_6 = (0 \quad 1 \quad 0; \quad 0 \quad 0 \quad 0);$$
$$\mathbf{S}_7 = (0 \quad 1 \quad 0; \quad 0 \quad 0 \quad -e);$$

where h is the height of the moving platform, h_1 is the height from the moving platform to R_2, and e is the distance between R_4 and R_5, R_6 and R_7. They are shown in Fig. 10. The distance between R_1 and the z-axis is a, between R_2 and the z-axis is b, and between R_5 and the z-axis is c.

The seven motion screws form a limb motion screw system. We can check that there are only five linearly independent screws. A basis for the system is as follows:

$$\mathbf{S}_{1b} = (1 \quad 0 \quad 0; \quad 0 \quad 0 \quad 0);$$
$$\mathbf{S}_{2b} = (0 \quad 1 \quad 0; \quad 0 \quad 0 \quad 0);$$
$$\mathbf{S}_{3b} = (0 \quad 0 \quad 0; \quad 1 \quad 0 \quad 0);$$
$$\mathbf{S}_{4b} = (0 \quad 0 \quad 0; \quad 0 \quad 1 \quad 0);$$
$$\mathbf{S}_{5b} = (0 \quad 0 \quad 0; \quad 0 \quad 0 \quad 1).$$

Hence the limb constraint screw system is a 1 system, and we can find its basis as $\mathbf{S}^{\mathrm{r}} = (0\ 0\ 0;\ 0\ 0\ 1)$ by reciprocal condition. It is a couple vector along z-axis. Similarly, we can find the limb constraint screw systems for the other two limbs. They are also couple vectors. The union for these three couple vectors forms the platform constraint screw system. They are linearly independent since if we establish a moving frame at each limb, the z-axis will be different. Hence they can be a basis for the platform constraint screw system.

If we represent the three couple vectors in a single fixed coordinate frame OXYZ shown in Fig. 10(a), they can be as follows:

$$\mathbf{S}_{p1} = (0 \quad 0 \quad 0; \quad o_1 \quad o_2 \quad o_3);$$
$$\mathbf{S}_{p2} = (0 \quad 0 \quad 0; \quad p_1 \quad p_2 \quad p_3);$$
$$\mathbf{S}_{p3} = (0 \quad 0 \quad 0; \quad q_1 \quad q_2 \quad q_3);$$

where $o_i\ p_i$ and q_i $(i = 1, 2, 3)$ are the direction cosine for each of the basis couple vectors in the fixed coordinate frame. The platform motion screw system can be obtained by the reciprocal condition, and we can find a basis for them as

$$\mathbf{S}^{\mathrm{r}}_{p1} = (0 \quad 0 \quad 0; \quad 1 \quad 0 \quad 0);$$
$$\mathbf{S}^{\mathrm{r}}_{p2} = (0 \quad 0 \quad 0; \quad 0 \quad 1 \quad 0);$$
$$\mathbf{S}^{\mathrm{r}}_{p3} = (0 \quad 0 \quad 0; \quad 0 \quad 0 \quad 1);$$

which means the moving platform has three translational DOF.

- **Geometrical method**

By Grassmann geometry, all the seven line vectors form a 5 system. We need to find the reciprocal 1 system. From the figure, there cannot be any line vector coplanar with all the seven line vectors resulting in no reciprocal line vectors. However, there exists a couple vector perpendicular to them. This is shown in Fig. 10(b) by M along z-axis.

Each limb constraint screw system is a moment. Hence the moving platform is subject to three moments. Generally, the three moments are linearly independent. Since it is the same situation in example 2, the platform has three translational DOF.

Example 5: 3-UPU manipulator with four DOF

For this manipulator, two axes of each universal joint form a plane. The manipulator was first proposed by Tsai[13] to achieve three translational DOF. However, there indeed exist various types of 3-UPU manipulators due to different relationships between two universal joint planes. In fact, Li[14] has analyzed a number of different 3-UPU manipulators. The typical 3-UPU manipulator analyzed in this example is a four-DOF manipulator shown in Fig. 11(a). For this manipulator, in the initial configuration, the fixed base and the moving platform are parallel to each other. The two universal joint planes for every limb are also parallel to each other, and both of them are perpendicular to the fixed base.

Establish a coordinate system in a limb with its x-axis along with R_2, z-axis perpendicular to the fixed base, and y-axis determined by the right-hand rule shown in Fig. 11(a). The geometrical presentation of the joints in this limb and its reciprocal screw is shown in Fig. 11(b). In the figure, we have drawn the two universal joint planes. By observations 1 and 3, there cannot be any reciprocal line vector for the limb. By propositions 2 and 3, a reciprocal couple vector exists. It is M_1 as shown in Fig. 11(b), which is perpendicular to the two universal joint planes.

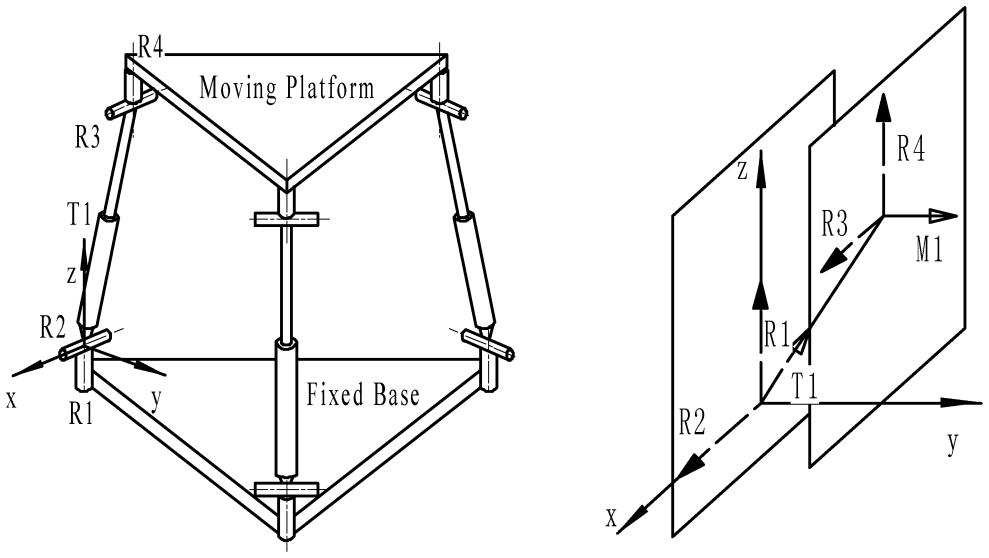
For all the limbs, there will be three couple vectors, and they are coplanar since couple vectors are free vectors. As there are only two linearly independent couple vectors in a plane, the platform constraint screw system is a 2 system. As a result, the platform motion screw system is a 4 system, and the manipulator has four DOF. Applying the basic observations again, we can get the platform motion screw system, and conclude that the moving platform has three translational and one rotational DOF about the z-axis which is perpendicular to the couple vector plane. This is the DOF for the initial configuration.

When the manipulator translates in any direction or rotates about the z-axis, the fixed base and the moving platform will constantly be parallel to each other. In addition, the two universal joint planes will also be parallel to each other all the time. Therefore, this type of 3-UPU manipulator has four DOF. It is underactuated because only three actuators are used to actuate four DOF.

4.3. Jacobian formulation for parallel manipulators

Jacobian relates the actuators' velocity to end-effector's velocity which is crucial for velocity analysis. For parallel manipulator, various methods can be used to obtain the Jacobian matrix. Since the inverse kinematics relates the end-effector's position to the actuators' displacement, the most intuitive method is the direct derivation method which

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(a) The architecture (b) Geometrical presentation for the joints in a limb

Fig. 11. Mobility analysis for 3-UPU manipulator.

differentiates the inverse kinematic equations. Though straightforward, it is rather cumbersome for complex manipulators. Another method is the velocity vector loop method that forms velocity equations for different loops in the manipulator. Then the passive joints' velocity will be eliminated by equation manipulation.[5] This method is also very complicated when applying to complex manipulators.

The screw-based method is the most efficient one. As mentioned before, all the joints in common manipulator can be either one-DOF joints or can be decomposed into several one-DOF joints. Without loss of generality, suppose the manipulator has n limbs and λ one-DOF joints in each limb. Each limb has only one actuator; thus the number of limbs is equal to the DOF of the manipulator. Then for every limb, the twist of the moving platform $\mathbf{S}_\mathrm{p}$ is the linear combination of the twist associated with every joint $\mathbf{S}_{i,j}$:

$$\mathbf{S}_\mathrm{p} = \sum_{j=1}^{\lambda} \omega_{i,j}\mathbf{S}_{i,j} \qquad i = 1, 2, \ldots, n, \tag{1}$$

where i denotes the limb index, j denotes the joint index, and $\omega_{i,j}$ is the velocity of the jth joint in the ith limb. In order to obtain the Jacobian, the velocity of passive joints must be eliminated in Eq. (1). Hence we need to find screws reciprocal to all the passive joints while not reciprocal to the actuator joints. Then by applying the reciprocal product to both sides of Eq. (1), we can achieve the elimination goal.

As a matter of fact, we can always obtain a screw reciprocal to all the other joints except a typical chosen actuator. λ linearly independent twists associated with individual joints of a limb will have a reciprocal screw system of order $6 - \lambda$. If we lock a chosen actuator in the limb, the $6 - \lambda$ linearly independent screws will have a reciprocal screw system of order $7 - \lambda$. As a result, we can always find a screw in the $7 - \lambda$ system but not in the $6 - \lambda$ one. We call this desired screw the *elimination screw* for the actuator. For an n-DOF parallel manipulator, if $\mathbf{S}_{ei}$ is the elimination screw for limb i, applying the reciprocal product to Eq. (1) we can get

$$\mathbf{S}_\mathrm{p} \circ \mathbf{S}_{ei} = \omega_i \mathbf{S}_{ai} \circ \mathbf{S}_{ei} \qquad i = 1, 2, \ldots, n, \tag{2}$$

where $\mathbf{S}_{ai}$ is the screw for actuator i, and ω_i is the velocity for actuator i. Then the Jacobian matrix can be formulated by Eq. (2).

From the above statement, we can see that the critical step in obtaining the Jacobian is to find the elimination screw for each limb. By the three basic observations, we can easily get the result as shown in the following two examples. Without otherwise stated, the actuator joint in each limb is denoted by an underlined letter in the manipulator abbreviation.

Example 6: Elimination screw for 3-<u>R</u>RR planar manipulator

In this manipulator, each of the three identical limbs has three parallel revolute joints. The reciprocal screws for each limb are the same as in Fig. 4. They form a 3 system. In order to obtain the elimination screw, we lock the first actuator joint. Then the reciprocal screws for the remaining two line vectors form a 4 system. Besides the screws shown in Fig. 4, by observation 1, the line vectors lying in the plane formed by the two remaining line vectors also belong to the 4 system. For simplicity, we can choose the line vector passing through the two joint centers as the elimination screw. The elimination screw in one limb is shown in Fig. 12.

Example 7: Elimination screw for 3-R<u>P</u>S manipulator

Each limb of this manipulator contains a revolute joint, a prismatic joint, and a spherical joint. The axis of the prismatic joint is perpendicular to the axis of the revolute joint. The screws for three joints form a 5 system and the reciprocal screw is a 1 system, which has only one linearly independent screw. By observations 1 and 2, the line vector which passes through the spherical joint center and parallel to the axis of revolute joint is a reciprocal screw. In order to find

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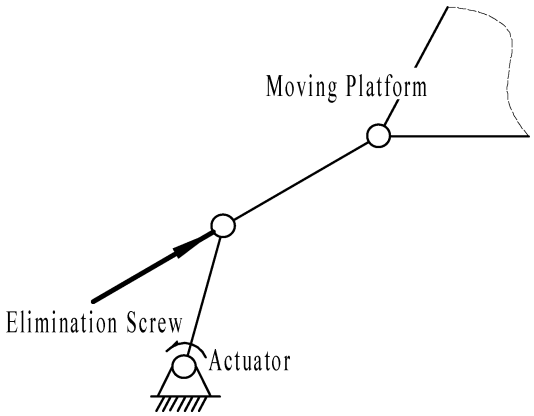


Fig. 12. The elimination screw in one limb for 3-RRR planar manipulator.

the elimination screw, the actuation prismatic joint is locked. The reciprocal screws for the remaining revolute-spherical dyad have been discussed and shown in Fig. 7. They form a 2 system. Therefore, the elimination screw for a limb is any line vector shown in that figure except the one parallel to the axis of the revolute joint. For simplicity, we choose the one along the prismatic joint as shown in Fig. 13

4.4. Singularity analysis for parallel manipulators

In the workspace of parallel manipulator, there exist some configurations called singular configurations where the manipulator cannot move in certain directions or have extra uncontrollable DOF. Moreover, the manipulator also exhibits bad performance near singular configurations. Thus it is necessary to study the singularities in order to keep the manipulator far away from them.[15]

There are a variety of classifications for singularities. The most popular classification is based on the Jacobian matrix.[5] By differentiating the inverse kinematic equations, the following equation can be obtained:

$$\mathbf{J}_x \dot{\mathbf{x}} = \mathbf{J}_\theta \dot{\boldsymbol{\theta}}, \tag{3}$$

where $\mathbf{J}_x$ is called the inverse kinematic Jacobian, $\dot{\mathbf{x}}$ is the velocity of end-effector, $\mathbf{J}_\theta$ is called the direct kinematic Jacobian, and $\dot{\boldsymbol{\theta}}$ is the velocity of actuators. The classification is based on the determinants of two Jacobians as follows:

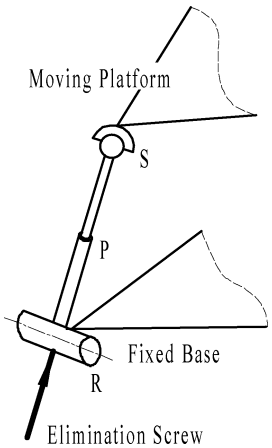


Fig. 13. The elimination screw in one limb for 3-RPS manipulator.

(1) If $\det(\mathbf{J}_\theta) = 0$, the manipulator is at an inverse kinematic singularity (IKS) configuration.
(2) If $\det(\mathbf{J}_x) = 0$, the manipulator undergoes a direct kinematic singularity (DKS) configuration.
(3) If $\det(\mathbf{J}_x) = 0$ and $\det(\mathbf{J}_\theta) = 0$, the manipulator undergoes a combined singularity configuration.

However, the passive joints' velocities are not considered in this classification since the Jacobian only relates the end-effector's velocity to the actuators' velocity. As a result, it cannot contain all the singularity types. A particular type in this category is the constraint singularity (CS).

If a manipulator has n DOF in ordinary configurations, then a CS happens when it has more than n DOF.[16] The manipulator will thus gain additional DOF. For both DKS and CS, the geometrical approach for determining the reciprocal screw for each limb can be used to analyze the constraints imposed on the moving platform, and thus can decide whether the configuration is singular or not.

4.4.1. Direct kinematic singularity. For DKS, there exists $\dot{\mathbf{x}} \neq 0$ such that $\dot{\boldsymbol{\theta}} = 0$ which means the moving platform can still move if all the actuators are locked. If we lock the actuators, the constraint wrenches for the moving platform imposed by the individual limbs should form a 6 system, otherwise it can move in certain directions. Thus if all the wrenches form a screw system with order less than 6, then the manipulator is at a DSK configuration. In the next two examples, some of the DKS configurations are identified by the geometrical approach.

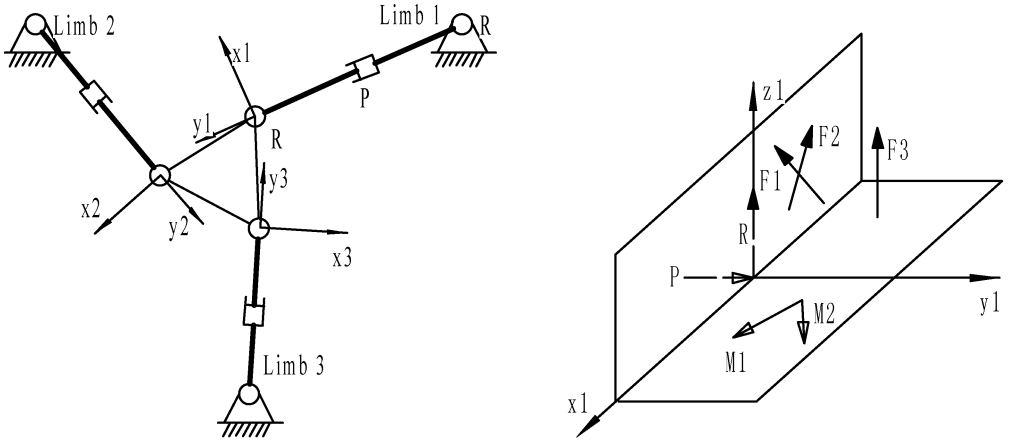
Example 8: DKS for 3-RPR planar manipulator

For this manipulator, the revolute joint connecting to the fixed base is the actuator. If the actuator is locked, then every limb is just a prismatic-revolute dyad. Establish a coordinate frame in limb 1 as shown in Fig. 14(a). Let y_1-axis along the direction of prismatic joint, x_1 in the plane and point out from the moving platform, and z_1, not shown in the figure, be determined by the right-hand rule. By observations 1 and 3, the reciprocal line vectors should be perpendicular to the prismatic joint axis and coplanar with the remaining revolute joint axis. Hence as shown in Fig. 14(b), all the line vectors in x_1z_1 plane are reciprocal screws. Note that only two of them, F_1 and F_2, are depicted in the figure.

In addition to the line vectors in x_1z_1 plane, any line vector parallel to the revolute joint axis is also a reciprocal screw. Only one of them, F_3, is shown in Fig. 14(b). By observations 2 and 3, the reciprocal couple vectors should be perpendicular to the revolute joint axis. Thus all the couple vectors in the x_1y_1 plane are also reciprocal screws. Only two of the couple vectors M_1 and M_2 are shown in Fig. 14(b). All the above reciprocal screws form a 4 system for the dyad after locking the actuator.

All the reciprocal screws for three limbs constitute the constraint wrenches for the moving platform. When the manipulator is at a general configuration, we cannot find a screw reciprocal to all the constraint wrenches, which means the moving platform is motionless when the actuators are locked. However, at two configurations such reciprocal

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(a) The 3-RPR manipulator (b) Reciprocal screws after locking the actuation joint

Fig. 14. DKS for 3-RPR planar manipulator.

screws exist:

(1) When the three planes x_1z_1, x_2z_2, and x_3z_3 intersect at a common line, the line vector along this common line is coplanar to all the constraint line vectors and perpendicular to all constraint couple vectors. Hence the moving platform can instantaneously rotate about it. It is a DKS configuration.
(2) When the aforementioned three planes are parallel to each other, there exists a couple vector perpendicular to all of them. The moving platform gains an uncontrollable translational DOF along the couple vector. It is also a DKS configuration.

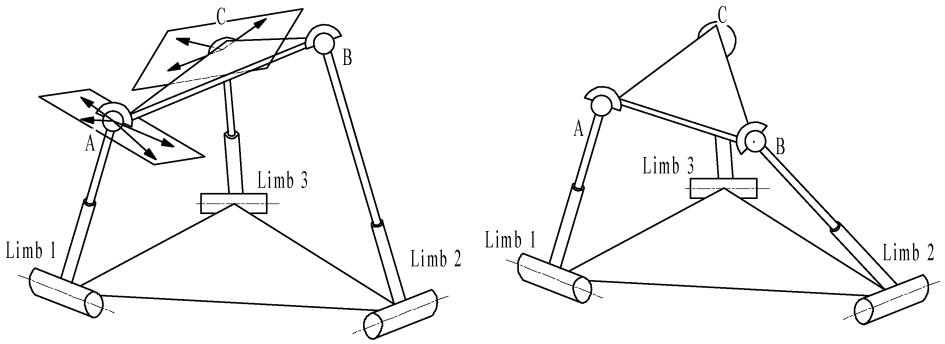
Example 9: DKS for 3-RPS manipulator

In this manipulator, the actuator is not specified because we want to show that for different actuators, different DKS can be obtained.

First of all, suppose the actuators are the revolute joints. In order to examine the DKS, we lock the revolute joint actuators. Then for each limb, the reciprocal line vectors are the planar pencil lying in the plane perpendicular to the prismatic joint axis and passing through the spherical joint center as shown in Fig. 15(a). If the plane ABC is perpendicular to limb 2 as shown in the figure, then the line vector AC will be coplanar with all the line vectors in the three planes. It is a platform motion screw reciprocal to all the limb constraint screw systems. Hence the moving platform has an uncontrollable rotation about AC. It is a DKS configuration. Note that similar phenomenon can also happen to limbs 1 and 3.

Alternatively, we can choose the prismatic joint to actuate the manipulator. If we lock this joint, then for each limb, the reciprocal line vectors are also planar pencil lying in the plane formed by the revolute joint axis and the spherical joint center. This plane is perpendicular to the plane in the revolute actuation case. Hence the DKS configuration ought to be different. If limb 2 lies in the moving platform plane ABC, then AC is coplanar with all the three planar pencils. As a result, it is a DKS configuration which is shown in Fig. 15(b). The moving platform can rotate about AC after all the actuators are locked.

4.4.2. Constraint singularity. Suppose an n-DOF manipulator is under study; then the moving platform is subject to $6-n$ linearly independent constraint wrenches. If the manipulator undergoes a CS configuration, the original $6-n$ constraint wrenches become linearly dependent, and the moving platform will have less than $6-n$ independent constraint wrenches. Therefore, we can use this criterion



(a) Revolute joints are actuators (b) Prismatic joints are actuators

Fig. 15. DKS for 3-RPS manipulator.

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to determine the CS configurations. One example is given to demonstrate the basic idea.

Example 10: CS for 3-RPS manipulator

For each limb, the constraint wrench applying to the moving platform is a line vector passing through the spherical joint center and parallel to the revolute joint axis. This is the only line vector which is both coplanar to all the line vectors associated with the revolute and spherical joints and perpendicular to the prismatic joint axis. Generally, the three constraint line vectors lie in three planes parallel to each other. They form a 3 system and its reciprocal system is also a 3 system. Hence the moving platform has three DOF.

However, if the three line vectors become linearly dependent, the moving platform will gain extra DOF and it is at a CS configuration. Zlatanov[16] shows one of the CS configurations when the moving platform goes upside down and the three line vectors all lie in the moving platform plane and pass through a common point. Since there are only two linearly independent line vectors for coplanar line pencil by Grassmann geometry,[8] the three line vectors become linearly dependent. Hence it is a CS configuration. Note that CS is different from DKS since it has nothing to do with the actuators.

5. Conclusions

In this paper, a geometrical approach to determine the reciprocal screws for a given screw system is developed. First of all, three basic geometrical observations are proposed which describe the reciprocity between line vectors and couple vectors. Based on the observations, reciprocal screws for common kinematic elements are first illustrated which pave way for the kinematic analysis for parallel manipulator. Different aspects of mechanism kinematics demonstrate the usefulness of the geometrical approach. The mobility for a rigid body subjected to different constraints was first analyzed, and then the mobility analysis of parallel manipulator is performed based on the rigid-body mobility analysis. It is shown that one can directly obtain the mobility by examining the structure of the manipulator. In addition to mobility analysis, Jacobian matrix formulation for parallel manipulator is also discussed by the geometrical approach. The novel elimination screw is proposed for the screw-based approach. Finally, two types of singularities, DKS and CS for parallel manipulator, are analyzed based on the three observations. By applying the geometrical method to parallel manipulator analysis, we conclude that this method has many advantages such as intuitive and simple compared with the traditional Plücker coordinate method. Therefore, it can be used to facilitate the parallel manipulator design process and give insights to existing manipulators.

Acknowledgments

This work is supported by Natural Science Foundation of China (Project No.: 60875060) and State Key Laboratory of Robotics and System (HIT) (Project No: SKLRS200719). The work is also partly supported by 863 Hi-Tech Research and Development Program of China (Project No: 2006AA040205). The authors would also like to thank the reviewers' comments on adding the coordinate method for comparison with our geometrical approach.

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