A NOTE ON UNIFORM CONTINUITY OF MONOTONE FUNCTIONS

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Abstract. We prove that it is consistent with ZFC that for every non-decreasing function $f : [0, 1] \to [0, 1]$, each subset of $[0, 1]$ of cardinality $\mathfrak{c}$ contains a set of cardinality $\mathfrak{c}$ on which f is uniformly continuous. We show that this statement follows from the assumptions that $\mathfrak{d}^* < \mathfrak{c}$ and $\mathfrak{c}$ is regular, where $\mathfrak{d}^* \leq \mathfrak{d}$ is the smallest cardinality κ such that any two disjoint countable dense sets in $2^{\mathbb{N}}$ can be separated by sets each of which is an intersection of at most κ-many open sets in $2^{\mathbb{N}}$. We establish also that $\mathfrak{d}^* = \min\{\mathfrak{u}, \mathfrak{d}\} = \min\{\mathfrak{r}, \mathfrak{d}\}$, thus giving an alternative proof of the latter equality established by J. Aubrey in 2004.

§1. Introduction.

The main subject of this note is the following theorem.

THEOREM 1.1. *The statement*

(∗) *For every non-decreasing function $f : [0, 1] \to [0, 1]$, each subset of $[0, 1]$ of cardinality $\mathfrak{c}$ contains a set of cardinality $\mathfrak{c}$ on which* f *is uniformly continuous.*

is independent of ZFC.

Sierpinski [11, Théorème 6] proved that CH implies the negation of (∗). Sierpinski considered an increasing function $f : [0, 1] \to [0, 1]$ which is discontinuous precisely at the rationals in $(0, 1)$ (a well-known example of such a function is due to Lebesgue and is defined by letting $L(0) = 0$ and $L(x) = \sum_{\{n\in\mathbb{N}:\, q_n<x\}} 2^{-n}$ for $x \in (0, 1]$, where $\{q_n : n \in \mathbb{N}\}$ is an injective enumeration of $\mathbb{Q} \cap [0, 1)$). He proved that the (continuous) restriction of f to the set $\mathbb{P}$ of irrationals in $[0, 1]$ is not uniformly continuous on any Lusin set in $\mathbb{P}$.

The fact that in some models of ZFC statement (∗) can be true will be explained in Section 2, where a new cardinal coefficient $\mathfrak{d}^*$ which plays a crucial role in our considerations is introduced. A combinatorial analysis, giving a precise description of $\mathfrak{d}^*$ in terms of some other known cardinal characteristics of the continuum, will be presented in Section 3. Some additional comments will be gathered in Section 4.

§2. The statement (∗) is consistently true.

DEFINITION 2.1. Let Fin and cFin be the families of all finite and co-finite subsets of $\mathbb{N}$, respectively. Then $\mathfrak{d}^*$ is the minimal cardinality of a cover K of $\mathcal{P}(\mathbb{N})$ by compact subspaces[1] such that for each $\mathcal{K} \in \mathsf{K}$, either $\mathcal{K} \cap \mathsf{Fin} = \emptyset$ or $\mathcal{K} \cap \mathsf{cFin} = \emptyset$.

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[1] As usual, we identify $\mathcal{P}(\mathbb{N})$ with $2^{\mathbb{N}}$ using characteristic functions.

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Since both $\mathcal{P}(\mathbb{N}) \setminus \mathsf{Fin}$ and $\mathcal{P}(\mathbb{N}) \setminus \mathsf{cFin}$ are homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$ which can be covered by $\mathfrak{d}$ many compact subspaces, we have $\mathfrak{d}^* \leq \mathfrak{d}$ (here $\mathfrak{d}$ denotes, as usual, the *dominating number*, i.e., the smallest cardinality of a dominating family in $\mathbb{N}^{\mathbb{N}}$ corresponding to the ordering of eventual domination $\leq^*$, cf. [3]). It follows that it is consistent with ZFC to assume that $\mathfrak{d}^* < \mathfrak{c}$ and $\mathfrak{c}$ is regular, cf. [3], and we shall show that these conditions yield $(*)$.

Let us start from the following observation.

LEMMA 2.2. *Given two disjoint countable sets D_0, D_1 in a compact metrizable space* X, *there is a collection $\mathcal{K}$ of compact sets in* X *such that $|\mathcal{K}| \leq \mathfrak{d}^*$, $\bigcup \mathcal{K} = X$, and each element of $\mathcal{K}$ hits at most one D_i.*

PROOF. Using the Cantor–Bendixson theorem, we write X in the form $Y \cup Z$, $Y \cap Z = \emptyset$, such that Y is a compact space without isolated points and Z is countable. Now, to get $\mathcal{K}$, it suffices to find a relevant cover of Y and then to extend it by the singletons of Z. Instead, we simply assume that X has no isolated points.

There is a continuous surjection $u : \mathcal{P}(\mathbb{N}) \to X$ such that $u^{-1}(d)$ is a singleton, for $d \in D_0 \cup D_1$. Indeed, if we expand each $d \in D = D_0 \cup D_1$ to a copy P_d of the Cantor set in X, then [7, Section 45, II, Theorem 4] gives us a continuous surjection $u : \mathcal{P}(\mathbb{N}) \to X$ which is injective on $u^{-1}(\bigcup_{d \in D} P_d)$. In particular, since d is in P_d, the preimage $u^{-1}(d)$ is a singleton.

Now, the disjoint sets $u^{-1}(D_i)$, $i = 0, 1$, are countable, and by adopting standard arguments concerning countable dense homogeneity of $2^{\mathbb{N}}$ (cf. [9, Theorem 1.6.9]) one can find a homeomorphism $h : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that $u^{-1}(D_0) \subseteq h(\mathsf{Fin})$ and $u^{-1}(D_1) \subseteq h(\mathsf{cFin})$.

Finally, if K is a collection of compact sets in $\mathcal{P}(\mathbb{N})$ satisfying the conditions in the definition of $\mathfrak{d}^*$, with $|\mathsf{K}| = \mathfrak{d}^*$, the collection $\mathcal{K} = \{(u \circ h)(\mathcal{K}) : \mathcal{K} \in \mathsf{K}\}$ satisfies the assertion of the lemma. ⊣

Let us indicate that the possibility of $\mathfrak{d}^*$ being the minimal size of a covering $\mathcal{K}$ as above will be discussed in Subsection 4.3.

To complete the proof of the consistency of statement $(*)$ (hence also the proof of Theorem 1.1) it is enough to prove the following result.

PROPOSITION 2.3. *If $\mathfrak{d}^* < \mathrm{cf}(\mathfrak{c})$, then statement $(*)$ is true.*

PROOF. Let a function $f : [0, 1] \to [0, 1]$ be non-decreasing. Let D be the set of discontinuity points of f and let X be the closure in the square $[0, 1] \times [0, 1]$ of the graph of f restricted to the set $[0, 1] \setminus D$. For each $t \in [0, 1]$ let $X(t) = X \cap (\{t\} \times [0, 1])$.

Since f is non-decreasing, the set D of discontinuity points of f is countable and for each $d \in D$ we have $X(d) = \{(d, d_0), (d, d_1)\}$, where $d_0 < d_1$, while $X(t) = \{(t, f(t))\}$ for $t \in [0, 1] \setminus D$.

Let

$$Q_i = \{(d, d_i) : d \in D\}, \ i = 0, 1.$$

Since X is compact, Lemma 2.2 provides a covering $\mathcal{K}$ of X by compact sets such that $|\mathcal{K}| \leq \mathfrak{d}^*$ and each element of $\mathcal{K}$ intersects at most one of the sets $Q_0, \ Q_1$.

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Let $\pi : X \to [0,1]$ be the projection onto the first coordinate. Since each $K \in \mathcal{K}$ contains at most one point from every pair $X(d) = \{(d,d_0),(d,d_1)\}$, $d \in D \cap \pi(K)$, K is the graph of a continuous function $f_K : \pi(K) \to [0,1]$. Since K is compact, f_K is uniformly continuous. Moreover, f and f_K coincide on $\pi(K) \setminus D$, so f is uniformly continuous on $\pi(K) \setminus D$.

We are ready to address statement $(*)$. Let $E \subseteq [0,1]$ have cardinality $\mathfrak{c}$. Since the sets $\pi(K)$, $K \in \mathcal{K}$, cover $[0,1]$ and $|\mathcal{K}| \leq \mathfrak{d}^* < \mathrm{cf}(\mathfrak{c})$ (cf. Lemma 2.2), there is $K \in \mathcal{K}$ with $|\pi(K) \cap E| = \mathfrak{c}$, and since f is uniformly continuous on the set $(\pi(K) \cap E) \setminus D$, we get $(*)$. ⊣

§3. $\mathfrak{d}^* = \min\{\mathfrak{u},\mathfrak{d}\} = \min\{\mathfrak{r},\mathfrak{d}\}$. The aim of this section is to prove the equalities announced in its title. Let us recall that the *ultrafilter number* $\mathfrak{u}$ is the minimal size of a base of a free ultrafilter in $\mathcal{P}(\mathbb{N})$, and the *reaping number* $\mathfrak{r}$ is the minimal size of a *reaping subfamily* $\mathcal{B} \subseteq \mathcal{P}(\mathbb{N})$, that is, a family $\mathcal{B}$ of infinite subsets of $\mathbb{N}$ such that for every $X \subseteq \mathbb{N}$ there exists $B \in \mathcal{B}$ such that either $X \supset B$ or $B \cap X = \emptyset$, see [3] for more information on these cardinals.

Clearly, each ultrafilter base is reaping, hence $\mathfrak{r} \leq \mathfrak{u}$, and the strict inequality is consistent (cf. [6]). However, by [1, Corollary 6.3] we cannot have $\mathfrak{r} < \mathfrak{u}$ if $\mathfrak{r} < \mathfrak{d}$, which is clearly equivalent to $\min\{\mathfrak{u},\mathfrak{d}\} = \min\{\mathfrak{r},\mathfrak{d}\}$. In the course of our proof of Theorem 3.4 below we reestablish the latter equality, this way giving an alternative and more streamlined proof thereof, as well as giving a natural topological interpretation of $\min\{\mathfrak{u},\mathfrak{d}\}$.

Definition 3.1.

- A family $\mathcal{F}$ of infinite subsets of $\mathbb{N}$ is called a *semifilter*, if it is closed under taking almost-supersets of its elements, that is, if $F \in \mathcal{F}$ and $F \subseteq^* X \subseteq \mathbb{N}$, then $X \in \mathcal{F}$.
- A family $\mathcal{C}$ of infinite subsets of $\mathbb{N}$ is called *centered* if $\cap\mathcal{C}'$ is infinite for any finite subfamily $\mathcal{C}' \subseteq \mathcal{C}$.

For a semifilter $\mathcal{F}$ we denote by $\mathcal{F}^+$ the family $\{X \subseteq \mathbb{N} : \forall F \in \mathcal{F}\ (F \cap X \neq \emptyset)\}$. It is clear that $\mathcal{F}^+$ is also a semifilter.

For $\mathcal{X} \subseteq \mathcal{P}(\mathbb{N})$ we denote by $\sim\mathcal{X}$ the family $\{\mathbb{N} \setminus X : X \in \mathcal{X}\}$. It is easy to check that $\mathcal{F}^+ = \mathcal{P}(\mathbb{N}) \setminus \sim\mathcal{F}$. Note that if $\mathcal{F}_0 \subseteq \mathcal{F}_1$ are semifilters, then $\mathcal{F}_0^+ \supseteq \mathcal{F}_1^+$.

The following two facts are probably well-known, but we give their proofs for the sake of completeness. The first one is reminiscent of [3, Theorem 2.10].

Lemma 3.2. *Let $\kappa < \mathfrak{d}$ be a cardinal and*

$$\left\{\langle n_i^\alpha : i \in \mathbb{N}\rangle \ :\ \alpha < \kappa\right\}$$

be a family of increasing number sequences. Then there exists an increasing number sequence $\langle h(j) : j \in \mathbb{N}\rangle$ such that for every finite $A \subset \kappa$ the set

$$J_A = \{j \in \mathbb{N} \ :\ \forall \alpha \in A\, \exists i \in \mathbb{N}\left([n_i^\alpha, n_{i+1}^\alpha) \subseteq [h(j), h(j+1))\right)\}$$

is infinite.

Proof. It is enough to find h satisfying the statement of the lemma for singletons $A \subseteq \kappa$, that is, A of the form $\{\alpha\}$ for some $\alpha \in \kappa$, since without loss of generality

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we may assume that for every finite $A \subseteq \kappa$ there exists $\gamma \in \kappa$ such that

$$\forall j \in \mathbb{N}\, \forall \alpha \in A\, \exists i \in \mathbb{N}\, \big([n_i^\alpha, n_{i+1}^\alpha) \subseteq [n_j^\gamma, n_{j+1}^\gamma)\big).$$

For every $\alpha \in \kappa$ and $i \in \mathbb{N}$ set $f_\alpha(i) = n_{2i}^\alpha$ and find a strictly increasing $h \in \mathbb{N}^\mathbb{N}$ such that $h(0) = 0$ and $h \not\leq^* f_\alpha$ for all $\alpha \in \kappa$. We claim that $\langle h(j) : j \in \mathbb{N}\rangle$ is as required. Indeed, let us fix $i > 0$ such that $h(i) \geq f_\alpha(i) = n_{2i}^\alpha$. Thus,

$$\{n_k^\alpha : k \leq 2i\} \subseteq \bigcup_{l<i} [h(l), h(l+1)),$$

which gives that $\big|\big\{k < 2i : n_k^\alpha, n_{k+1}^\alpha \text{ belong to the same interval } [h(l), h(l+1))\big\}\big| \geq i$, and consequently $J_{\{\alpha\}}$ defined in the formulation must be infinite because i with $h(i) \geq f_\alpha(i) = n_{2i}^\alpha$ may be taken arbitrarily large. ⊣

COROLLARY 3.3. *Let* K *be a family of* $< \mathfrak{d}$*-many compact subspaces of* $\mathcal{P}(\mathbb{N}) \setminus \mathsf{Fin}$. *Then there exists a monotone surjection* $\phi : \mathbb{N} \to \mathbb{N}$ *and a centered family* $\mathcal{C}$ *of size* $|\mathcal{C}| \leq |\mathsf{K}|$ *such that* $\{\phi[X] : X \in \bigcup K\} \subseteq \mathcal{F}$, *where* $\mathcal{F}$ *is the smallest free filter generated by* $\mathcal{C}$.

PROOF. For every $\mathcal{K} \in \mathsf{K}$, $\mathcal{K}$ being compact, there is an increasing number sequence $\langle n_i^\mathcal{K} : i \in \mathbb{N}\rangle$ such that $X \cap [n_i^\mathcal{K}, n_{i+1}^\mathcal{K}) \neq \emptyset$ for any $X \in \mathcal{K}$ and $i \in \mathbb{N}$. Since $|\mathsf{K}| < \mathfrak{d}$, by Lemma 3.2 there exists a strictly increasing function $h : \mathbb{N} \to \mathbb{N}$ such that $h(0) = 0$ and for any finite $\mathsf{K}' \subseteq \mathsf{K}$ the set $C(\mathsf{K}')$ of those $j \in \mathbb{N}$ such that for every $\mathcal{K} \in \mathsf{K}'$ there exists $i \in \mathbb{N}$ with

$$[h(j), h(j+1)) \supseteq [n_i^\mathcal{K}, n_{i+1}^\mathcal{K}) \tag{3.1}$$

is infinite. Note that the family

$$\mathcal{C} = \{C(\mathsf{K}') : \mathsf{K}' \text{ is a finite subset of } \mathsf{K}\}$$

is centered because by the definition of $C(\mathsf{K}')$ we have

$$C(\mathsf{K}'_0 \cup \mathsf{K}'_1 \cup \cdots \cup \mathsf{K}'_n) = \bigcap_{i \leq n} C(\mathsf{K}'_i)$$

for any finite family $\{\mathsf{K}'_0, \mathsf{K}'_1, \ldots, \mathsf{K}'_n\}$ of finite subsets of K.

We claim that $\mathcal{C}$ and $\phi : \mathbb{N} \to \mathbb{N}$ such that $\phi^{-1}(j) = [h(j), h(j+1))$ for all $j \in \mathbb{N}$, are as required. Indeed, given $X \in \bigcup \mathsf{K}$, let $\mathcal{K} \in \mathsf{K}$ be such that $X \in \mathcal{K}$. Then

$$\begin{aligned}\phi[X] &= \{j \in \mathbb{N} : X \cap [h(j), h(j+1)) \neq \emptyset\} \supseteq \\ &\supseteq \big\{j \in \mathbb{N} : \exists i \in \mathbb{N}\, \big([h(j), h(j+1)) \supseteq [n_i^\mathcal{K}, n_{i+1}^\mathcal{K})\big)\big\} \supseteq C(\{\mathcal{K}\}) \in \mathcal{C},\end{aligned}$$

which completes the proof. ⊣

For every $X \subseteq \mathbb{N}$ we adopt the following notation:

$$\begin{aligned}&\downarrow X = \mathcal{P}(X),\ \uparrow X = \{Y \subseteq \mathbb{N} : X \subseteq Y\},\\ &\downarrow^* X = \{Y \subseteq \mathbb{N} : Y \subseteq^* X\},\ \&\ \uparrow^* X = \{Y \subseteq \mathbb{N} : X \subseteq^* Y\}.\end{aligned}$$

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For a subset $\mathcal{L} \subseteq \mathcal{P}(\mathbb{N})$ we set

$$\uparrow\mathcal{L} = \bigcup\{\uparrow X : X \in \mathcal{L}\},\ \&\ \downarrow\mathcal{L} = \bigcup\{\downarrow X : X \in \mathcal{L}\},$$
$$\uparrow^*\mathcal{L} = \bigcup\{\uparrow^* X : X \in \mathcal{L}\},\ \&\ \downarrow^*\mathcal{L} = \bigcup\{\downarrow^* X : X \in \mathcal{L}\}.$$

It is easy to see that if $\mathcal{L}$ is compact, then both $\uparrow\mathcal{L}$ and $\downarrow\mathcal{L}$ are compact, and $\uparrow^*\mathcal{L}$ and $\downarrow^*\mathcal{L}$ are σ-compact.

We are now ready to prove the main result of this section.

THEOREM 3.4. $\mathfrak{d}^* = \min\{\mathfrak{u}, \mathfrak{d}\} = \min\{\mathfrak{r}, \mathfrak{d}\}$.

PROOF. As we have already noticed, since both $\mathcal{P}(\mathbb{N}) \setminus \mathsf{Fin}$ and $\mathcal{P}(\mathbb{N}) \setminus \mathsf{cFin}$ are homeomorphic to the Baire space $\mathbb{N}^\mathbb{N}$ which can be covered by $\mathfrak{d}$ many compact subspaces, we have $\mathfrak{d}^* \leq \mathfrak{d}$.

Let $\mathcal{B}$ be a reaping family of cardinality $\mathfrak{r}$. For every $B \in \mathcal{B}$ set $\mathcal{K}(B)_d = \downarrow(\mathbb{N} \setminus B)$ and $\mathcal{K}(B)_u = \uparrow B$.

It follows that

$$\mathcal{P}(\mathbb{N}) = \bigcup\{\mathcal{K}(B)_d : B \in \mathcal{B}\} \cup \bigcup\{\mathcal{K}(B)_u : B \in \mathcal{B}\}$$

and $\mathcal{K}(B)_d \cap \mathsf{cFin} = \mathcal{K}(B)_u \cap \mathsf{Fin} = \emptyset$ for all $B \in \mathcal{B}$, hence $\mathfrak{d}^* \leq \mathfrak{r}$, which together with the first paragraph yields $\mathfrak{d}^* \leq \min\{\mathfrak{r}, \mathfrak{d}\}$.

To show that $\mathfrak{d}^* \geq \min\{\mathfrak{u}, \mathfrak{d}\}$, suppose, towards a contradiction, that there is a family $\mathsf{K} = \mathsf{K}_d \cup \mathsf{K}_u$ of compact subspaces of $\mathcal{P}(\mathbb{N})$ of size $\kappa = |\mathsf{K}| < \min\{\mathfrak{u}, \mathfrak{d}\}$ such that $\bigcup\mathsf{K} = \mathcal{P}(\mathbb{N})$ and

$$\mathcal{K}_0 \cap \mathsf{cFin} = \mathcal{K}_1 \cap \mathsf{Fin} = \emptyset \tag{3.2}$$

for any $\mathcal{K}_0 \in \mathsf{K}_d$ and $\mathcal{K}_1 \in \mathsf{K}_u$.

Replacing each $\mathcal{K} \in \mathsf{K}_d$ (resp. $\mathcal{K} \in \mathsf{K}_u$) with a countable family of compact subspaces of $\mathcal{P}(\mathbb{N})$ covering $\downarrow^*\mathcal{K}$ (resp. $\uparrow^*\mathcal{K}$), we can assume that $\mathcal{U}_u = \bigcup\mathsf{K}_u$ as well as $\mathcal{U}_d = \sim\bigcup\mathsf{K}_d$ are semifilters on $\mathbb{N}$. Set $\mathcal{U} = \mathcal{U}_u \cup \mathcal{U}_d$. It follows that

$$\mathcal{U}_d^+ = \mathcal{P}(\mathbb{N}) \setminus \sim\mathcal{U}_d = \mathcal{P}(\mathbb{N}) \setminus \sim(\sim\bigcup\mathsf{K}_d) = \mathcal{P}(\mathbb{N}) \setminus \bigcup\mathsf{K}_d \subseteq \bigcup\mathsf{K}_u = \mathcal{U}_u,$$

and hence

$$\mathcal{U}^+ \subseteq \mathcal{U}_d^+ \subseteq \mathcal{U}_u \subseteq \mathcal{U}. \tag{3.3}$$

By the construction, the semifilter $\mathcal{U}$ can be covered by κ many compact subspaces. Since $\kappa < \mathfrak{d}$, by Corollary 3.3, there exists a free filter $\mathcal{F}$ generated by at most κ many subsets of $\mathbb{N}$ and a monotone surjection $\phi : \mathbb{N} \to \mathbb{N}$ with $\mathcal{F} \supseteq \{\phi[X] : X \in \mathcal{U}\}$. Moreover, the family $\{\phi[X] : X \in \mathcal{U}\}$ is a semifilter and one easily checks that

$$\mathcal{F}^+ \subseteq \{\phi[X] : X \in \mathcal{U}\}^+ = \{\phi[X] : X \in \mathcal{U}^+\} \subseteq \{\phi[X] : X \in \mathcal{U}\} \subseteq \mathcal{F},$$

the second inclusion being a consequence of (3.3). This shows that $\mathcal{F}$ is an ultrafilter generated by at most κ many sets, which is impossible because $\kappa < \mathfrak{u}$.

It follows from the above that

$$\min\{\mathfrak{u}, \mathfrak{d}\} \leq \mathfrak{d}^* \leq \min\{\mathfrak{r}, \mathfrak{d}\},$$

and therefore these two inequalities must actually be equalities because $\mathfrak{r} \leq \mathfrak{u}$. ⊣

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§4. Comments.

4.1. Non-decreasing functions versus arbitrary functions. In [10, Proposition 3.5] the following result is obtained.

Let $g : \mathbb{P} \to [0,1]$ be a continuous function such that the closure of the graph of g hits each section $\{q\} \times [0,1]$, $q \in \mathbb{Q} \cap [0,1]$, in an uncountable set. Then $\mathbb{P}$ cannot be covered by less than $\mathfrak{d}$ sets on which g is uniformly continuous. In particular, if $\mathfrak{d} = \mathfrak{c}$, there is a subset E of $\mathbb{P}$ of cardinality $\mathfrak{c}$ such that g is not uniformly continuous on any subset of E of cardinality $\mathfrak{c}$ (cf. the proof of [10, Theorem 3.4]).

In contrast, Proposition 2.3 yields the following.

REMARK 4.1. Let $h : E \to [0,1]$ be a monotone function on a set $E \subseteq [0,1]$ of cardinality $\mathfrak{c}$. If $\mathfrak{d}^* < \mathfrak{c}$ and $\mathfrak{c}$ is regular, then h is uniformly continuous on a set of cardinality $\mathfrak{c}$.

Indeed, if $h : E \to [0,1]$ is, say, non-decreasing, then we can extend it to $f : [0,1] \to [0,1]$ which is also non-decreasing (by setting $f(y) = \sup h[E \cap [0,y]]$ for all $y \in [0,1]$, where we set $\sup(\emptyset) = 0$) and then we can apply Proposition 2.3.

Remark 4.1 applied to the continuous function g and the set E described in the opening statement of this subsection, leads to the following.

REMARK 4.2. If $\mathfrak{d}^* < \mathfrak{d} = \mathfrak{c}$ and $\mathfrak{c}$ is regular, then there exists a continuous function $f : E \to [0,1]$ on a set $E \subseteq \mathbb{R}$ of cardinality $\mathfrak{c}$ such that no restriction of f to a subset of E of cardinality $\mathfrak{c}$ is monotone.

4.2. Models of ZFC with $\mathfrak{d}^* < \mathfrak{d}$. In view of the remarks from the preceding subsection and for the sake of completeness, we would like to mention that an example of a model of ZFC with $\mathfrak{d}^* = \mathfrak{u} = \aleph_1 < \aleph_2 = \mathfrak{d}$ (cf. Theorem 3.4) is provided by the Miller model resulting from the $\aleph_2$-length countable support iteration of Miller forcing (see [2, Model 7.5.2][2]). Another model which has the advantage that $\mathfrak{d}^* = \mathfrak{u} < \mathfrak{d}$ can be any prescribed uncountable regular cardinals is presented by Blass and Shelah in [4].

4.3. A generalization of $\mathfrak{d}^*$. In view of Lemma 2.2 it is natural to generalize $\mathfrak{d}^*$ as follows.

DEFINITION 4.3. For a compact metrizable space X and countable disjoint sets D_0, D_1 in X let $\mathfrak{d}^*(X, D_0, D_1)$ be the minimal cardinality of a cover K of X by compact subspaces such that for each $\mathcal{K} \in \mathsf{K}$, either $\mathcal{K} \cap D_0 = \emptyset$ or $\mathcal{K} \cap D_1 = \emptyset$.

In this notation we have $\mathfrak{d}^* = \mathfrak{d}^*(\mathcal{P}(\mathbb{N}), \mathsf{Fin}, \mathsf{cFin})$ and Lemma 2.2 states that $\mathfrak{d}^*(X, D_0, D_1) \leq \mathfrak{d}^*$ for any two disjoint countable sets D_0, D_1 in an arbitrary compact metrizable space X. We complement this lemma with the following observation (whose proof gives also another method for establishing Lemma 2.2).

[2] As was pointed out by the referee, the inequality $\aleph_1 = \mathfrak{u} < \mathfrak{d} = \aleph_2 = \mathfrak{c}$ in the forcing extension requires only CH in the ground model. Moreover, the model obtained after iterating the Miller forcing with countable supports $\aleph_1$ many steps over an arbitrary ground model satisfies CH, so treating this intermediate model as our new ground model of CH shows that actually we do not even have to demand CH in V in order to obtain $\aleph_1 = \mathfrak{u} < \mathfrak{d} = \aleph_2 = \mathfrak{c}$ after $\aleph_2$-step iteration.

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PROPOSITION 4.4. *Let* X *be a compact metrizable space with no isolated points and let* $D_0,\ D_1$ *be two disjoint countable dense subspaces of* X. *Then* $\mathfrak{d}^*(X, D_0, D_1) = \mathfrak{d}^*$ (*equivalently,* $\mathfrak{d}^*$ *is the smallest cardinality* κ *such that* $D_0,\ D_1$ *can be separated by two sets each of which is the intersection of a collection of at most* κ*-many open sets in* X).[3]

In the proof we shall need the following well-known fact which can be proved by the standard back-and-forth argument.

LEMMA 4.5. *Let* $\mathbb{Q}$ *be the set of the rational numbers and* $\mathbb{Q} = Q_0^0 \cup Q_1^0$, $\mathbb{Q} = Q_0^1 \cup Q_1^1$ *be two decompositions of* $\mathbb{Q}$ *into disjoint dense subspaces. Then there exists an automorphism of* $\langle \mathbb{Q}, \leq \rangle$ (*hence a homeomorphism*) $h : \mathbb{Q} \to \mathbb{Q}$ *such that* $h[Q_0^0] = Q_0^1$ *and* $h[Q_1^0] = Q_1^1$.

PROOF OF PROPOSITION 4.4. As X has no isolated points and $D_0,\ D_1$ are dense in X, both $A = \mathsf{Fin} \cup \mathsf{cFin} \subseteq \mathcal{P}(\mathbb{N})$ and $B = D_0 \cup D_1$ are homeomorphic to the rationals $\mathbb{Q}$ being countable metrizable spaces without isolated points. Let $h : A \to B$ be a homeomorphism such that $h[\mathsf{Fin}] = D_0$ and $h[\mathsf{cFin}] = D_1$ (its existence follows from Lemma 4.5) and let $\beta h : \beta A \to \beta B$ be its homeomorphic extension to the Čech–Stone compactification βA of A.

By [5, Theorem 3.5.7] we have $\beta h[\beta A \setminus A] = \beta B \setminus B$ and analogously, the extensions $f_A : \beta A \to \mathcal{P}(\mathbb{N})$ and $f_B : \beta B \to X$ of the identity maps $i_A : A \to \mathcal{P}(\mathbb{N})$ and $i_B : B \to X$ have the property that $f_A[\beta A \setminus A] = \mathcal{P}(\mathbb{N}) \setminus A$ and $f_B[\beta B \setminus B] = X \setminus B$.

It follows that if K is a family of compact subspaces of $\mathcal{P}(\mathbb{N})$ such as in the definition of $\mathfrak{d}^*$, then

$$\{f_B[\beta h[f_A^{-1}[\mathcal{K}]]] : \mathcal{K} \in \mathsf{K}\}$$

is such as in the definition of $\mathfrak{d}^*(X, D_0, D_1)$; and vice versa, if K is a family of compact subspaces of X such as in the definition of $\mathfrak{d}^*(X, D_0, D_1)$, then

$$\{f_A[\beta h^{-1}[f_B^{-1}[\mathcal{K}]]] : \mathcal{K} \in \mathsf{K}\}$$

is such as in the definition of $\mathfrak{d}^*$. Thus, $\mathfrak{d}^*(X, D_0, D_1) = \mathfrak{d}^*$, which completes the proof. ⊣

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[3] As the referee pointed out, this conclusion can be no longer valid if we drop either the assumption that X has no isolated points (consider a compact metrizable space X containing a discrete (open) dense subset D_0 such that $X \setminus D_0$ has no isolated points (see, e.g., [8]) and a countable dense subset D_1 of $X \setminus D_0$) or the assumption of both D_0 and D_1 being dense subsets of X (consider a countable compact set D_0 in $X = [0, 1]$ and a countable dense subset D_1 of $X \setminus D_0$). In both cases we have $\mathfrak{d}^*(X, D_0, D_1) = \aleph_0$.

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