Pascal-points quadrilaterals inscribed in a cyclic quadrilateral

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1. *The Pascal points of a quadrilateral associated with a circle*

This paper presents some new theorems about the Pascal points of a quadrilateral. We shall begin by explaining what these are.

Let $ABCD$ be a convex quadrilateral, with AC and BD intersecting at E and DA and CB intersecting at F. Let ω be a circle through E and F which meets CB internally at M and DA internally at N. Let CA meet ω again at L and let DB meet ω again at K. By using Pascal's theorem for the crossed hexagons $EKNFML$ and $EKMFNL$ which are circumscribed by ω, the following results can be proved [1]:–

(a) NK, ML and AB are concurrent (at a point P internal to AB)

(b) NL, KM and CD are concurrent (at a point Q internal to CD)

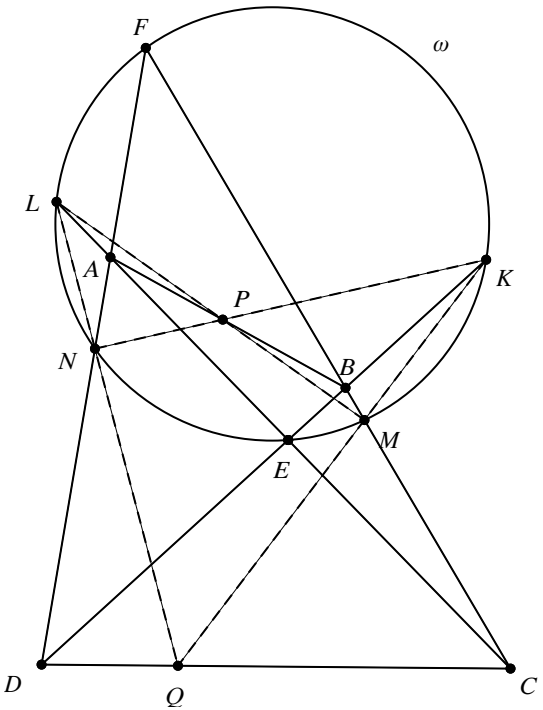


FIGURE 1

This configuration is shown in Figure 1. We call P and Q the *Pascal points* of the quadrilateral associated with circle ω. It will also be convenient to call $PMQN$ the *Pascal-points quadrilateral* associated with circle ω.

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Different circles which satisfy the conditions will produce different Pascal-points quadrilaterals, and we now turn attention to some of these.

Let ω_A be the circumcircle of triangle AEF, with centre O_A and Pascal points A and Q_A, and let ω_B be the circumcircle of triangle BEF, with centre O_B and Pascal points B and Q_B. Let ω_1 and ω_2 be circles through E and F (which satisfy the intersection conditions) with centres O_1 and O_2 on O_AO_B and let $P_1M_1Q_1N_1$ and $P_2M_2Q_2N_2$ be Pascal-points quadrilaterals defined by ω_1 and ω_2.

This configuration is shown in Figure 2. Requiring that ω_1 and ω_2 intersect sides BC and AD at interior points ensures that quadrilaterals $P_1M_1Q_1N_1$ and $P_2M_2Q_2N_2$ are inscribed in quadrilateral $ABCD$.

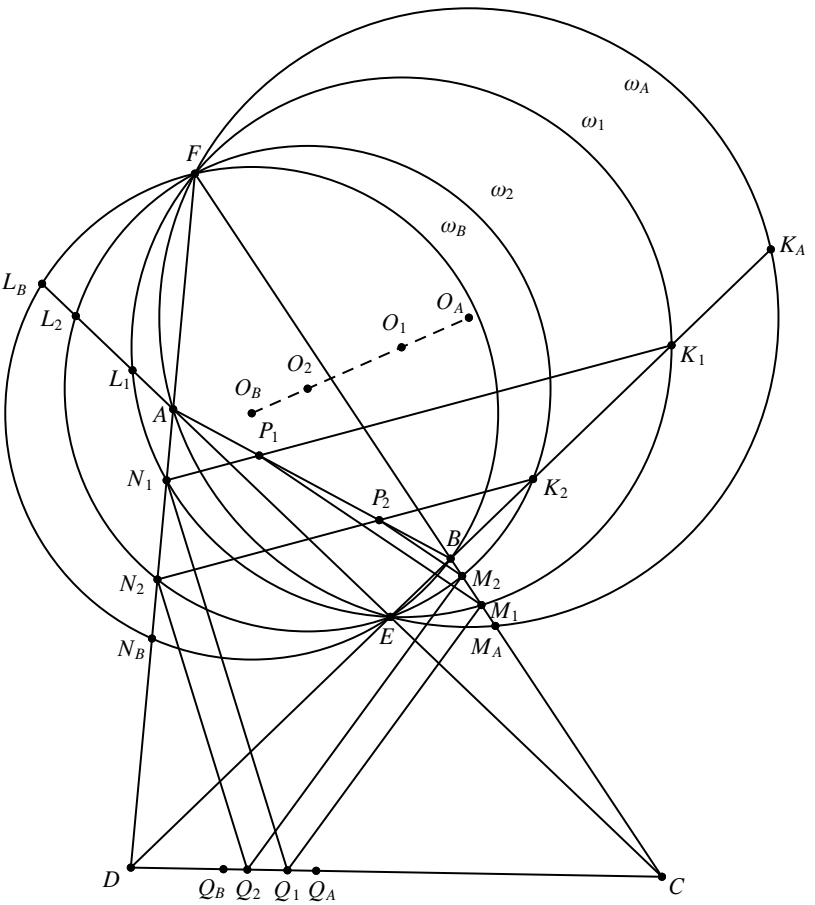


FIGURE 2

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In what follows we will quote some properties proved in [1].

Property 1: $\frac{O_AO_1}{O_1O_2} = \frac{AP_1}{P_1P_2} = \frac{Q_AQ_1}{Q_1Q_2}$ and $\frac{O_BO_2}{O_1O_2} = \frac{BP_2}{P_1P_2} = \frac{Q_BQ_2}{Q_1Q_2}$.

Property 2: The corresponding sides of Pascal-points quadrilaterals $P_1M_1Q_1N_1$ and $P_2M_2Q_2N_2$ are parallel.

Now we consider what happens when $ABCD$ is cyclic and state some further properties. Figure 3 shows an additional circle ω on diameter EF with centre O and Pascal points P and Q.

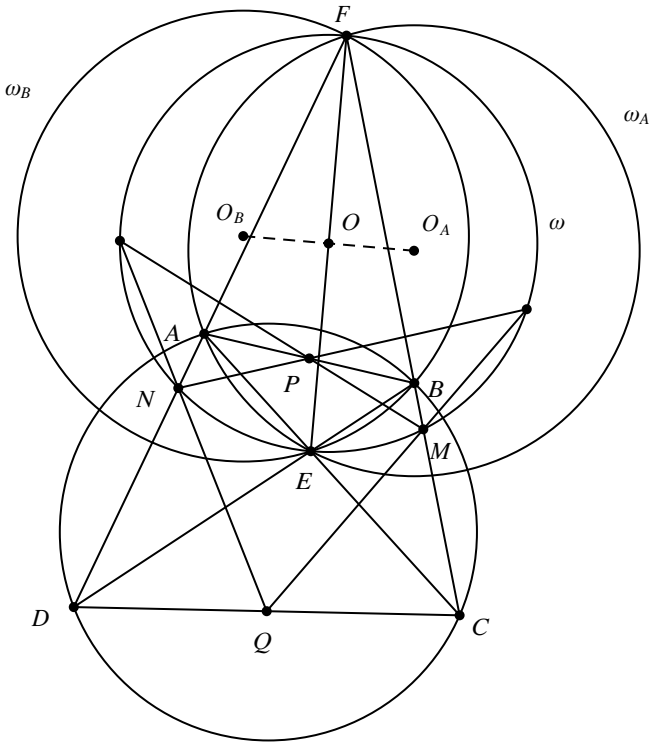


FIGURE 3

Property 3: Points P, Q, O are the midpoints of AB, CD, O_AO_B respectively.

Property 4: The Pascal-points quadrilateral $PMQN$ is a kite.

Property 5: The Pascal-points quadrilaterals $P_1M_1Q_1N_1$ and $P_2M_2Q_2N_2$ have equal perimeters.

Properties 3, 4 and 5 are proved in [2].

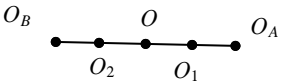
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2. *New theorems*

We now prove two new theorems in the case when the quadrilateral *ABCD* is cyclic.

The first concerns the circles ω_1 and ω_2 from Figure 2 and the circle ω from Figure 3. We make the additional assumption that $O_1O = OO_2$. For clarity, the circles and the diameter *EF* have been omitted in Figure 4.

Theorem 1: The Pascal-points quadrilaterals $P_1M_1Q_1N_1$ and $P_2M_2Q_2N_2$ are congruent.



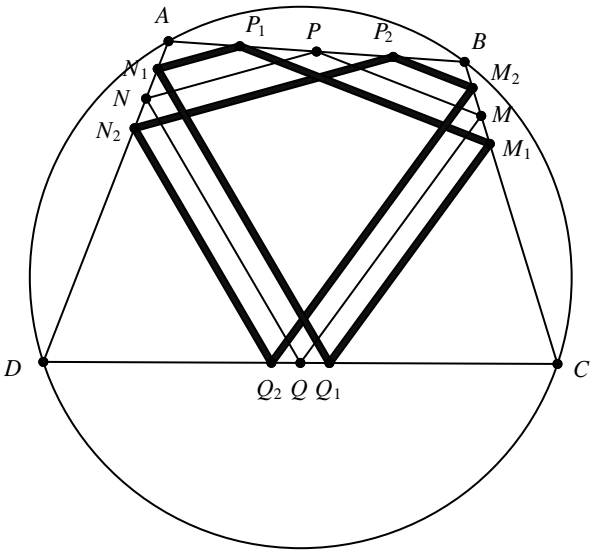


FIGURE 4

Proof: From Property 1, we have

$$\frac{AP_1}{P_1P} = \frac{O_AO_1}{O_1O} \text{ and } \frac{BP_2}{P_2P} = \frac{O_BO_2}{O_2O}$$

and so

$$\frac{AP_1}{AP} = \frac{O_AO_1}{O_AO} \text{ and } \frac{BP_2}{BP} = \frac{O_BO_2}{O_BO}. \quad (1)$$

From the midpoints in Property 3 and (1), it follows that $O_AO_1 = O_BO_2$ and

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so $\frac{AP_1}{AP} = \frac{BP_2}{BP}$. From Property 2, the corresponding sides of the three quadrilaterals are parallel, and, specifically, P_1N_1 is parallel to PN and P_2M_2 is parallel to PM. Hence we obtain

$$\frac{AP_1}{AP} = \frac{P_1N_1}{PN} \text{ and } \frac{BP_2}{BP} = \frac{P_2M_2}{PM}$$

and therefore

$$\frac{P_1N_1}{PN} = \frac{P_2M_2}{PM}. \tag{2}$$

From Property 4, quadrilateral $PMQN$ is a kite with $PM = PN$ and $QM = QN$. Therefore, from (2), $P_1N_1 = P_2M_2$.

By similar arguments, we obtain $P_1M_1 = P_2N_2$, $Q_1M_1 = Q_2N_2$ and $Q_1N_1 = Q_2M_2$.

In the three quadrilaterals $P_1M_1Q_1N_1$, $PMQN$ and $P_2M_2Q_2N_2$, the corresponding sides are parallel and so the corresponding angles at the vertices are also equal. Since quadrilateral $PMQN$ is a kite, the opposite angles $\angle M$ and $\angle N$ are equal. Hence, in the quadrilaterals $P_1M_1Q_1N_1$ and $P_2M_2Q_2N_2$ we have $\angle M_1 = \angle M = \angle N = \angle N_2$ and similarly $\angle N_1 = \angle M_2$.

Since corresponding sides and angles are equal, quadrilaterals $P_1M_1Q_1N_1$ and $P_2M_2Q_2N_2$ are congruent.

The second theorem concerns the Pascal points from Figure 2 for a cyclic quadrilateral.

Theorem 2: The triangles AM_AQ_A and BN_BQ_B are congruent.

Proof: We choose a sequence of points $\{O_i\}$ on the segment O_AO_B. O_1 is any point on the segment and the other points are defined recursively by the rule $|O_AO_{i+1}| = \frac{1}{2}|O_AO_i|$. Clearly the sequence $\{O_i\}$ converges to O_A. Therefore the sequence of circles through E with centres O_i converges to circle ω_A. Each circle in the sequence defines four points M_i, N_i, P_i and Q_i as defined in Figure 2. It is clear that $\{M_i\}$ converges to M_A and $\{N_i\}$ and $\{P_i\}$ converge to A. Hence the sequence of quadrilaterals $\{P_iM_iQ_iN_i\}$ converges to the triangle AM_AQ_A. Now let each point in the sequence $\{O_i'\}$ be the reflection of O_i in O, the midpoint of O_AO_B. It is clear that the sequence of points $\{O_i'\}$ converges to O_B and produces a sequence of quadrilaterals $\{P_i'M_i'Q_i'N_i'\}$ converging to the triangle BN_BQ_B.

From Theorem 1, the Pascal-points quadrilaterals $P_iM_iQ_iN_i$ and $P_i'M_i'Q_i'N_i'$ are congruent, and so, taking the limit, the triangles AM_AQ_A and BN_BQ_B are also congruent.

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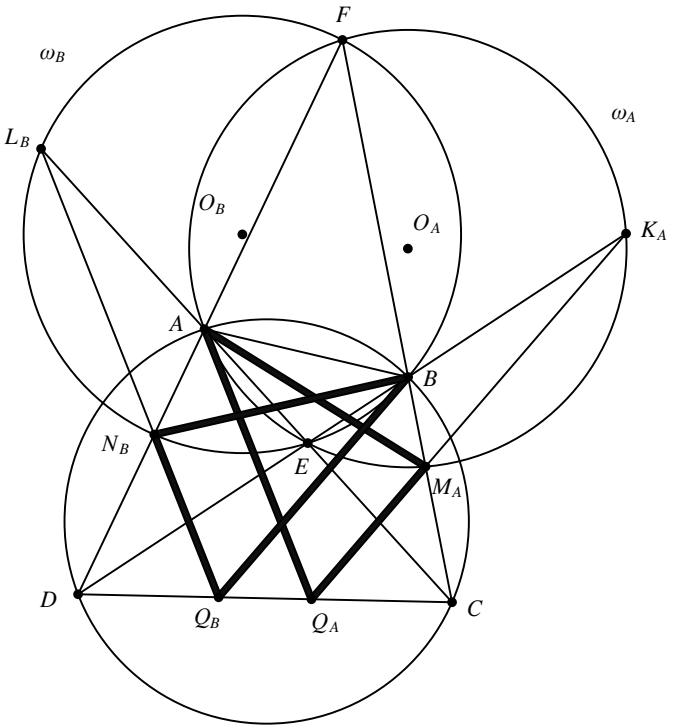


FIGURE 5

3. *Construction problems that can be solved using Theorems* 1 *and* 2 *and Properties* 4 *and* 5

A cyclic quadrilateral *ABCD* is given in which the opposite sides are not parallel and $\angle AEB$ between the diagonals is acute.

The condition on $\angle AEB$ ensures that circles exist that pass through points *E* and *F* and meet the conditions regarding their intersections with *BC* and *AD*. This condition therefore guarantees that Pascal-points quadrilaterals inscribed in quadrilateral *ABCD* do indeed exist.

Problem 1: Construct a kite and a quadrilateral with equal perimeters inscribed in *ABCD*.

Solution: Let circles ω and ω_1 be defined as in Figure 4 with Pascal-points quadrilaterals *PMQN* and $P_1M_1Q_1N_1$. From Property 4, *PMQN* is a kite, and by Property 5, the quadrilaterals have equal perimeters.

Problem 2: Construct two congruent quadrilaterals inscribed in *ABCD* such that each side of the first quadrilateral is parallel to one of the sides of the second.

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Solution: Let O_1, O_2, ω_1 and ω_2 be defined as in Theorem 1. It follows from Property 2 that the Pascal-points quadrilaterals of ω_1 and ω_2 conform to the property of parallel sides and from Theorem 1 that they are congruent.

Problem 3: Construct two congruent triangles inscribed in *ABCD* which satisfy the following requirement: for each triangle, one vertex coincides with a vertex of *ABCD* and the other two vertices lie on the two sides which do not contain this vertex.

Solution: Let O_A, O_B, ω_A and ω_B be defined as in Theorem 2. The Pascal-points quadrilaterals of ω_A and ω_B have degenerated to triangles with the desired properties.

Comment: These three problems are not necessarily restricted to cyclic quadrilaterals. If formulated for general quadrilaterals, each has a solution giving an infinite number of pairs which satisfy the requirements of the problem. However, in view of the purpose of this paper – to give applications of Pascal points – we have presented solutions for cyclic quadrilaterals which use these applications. The constructions we present are, in some cases, simpler than those for a general quadrilateral.

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