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ARTICLE

On higher dimensional point sets in general position

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Abstract

A finite point set in $\mathbb{R}^d$ is in general position if no $d+1$ points lie on a common hyperplane. Let $\alpha_d(N)$ be the largest integer such that any set of N points in $\mathbb{R}^d$, with no $d+2$ members on a common hyperplane, contains a subset of size $\alpha_d(N)$ in general position. Using the method of hypergraph containers, Balogh and Solymosi showed that $\alpha_2(N) < N^{5/6+o(1)}$. In this paper, we also use the container method to obtain new upper bounds for $\alpha_d(N)$ when $d \geq 3$. More precisely, we show that if d is odd, then $\alpha_d(N) < N^{\frac{1}{2}+\frac{1}{2d}+o(1)}$, and if d is even, we have $\alpha_d(N) < N^{\frac{1}{2}+\frac{1}{d-1}+o(1)}$. We also study the classical problem of determining $a(d,k,n)$, the maximum number of points selected from the grid $[n]^d$ such that no $k+2$ members lie on a k-flat, and improve the previously best known bound for $a(d,k,n)$, due to Lefmann in 2008, by a polynomial factor when $k = 2$ or $3 \pmod 4$.

Keywords: General position sets; hypergraph container method; generalised Sidon sets

2020 MSC Codes: 52C35, 52C10

1. Introduction

A finite point set in $\mathbb{R}^d$ is said to be in *general position* if no $d+1$ members lie on a common hyperplane. Let $\alpha_d(N)$ be the largest integer such that any set of N points in $\mathbb{R}^d$, with no $d+2$ members on a hyperplane, contains $\alpha_d(N)$ points in general position.

In 1986, Erdős [9] proposed the problem of determining $\alpha_2(N)$ and observed that a simple greedy algorithm shows $\alpha_2(N) \geq \Omega(\sqrt{N})$. A few years later, Füredi [11] showed that

$$\Omega(\sqrt{N \log N}) < \alpha_2(N) < o(N),$$

where the lower bound uses a result of Phelps and Rödl [22] on partial Steiner systems, and the upper bound relies on the density Hales–Jewett theorem [12, 13]. In 2018, a breakthrough was made by Balogh and Solymosi [3], who showed that $\alpha_2(N) < N^{5/6+o(1)}$. Their proof was based on the method of hypergraph containers, a powerful technique introduced independently by Balogh, Morris, and Samotij [1] and by Saxton and Thomason [26], that reveals an underlying structure of the independent sets in a hypergraph. We refer interested readers to [2] for a survey of results based on this method.

In higher dimensions, the best lower bound for $\alpha_d(N)$ is due to Cardinal, Tóth, and Wood [5], who showed that $\alpha_d(N) \geq \Omega((N \log N)^{1/d})$, for every fixed $d \geq 2$. For upper bounds, Milićević [19] used the density Hales–Jewett theorem to show that $\alpha_d(N) = o(N)$ for every fixed $d \geq 2$. However, these upper bounds in [19], just like those in [11], are still almost linear in N. Our main result is the following.

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Theorem 1.1. *Let $d \geq 3$ be a fixed integer. If d is odd, then $\alpha_d(N) < N^{\frac{1}{2}+\frac{1}{2d}+o(1)}$. If d is even, then $\alpha_d(N) < N^{\frac{1}{2}+\frac{1}{d-1}+o(1)}$.*

Our proof of Theorem 1.1 is also based on the hypergraph container method. A key ingredient in the proof is a new supersaturation lemma for $(k+2)$-tuples of the grid $[n]^d$ that lie on a k-flat, which we shall discuss in the next section. Here, by a *k-flat* we mean a k-dimensional affine subspace of $\mathbb{R}^d$.

One can consider a generalisation of the quantity $\alpha_d(N)$. We let $\alpha_{d,s}(N)$ be the largest integer such that any set of N points in $\mathbb{R}^d$, with no $d+s$ members on a hyperplane, contains $\alpha_{d,s}(N)$ points in general position. Hence, $\alpha_d(N) = \alpha_{d,2}(N)$. A simple argument of Erdős [9] shows that $\alpha_{d,s}(N) \geq \Omega(N^{1/d})$ for fixed d and s (see Section 6, or [5] for large s). In the other direction, following the arguments in our proof of Theorem 1.1 with a slight modification, we show the following.

Theorem 1.2. *Let $d, s \geq 3$ be fixed integers. If d is odd and $ds + 2 > 2d + 2s$, then $\alpha_{d,s}(N) \leq N^{\frac{1}{2}+o(1)}$. If d is even and $ds + 2 > 2d + 3s$, then $\alpha_{d,s}(N) \leq N^{\frac{1}{2}+o(1)}$.*

For example, when we fix $d = 3$ and $s \geq 5$, we have $\alpha_{d,s}(N) \leq N^{\frac{1}{2}+o(1)}$.

We also study the classical problem of determining the maximum number of points selected from the grid $[n]^d$ such that no $k+2$ members lie on a k-flat. The key ingredient of Theorem 1.1 mentioned above can be seen as a supersaturation version of this Turán-type problem. When $k = 1$, this is the famous *no-three-in-line problem* raised by Dudeney [7] in 1917: Is it true that one can select $2n$ points in $[n]^2$ such that no three are collinear? Clearly, $2n$ is an upper bound as any vertical line must contain at most 2 points. For small values of n, many authors have published solutions to this problem obtaining the bound of $2n$ (e.g. see [10]), but for large n, the best known general construction is due to Hall–Jackson–Sudbery–Wild [14] with slightly fewer than $3n/2$ points.

More generally, we let $a(d, k, r, n)$ denote the maximum number of points from $[n]^d$ such that no r points lie on a k-flat. Since $[n]^d$ can be covered by n^{d-k} many k-flats, we have the trivial upper bound $a(d, k, r, n) \leq (r-1)n^{d-k}$. For certain values d, k, and r fixed and n tends to infinity, this bound is known to be asymptotically best possible: Many authors [4, 18, 24] noticed that $a(d, d-1, d+1, n) = \Theta(n)$ by looking at the modular moment curve over a finite field $\mathbb{Z}_p$; In [23], Pór and Wood proved that $a(3, 1, 3, n) = \Theta(n^2)$; Dvir and Lovett [8] showed that $a(d, k, r, n) = \Theta(n^{d-k})$ when $r > d^k$ (see also [27]).

We shall focus on the case when $r = k+2$ and write $a(d, k, n) := a(d, k, k+2, n)$. Surprisingly, Lefmann [18] (see also [17]) showed that $a(d, k, n)$ behaves much differently than $\Theta(n^{d-k})$. In particular, he showed that

$$a(d, k, n) \leq O\left(n^{\frac{d}{\lfloor (k+2)/2 \rfloor}}\right).$$

Our next result improves this upper bound when $k+2$ is congruent to 0 or 1 mod 4.

Theorem 1.3. *For fixed d and k, as $n \to \infty$, we have*

$$a(d, k, n) \leq O\left(n^{\frac{d}{2\lfloor (k+2)/4 \rfloor}(1-\frac{1}{2\lfloor (k+2)/4 \rfloor d+1})}\right).$$

For example, we have $a(4, 2, n) \leq O(n^{\frac{16}{9}})$ while Lefmann's bound in [18] gives us $a(4, 2, n) \leq O(n^2)$, which coincides with the trivial upper bound. In particular, Theorem 1.3 tells us that, if 4 divides $k+2$, then $a(d, k, n)$ only behaves like $\Theta(n^{d-k})$ when $d = k+1$. This is quite interesting compared to the fact that $a(3, 1, n) = \Theta(n^2)$ proved in [23]. Lastly, let us note that the

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current best lower bound for $a(d,k,n)$ is also due to Lefmann [18], who showed that $a(d,k,n) \geq \Omega\left(n^{\frac{d}{k+1}-k-\frac{k}{k+1}}\right)$.

For integer $n>0$, we let $[n]=\{1,\dots,n\}$, and $\mathbb{Z}_n=\{0,1,\dots,n-1\}$. We systemically omit floors and ceilings whenever they are not crucial for the sake of clarity in our presentation. All exponentials and logarithms are in base two.

2. Supersaturation of non-degenerate coplanar tuples

In this section, we establish some lemmas for the proofs of Theorems 1.1 and 1.2.

Given a set T of $k+2$ points in $\mathbb{R}^d$ that lie on a k-flat, we say that T is *degenerate* if there is a subset $S\subset T$ of size j, where $3\leq j\leq k+1$, such that S lies on a $(j-2)$-flat. Otherwise, we say that T is *non-degenerate*. We establish a supersaturation lemma for non-degenerate $(k+2)$-tuples of $[n]^d$.

Lemma 2.1. *For real number $\delta>0$ and fixed positive integers d,k, such that k is even and $d-2\delta>(k-1)(k+2)$, any subset $V\subset[n]^d$ of size $n^{d-\delta}$ spans at least $\Omega(n^{(k+1)d-(k+2)\delta})$ non-degenerate $(k+2)$-tuples that lie on a k-flat.*

Proof. Let $V\subset[n]^d$ such that $|V|=n^{d-\delta}$. Set $r=\frac{k}{2}+1$ and $E_r=\binom{V}{r}$ to be the collection of r-tuples of V. Notice that the sum of an r-tuple from V belongs to $[rn]^d$. For each $v\in[rn]^d$, we define

$$E_r(v)=\{\{v_1,\dots,v_r\}\in E_r : v_1+\ldots+v_r=v\}.$$

Then for $T_1,T_2\in E_r(v)$, where $T_1=\{v_1,\dots,v_r\}$ and $T_2=\{u_1,\dots,u_r\}$, we have

$$v_1+\cdots+v_r=v=u_1+\cdots+u_r,$$

which implies that $T_1\cup T_2$ lies on a common k-flat. Let

$$E_{2r}=\bigcup_{v\in[rn]^d}\ \bigcup_{T_1,T_2\in E_r(v)}\{T_1,T_2\}.$$

Hence, for each $\{T_1,T_2\}\in E_{2r}$, $T_1\cup T_2$ lies on a k-flat. Moreover, by Jensen's inequality, we have

$$|E_{2r}|=\sum_{v\in[rn]^d}\binom{|E_r(v)|}{2}\geq (rn)^d\binom{\frac{\sum_v|E_r(v)|}{(rn)^d}}{2}=(rn)^d\binom{|E_r|/(rn)^d}{2}\geq\frac{|E_r|^2}{4(rn)^d}.$$

Since k and d are fixed and $r=\frac{k}{2}+1$ and $|V|=n^{d-\delta}$,

$$|E_r|^2=\binom{|V|}{r}^2=\binom{|V|}{(k/2)+1}^2\geq\Omega(n^{(k+2)(d-\delta)}).$$

Combining the two inequalities above gives

$$|E_{2r}|\geq\Omega(n^{(k+1)d-(k+2)\delta}).$$

We say that $\{T_1,T_2\}\in E_{2r}$ is *good* if $T_1\cap T_2=\emptyset$, and the $(k+2)$-tuple $(T_1\cup T_2)$ is non-degenerate. Otherwise, we say that $\{T_1,T_2\}$ is *bad*. In what follows, we will show that at least half of the pairs (i.e. elements) in E_{2r} are good. To this end, we will need the following claim. □

Claim 2.2. *If $\{T_1,T_2\}\in E_{2r}$ is bad, then $T_1\cup T_2$ lies on a $(k-1)$-flat.*

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Proof of Claim. Write $T_1 = \{v_1, \ldots, v_r\}$ and $T_2 = \{u_1, \ldots, u_r\}$. Let us consider the following cases.

Case 1. Suppose $T_1 \cap T_2 \neq \emptyset$. Then, without loss of generality, there is an integer $j < r$ such that

$$v_1 + \cdots + v_j = u_1 + \cdots + u_j,$$

where $v_1, \ldots, v_j, u_1, \ldots, u_j$ are all distinct elements, and $v_t = u_t$ for $t > j$. Thus, $|T_1 \cup T_2| = 2j + (r - j)$. The $2j$ elements above lie on a $(2j - 2)$-flat. Adding the remaining $r - j$ points implies that $T_1 \cup T_2$ lies on a $(j - 2 + r)$-flat. Since $r = \frac{k}{2} + 1$ and $j \leq \frac{k}{2}$, $T_1 \cup T_2$ lies on a $(k - 1)$-flat.

Case 2. Suppose $T_1 \cap T_2 = \emptyset$. Then $T_1 \cup T_2$ must be degenerate, which means there is a subset $S \subset T_1 \cup T_2$ of j elements such that S lies on a $(j - 2)$-flat, for some $3 \leq j \leq k + 1$. Without loss of generality, we can assume that $v_1 \notin S$. Hence, $(T_1 \cup T_2) \setminus \{v_1\}$ lies on a $(k - 1)$-flat. On the other hand, we have

$$v_1 = u_1 + \cdots + u_r - v_2 - \cdots - v_r.$$

Hence, v_1 is in the affine hull of $(T_1 \cup T_2) \setminus \{v_1\}$ which implies that $T_1 \cup T_2$ lies on a $(k - 1)$-flat. □

We are now ready to prove the following claim.

Claim 2.3. *At least half of the pairs in E_{2r} are good.*

Proof of Claim. For the sake of contradiction, suppose at least half of the pairs in E_{2r} are bad. Let H be the collection of all the j-flats spanned by subsets of V for all $j \leq k - 1$. Notice that if $S \subset V$ spans a j-flat h, then h is also spanned by only $j + 1$ elements from S. So we have

$$|H| \leq \sum_{j=0}^{k-1} |V|^{j+1} \leq k n^{k(d-\delta)}.$$

For each bad pair $\{T_1, T_2\} \in E_{2r}$, $T_1 \cup T_2$ lies on a j-flat from H by Claim 2.2. By the pigeonhole principle, there is a j-flat h with $j \leq k - 1$ such that at least

$$\frac{|E_{2r}|/2}{|H|} \geq \frac{\Omega(n^{(k+1)d-(k+2)\delta})}{2k n^{k(d-\delta)}} = \Omega(n^{d-2\delta})$$

bad pairs from E_{2r} have the property that their union lies in h. On the other hand, since h contains at most n^{k-1} points from $[n]^d$, h can correspond to at most $O(n^{(k-1)(k+2)})$ bad pairs from E_{2r}. Since we assumed $d - 2\delta > (k - 1)(k + 2)$, we have a contradiction for n sufficiently large. □

Each good pair $\{T_1, T_2\} \in E_{2r}$ gives rise to a non-degenerate $(k + 2)$-tuple $T_1 \cup T_2$ that lies on a k-flat. On the other hand, any such $(k + 2)$-tuple in V will correspond to at most $\binom{k+2}{r}$ good pairs in E_{2r}. Hence, by Claim 2.3, there are at least

$$\frac{|E_{2r}|}{2} \Big/ \binom{k+2}{r} = \Omega(n^{(k+1)d-(k+2)\delta})$$

non-degenerate $(k + 2)$-tuples that lie on a k-flat, concluding the proof.

In the other direction, we will use the following upper bounds.

Lemma 2.4. *For real number $\delta > 0$ and fixed positive integers d, k, i, such that $i < k + 2$, suppose $U, V \subset [n]^d$ satisfy $|U| = i$ and $|V| = n^{d-\delta}$, then V contains at most $n^{(k+1-i)(d-\delta)+k}$ non-degenerate $(k + 2)$-tuples that lie on a k-flat and contain U.*

Proof. If U spans a j-flat for some $j < i - 1$, then by definition no non-degenerate $(k + 2)$-tuple contains U. Hence we can assume U spans a $(i - 1)$-flat. Observe that a non-degenerate $(k + 2)$-tuple T, which lies on a k-flat and contains U, must contain a $(k + 1)$-tuple $T' \subset T$ such that T'

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spans a k-flat and $U \subset T'$. Then there are at most $n^{(k+1-i)(d-\delta)}$ ways to add $k+1-i$ points to U from V to obtain such T'. After T' is determined, there are at most n^k ways to add a final point from the affine hull of T' to obtain T. So we conclude the proof by multiplication. □

Lemma 2.5. *For positive integers $\ell \le d$, the grid $[n]^d$ contains at most $\ell \cdot n^{(\ell+1)d+(s-1)\ell}$ many $(\ell+s)$-tuples that lie on an ℓ-flat.*

Proof. We count the number of ways to choose an $(\ell+s)$-tuple T that spans a j-flat. There are at most $n^{(j+1)d}$ ways to choose a subset $T' \subset T$ of size $j+1$ that spans the affine hull of T. After this T' is determined, there are at most $n^{(\ell+s-1-j)j}$ ways to add the remaining $\ell+s-1-j$ points from the j-flat spanned by T'. Then the total number of $(\ell+s)$-tuples that lie on an ℓ-flat is at most

$$\sum_{j=1}^{\ell} n^{(j+1)d+(\ell+s-1-j)j} \le \sum_{j=1}^{\ell} n^{(j+1)d+(\ell+s-1-j)\ell} \le \sum_{j=1}^{\ell} n^{(\ell+1)d+(s-1)\ell} \le \ell \cdot n^{(\ell+1)d+(s-1)\ell},$$

where the second inequality uses $\ell \le d$. □

3. Proof of Theorem 1.1

In this section, we use the hypergraph container method to prove Theorem 1.1. We shall assume basic notions about hypergraphs and follow the strategy outlined in [3]. Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ denote a r-uniform hypergraph. For any $U \subset V(\mathcal{H})$, its *degree* is the number of edges containing U. For each $i \in [r]$, we use $\Delta_i(\mathcal{H})$ to denote the maximum degree among all U of size i. For $S \subset V(\mathcal{H})$, we use $\mathcal{H}[S]$ to denote the *induced sub-hypergraph* on S. We shall use the following version of the hypergraph container lemma, which is Theorem 4.2 in [20].

Lemma 3.1. *Let $r \ge 2$ be an integer and $c > 0$ be sufficiently small with respect to r. If $\mathcal{H} = (V, E)$ is an r-uniform hypergraph and $0 < \tau < 1/2$ is a real number such that*

$$\Delta_i(\mathcal{H}) \le c \cdot \tau^{i-1} \frac{|E|}{|V|} \quad \text{for all } 2 \le i \le r,$$

then there exists a family $\mathcal{C}$ of vertex subsets of $\mathcal{H}$ with the following properties:

- *(a) Every independent set of $\mathcal{H}$ is contained in some $C \in \mathcal{C}$.*
- *(b) $|\mathcal{C}| \le \exp\left(c^{-1} \cdot \tau |V| \cdot \log(1/\tau)\right)$.*
- *(c) For every $C \in \mathcal{C}$, we have $|E(\mathcal{H}[C])| \le (1-c)|E|$.*

The main result of this section is the following theorem.

Theorem 3.2. *Let k, ℓ be fixed integers such that $\ell \ge k \ge 2$ and k is even. Then for any $\epsilon > 0$, there is a constant $d = d(\epsilon, k, \ell)$ such that the following holds. For infinitely many values of N, there is a set V of N points in $\mathbb{R}^d$ such that no $\ell+3$ members of V lie on an ℓ-flat, and every subset of V without $k+2$ members on a k-flat has size at most $O\left(N^{\frac{\ell+2}{2(k+1)}+\epsilon}\right)$.*

Before we prove Theorem 3.2, let us show that it implies Theorem 1.1.

Proof of Theorem 1.1. In dimensions $d' \ge 3$ where d' is odd, we apply Theorem 3.2 with $k = \ell = d'-1$ to obtain a point set V of size N in $\mathbb{R}^d$ with the property that no $d'+2$ members lie on a $(d'-1)$-flat, and every subset of size $\Omega\left(N^{\frac{1}{2}+\frac{1}{2d'}+\epsilon}\right)$ contains $d'+1$ members on a $(d'-1)$-flat. By projecting V to a generic d'-dimensional subspace of $\mathbb{R}^d$, we obtain N points in $\mathbb{R}^{d'}$ with no $d'+2$ members on a common hyperplane, and every subset in general position has size $O\left(N^{\frac{1}{2}+\frac{1}{2d'}+\epsilon}\right)$.

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In dimensions $d' \geq 4$ where d' is even, we apply Theorem 3.2 with $k = d' - 2$ and $\ell = d' - 1$ to obtain a point set V of size N in $\mathbb{R}^d$ with the property that no $d' + 2$ members on a $(d' - 1)$-flat, and every subset of size $\Omega\left(N^{\frac{1}{2}+\frac{1}{d'-1}+\epsilon}\right)$ contains d' members on a $(d' - 2)$-flat. By adding another point from this subset, we obtain $d' + 1$ members on a $(d' - 1)$-flat. Hence, by projecting to V a generic d'-dimensional subspace of $\mathbb{R}^d$, we obtain N points in $\mathbb{R}^{d'}$ with no $d' + 2$ members on a common hyperplane, and every subset in general position has size $O\left(N^{\frac{1}{2}+\frac{1}{d'-1}+\epsilon}\right)$.

Since ϵ is arbitrary and N grows to infinity, we can conclude the proof of Theorem 1.1 after renaming d' to d. □

Proof of Theorem 3.2. Let d be a sufficiently large integer and n tend to infinity. We denote $\mathcal{H}$ as the hypergraph with $V(\mathcal{H}) = [n]^d$ and $E(\mathcal{H})$ consisting of non-degenerate $(k + 2)$-tuples T such that T lies on a k-flat. We shall construct a rooted tree $\mathfrak{T}$ whose nodes are labelled with vertex subsets of $\mathcal{H}$ as follows. We start with $\mathfrak{T}$ consisting of one root node labelled with $V(\mathcal{H})$. Iteratively, if there is a leaf $x \in \mathfrak{T}$ whose labelled set C_x has size at least $n^{\frac{k}{k+1}d+k}$, we apply Lemma 3.1 to $\mathcal{H}[C_x]$ with $\tau = n^{-\frac{k}{k+1}d+\delta+\epsilon}$ where δ is defined by $|C_x| = n^{d-\delta}$. As a consequence, Lemma 3.1 produces a collection $\mathcal{C}$ of subsets of C_x. Then we create a child of x in $\mathfrak{T}$ labelled by C for each $C \in \mathcal{C}$. The iteration continues until there is no leaf $x \in \mathfrak{T}$ with $|C_x| \geq n^{\frac{k}{k+1}d+k}$.

During the interative construction of $\mathfrak{T}$, we need to verify the hypothesis of Lemma 3.1, that is,

$$\Delta_i(\mathcal{H}[C_x]) \leq c \cdot \tau^{i-1} \frac{|E(\mathcal{H}[C_x])|}{|V(\mathcal{H}[C_x])|} \quad \text{for all } 2 \leq i \leq k + 2.$$

To check this, we use Lemma 2.4 to upper bound $\Delta_i(\mathcal{H}[C_x])$ for $2 \leq i < k + 2$ and use the trivial bound $\Delta_i(\mathcal{H}[C_x]) \leq 1$ for $i = k + 2$. On the other hand, we use Lemma 2.1 to lower bound $|E(\mathcal{H}[C_x])|$. We shall use $n^{d-\delta} = |V(\mathcal{H}')| \geq n^{\frac{k}{k+1}d+k}$ as well. Since this is a straightforward computation, whose detail will be given as Claim 4.2 in the proof of Theorem 1.2, we skip it here.

Now, we analyse this rooted tree $\mathfrak{T}$. According to Lemma 3.1(c), if y (labelled with C_y) is a child of x (labelled with C_x) in $\mathfrak{T}$, the number of edges induced by C_y shrinks from that by C_x by a constant factor $(1 - c)$. On the other hand, a reasonably large set induces many edges in $\mathcal{H}$ by Lemma 2.1 (assuming d is large). This means the height of $\mathfrak{T}$ is upper bounded by $O(\log n)$, and in particular our iterative construction ends. According to Lemma 3.1(b), the number of children of any node x in $\mathfrak{T}$ is at most

$$|\mathcal{C}| \leq \exp\left(c^{-1} \cdot \tau |C_x| \cdot \log(1/\tau)\right) \leq \exp\left(O\left(n^{\frac{d}{k+1}+\epsilon} \cdot \log n\right)\right).$$

Therefore, let $\mathfrak{C}$ be the collection of sets labelling the leaves of $\mathfrak{T}$. Hence, we have

$$|\mathfrak{C}| \leq \exp\left(O\left(n^{\frac{d}{k+1}+\epsilon} \cdot \log^2 n\right)\right) \quad \text{and} \quad |C| \leq n^{\frac{k}{k+1}d+k} \text{ for all } C \in \mathfrak{C}.$$

Furthermore, if I is an independent set of $\mathcal{H}$ that is contained in a vertex subset C_x labelling a non-leaf node x, then by the construction of $\mathfrak{T}$ and Lemma 3.1(a), there exists a child y of x in $\mathfrak{T}$ whose labelling set C_y contains I. This implies every independent set of $\mathcal{H}$ is contained in some member of $\mathfrak{C}$. Elements in this collection $\mathfrak{C}$ are called containers.

Next, we randomly select a subset of $[n]^d$ by keeping each point independently with probability p. Let S be the set of selected elements. Then for each $(\ell + 3)$-tuple T in S that lies on an ℓ-flat, we delete one point from T. We denote the resulting set of points by S'. By Lemma 2.5, we have

$$\mathbb{E}[|S'|] \geq pn^d - p^{\ell+3}\ell n^{(\ell+1)d+2\ell}.$$

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By setting $p = (2\ell)^{-\frac{1}{\ell+2}} n^{-\frac{\ell}{\ell+2}(d+2)}$, we have

$$\mathbb{E}[|S'|] \geq \frac{pn^d}{2} = \Omega\left(n^{\frac{2(d-\ell)}{\ell+2}}\right).$$

Finally, we set $m = n^{\frac{d}{k+1}+2\epsilon}$. Let X denote the number of independent sets of $\mathcal{H}$ in S' with cardinality m. Using the family of containers, we have

$$\begin{aligned}
\mathbb{E}[X] &\leq |\mathfrak{C}| \cdot \binom{n^{\frac{k}{k+1}d+k}}{m} \cdot p^m \\
&\leq \exp\left(O\left(n^{\frac{d}{k+1}+\epsilon} \cdot \log^2 n\right)\right) \cdot \left(\frac{e \cdot n^{\frac{k}{k+1}d+k}}{m}\right)^m p^m \\
&\leq \exp\left(O\left(n^{\frac{d}{k+1}+\epsilon} \cdot \log^2 n\right)\right) \cdot \left(e \cdot n^{\frac{k-1}{k+1}d+k-2\epsilon}\right)^m \left((2\ell)^{-\frac{1}{\ell+2}} \cdot n^{-\frac{\ell}{\ell+2}(d+2)}\right)^m \\
&\leq \exp\left(O\left(n^{\frac{d}{k+1}+\epsilon} \cdot \log^2 n\right)\right) \cdot \left(\frac{1}{2}\right)^m \\
&\leq o(1).
\end{aligned}$$

Here, the fourth inequality uses the following consequence of $k \leq \ell$ and d being large:

$$\frac{k-1}{k+1}d + k - 2\epsilon - \frac{\ell}{\ell+2}(d+2) < 0.$$

Notice that $|S'|$ is exponentially concentrated around its mean by Chernoff's inequality. Therefore, some realisation of S' satisfies: $|S'| = N = \Omega(n^{2(d-\ell)/(\ell+2)})$; S' contains no $(\ell+3)$-tuples on a ℓ-flat; and $\mathcal{H}[S']$ does not contain an independent set of $\mathcal{H}$ with cardinality

$$m = n^{\frac{d}{k+1}+2\epsilon} = O\left(N^{\frac{\ell+2}{2(k+1)} + \frac{(\ell+2)\ell}{2(k+1)(d-\ell)} + \frac{\ell+2}{d-\ell}\epsilon}\right) \leq O\left(N^{\frac{\ell+2}{2(k+1)}+\epsilon}\right).$$

Here, we assume $d = d(\epsilon, k, \ell)$ is sufficiently large so that

$$\frac{(\ell+2)\ell}{2(k+1)(d-\ell)} + \frac{\ell+2}{d-\ell}\epsilon \leq \epsilon.$$

Notice that S' not containing an independent set of size m means every subset of S' of size m contains $k+2$ points on a k-flat. We conclude the proof by renaming S' to V. □

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. The proof is essentially the same as in the previous section with a different choice of parameters. For the reader's convenience, we include the details here. We start by proving the following theorem.

Theorem 4.1. *Let k, ℓ, s be fixed integers such that $\ell \geq k \geq 2$, $s \geq 2$, k is even, and $\frac{2\ell+s-1}{\ell+s-1} < \frac{2k}{k+1}$. Then for any $\epsilon > 0$, there is a constant $d = d(\epsilon, k, \ell, s)$ such that the following holds. For infinitely many values of N, there is a set V of N points in $\mathbb{R}^d$ such that no $\ell+s$ members of V lie on an ℓ-flat, and every subset of V without $k+2$ members on a k-flat has size at most $O\left(N^{\frac{1}{2}+\epsilon}\right)$.*

Proof. Just as before, let $\mathcal{H}$ be the hypergraph with $V(\mathcal{H}) = [n]^d$ and $E(\mathcal{H})$ consisting of non-degenerate $(k+2)$-tuples T such that T lies on a k-flat. We let $q = q(k, r, s)$ be a quantity that will be determined later. We again construct a rooted tree $\mathfrak{T}$ whose nodes are labelled with vertex

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subsets of $\mathcal{H}$. We start with $\mathfrak{T}$ consisting of one root node labelled with $V(\mathcal{H})$. Iteratively, if there is a leaf $x \in \mathfrak{T}$ whose labelled set C_x has size at least n^{qd+k}, we apply Lemma 3.1 to $\mathcal{H}' = \mathcal{H}[C_x]$ with $\tau = n^{-qd+\delta+\epsilon}$ where δ is defined by $|C_x| = n^{d-\delta}$. We shall use the claim below to verify the hypothesis of Lemma 3.1. As a consequence, Lemma 3.1 produces a collection $\mathcal{C}$ of subsets of C_x. Then we create a child of x in $\mathfrak{T}$ labelled by C for each $C \in \mathcal{C}$. The iteration continues until there is no leaf $x \in \mathfrak{T}$ with $|C_x| \geq n^{qd+k}$.

Claim 4.2. *If* $\frac{1}{2} < q \leq \frac{k}{k+1}$ *and* $\mathcal{H}'$ *defined as above, then*

$$\Delta_i(\mathcal{H}') \leq c \cdot \tau^{i-1} \frac{|E(\mathcal{H}')|}{|V(\mathcal{H}')|} \quad \text{for all } 2 \leq i \leq k+2,$$

where c is the constant in Lemma 3.1 depending only on k.

Proof of Claim. First, we notice that

$$n^{d-\delta} = |V(\mathcal{H}')| \geq n^{qd+k} \implies \delta \leq d - qd - k. \tag{4.1}$$

Assuming d is large, we have $|E(\mathcal{H}')| \geq \Omega(n^{(k+1)d-(k+2)\delta})$ by Lemma 2.1.

For $2 \leq i < k+2$, Lemma 2.4 gives us $\Delta_i(\mathcal{H}') \leq n^{(k+1-i)(d-\delta)+k}$. Hence, it suffices to check

$$n^{(k+1-i)(d-\delta)+k} \ll \left(n^{-qd+\delta+\epsilon}\right)^{i-1} \cdot \frac{n^{(k+1)d-(k+2)\delta}}{n^{d-\delta}}.$$

Simplifying and comparing the exponents over n, this is implied by

$$(i-1)d + k + (i-1)\epsilon > (i-1)qd + \delta.$$

Since d is sufficiently large, it suffices to compare the coefficients of d. Applying (4.1) and simplifying the terms, the inequality above is implied by $i - 1 \geq (i-2)q + 1$, which is true by our hypothesis.

For $i = k+2$, we have $\Delta_i(\mathcal{H}') \leq 1$ trivially. Hence, it suffices to check

$$1 \ll \left(n^{-qd+\delta+\epsilon}\right)^{k+1} \cdot \frac{n^{(k+1)d-(k+2)\delta}}{n^{d-\delta}}.$$

Simplifying and comparing the exponents over n, this is implied by

$$(k+1)qd < kd + (k+1)\epsilon.$$

Again, since d is sufficiently large, it suffices to compare the coefficients of d. The inequality above is implied by $(k+1)q \leq k$, which is true by our hypothesis. □

We can analyse this rooted tree $\mathfrak{T}$ using arguments similar to the previous section. We can conclude that there exists a collection $\mathfrak{C}$ of vertex subsets of $\mathcal{H}$ with

$$|\mathfrak{C}| \leq \exp\left(O\left(n^{d-qd+\epsilon} \cdot \log^2 n\right)\right) \quad \text{and} \quad |C| \leq n^{qd+k} \text{ for all } C \in \mathfrak{C}.$$

and every independent set of $\mathcal{H}$ is contained in some member of $\mathfrak{C}$.

Next, we randomly select a subset of $[n]^d$ by keeping each point independently with probability p. Let S be the set of selected elements. Then for each $(\ell+s)$-tuple T in S that lies on an ℓ-flat, we delete one point from T. We denote the resulting set of points by S'. By Lemma 2.5, we have

$$\mathbb{E}[|S'|] \geq pn^d - p^{\ell+s}\ell n^{(\ell+1)d+(s-1)\ell}.$$

By setting $p = (2\ell)^{-\frac{1}{\ell+s-1}} n^{-\frac{\ell}{\ell+s-1}(d+s-1)}$, we have

$$\mathbb{E}[|S'|] \geq \frac{pn^d}{2} = \Omega\left(n^{\frac{(s-1)(d-\ell)}{\ell+s-1}}\right).$$

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Finally, we set $m = n^{d-qd+2\epsilon}$. Let X denote the number of independent sets of $\mathcal{H}$ in S' with cardinality m. With a foresight soon to be self-evident, we choose

$$q = \frac{1}{2} \cdot \frac{2\ell + s - 1}{\ell + s - 1} + \frac{1}{2d} \cdot \frac{\ell(s-1)}{\ell + s - 1} - \frac{k}{2d}. \tag{4.2}$$

We remark that our hypothesis on k, ℓ, s implies $\frac{1}{2} < q \leq \frac{k}{k+1}$ assuming d is large, hence Claim 4.2 can be applied in construction of $\mathfrak{T}$.

Using the family $\mathfrak{C}$, we can estimate

$$\begin{aligned}
\mathbb{E}[X] &\leq |\mathfrak{C}| \cdot \binom{n^{qd+k}}{m} \cdot p^m \\
&\leq \exp\left(O\left(n^{d-qd+\epsilon} \cdot \log^2 n\right)\right) \cdot \left(\frac{e \cdot n^{qd+k}}{m}\right)^m p^m \\
&\leq \exp\left(O\left(n^{d-qd+\epsilon} \cdot \log^2 n\right)\right) \cdot \left(e \cdot n^{(2q-1)d+k-2\epsilon}\right)^m \left((2\ell)^{-\frac{1}{\ell+s-1}} n^{-\frac{\ell}{\ell+s-1}(d+s-1)}\right)^m \\
&\leq \exp\left(O\left(n^{d-qd+\epsilon} \cdot \log^2 n\right)\right) \cdot \left(\frac{1}{2}\right)^m \\
&\leq o(1).
\end{aligned}$$

Here, the fourth inequality uses the following consequence of (4.2):

$$(2q-1)d + k - 2\epsilon - \frac{\ell}{\ell + s - 1}(d + s - 1) < 0.$$

Notice that $|S'|$ is exponentially concentrated around its mean by Chernoff's inequality. Therefore, some realisation of S' satisfies: $|S'| = N = \Omega\left(n^{\frac{(s-1)(d-\ell)}{\ell+s-1}}\right)$; S' contains no $(\ell + s)$-tuples on a ℓ-flat; and $\mathcal{H}[S']$ does not contain an independent set of $\mathcal{H}$ with cardinality

$$m = n^{d-qd+2\epsilon} = O\left(N^{\frac{1}{2} + \left(\frac{k}{2} + 2\epsilon\right) \cdot \frac{\ell+s-1}{(s-1)(d-\ell)}}\right) \leq O\left(N^{\frac{1}{2}+\epsilon}\right).$$

Here, we assume $d = d(\epsilon, k, \ell, s)$ is sufficiently large so that

$$\left(\frac{k}{2} + 2\epsilon\right) \cdot \frac{\ell + s - 1}{(s-1)(d-\ell)} \leq \epsilon.$$

Since S' does not contain an independent set of size m, every subset of S' of size m contains $k+2$ points on a k-flat. We conclude the proof by renaming S' to V. □

Proof of Theorem 1.2. In dimensions $d' \geq 3$ where d' is odd, we obtain an upper bound for $\alpha_{d',s'}(N)$ with $d's' + 2 > 2d' + 2s'$. We set $k = \ell = d' - 1$ and $s = s' + 1$, so we can verify $\frac{2\ell+s-1}{\ell+s-1} < \frac{2k}{k+1}$. Hence we can apply Theorem 4.1 to obtain a point set V of size N in $\mathbb{R}^d$ with the property that no $d' + s'$ members lie on a $(d'-1)$-flat, and every subset of size $\Omega(N^{\frac{1}{2}+\epsilon})$ contains $d'+1$ members on a $(d'-1)$-flat. By projecting V to a generic d'-dimensional subspace of $\mathbb{R}^d$, we obtain N points in $\mathbb{R}^{d'}$ with no $d' + s'$ members on a common hyperplane, and every subset in general position has size $O(N^{\frac{1}{2}+\epsilon})$.

In dimensions $d' \geq 4$ where d' is even, we obtain an upper bound for $\alpha_{d',s'}(N)$ with $d's' + 2 > 2d' + 3s'$. We set $k = d' - 2$, $\ell = d' - 1$, and $s = s' + 1$, so we can verify $\frac{2\ell+s-1}{\ell+s-1} < \frac{2k}{k+1}$. Hence we can apply Theorem 4.1 to obtain a point set V of size N in $\mathbb{R}^d$ with the property that no $d' + s'$ members on a $(d'-1)$-flat, and every subset of size $\Omega(N^{\frac{1}{2}+\epsilon})$ contains d' members on a $(d'-2)$-flat. By adding another point from this subset, we obtain $d'+1$ members on a $(d'-1)$-flat. Hence, by projecting to V a generic d'-dimensional subspace of $\mathbb{R}^d$, we obtain N points in $\mathbb{R}^{d'}$

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with no $d'+s'$ members on a common hyperplane, and every subset in general position has size $O(N^{\frac{1}{2}+\epsilon})$.

Since ϵ is arbitrary and N grows to infinity, we can conclude the proof of Theorem 1.2 after renaming d' to d and s' to s. □

5. Proof of Theorem 1.3

In this section, we will give a proof of Theorem 1.3. Let $V \subset [n]^d$ such that there are no $k+2$ points that lie on a k-flat. In [18], Lefmann showed that $|V| \leq O\left(n^{\frac{d}{\lfloor (k+2)/2 \rfloor}}\right)$. To see this, assume that k is even and consider all elements of the form $v_1 + \cdots + v_{\frac{k}{2}+1}$, where $v_i \neq v_j$ and $v_i \in V$. All of these elements are distinct, since otherwise we would have $k+2$ points on a k-flat. In other words, the equation

$$\left(\mathbf{x}_1 + \cdots + \mathbf{x}_{\frac{k}{2}+1}\right) - \left(\mathbf{x}_{\frac{k}{2}+2} + \cdots + \mathbf{x}_{k+2}\right) = \mathbf{0},$$

does not have a solution with $\{\mathbf{x}_1, \ldots, \mathbf{x}_{\frac{k}{2}+1}\}$ and $\{\mathbf{x}_{\frac{k}{2}+2}, \ldots, \mathbf{x}_{k+2}\}$ being two different $(\frac{k}{2}+1)$-tuples of V. Therefore, we have $\binom{|V|}{\frac{k}{2}+1} \leq (kn)^d$, and this implies Lefmann's bound.

More generally, let us consider the equation

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_r\mathbf{x}_r = \mathbf{0}, \tag{5.1}$$

with constant coefficients $c_i \in \mathbb{Z}$ and $\sum_i c_i = 0$. Here, the variables $\mathbf{x}_i$ takes value in $\mathbb{Z}^d$. A solution $(\mathbf{x}_1, \ldots, \mathbf{x}_r)$ to equation (5.1) is called *trivial* if there is a partition $\mathcal{P} : [r] = \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_t$, such that $\mathbf{x}_j = \mathbf{x}_\ell$ if and only if $j, \ell \in \mathcal{I}_i$, and $\sum_{j \in \mathcal{I}_i} c_j = 0$ for all $i \in [t]$. In other words, being trivial means that, after combining like terms, the coefficient of each $\mathbf{x}_i$ becomes zero. Otherwise, we say that the solution $(\mathbf{x}_1, \ldots, \mathbf{x}_r)$ is *non-trivial*. A natural extremal problem is to determine the maximum size of a set $A \subset [n]^d$ with only trivial solutions to (5.1). When $d = 1$, this is a classical problem in additive number theory, and we refer the interested reader to [6, 16, 21, 25].

By combining the arguments of Cilleruelo and Timmons [6] and Jia [15], we establish the following theorem.

Theorem 5.1. *Let d, r be fixed positive integers. Suppose $V \subset [n]^d$ has only trivial solutions to each equation of the form*

$$c_1\left((\boldsymbol{x}_1 + \cdots + \boldsymbol{x}_r) - (\boldsymbol{x}_{r+1} + \cdots + \boldsymbol{x}_{2r})\right) = c_2\left((\boldsymbol{x}_{2r+1} + \cdots + \boldsymbol{x}_{3r}) - (\boldsymbol{x}_{3r+1} + \cdots + \boldsymbol{x}_{4r})\right), \tag{5.2}$$

for integers c_1, c_2 such that $1 \leq c_1, c_2 \leq n^{\frac{d}{2rd+1}}$. Then we have

$$|V| \leq O\left(n^{\frac{d}{2r}\left(1-\frac{1}{2rd+1}\right)}\right).$$

Notice that Theorem 1.3 follows from Theorem 5.1. Indeed, when $k+2$ is divisible by 4, we set $r = (k+2)/4$. If $V \subset [n]^d$ contains $k+2$ points $\{v_1, \ldots, v_{k+2}\}$ that is a non-trivial solution to (5.2) with $\mathbf{x}_i = v_i$, then $\{v_1, \ldots, v_{k+2}\}$ must lie on a k-flat. Hence, when $k+2$ is divisible by 4, we have

$$a(d,k,n) \leq O\left(n^{\frac{d}{(k+2)/2}\left(1-\frac{1}{(k+2)d/2+1}\right)}\right).$$

Since we have $a(d,k,n) < a(d,k-1,n)$, this implies that for all $k \geq 2$, we have

$$a(d,k,n) \leq O\left(n^{\frac{d}{2\lfloor (k+2)/4 \rfloor}\left(1-\frac{1}{2\lfloor (k+2)/4 \rfloor d+1}\right)}\right).$$

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In the proof of Theorem 5.1, we need the following well-known lemma (see e.g. Lemma 2.1 in [6] and Theorem 4.1 in [25]). For $U, T \subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, we define

$$\Phi_{U-T}(x) = \{(u,t) : u - t = x, u \in U, t \in T\}.$$

Lemma 5.2. *For finite sets $U, T \subset \mathbb{Z}^d$, we have*

$$\frac{(|U||T|)^2}{|U+T|} \leq \sum_{x \in \mathbb{Z}^d} |\Phi_{U-U}(x)| \cdot |\Phi_{T-T}(x)|.$$

Proof of Theorem 5.1. Let d, r, and V be as given in the hypothesis. Let $m \geq 1$ be an integer that will be determined later. We define

$$S_r = \{v_1 + \cdots + v_r : v_i \in V, v_i \neq v_j\},$$

and a function

$$\sigma : \binom{V}{r} \to S_r, \ \{v_1, \ldots, v_r\} \mapsto v_1 + \cdots + v_r.$$

Notice that σ is a bijection. Indeed, suppose on the contrary that

$$v_1 + \cdots + v_r = v_1' + \cdots + v_r'$$

for two different r-tuples in V. Then by setting $(\mathbf{x}_1, \ldots, \mathbf{x}_r) = (v_1, \ldots, v_r)$, $(\mathbf{x}_{r+1}, \ldots, \mathbf{x}_{2r}) = (v_1', \ldots, v_r')$, $(\mathbf{x}_{2r+1}, \ldots, \mathbf{x}_{3r}) = (\mathbf{x}_{3r+1}, \ldots, \mathbf{x}_{4r})$ arbitrarily, and $c_1 = c_2 = 1$, we obtain a non-trivial solution to (5.2), which is a contradiction. In particular, we have $|S_r| = \binom{|V|}{r}$.

For $j \in [m]$ and $w \in \mathbb{Z}_j^d$, we let

$$U_{j,w} = \{u \in \mathbb{Z}^d : ju + w \in S_r\}.$$

Notice that for fixed $j \in [m]$, we have

$$\sum_{w \in \mathbb{Z}_j^d} |U_{j,w}| = \sum_{w \in \mathbb{Z}_j^d} |\{v \in S_r : v \equiv w \bmod j\}| = |S_r|.$$

Applying Jensen's inequality to above, we have

$$\sum_{w \in \mathbb{Z}_j^d} |U_{j,w}|^2 \geq |S_r|^2 / j^d. \tag{5.3}$$

For $i \geq 0$, we define

$$\Phi^i_{U_{j,w}-U_{j,w}}(x) = \{(u_1, u_2) \in \Phi_{U_{j,w}-U_{j,w}}(x) : |\sigma^{-1}(ju_1 + w) \cap \sigma^{-1}(ju_2 + w)| = i\}.$$

It's obvious that these sets form a partition of $\Phi_{U_{j,w}-U_{j,w}}(x)$. We also make the following claims.

Claim 5.3. *For a fixed $x \in \mathbb{Z}^d$, we have*

$$\sum_{j \in [m]} \sum_{w \in \mathbb{Z}_j^d} |\Phi^0_{U_{j,w}-U_{j,w}}(x)| \leq 1,$$

Proof. For the sake of contradiction, suppose the summation above is at least two, then we have $(u_1, u_2) \in \Phi^0_{U_{j,w}-U_{j,w}}(x)$ and $(u_3, u_4) \in \Phi^0_{U_{j',w'}-U_{j',w'}}(x)$ such that either $(u_1, u_2) \neq (u_3, u_4)$ or $(j, w) \neq (j', w')$.

Let $s_1, s_2, s_3, s_4 \in S_r$ such that $s_1 = ju_1 + w$, $s_2 = ju_2 + w$, $s_3 = j'u_3 + w'$, $s_4 = j'u_4 + w'$ and write $\sigma^{-1}(s_i) = \{v_{i,1}, \ldots, v_{i,r}\}$. Notice that $u_1 - u_2 = x = u_3 - u_4$. Putting these equations together

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gives us

$$j'((v_{1,1}+\cdots+v_{1,r})-(v_{2,1}+\cdots+v_{2,r}))=j((v_{3,1}+\cdots+v_{3,r})-(v_{4,1}+\cdots+v_{4,r})). \tag{5.4}$$

It suffices to show that (5.4) can be seem as a non-trivial solution to (5.2). The proof now falls into the following cases.

Case 1. Suppose $j\neq j'$. Without loss of generality we can assume $j'>j$. Notice that $(u_1,u_2)\in \Phi^0_{U_{j,w}-U_{j,w}}(x)$ implies

$$\{v_{1,1},\ldots,v_{1,r}\}\cap\{v_{2,1},\ldots,v_{2,r}\}=\emptyset.$$

Then after combining like terms in (5.4), the coefficient of v_1^1 is at least $j'-j$, which means this is indeed a non-trivial solution to (5.2).

Case 2. Suppose $j=j'$, then we must have $s_1\neq s_3$. Indeed, if $s_1=s_3$, we must have $w=w'$ (as s_1 modulo j equals s_3 modulo j') and $s_2=s_4$ (as $j'(s_1-s_2)=j(s_3-s_4)$). This is a contradiction to either $(u_1,u_2)\neq(u_3,u_4)$ or $(j,w)\neq(j',w')$.

Given $s_1\neq s_3$, we can assume, without loss of generality, $v_{1,1}\notin\{v_{3,1},\ldots,v_{3,r}\}$. Again, we have $\{v_{1,1},\ldots,v_{1,r}\}\cap\{v_{2,1},\ldots,v_{2,r}\}=\emptyset$. Hence, after combining like terms in (5.4), the coefficient of v_1^1 is positive and we have a non-trivial solution to (5.2). □

Claim 5.4. *For a finite set $T\subset\mathbb{Z}^d$, and fixed integers $i,j\geq 1$, we have*

$$\sum_{w\in\mathbb{Z}_j^d}\sum_{x\in\mathbb{Z}^d}|\Phi^i_{U_{j,w}-U_{j,w}}(x)|\cdot|\Phi_{T-T}(x)|\leq|V|^{2r-i}|T|.$$

Proof. The summation on the left-hand side counts all (ordered) quadruples (u_1,u_2,t_1,t_2) such that $(u_1,u_2)\in\Phi^i_{U_{j,w}-U_{j,w}}(t_1-t_2)$. For each such a quadruple, let $s_1,s_2\in S_r$ such that

$$s_1=ju_1+w\quad\text{and}\quad s_2=ju_2+w.$$

There are at most $|V|^{2r-i}$ ways to choose a pair (s_1,s_2) satisfying $|\sigma^{-1}(s_1)\cap\sigma^{-1}(s_2)|=i$. Such a pair (s_1,s_2) determines (u_1,u_2) uniquely. Moreover, (s_1,s_2) also determines the quantity

$$t_1-t_2=u_1-u_2=\frac{s_1-w}{j}-\frac{s_2-w}{j}=\frac{1}{j}(s_1-s_2).$$

After such a pair (s_1,s_2) is chosen, there are at most $|T|$ ways to choose t_1 and this will also determine t_2. So we conclude the claim by multiplication. □

Now, we set $T=\mathbb{Z}_\ell^d$ for some integer ℓ to be determined later. Notice that $U_{j,w}+T\subset\{0,1,\ldots,\lfloor rn/j\rfloor+\ell-1\}^d$, which implies

$$|U_{j,w}+T|\leq(rn/j+\ell)^d. \tag{5.5}$$

By Lemma 5.2, we have

$$\frac{|U_{j,w}|^2|T|^2}{|U_{j,w}+T|}\leq\sum_{x\in\mathbb{Z}^d}|\Phi_{U_{j,w}-U_{j,w}}(x)|\cdot|\Phi_{T-T}(x)|.$$

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Summing over all $j \in [m]$ and $w \in \mathbb{Z}_j^d$, and using Claims 5.3 and 5.4, we can compute

$$\begin{aligned}
\sum_{j\in[m]}\sum_{w\in\mathbb{Z}_j^d}\frac{|U_{j,w}|^2||T|^2}{|U_{j,w}+T|} &\le \sum_{j\in[m]}\sum_{w\in\mathbb{Z}_j^d}\sum_{x\in\mathbb{Z}^d}|\Phi_{U_{j,w}-U_{j,w}}(x)|\cdot|\Phi_{T-T}(x)| \\
&= \sum_{x\in\mathbb{Z}^d}\sum_{j\in[m]}\sum_{w\in\mathbb{Z}_j^d}\left(|\Phi^0_{U_{j,w}-U_{j,w}}(x)|+\sum_{i=1}^{r}|\Phi^i_{U_{j,w}-U_{j,w}}(x)|\right)|\Phi_{T-T}(x)| \\
&\le \sum_{x\in\mathbb{Z}^d}|\Phi_{T-T}(x)|\sum_{j\in[m]}\sum_{w\in\mathbb{Z}_j^d}|\Phi^0_{U_{j,w}-U_{j,w}}(x)|+\sum_{j\in[m]}\sum_{i=1}^{r}|V|^{2r-i}\ell^d \\
&\le \sum_{x\in\mathbb{Z}^d}\Phi_{T-T}(x)+\sum_{j\in[m]}\sum_{i=1}^{r-1}|V|^{2r-i}\ell^d \\
&\le \ell^{2d}+rm|V|^{2r-1}\ell^d,
\end{aligned}$$

On the other hand, using (5.3) and (5.5), we can compute

$$\begin{aligned}
\sum_{j\in[m]}\sum_{w\in\mathbb{Z}_j^d}\frac{|U_{j,w}|^2||T|^2}{|U_{j,w}+T|} &\ge \sum_{j\in[m]}\sum_{w\in\mathbb{Z}_j^d}\frac{|U_{j,w}|^2\ell^{2d}}{(rn/j+\ell)^d} \\
&\ge \sum_{j\in[m]}\frac{|S_r|^2\ell^{2d}}{j^d(rn/j+\ell)^d} \\
&= \sum_{j\in[m]}\frac{|S_r|^2\ell^{2d}}{(rn+j\ell)^d} \\
&\ge \frac{m|S_r|^2\ell^{2d}}{(rn+m\ell)^d},
\end{aligned}$$

Combining the two inequalities above gives us

$$\begin{aligned}
&\frac{m|S_r|^2\ell^{2d}}{(rn+m\ell)^d} \le \ell^{2d}+rm|V|^{2r-1}\ell^d \\
\Longrightarrow\ &|S_r|^2 \le \frac{(rn+m\ell)^d}{m}+r|V|^{2r-1}\frac{(rn+m\ell)^d}{\ell^d}.
\end{aligned}$$

By setting $m = n^{\frac{d}{2rd+1}}$ and $\ell = n^{1-\frac{d}{2rd+1}}$, we get

$$\binom{|V|}{r}^2 = |S_r|^2 \le cn^{d-\frac{d}{2rd+1}}+c|V|^{2r-1}n^{\frac{d^2}{2rd+1}},$$

for some constant c depending only on d and r. We can solve from this inequality that

$$|V| = O\left(n^{\frac{d}{2r}\left(1-\frac{1}{2rd+1}\right)}\right),$$

completing the proof. □

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6. Concluding remarks

1. It is easy to see that $\alpha_{d,s}(N) \geq \Omega(N^{1/d})$ for any fixed $d, s \geq 2$. Let S be a set consisting of N points in $\mathbb{R}^d$ with no $d+s$ members on a hyperplane. Suppose V is a maximal subset of S in general position, then V generates at most $\binom{|V|}{d}$ hyperplanes and each of them covers at most s points from $S \setminus V$. Hence, we have the inequality

$$s\binom{|V|}{d} + |V| \geq |S| = N,$$

which justifies the claimed lower bound of $\alpha_{d,s}(N)$.

Problem 6.1. *Are there fixed integers $d, s \geq 3$ such that $\alpha_{d,s}(N) \leq o(N^{1/2})$?*

2. We call a subset $V \subset [n]^d$ a *m-fold B_g-set* if V only contains trivial solutions to the equations

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_g\mathbf{x}_g = c_1\mathbf{x}'_1 + c_2\mathbf{x}'_2 + \ldots + c_g\mathbf{x}'_g,$$

with constant coefficients $c_i \in [m]$. We call 1-fold B_g-sets simply *B_g-sets*. By counting distinct sums, we have an upper bound $|V| \leq O(n^{d/g})$ for any B_g-set $V \subset [n]^d$.

Our Theorem 5.1 can be interpreted as the following phenomenon: by letting m grow as some proper polynomial in n, we have an upper bound for m-fold B_g-sets, where g is even, which gives a polynomial-saving improvement from the trivial $O(n^{d/g})$ bound. We believe this phenomenon should also hold without the parity condition on g.

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