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Equivariant shape theory

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Abstract

Equivariant shape theory is an improvement of equivariant homotopy theory which could be regarded as an equivariant version of Borsuk's shape theory. The main result in this paper is a description of the equivariant shape category Sh^G whose objects are equivariant spaces or G-spaces, i.e. topological spaces endowed with an action of a given topological group G, and whose morphisms are equivariant homotopy or G-homotopy classes of families of multi-valued functions which we call equivariant multi-nets or G-multi-nets. Previously equivariant shape theories have been described only under the assumptions that the group G is either finite or compact. We also study classes of G-spaces on which equivariant shape and equivariant homotopy coincide, look for conditions under which a G-map $f: X \to Y$ is an equivariant shape equivalence, and give some characterizations of G-spaces with trivial equivariant shape.

1. *Introduction*

The aim of this paper is to define shape category for G-spaces, i.e. topological spaces endowed with an action of a given topological group G. In our development we follow the method of multi-valued functions with smaller and smaller images of points, introduced in the case of ordinary shape by Sanjurjo[**13**] (see also [**5**]).

This approach is formally very similar to the original Borsuk's description of shape theory of compact metric spaces [**3**]. The role of fundamental sequences take equivariant multi-nets which are families of multi-valued functions indexed by a certain cofinite directed set determined by the codomain. The multi-valued functions that make up an equivariant multi-net are increasingly more and more resembling equivariant single-valued continuous functions. Then we shall define the notion of equivariant homotopy for equivariant multi-nets that corresponds to Borsuk's concept of homotopy for fundamental sequences. The morphisms of our equivariant shape category Sh^G are equivariant homotopy classes of equivariant multi-nets.

When G is the trivial group, the equivariant multi-net and equivariant homotopy for equivariant multi-nets agree with the multi-net and homotopy for multi-nets from [**5**]. Hence, the main result in [**5**] implies that in this case the category Sh^G is isomorphic to the shape category Sh [**10**].

The following features of our equivariant shape theory deserve to be emphasised. It is intrinsic in the sense that we do not use any outside objects (like G-ANRs, equivariant embeddings, equivariant resolutions, or equivariant expansions). There

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are no restrictions on the topological group G. In fact, our description applies to continuous actions of an arbitrary pointed space with the sole requirement that the base point keeps each element fixed. In particular, it applies to spaces endowed with an action of a topological semigroup with identity. Finally, it is elementary and the only tricky part is in defining the composition of equivariant homotopy classes of equivariant multi-nets.

In the special case of metrizable G-spaces an equivariant shape category for compact groups has been previously announced by Yu. M. Smirnov [**14**], [**15**], [**16**], who used an equivariant version of the R. H. Fox approach to shape [**8**]. Modifying K. Morita's description of shape I. Pop [**12**] obtained an equivariant shape category for arbitrary topological spaces but only for finite groups. Using the method of resolutions, S. A. Antonian and S. Mardešić [**2**] constructed an equivariant shape category Sh^G of topological spaces endowed with an action of a given compact group G such that Smirnov's category is the restriction of Sh^G to metrizable G-spaces.

The natural questions of relationship of our equivariant shape category with those of Pop and Antonian and Mardešić are deferred to another paper. We only give three examples of applications of our approach to equivariant shape theory. In the first we discuss the problem of identifying G-spaces on which equivariant shape theory and equivariant homotopy theory coincide. The second application addresses the problem to find necessary and sufficient conditions under which an equivariant map between G-spaces is an equivariant shape equivalence. Finally, we give some characterizations of G-spaces with trivial equivariant shape.

2. *Equivariant homotopy theory*

Throughout this paper G denotes a topological group which we keep fixed. An action of G on a topological space X is a (continuous) map $(g,x)\to gx$ of the direct product $G\times X$ into X such that $(gh)x = g(hx)$ and $ex = x$, where $x\in X$, $g,h\in G$ and e is the unity of G. A very special example is the trivial action of G on X, where $gx = x$ for all $g\in G$, $x\in X$.

By an equivariant space or a G-space we mean a topological space X together with an action of G on X. If X and Y are equivariant spaces then so is $X\times Y$, where $g(x,y) = (gx,gy)$ for all $g\in G$ and $(x,y)\in X\times Y$.

A map $\phi: X\to Y$ between equivariant spaces is called an equivariant map or a G-map provided $\phi(gx) = g(\phi(x))$ for every $g\in G$ and every $x\in X$. Note that the identity map is equivariant and that the composition of equivariant maps is equivariant. Therefore, equivariant spaces and equivariant maps form a category, which we denote $\mathscr{T}^G$.

Let X and Y be equivariant spaces and let $\phi,\psi: X\to Y$ be equivariant maps. An equivariant homotopy or a G-homotopy from ϕ to ψ is by definition a homotopy $H: X\times I\to Y$ from ϕ to ψ which is an equivariant map. Here we assume that G acts trivially on the unit segment I so that $g(x,t) = (gx,t)$ for every $g\in G$, every $x\in X$, and every $t\in I$. If for equivariant maps ϕ and ψ there is an equivariant homotopy from ϕ to ψ, we say that ϕ and ψ are equivariantly homotopic or G-homotopic and we write $\phi\simeq_G\psi$.

The relation $\simeq_G$ of equivariant homotopy is an equivalence relation and we denote the class of an equivariant map ϕ by $[\phi]_G$. The relation $\simeq_G$ is also compatible with

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the composition so that one can define composition of equivariant homotopy classes of equivariant maps by $[\psi]_G \circ [\phi]_G = [\psi \circ \phi]_G$, i.e. by composing representatives. In this way one obtains a category $\mathscr{H}^G$, whose objects are equivariant spaces and whose morphisms are classes of equivariantly homotopic equivariant maps. There is a homotopy functor from $\mathscr{T}^G$ to $\mathscr{H}^G$ which keeps the objects fixed and takes an equivariant map ϕ into its equivariant homotopy class $[\phi]_G$.

For further information concerning G-spaces see [**11**], [**6**] and [**4**].

3. *Normal covers and multi-valued functions*

In this section we shall introduce notions that are required for our theory and prove a useful technical result (Lemma 4). We begin with the definition of an increasing function between partially ordered sets. Those functions are used as a connection between indexing sets of families of multi-valued functions which make up equivariant multi-nets.

Let us agree that an *increasing* function $f: P \to P$ of a partially ordered set $(P, <)$ into itself is a function which satisfies $x < f(x)$ for every $x \in P$ and $x < y$ in P implies $f(x) < f(y)$. In the case when the domain and the codomain of a function f are different, the first requirement is dropped.

We shall repeatedly use the following lemma (see [**10**, p. 9]).

LEMMA 1. *Let $\{f_1, \ldots, f_n\}$ be functions from a cofinite directed set $(M, <)$ into a directed set $(L, <)$. Then there is an increasing function $g: M \to L$ such that $g(x) > f_1(x), \ldots, f_n(x)$ for every $x \in M$.*

We shall be using normal covers of a topological space to measure the size and distance of multi-valued functions. Certain cofinite directed sets build from the collection of all normal covers of a space will serve as indexing sets for equivariant multi-nets.

Let $Cov(Y)$ denote the collection of all normal covers of a topological space Y [**1**]. With respect to the refinement relation $>$ the set $Cov(Y)$ is a directed set. Two normal covers σ and τ of Y are equivalent provided $\sigma > \tau$ and $\tau > \sigma$. In order to simplify our notation we denote a normal cover and its equivalence class by the same symbol. Consequently, $Cov(Y)$ also stands for the associated quotient set.

Let $Inc(Y)$ denote the collection of all finite subsets c of $Cov(Y)$ which have a unique (with respect to the refinement relation) maximal element which we denote by $[c]$. We consider $Inc(Y)$ ordered by the inclusion relation and regard $Cov(Y)$ as a subset of single-element subsets of $Cov(Y)$. Notice that $Inc(Y)$ is a cofinite directed set. These sets will serve as indexing sets of equivariant multi-nets.

The following notation for normal covers will be used frequently.

If σ is a normal cover of a space Y, let σ^+ be the collection of all normal covers of Y which refine σ while σ^* denotes the set of all normal overs τ of Y such that the star $st(\tau)$ of τ refines σ. Similarly, for a natural number n, σ^{*n} denotes the set of all normal covers τ of Y such that the n-th star $st^n(\tau)$ of τ refines σ.

Instead of continuous single-valued functions we shall use multi-valued functions with small images of points which we define next.

Let X and Y be topological spaces. By a *multi-valued function* $F: X \to Y$ we mean a rule which associates a non-empty subset $F(x)$ of Y to every point x of X.

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For our approach to equivariant shape theory the following notion of size for multi-valued functions will play the most important role.

Let $F: X \to Y$ be a multi-valued function and let $\alpha \in Cov(X)$ and $\sigma \in Cov(Y)$. We shall say that F is an *equivariant multi-valued* (α, σ)*-function* or a *multi-valued* G^{σ}_{α}*-function* provided for every $A \in \alpha$ there is a $C_A \in \sigma$ with $F(gA) \subset gC_A$ for every $g \in G$. On the other hand, F is an *equivariant multi-valued* σ*-function* or a *multi-valued* G^{σ}*-function* provided there is an $\alpha \in Cov(X)$ such that F is an equivariant multi-valued (α, σ)-function.

LEMMA 2. *Let X and Y be equivariant spaces. An equivariant map $\phi: X \to Y$ is an equivariant multi-valued σ-function for every $\sigma \in Cov(Y)$.*

Proof. Let $\sigma \in Cov(Y)$. Put $\alpha = \phi^{-1}(\sigma)$. For an $A = \phi^{-1}(S)$ in α, where $S \in \sigma$, and an arbitrary $g \in G$ we have

$$\phi(gA) = \bigcup_{a \in A} \phi(ga) = \bigcup_{a \in A} g\phi(a) = \bigcup_{s \in S} gs = gS. \quad \blacksquare$$

LEMMA 3. *Let X, Y and Z be equivariant spaces and let α, β and γ be normal covers of X, Y and Z, respectively. If $H: X \to Y$ is an equivariant multi-valued (α, β)-function and $K: Y \to Z$ is an equivariant multi-valued (β, γ)-function, then $K \circ H$ is an equivariant multi-valued (α, γ)-function.*

Proof. Let $A \in \alpha$. Since H is an equivariant multi-valued (α, β)-function, there is a $B \in \beta$ with $H(gA) \subset gB$ for every $g \in G$. On the other hand, since K is an equivariant multi-valued (β, γ)-function, there is a $C \in \gamma$ such that $K(gB) \subset gC$ for every $g \in G$. Let $g \in G$. Then we obtain the following chain of inclusions $K \circ H(gA) \subset K(gB) \subset gC$. $\blacksquare$

Next we introduce the notions which correspond to the equivalence relation above of equivariant homotopy for equivariant maps.

Let H and K be multi-valued functions between equivariant spaces X and Y and let σ be a normal cover of Y. We shall say that H and K are *equivariantly* σ*-homotopic* or G^{σ}*-homotopic* and write $H \overset{\sigma}{\simeq}_G K$ provided there is an equivariant multi-valued σ-function M from the product $X \times I$ of X and the unit segment $I = [0, 1]$ into Y such that $H(x) = M(x, 0)$ and $K(x) = M(x, 1)$ for every $x \in X$. We shall say that M is an equivariant σ-homotopy that joins H and K or that it realizes the relation (or equivariant σ-homotopy) $H \overset{\sigma}{\simeq}_G K$.

The following lemma is crucial because it provides an adequate substitute for the transitivity of the relation of equivariant homotopy.

LEMMA 4. *Let H, K, and L be multi-valued functions between equivariant spaces X and Y. Let $\sigma \in Cov(Y)$ and $\tau \in \sigma^*$. If $H \overset{\tau}{\simeq}_G K$ and $K \overset{\tau}{\simeq}_G L$, then $H \overset{\sigma}{\simeq}_G L$.*

Proof. By assumption there are normal covers α and β of $X \times I$, an equivariant multi-valued (α, τ)-function $P: X \times I \to Y$, and an equivariant multi-valued (β, τ)-function $Q: X \times I \to Y$ such that $H(x) = P(x, 0)$, $K(x) = P(x, 1)$, $K(x) = Q(x, 0)$, and $L(x) = Q(x, 1)$ for every $x \in X$. Let a normal cover γ of $X \times I$ be a common refinement of α and β. Observe that P and Q are both equivariant multi-valued (γ, τ)-functions. Define a multi-valued function $M: X \times I \to Y$ by the rule

$$M(x,t) = \begin{cases} P(x, 2t), & x \in X, \quad 0 \leqslant t \leqslant \frac{1}{2} \\ Q(x, 2t-1), & x \in X, \quad \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

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Since $H(x) = M(x, 0)$ and $L(x) = M(x, 1)$ for every $x \in X$, it remains to see that M is an equivariant multi-valued σ-function.

By [7, p. 358], there is a normal cover δ of X and a function $r\colon \delta \to \{2, 3, 4, \ldots\}$ such that the set $D \times D_i$ is contained in a member $C_{D,i}$ of γ for every $D \in \delta$ and every $i = 1, \ldots, rD$, where $D_i = [(i-1)/rD,\ i/rD]$ $(i = 1, \ldots, rD)$. Let

$$E_i = \begin{cases} [0, 1/2rD), & i = 0, \\ (i/4rD, (i+2)/4rD), & i = 1, \ldots, 4rD-3, \\ (1-(1/2rD), 1], & i = 4rD-2. \end{cases}$$

The collection $\epsilon = \{D \times E_i \mid D \in \delta, i = 0, 1, \ldots, 4rD-2\}$ is a normal cover of $X \times I$. In order to prove that M is an equivariant multi-valued σ-function, we shall show that it is in fact an equivariant multi-valued (ϵ, σ)-function.

Let $E = D \times E_i$ be a member of ϵ. We must find an $S_E \in \sigma$ such that $M(gE) \subset gS_E$ for every $g \in G$.

Case I ($i = 2k$ for $0 \leqslant k \leqslant rD-1$). Then $E_i = E_{2k}$ is below $\frac{1}{2}$ so that for every $g \in G$ we get

$$\begin{aligned} M(gE) = M((gD) \times E_i) &= P((gD) \times (2E_i)) \\ &= P(g(D \times (2E_i))) \subset P(gV_k) \subset P(gC_{D,k}) \subset gW_k \subset gS_k, \end{aligned}$$

where $V_k = D \times D_k$, $W_k = \tau(P, C_{D,k})$ is a member of τ which we obtain from the fact that P is an equivariant multi-valued (γ, τ)-function, and S_k is a member of σ which contains W_k. Thus, in this case we can take S_k for S_E.

Case II ($i = 2k$ for $rD \leqslant k \leqslant 2rD-1$). Then $E_i = E_{2k}$ is above $\frac{1}{2}$ so that for every $g \in G$ we get

$$\begin{aligned} M(gE) = M((gD) \times E_i) &= Q((gD) \times (2E_i - 1)) \\ &= Q(g(D \times (2E_i - 1))) \subset Q(gV_{k-rD}) \subset Q(gC_{D,k-rD}) \subset gW_{k-rD} \subset gS_{k-rD}. \end{aligned}$$

Thus, in this case we can take S_{k-rD} for S_E.

Case III ($i = 2k+1$ for $0 \leqslant k \leqslant rD-2$). Then $E_i = E_{2k+1}$ is below $\frac{1}{2}$ so that for every $g \in G$ we get

$$\begin{aligned} M(gE) = M((gD) \times E_i) &= P((gD) \times (2E_i)) \\ &\subset P((gD) \times D_k) \cup P((gD) \times D_{k+1}) = P(gV_k) \cup P(gV_{k+1}) \subset g[W_k \cup W_{k+1}]. \end{aligned}$$

Observe that $D \times \{(k+1)/rD\} \subset C_{D,k} \cap C_{D,k+1}$ so that for $g = e$ (the neutral element) the relations above give

$$\varnothing \neq P(D \times \{(k+1)/rD\}) \subset W_k \cap W_{k+1}.$$

Let S be a member of σ which contains the union of W_k and W_{k+1}. For every $g \in G$ we obtain $g[W_k \cup W_{k+1}] \subset gS$, so that $M(gE) \subset gS$. Thus, in this case we can take S for S_E.

Case IV ($i = 2k+1$ for $rD \leqslant k \leqslant 2rD-1$). This case is analogous to the Case III. This time E_i is above $\frac{1}{2}$ and we must use Q instead of P.

Case V ($i = 2k+1$ and $k = rD-1$). Then $E_i = E^- \cup E^+$, where $E^- = ((2rD-1)/4rD, \frac{1}{2}]$ and $E^+ = [\frac{1}{2}, ((2rD+1)/4rD)$. For every $g \in G$ we get

$$\begin{aligned} M(gE) &= M((gD) \times E^-) \cup M((gD) \times E^+) \\ &= P((gD) \times (2E^-)) \cup Q((gD) \times (2E^+ - 1)) \subset P(gV_{rD-1}) \cup Q(gV_0) \subset gW_{rD-1} \cup gW_0. \end{aligned}$$

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In particular, for $g = e$, we get

$$P(D \times \{1\}) = K(D) \subset W_{rD-1} \quad \text{and} \quad Q(D \times \{0\}) = K(D) \subset W_0.$$

We conclude that $W_0 \cap W_{rD-1} \neq \emptyset$ so that there is a member S of σ which contains both W_0 and W_{rD-1}. It follows that $gW_0 \cup gW_{rD-1} \subset gS$ and therefore $M(gE) \subset gS$ for every $g \in G$. Thus, in this case we can take S for S_E. ∎

Since we shall be using [7, p. 358] quite often, for a space Y and a normal cover α of the product $Y \times I$, we let $D(Y, \alpha)$ denote all normal covers β of Y such that some stacked normal cover over β refines α.

4. *Equivariant multi-nets*

The following two definitions correspond to Borsuk's definitions of fundamental sequence and homotopy for fundamental sequences.

Let X and Y be equivariant spaces. By an equivariant multi-net or a G-multi-net from X into Y we shall mean a collection $\phi = \{F_c \mid c \in Inc(Y)\}$ of multi-valued functions $F_c : X \to Y$ such that for every $\sigma \in Cov(Y)$ there is a $c \in Inc(Y)$ with $F_d \overset{\sigma}{\simeq}_G F_c$ for every $d > c$. We use functional notation $\phi : X \to Y$ to indicate that ϕ is an equivariant multi-net from X into Y. Let $\mathcal{M}n^G(X, Y)$ denote all equivariant multi-nets $\phi : X \to Y$.

Two equivariant multi-nets $\phi = \{H_c\}$ and $\psi = \{K_c\}$ between equivariant spaces X and Y are said to be equivariantly homotopic or G-homotopic and we write $\phi \simeq_G \psi$ provided for every $\sigma \in Cov(Y)$ there is a $c \in Inc(Y)$ such that $H_d \overset{\sigma}{\simeq}_G K_d$ for every $d > c$.

It follows from Lemma (3·4) that the relation of equivariant homotopy is an equivalence relation on the set $\mathcal{M}n^G(X, Y)$. The equivariant homotopy class of an equivariant multi-net ϕ is denoted by $[\phi]_G$ and the set of all equivariant homotopy classes by $\mathcal{S}h^G(X, Y)$.

5. *Composition of equivariant homotopy classes*

In order to show that $\mathcal{S}h^G$ with G-spaces as objects and with equivariant homotopy classes of G-multi-nets as morphisms is our desired equivariant shape category we must define a composition for equivariant homotopy classes of equivariant multi-nets and establish that it is associative. The present section is only for this task.

Our principal goal now is to define a composition for equivariant homotopy classes of equivariant multi-nets. This is the only tricky part in setting up the category $\mathcal{S}h^G$. Our idea is to associate to every equivariant multi-net $\phi : X \to Y$ two increasing functions $\phi : Inc(Y) \to Inc(Y)$ and $\phi^* : Inc(Y) \to Inc(X)$. The first function associates to an index $c \in Inc(Y)$ of the family $\phi = \{F_c\}$ a much larger index $\phi(c)$ in $Inc(Y)$ such that F_d and F_e are joined by a multi-valued $G^{[c]}_{\phi(c,d,e)}$-homotopy for a suitable normal cover $\phi(c, d, e)$ of the cylinder $X \times I$ whenever $d, e > \phi(c)$. The second function associates to a $c \in Inc(Y)$ an element $\phi^*(c)$ of $Inc(X)$ such that the normal cover $[\phi^*(c)]$ refines some normal cover of X which we get from the normal cover $\phi(c, \phi(c), \phi(c))$ by [5, p. 358]. This is a rough description of these functions and now we proceed with the details.

Let $\phi = \{F_c\} : X \to Y$ be an equivariant multi-net. Let $\phi : Inc(Y) \to Inc(Y)$ be an increasing function such that for every $c \in Inc(Y)$ the relation $d, e > \phi(c)$ implies the relation $F_d \overset{[c]}{\simeq}_G F_e$.

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Let $\mathscr{C}_\phi = \{(c,d,e) \mid c \in Inc(Y), d, e > \phi(c)\}$. Then the set $\mathscr{C}_\phi$ is the subset of the product $Inc(Y) \times Inc(Y) \times Inc(Y)$ that becomes a cofinite directed set when we define that $(c,d,e) > (c',d',e')$ if and only if $c > c'$, $d > d'$, and $e > e'$.

We shall use the same notation ϕ for an increasing function $\phi: \mathscr{C}_\phi \to Cov(X \times I)$ such that F_d and F_e are joined by an equivariant $(\phi(c,d,e),[c])$-homotopy whenever $(c,d,e) \in \mathscr{C}_\phi$.

Let $\bar{\phi}: \mathscr{C}_\phi \to Inc(X)$ be an increasing function such that the normal cover $[\bar{\phi}(c,d,e)]$ belongs to the set $D(X, \phi(c,d,e))$ for every $(c,d,e) \in \mathscr{C}_\phi$.

Claim 1. *There is an increasing function $\phi^*: Inc(Y) \to Inc(X)$ such that*

$$\phi^*(c) > \bar{\phi}(c, \phi(c), \phi(c)) \text{ for every } c \in Inc(Y), \tag{1}$$

and

ϕ^* is cofinal in $\bar{\phi}$, i.e. for every $(c,d,e) \in \mathscr{C}_\phi$ there is an $m \in Inc(Y)$

$$\text{with } \phi^*(m) > \bar{\phi}(c,d,e). \tag{2}$$

Proof. Let $\mathscr{D} = \{\bar{\phi}(c,d,e) \mid (c,d,e) \in \mathscr{C}_\phi\}$.

If $Inc(Y)$ is a finite set, then $\mathscr{D}$ is a finite collection of elements of $Inc(X)$. Let $a \in Inc(X)$ be greater than all members of $\mathscr{D}$. Let $\phi^*: Inc(Y) \to Inc(X)$ be a constant function into a.

If $Inc(Y)$ is an infinite set, then the cardinality of $\mathscr{D}$ does not exceed the cardinality of $Inc(Y)$. Hence, there is a surjection $g: Inc(Y) \to \mathscr{D}$. Let $\phi^*: Inc(Y) \to Inc(X)$ be an increasing function such that $\phi^*(c) > g(c)$, $\bar{\phi}(c, \phi(c), \phi(c))$ for every $c \in Inc(Y)$. ∎

The discussion above shows that every equivariant multi-net $\phi: X \to Y$ determines two increasing functions $\phi: Inc(Y) \to Inc(Y)$ and $\phi^*: Inc(Y) \to Inc(X)$. With the help of these functions we shall define the composition of equivariant homotopy classes of equivariant multi-nets as follows.

Let X, Y and Z be equivariant spaces and let $\phi = \{F_c\}: X \to Y$ and $\psi = \{G_s\}: Y \to Z$ be equivariant multi-nets. Let $\chi = \{H_s\}$, where $H_s = G_{\psi(s)} \circ F_{\phi(\psi^*(s))}$ for every $s \in Inc(Z)$. Observe that each H_s is a multi-valued function because the composition of two multi-valued functions is a multi-valued function. In the following claim we shall prove that the increasing functions ϕ and ϕ^* have been well selected because the collection χ is an equivariant multi-net.

Claim 2. *The collection χ is an equivariant multi-net from X into Z.*

Proof. Let $\sigma \in Cov(Z)$. We must find a $u \in Inc(Z)$ such that

$$H_v \overset{\sigma}{\simeq}_G H_u \quad \text{for every } v > u. \tag{3}$$

Let $\tau \in \sigma^{*2}$. Let $u = \{\tau\} \in Inc(Z)$.

Consider an index $v > u$. We shall find an index $e \in Inc(Y)$ so that

$$H_v \overset{\tau}{\simeq}_G G_y \circ F_e, \tag{4}$$

$$G_y \circ F_e \overset{\tau}{\simeq}_G G_x \circ F_e \tag{5}$$

and

$$G_x \circ F_e \overset{\tau}{\simeq}_G H_u, \tag{6}$$

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where $x = \psi(u)$, $y = \psi(v)$, $a = \psi^*(u)$, $b = \psi^*(v)$, $c = \phi(a)$, and $d = \phi(b)$. Repeated use of Lemma (3·4) will give (3) from the relations (4)–(6).

Add (4). Since $(v, y, y) \in \mathscr{C}_\psi$, we see that there is an equivariant multi-valued (α, τ)-function $K: Y \times I \to Z$ such that $\alpha = \psi(v, y, y)$, $K_0 = G_y$ and $K_1 = G_y$. Let $s = \bar{\psi}(v, y, y)$ and $\pi = [s]$. Observe that $\pi \in D(Y, \alpha)$. We claim that G_y is an equivariant multi-valued (π, τ)-function.

Indeed, let P be a member of π. Then there is a number $t > 0$ such that the product $P \times [0, t)$ lies in a member A of α. Since K is an equivariant multi-valued (α, τ)-function, there is a $B \in \tau$ such that $K(gA) \subset gB$ for every $g \in G$. It follows that

$$G_y(gP) = K_0(gP) = K((gP) \times \{0\}) = K(g(P \times \{0\})) \subset K(gA) \subset gB$$

for every $g \in G$.

Once we know that G_y is an equivariant multi-valued (π, τ)-function, we see that it suffices to take $e > d$ because then F_d and F_e are joined by an equivariant $[b]$-homotopy $L: X \times I \to Y$. But, by construction, the cover $[b]$ refines π so that the composition $G_y \circ L$ is an equivariant τ-homotopy joining H_v and $G_y \circ F_e$.

Add (5). Since $(u, x, y) \in \mathscr{C}_\psi$, it follows that the multi-valued functions G_x and G_y are joined by an equivariant (α, τ)-homotopy $K: Y \times I \to Z$, where α denotes the normal cover $\psi(u, x, y)$ of $Y \times I$. Choose a normal cover β of Y and a function $r: \beta \to \{4, 5, 6, \ldots\}$ such that every set $B \times [(i-1)/rB, (i+1)/rB]$, where $B \in \beta$ and $i = 1, \ldots, rB-1$, is contained in a member of α. Then F_e is an equivariant multi-valued (δ, β)-function for some $\delta \in Cov(X)$ so that for every $D \in \delta$ there is a $B_D \in \beta$ such that $F_e(gD) \subset gB_D$ for every $g \in G$.

For every $D \in \delta$, let

$$\gamma_D = \left\{\left[0, \frac{2}{rB_D}\right), \left(\frac{rB_D - 2}{rB_D}, 1\right]\right\} \cup \left\{\left(\frac{i}{rB_D}, \frac{i+2}{rB_D}\right) \middle| i = 1, \ldots, rB_D - 3\right\}.$$

Put $\epsilon = \{D \times C \mid D \in \delta, C \in \gamma_D\}$. Observe that ϵ is a normal cover of $X \times I$. We claim that the product $F_e \times id_I$ is an equivariant multi-valued (ϵ, α)-function.

Indeed, let $E = D \times C$, where $D \in \delta$ and $C \in \gamma_D$. For every $g \in G$, we obtain

$$(F_e \times id_I)(gE) = (F_e \times id_I)((gD) \times C) \subset (gB_D) \times C = g(B_D \times C) \subset gA,$$

where A is a member of α which contains the product $B_D \times C$.

Once we know that $F_e \times id_I$ is an equivariant multi-valued α-function we conclude that the composition $K \circ (F_e \times id_I)$ is an equivariant τ-homotopy which joins $G_x \circ F_e$ and $G_y \circ F_e$.

Add (6). This is analogous to the proof of (4). ∎

Now that we know that the family χ is an equivariant multi-net, we can define the composition of equivariant homotopy classes of equivariant multi-nets by the rule $[\{G_s\}]_G \circ [\{F_c\}]_G = [\{G_{\psi(s)} \circ F_{\phi(\psi^*(s))}\}]_G$.

In the following claim we shall show that this definition is correct, i.e. that it is independent of the selection of representatives.

CLAIM 3. *The composition of equivariant homotopy classes of equivariant multi-nets is well-defined.*

Proof. Let $\kappa = \{K_c\}$ and $\lambda = \{L_s\}$ be equivariant multi-nets equivariantly homotopic to ϕ and ψ, respectively, and let $\mu = \{M_s\}$, where $M_s = L_{\lambda(s)} \circ K_{\kappa(\lambda^*(s))}$ for every

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$s \in Inc(Z)$. We must show that equivariant multi-nets χ and μ are equivariantly homotopic. In other words, that for every $\sigma \in Cov(Z)$ there is an $s \in Inc(Z)$ such that

$$H_t \overset{\sigma}{\simeq}_G M_t \quad \text{for every} \quad t > s. \tag{7}$$

Let $\sigma \in Cov(Z)$ and $\tau \in \sigma^{*4}$. Let $s = \{\tau\} \in Inc(Z)$. In order to prove (7), we shall argue that for every $t > s$ we can find indices $e \in Inc(Y)$ and $u \in Inc(Z)$ such that

$$H_t \overset{\tau}{\simeq}_G G_x \circ F_e, \tag{8}$$

$$G_x \circ F_e \overset{\tau}{\simeq}_G G_u \circ F_e, \tag{9}$$

$$G_u \circ F_e \overset{\tau}{\simeq}_G L_u \circ F_e, \tag{10}$$

$$L_u \circ F_e \overset{\tau}{\simeq}_G L_u \circ K_e, \tag{11}$$

$$L_u \circ K_e \overset{\tau}{\simeq}_G L_y \circ K_e, \tag{12}$$

$$L_y \circ K_e \overset{\tau}{\simeq}_G M_t, \tag{13}$$

where we put $x = \psi(t)$, $a = \psi^*(t)$, $b = \phi(a)$, $y = \lambda(t)$, $c = \lambda^*(t)$, and $d = \kappa(c)$. From the relations (8)–(13), with the help of Lemma (3·4), we shall get (7).

We shall now describe how big e and u must be chosen for relations (8), (9), (10) and (11) to hold separately. The relations (12) and (13) are analogous to relations (9) and (8), respectively. We leave to the reader the task of making a cumulative choice for e and u which accomplishes our goal. It is important to notice that u is selected first while e is selected only once u is already known.

Add (8). We know from the proof of Claim 2 that G_x is an equivariant multi-valued $([s], \sigma)$-function, where $s = \bar{\psi}(t, x, x)$. Since $a > s$ by the property (1) of Claim 1, it suffices to take $e > b$.

Add (9). If $u > x$, then G_x and G_u are joined by an equivariant (π, τ)-homotopy $P: Y \times I \to Z$, where $\pi = \bar{\psi}(u, m, m)$ and $m = \psi(u)$. Choose a normal cover β of Y and a function $r: \beta \to \{4, 5, 6, \ldots\}$ such that every set $B \times [(i-1)/rB, (i+1)/rB]$, where $B \in \beta$ and $i = 1, \ldots, rB-1$, is contained in a member of π. Let $e > \phi(\{\beta\})$. Just as in the proof of (15) we can see that $F_e \times id_I$ is an equivariant multi-valued π-function, so that the composition $P \circ (F_e \times id_I)$ is an equivariant τ-homotopy joining the left and right side of the relation (9).

Add (10). Since the equivariant multi-nets ψ and λ are equivariantly homotopic, there is a $u \in Inc(Z)$, a normal cover π of $Y \times I$, and an equivariant (π, τ)-homotopy $S: Y \times I \to Z$ joining G_u and L_u. Choose a β, an r and an e as above. Then $S \circ (F_e \times id_I)$ is an equivariant τ-homotopy joining compositions which appear in (10).

Add (11). Let $u > y$. Then L_u is an equivariant multi-valued (π, τ)-function, where $\pi = [p]$ and $p = \bar{\lambda}(t, y, y)$. Since the equivariant multi-nets ϕ and κ are equivariantly homotopic, there is an index $e \in Inc(Y)$ so that F_e and K_e are joined by an equivariant π-homotopy $T: X \times I \to Y$. The composition $L_u \circ T$ realizes the relation (11). ∎

The following theorem is the main result of this section.

THEOREM 1. *The composition of equivariant homotopy classes of equivariant multi-nets is associative.*

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Proof. Let $\phi = \{F_c\}$, $\psi = \{G_s\}$, and $\chi = \{H_p\}$ be equivariant multi-nets from X into Y, from Y into Z and from Z into W, respectively. Let $\mu = \{M_s\}$, $\nu = \{N_p\}$, $\kappa = \{K_p\}$ and $\lambda = \{L_p\}$, where $M_s = G_{\psi(s)} \circ F_{\phi(\psi^*(s))}$ for every $s \in Inc(Z)$ and $N_p = H_{\chi(p)} \circ G_{\psi(\chi^*(p))}$, $K_p = H_{\chi(p)} \circ M_{\mu(\chi^*(p))}$ and $L_p = N_{\nu(p)} \circ F_{\phi(\nu^*(p))}$, for every $p \in Inc(W)$. We must show that equivariant multi-nets κ and λ are equivariantly homotopic, i.e. that for every $\pi \in Cov(W)$ there is a $p \in Inc(W)$ such that

$$K_q \overset{\pi}{\simeq}_G L_q \quad \text{for every} \quad q > p. \tag{14}$$

Let $\pi \in Cov(W)$. Let $\rho \in \pi^{*4}$. Let $p = \{\rho\} \in Inc(W)$. In order to prove (14), we shall show that for every $q > p$ we can find indices $e \in Inc(Y)$ and $s \in Inc(Z)$ such that

$$K_q \overset{\rho}{\simeq}_G H_x \circ G_y \circ F_e, \tag{15}$$

$$H_x \circ G_y \circ F_e \overset{\rho}{\simeq}_G H_x \circ G_s \circ F_e, \tag{16}$$

$$H_x \circ G_s \circ F_e \overset{\rho}{\simeq}_G H_z \circ G_s \circ F_e, \tag{17}$$

$$H_z \circ G_s \circ F_e \overset{\rho}{\simeq}_G N_w \circ F_e, \tag{18}$$

and

$$N_w \circ F_e \overset{\rho}{\simeq}_G L_q, \tag{19}$$

where $x = \chi(q)$, $y = \psi(\mu(\chi^*(q)))$, $z = \chi(\nu(q))$ and $w = \nu(q)$. Repeated use of Lemma (3·4) will give (14) from the relations (15)–(19).

The method of proof is similar to the proof of Claim 3. We shall only describe for each of the relations (15)–(19) how large the indices u and e must be in order that this equivariant ρ-homotopy holds. An easy exercise of putting together all these selections is once again left to the reader. Since relations (18) and (19) are analogous with relations (16) and (15), respectively, it suffices to consider only relations (15)–(17).

Add (15). Observe that H_x is an equivariant multi-valued (θ, ρ)-function, where $\theta = [a]$, $a = \bar{\chi}(q, x, x)$, $\alpha = \chi(q, x, x)$ and $\theta \in D(Z, \alpha)$. Let $m = \chi^*(q)$, $n = \mu(m)$, $k = \psi(n)$, $d = \bar{\psi}(n, k, k)$, $c = \psi^*(n)$ and $\xi = [d]$.On the other hand, G_y is an equivariant multi-valued (ξ, θ)-function because $n > m$ so that $[n]$ refines θ. Let $b = \phi(c)$. If $e > b$, then F_e and F_b are joined by an equivariant $[c]$-homotopy P. But $[c]$ refines ξ. Hence $H_x \circ G_y \circ P$ is an equivariant ρ-homotopy between K_q and $H_x \circ G_y \circ F_e$.

Add (16). As above, H_x is an equivariant multi-valued (θ, ρ)-function. Since $\mu(s) > s$ for every $s \in Inc(Z)$, we get $y > k$. Therefore if we take $s > y$ then G_y and G_s are joined by an equivariant (ϵ, θ)-homotopy $Q: Y \times I \to Z$, where $\epsilon = \psi(n, y, s)$. Let a normal cover β of Y, a function $r: \beta \to \{4, 5, 6, \ldots\}$, and an index $e > \phi(\{\beta\})$ be selected with respect to ϵ similarly as in the proof of Claim 2 we have chosen them with respect to π. Then $H_x \circ Q \circ (F_e \times id_I)$ realizes the relation (16).

Add (17). Since $\nu(r) > r$ for every $r \in Inc(W)$, we get $z > x$, so that H_x and H_z are joined by an equivariant $([b], \rho)$-homotopy $T: Z \times I \to W$, where $b = \bar{\chi}(z, \eta(z), \eta(z))$. Let $v = \chi^*(z)$ and let $s > \psi(v)$. Then G_s is an equivariant multi-valued $([t], [v])$-function, where $t = \bar{\psi}(s, \psi(s), \psi(s))$. Let $u = \psi^*(t)$ and take $e > \phi(u)$. The composition $T \circ ((G_s \circ F_e) \times id_I)$ realizes the relation (17). ∎

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6. *The equivariant shape category* $\mathcal{S}h^G$

For an equivariant space X, let $\iota^X = \{I_a\}: X \to X$ be the identity equivariant multi-net defined by $I_a = id_X$ for every $a \in Inc(X)$. It is easy to show that for every equivariant multi-net $\phi: X \to Y$, the following relations hold:

$$[\phi]_G \circ [\iota^X]_G = [\phi]_G = [\iota^Y]_G \circ [\phi]_G.$$

We can summarize the above with the following theorem which is the main result and objective of this paper.

THEOREM 2. *The topological spaces endowed with an action of a given topological group G as objects together with equivariant homotopy classes of equivariant multi-nets as morphisms and the composition of equivariant homotopy classes form the category $\mathcal{S}h^G$. When G is a trivial group, the category $\mathcal{S}h^G$ is isomorphic to the shape category $\mathcal{S}h$.*

7. *Special equivariant multi-nets*

In this section we shall be looking for conditions under which an equivariant homotopy class has a representative of a special kind. As a corollary of these results we identify a class of spaces on which equivariant shape theory and equivariant homotopy theory coincide.

An equivariant space X is *internally G-movable* provided for every $\sigma \in Cov(X)$ there is a $\tau \in Cov(X)$ such that every equivariant multi-valued τ-function into X is equivariantly σ-homotopic to an equivariant map.

An equivariant space X is *internally G-calm* provided there is a $\sigma \in Cov(X)$ such that equivariant maps into X which are equivariantly σ-homotopic are equivariantly homotopic.

An equivariant space X is *G-calm* provided there is a $\sigma \in Cov(X)$ such that for every $\tau \in Cov(X)$ there is a $\rho \in Cov(X)$ with the property that equivariant multi-valued ρ-functions into X which are equivariantly σ-homotopic are also equivariantly τ-homotopic.

An equivariant multi-net $\phi = \{F_c\}_{c \in Inc(Y)}$ between equivariant spaces X and Y is *regular* provided each function F_c is an equivariant map. It is called *simple* when there is an equivariant map f such that $f = F_c$ for every $c \in Inc(Y)$.

THEOREM 3. *If an equivariant space Y is internally G-movable, then every equivariant multi-net ϕ from an equivariant space X into Y is equivariantly homotopic to a regular equivariant multi-net.*

Proof. Since Y is internally G-movable, for every $c \in Inc(Y)$ there is a $\chi(c) \in [c]^+$ such that every equivariant multi-valued $\chi(c)$-function into Y is equivariantly $[c]$-homotopic to an equivariant map. Let $\psi: Inc(Y) \to Inc(Y)$ be an increasing function such that $\psi(c) > \phi(\chi(c))$ for every $c \in Inc(Y)$. Then $F_{\psi(c)}$ is an equivariant multi-valued $\chi(c)$-function so that we can find an equivariant map $g_c: X \to Y$ with $g_c \overset{[c]}{\simeq}_G F_{\psi(c)}$, for every $c \in Inc(Y)$.

In order to verify that $\psi = \{g_c\}_{c \in Inc(Y)}$ is an equivariant multi-net, let a $\sigma \in Cov(Y)$ be given. Let $\mu \in \sigma^*$ and put $c = \{\mu\}$. For every $d > c$ we have

$$g_d \overset{\mu}{\simeq}_G F_{\psi(d)} \overset{\chi(c)}{\simeq}_G F_{\psi(c)} \overset{\mu}{\simeq}_G g_c.$$

Hence $g_d \overset{\sigma}{\simeq}_G g_c$ for every $d > c$.

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It remains to check that ϕ and ψ are equivariantly homotopic. Let a $\sigma \in Cov(Y)$ be given. Let $\mu \in \sigma^*$. Choose an index $c_0 \in Inc(Y)$ such that $F_d \overset{\mu}{\simeq}_G F_e$ for all $d, c > c_0$. Let $c > c_0, \{\mu\}$. For every $d > c$, we get $g_d \overset{\mu}{\simeq}_G F_{\psi(d)}$ by construction, while $F_{\psi(d)} \overset{\mu}{\simeq}_G F_d$ because $\psi(d) > d > c_0$. Hence, $g_d \overset{\sigma}{\simeq}_G F_d$ for every $d > c$.

THEOREM 4. *If an equivariant space Y is both internally G-movable and G-calm, then every equivariant multi-net ϕ into Y is equivariantly homotopic to a simple equivariant multi-net.*

Proof. Since Y is G-calm, there is a $\gamma \in Cov(Y)$ such that for every $\sigma \in Cov(Y)$ there is a $\tau \in Cov(Y)$ with the property that equivariantly γ-homotopic equivariant multi-valued τ-functions into Y are in fact equivariantly σ-homotopic.

Let $\delta \in \gamma^*$. Since Y is also internally G-movable, there is an $\eta \in \delta^+$ such that equivariant multi-valued η-functions into Y are equivariantly δ-homotopic to equivariant maps. Let $c = \phi(\{\eta\})$. Then F_c is an equivariant multi-valued η-function so that it is equivariantly δ-homotopic to an equivariant map g. Let ψ denote the simple equivariant multi-net determined by the equivariant map g.

In order to check that ϕ and ψ are equivariantly homotopic, let a $\sigma \in Cov(Y)$ be given. Choose a $\tau \in Cov(Y)$ as above. Since ϕ is an equivariant multi-net, there is an index $d > c$ such that F_e is an equivariant multi-valued τ-function for every $e > d$. Thus, for every $e > d$ we get $F_e \overset{\eta}{\simeq}_G F_c \overset{\delta}{\simeq}_G g$. Hence, $F_e \overset{\gamma}{\simeq}_G g$ so that $F_e \overset{\sigma}{\simeq}_G g$. ∎

THEOREM 5. *Let Y be an internally G-calm equivariant space and let $\phi = \{f\}$ and $\psi = \{g\}$ be simple equivariant multi-nets into Y. If ϕ and ψ are equivariantly homotopic, then the equivariant maps f and g are equivariantly homotopic.*

Proof. Since Y is internally G-calm, there is a $\sigma \in Cov(Y)$ such that equivariantly σ-homotopic equivariant maps into Y are in fact equivariantly homotopic. But, the assumption that equivariant multi-nets ϕ and ψ are equivariantly homotopic gives $f \overset{\sigma}{\simeq}_G g$. Hence, f and g are equivariantly homotopic. ∎

There is an obvious functor J^G from the category $\mathscr{H}^G$ of equivariant spaces and equivariant homotopy classes of equivariant maps into the category $\mathscr{S}h^G$. On objects the functor J^G is the identity while on morphisms it associates to an equivariant homotopy class $[f]_G$ of an equivariant map $f: X \to Y$ the equivariant homotopy class $[\underline{f}]_G$ of the equivariant multi-net $\underline{f} = \{F_c\}: X \to Y$, where $F_c = f$ for every $c \in Inc(Y)$.

The last three theorems imply that the functor J^G is an isomorphism of categories when we restrict to spaces that have the properties above.

Let $\mathscr{A}$ denote the collection of all equivariant spaces that are at the same time internally G-movable, internally G-calm and G-calm. One can show that every G-ANR is in the class $\mathscr{A}$.

Let $\mathscr{H}^G_{\mathscr{A}}$ be the full subcategory of $\mathscr{H}^G$ with objects precisely the members of the collection $\mathscr{A}$. The category $\mathscr{S}h^G_{\mathscr{A}}$ is defined similarly. Let $J^G_{\mathscr{A}}: \mathscr{H}^G_{\mathscr{A}} \to \mathscr{S}h^G_{\mathscr{A}}$ be the restriction of the functor J^G to the category $\mathscr{H}^G_{\mathscr{A}}$.

THEOREM 6. *The functor $J^G_{\mathscr{A}}$ is an isomorphism of categories.*

Proof. We shall construct a functor $K^G_{\mathscr{A}}: \mathscr{S}h^G_{\mathscr{A}} \to \mathscr{H}^G_{\mathscr{A}}$ which satisfies the relations

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$J^G_{\mathscr{A}} \circ K^G_{\mathscr{A}} = Id$ and $K^G_{\mathscr{A}} \circ J^G_{\mathscr{A}} = Id$. The functor $K^G_{\mathscr{A}}$ leaves the objects unchanged and on morphisms is defined as follows. Let C be an equivariant homotopy class of equivariant multi-nets between two members X and Y of $\mathscr{A}$. Let ϕ be a representative of C and let $g: X \to Y$ be an equivariant map such that the simple equivariant multi-net ψ determined by g is equivariantly homotopic to ϕ. The functor $K^G_{\mathscr{A}}$ associates to C the equivariant homotopy class of the equivariant map g. It follows from the above results that this definition is correct and that $K^G_{\mathscr{A}}$ has the required properties.

COROLLARY 1. *On equivariant spaces which are at the same time internally G-movable, internally G-calm and G-calm, the classical equivariant homotopy theory and the equivariant shape theory coincide.*

8. *Characterization of equivariant shape equivalences*

In this section we shall use classes of internally G-movable and internally G-calm spaces to obtain an analogue for equivariant maps of G. Kozlowski's characterization of shape equivalences in terms of induced functions of homotopy classes [**9**].

For G-spaces X and Y, let $[X, Y]_G$ denote the set of all G-homotopy classes of G-maps from X into Y. Every G-map $f: X \to Y$ induces for every G-space Z two set functions $f^\sharp_Z: [Z, X]_G \to [Z, Y]_G$ and $f^Z_\sharp: [Y, Z]_G \to [X, Z]_G$ which are defined by the rules $f^\sharp_Z([a]_G) = [f \circ a]_G$ for every G-map $a: Z \to X$ and $f^Z_\sharp([b]_G) = [b \circ f]_G$ for every G-map $b: Y \to Z$.

Recall that a morphism ϕ of a category $\mathscr{C}$ is called left $\mathscr{C}$-invertible provided there is another morphism ψ of $\mathscr{C}$ with the composition $\psi \circ \phi$ equal to the identity morphism. The right $\mathscr{C}$-invertible morphisms (or shortly $\mathscr{C}$-dominations) are defined similarly. A morphism of $\mathscr{C}$ is a $\mathscr{C}$-equivalence if and only if it is both left $\mathscr{C}$-invertible and right $\mathscr{C}$-invertible. When applied to G-maps and the equivariant shape category $\mathscr{S}h^G$ this notions mean that the equivariant shape morphism determined by this G-map has them.

The following six theorems are similar to the characterization of shape equivalences from [**9**]. We get information on conditions which imply that functions $f^Z_\sharp$ and $f^\sharp_Z$ induced by a G-map $f: X \to Y$ and a G-space Z are surjections and injections and we get information on what kind of G-spaces Z one can use.

THEOREM 7. *If a G-map $f: X \to Y$ between G-spaces is left $\mathscr{S}h^G$-invertible, then for every G-space Z which is at the same time internally G-movable and internally G-calm the induced function $f^Z_\sharp$ is a surjection.*

Proof. Let $h: X \to Z$ be a G-map. The assumptions about the G-space Z imply the existence of normal covers $\sigma \in Cov(Z)$, $\tau \in \sigma^*$, and $\mu \in \tau^+$ such that G^σ-homotopic G-maps into Z are G-homotopic and every multi-valued G^μ-function into Z is G^τ-homotopic to a G-map. Let $\rho = h^{-1}(\mu)$. Observe that ρ is a normal cover of X. Since f is left $\mathscr{S}h^G$-invertible, there is a G-multi-net $\psi: Y \to X$ with $\psi \circ f$ G-homotopic to the identity G-multi-net ι^X on X. In particular, there is an index $a > \{\rho\}$ in $Inc(X)$ such that $G_p \circ f \overset{\rho}{\simeq}_G 1_X$, where $p = \psi(a)$. Let $d = \psi^*(a)$.

The function G_p is a multi-valued $G^\rho_{[d]}$-function so that there is a G-map $k: Y \to Z$ which is G^τ-homotopic to the composition $h \circ G_p$. It follows that $k \circ f$ and h are G^σ-homotopic and therefore G-homotopic. In other words, $f^Z_\sharp([k]_G) = [h]_G$.

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THEOREM 8. *If $f: X \to Y$ is a right $\mathcal{S}h^G$-invertible G-map from an internally G-movable G-space X into an internally G-calm G-space Y, then for every G-space Z the induced function $f_Z^\sharp$ is a surjection.*

Proof. The assumptions about the G-spaces X and Y imply the existence of normal covers $\sigma \in Cov(Y)$, $\tau \in \sigma^*$, $\mu = f^{-1}(\tau)$ and $\nu \in \mu^+$ such that G^σ-homotopic G-maps into Y are G-homotopic and every multi-valued G^ν-function into X is G^μ-homotopic to a G-map. Since f is right $\mathcal{S}h^G$-invertible, there is a G-multi-net $\psi: Y \to X$ with $f \circ \psi$ G-homotopic to the identity G-multi-net ι^Y on Y. In particular, there is an index $a > \{\tau\}$ in $Inc(X)$ such that $f \circ G_{\psi(p)} \overset{\tau}{\simeq}_G 1_Y$, where $p = \phi^*(a)$ and ϕ denotes the G-multi-net determined by the G-map f. Pick an index $q > \psi(p)$ such that G_q is a multi-valued G^ν-function. We can assume that G_q and $G_{\psi(p)}$ are joined by a G^μ-homotopy.

Let Z be a G-space. In order to prove that the induced function $f_Z^\sharp$ is a surjection, let $h: Z \to Y$ be a G-map. The composition $G_q \circ h$ is a multi-valued G^ν-function of Z into X so that it is G^μ-homotopic to a G-map $k: Z \to X$. Then $f \circ k$ and h are G^σ-homotopic and therefore also G-homotopic. Hence the function $f_Z^\sharp$ is onto.

THEOREM 9. *If a G-map $f: X \to Y$ between G-spaces is right $\mathcal{S}h^G$-invertible, then for every internally G-calm G-space Z the induced function $f_\sharp^Z$ is an injection.*

Proof. Let $h, k: Y \to Z$ be G-maps and assume that $f_\sharp^Z([h]_G) = f_\sharp^Z([k]_G)$, i.e. that $h \circ f$ and $k \circ f$ are joined by a G-homotopy $m: X \times I \to Z$. Since Z is internally G-calm, there is a $\sigma \in Cov(Z)$ such that G^σ-homotopic G-maps into Z are G-homotopic. Let $\mu \in \sigma^*$. Let ρ be a common refinement of $h^{-1}(\mu)$ and $k^{-1}(\mu)$. Let $\pi \in \rho^*$. Since f is a $\mathcal{S}h^G$-domination, there is a G-multi-net $\psi: Y \to X$ with the composition $f \circ \psi$ G-homotopic to the identity G-multi-net ι^Y on Y. Hence, there is a multi-valued G^π-homotopy joining 1_Y and $f \circ G_p$, where $\lambda = f^{-1}(\pi)$, $n = \{\lambda\}$, and $p = \psi(n)$.

Let $\alpha \in D(X, m^{-1}(\rho))$. Pick an index $q > p$ such that G_q is a multi-valued G^α-function. Then $G_q \overset{\lambda}{\simeq}_G G_p$ so that

$$f \circ G_q \overset{\pi}{\simeq}_G f \circ G_p \overset{\pi}{\simeq}_G 1_Y.$$

It follows that $h \circ f \circ G_q \overset{\mu}{\simeq}_G h$ and similarly $k \circ f \circ G_q \overset{\mu}{\simeq}_G k$. On the other hand $m \circ (G_q \times 1_I)$ is a multi-valued G^μ-homotopy joining $h \circ f \circ G_q$ and $k \circ f \circ G_q$. Hence $h \overset{\sigma}{\simeq}_G k$ and therefore $h \simeq_G k$.

THEOREM 10. *If $f: X \to Y$ is a left $\mathcal{S}h^G$-invertible G-map from an internally G-calm G-space X into a G-space Y, then for every G-space Z the induced function $f_Z^\sharp$ is an injection.*

Proof. Since X is internally G-calm, there is a $\sigma \in Cov(X)$ such that G-maps into X which are G^σ-homotopic are also G-homotopic. Let $\tau \in \sigma^*$. By assumption, there is a G-multi-net $\psi: Y \to X$ such that $\psi \circ f \simeq_G \iota^X$.

Let Z be a G-space and assume that $a, b: Z \to X$ are G-maps with $f \circ a \simeq_G f \circ b$. Pick an index $p > \{\tau\}$ in $Inc(X)$ such that $G_{\psi(p)} \circ f \overset{\tau}{\simeq}_G 1_X$. Observe that $G_{\psi(p)}$ is a multi-valued G^τ-function, so we get

$$a \overset{\tau}{\simeq}_G G_{\psi(p)} \circ f \circ a \overset{\tau}{\simeq}_G G_{\psi(p)} \circ f \circ b \overset{\tau}{\simeq}_G b.$$

It follows that $a \overset{\sigma}{\simeq}_G b$ so that $a \simeq_G b$ and $f_Z^\sharp$ is injective. ∎

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In order to formulate a partial converse to Theorem 7 and for some other results below we shall need the following notion.

Let $\mathscr{P}$ be a class of G-spaces. By an inverse $G(\mathscr{P})$-exposition of a G-space X we mean a family $\{X_a\}_{a\in Inc(X)}$ of members of $\mathscr{P}$, a family $\{p_b^a \mid a,b\in Inc(X), a\leqslant b\}$ of G-maps $p_b^a: X_b\to X_a$, a family $\{p^a \mid a\in Inc(X)\}$ of G-maps $p^a: X\to X_a$, and a family $\{J_a \mid a\in Inc(X)\}$ of multi-valued $G^{[a]}$-functions such that:

(1) the relation $a\leqslant b$ in $Inc(X)$ implies $p^a \simeq_G p_b^a\circ p^b$,

(2) for every $a\in Inc(X)$, we have $J_a\circ p^a \overset{[a]}{\simeq}_G 1_X$, and

(3) the relation $a\leqslant b$ in $Inc(X)$ implies $J_b \overset{[a]}{\simeq}_G J_a\circ p_b^a$.

THEOREM 11. *Let $\mathscr{P}$ be a class of G-spaces. If a G-space X has an inverse $G(\mathscr{P})$-exposition and a G-map $f: X\to Y$ is such that the induced function $f_\sharp^Z$ is a bijection for every $Z\in\mathscr{P}$, then f is left $\mathscr{S}h^G$-invertible.*

Proof. By assumption the G-space X has an inverse $G(\mathscr{P})$-exposition formed by G-spaces X_a from $\mathscr{P}$, G-maps p_b^a and p^a and multi-valued $G^{[a]}$-functions J_a. We shall define a G-multi-net $\psi: Y\to X$ such that $\psi\circ f$ is G-homotopic to the identity G-multi-net ι^X on X.

Let $a\in Inc(X)$. Since the function $f_\sharp^{X_a}$ is a surjection, there is a G-map $r^a: Y\to X_a$ with $p^a\simeq_G r^a\circ f$. Put $G_a = J_a\circ r^a$ and $\psi = \{G_a\}_{a\in Inc(X)}$.

In order to verify that ψ is a G-multi-net, let a normal cover $\sigma\in Cov(X)$ be given. Let $\tau\in\sigma^*$ and put $a=\{\tau\}$. For every $b>a$, we have

$$r^a\circ f\simeq_G p^a\simeq_G p_b^a\circ p^b\simeq_G p_b^a\circ r^b\circ f.$$

Since $f_\sharp^{X_a}$ is also an injection, we obtain $r^a\simeq_G p_b^a\circ r^b$. It follows that

$$G_a = J_a\circ r^a \overset{\tau}{\simeq}_G J_a\circ p_b^a\circ r^b \overset{\tau}{\simeq}_G J_b\circ r^b = G_b$$

and therefore $G_a \overset{\sigma}{\simeq}_G G_b$.

We shall now check that $\psi\circ f$ is G-homotopic to ι^X, i.e. that for every $\sigma\in Cov(X)$ there is an $a\in Inc(X)$ such that $G_{\psi(b)}\circ f\overset{\sigma}{\simeq}_G 1_X$ for every $b>a$.

Let $\sigma\in Cov(X)$ and $\tau\in\sigma^*$. Put $a=\{\tau\}$. For every $b>a$, we get $\psi(b)>\psi(a)>a$ so that $G_{\psi(b)}\overset{\tau}{\simeq}_G G_{\psi(a)}$. Hence,

$$G_{\psi(b)}\circ f\overset{\tau}{\simeq}_G G_{\psi(a)}\circ f = J_{\psi(a)}\circ r^{\psi(a)}\circ f\overset{\tau}{\simeq}_G J_{\psi(a)}\circ p^{\psi(a)}\overset{\tau}{\simeq}_G 1_X.$$

Thus $G_{\psi(b)}\circ f\overset{\sigma}{\simeq}_G 1_X$.

THEOREM 12. *Let $\mathscr{P}$ and $\mathscr{R}$ be classes of G-spaces such that each member of $\mathscr{R}$ is both internally G-movable and internally G-calm. If a G-space X has an inverse $G(\mathscr{P})$-exposition, a G-space Y has an inverse $G(\mathscr{R})$-exposition, and a G-map $f: X\to Y$ is such that the function $f_\sharp^Z$ is a bijection for every $Z\in\mathscr{P}$ and an injection for every $Z\in\mathscr{R}$, then f is an equivariant shape equivalence.*

Proof. First we shall show that for every $\sigma\in Cov(Y)$ there is a $\tau\in Cov(Y)$ such that the relation $U\circ f\overset{\tau}{\simeq}_G V\circ f$ for multi-valued G^τ-functions $U, V: Y\to Y$ implies the relation $U\overset{\sigma}{\simeq} V$.

Let a $\sigma\in Cov(Y)$ be given. Let $\mu\in\sigma^*$ and $\xi\in\mu^*$. Put $c=\{\xi\}$. By assumption the G-space Y has an inverse $G(\mathscr{R})$-exposition formed by G-spaces Y_d from $\mathscr{R}$, G-maps q_e^d and q^d, and multi-valued $G^{[d]}$-functions K_d.

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Since the G-space Y_c is internally G-calm, there is a normal cover α of Y_c such that G^α-homotopic G-maps into Y_c are G-homotopic. Let $\beta \in \alpha^*$ be such that $K_c\colon Y_c \to Y$ is a multi-valued G^ξ_β-function. Since Y_c is also internally G-movable, there is a normal cover $\gamma \in \beta^+$ of Y_c with the property that every multi-valued G^γ-function into Y_c is G^β-homotopic to a G-map. Let $\tau = (q^c)^{-1}(\gamma)$.

Let $U, V\colon Y \to Y$ be multi-valued G^τ-functions and assume that $U \circ f \overset{\tau}{\simeq}_G V \circ f$. Then $q^c \circ U$ and $q^c \circ V$ are multi-valued G^γ-functions so that there are G-maps a and b from Y into Y_c with $a \overset{\beta}{\simeq}_G q^c \circ U$ and $b \overset{\beta}{\simeq}_G q^c \circ V$. Our choices imply the following chain of relations: $a \circ f \overset{\beta}{\simeq}_G q^c \circ U \circ f \overset{\gamma}{\simeq}_G q^c \circ V \circ f \overset{\beta}{\simeq}_G b \circ f$. It follows that $a \circ f \simeq_G b \circ f$ and therefore $a \simeq_G b$ because $f^{Y_c}_\sharp$ is an injection. Now, we have

$$U \overset{\mu}{\simeq}_G K_c \circ q^c \circ U \overset{\mu}{\simeq}_G K_c \circ a \overset{\mu}{\simeq}_G K_c \circ b \overset{\mu}{\simeq}_G K_c \circ q^c \circ V \overset{\mu}{\simeq}_G V.$$

Hence, $U \overset{\sigma}{\simeq}_G V$.

Now we are ready to prove the theorem. Let ψ be a G-multi-net constructed in the proof of Theorem 11. We shall show that $f \circ \psi$ is G-homotopic to ι^Y, i.e. that for every $\sigma \in Cov(Y)$ there is an index $a \in Inc(Y)$ such that $f \circ G_{\psi(\phi^*(b))} \overset{\sigma}{\simeq}_G 1_Y$ for every $b > a$. Since we can take $\phi^*(b) = \{f^{-1}([b])\}$, we must prove that $f \circ G_{\lambda(b)} \overset{\sigma}{\simeq}_G 1_Y$ for every $b > a$, where $\lambda(b) = \psi(\{f^{-1}([b])\})$.

Let a $\sigma \in Cov(Y)$ be given. Choose a normal cover τ of Y with the property described above. Let $\rho \in \tau^*$ and put $a = \{\rho\}$. For every $b > a$, $f \circ G_{\lambda(b)}$ and 1_Y are multi-valued G^τ-functions of Y into itself and

$$f \circ G_{\lambda(b)} \circ f = f \circ J_{\lambda(b)} \circ r^{\lambda(b)} \circ f \overset{\rho}{\simeq}_G f \circ J_{\lambda(b)} \circ p^{\lambda(b)} \overset{\rho}{\simeq}_G f = 1_Y \circ f.$$

It follows that $f \circ G_{\lambda(b)} \overset{\sigma}{\simeq}_G 1_Y$.

11. *Approximate G-contractibility and trivial equivariant shape*

As an illustration of the new insight offered in equivariant shape theory by our approach via G-multi-nets we will characterize G-spaces of trivial equivariant shape as approximately G-contractible G-spaces.

Let us agree that a G-space T is trivial provided T is a single-element space. A G-space X has trivial equivariant shape provided it is equivalent in the equivariant shape category $\mathcal{S}h^G$ to a trivial G-space.

As we shall see shortly, spaces of trivial equivariant shape coincide with approximately G-contractible G-spaces which we define next. A G-space X is *approximately G-contractible* provided for every normal cover σ of X there is a normal cover τ of X such that every two multi-valued G^τ-functions into X are G^σ-homotopic.

THEOREM 13. *A G-space X with a stationary point is approximately G-contractible if and only if X has trivial equivariant shape.*

Proof. ($\Rightarrow$). Let T be a trivial G-space and let $c\colon X \to T$ be the obvious G-map. Let $a \in X$ be a stationary point of X. Define a G-map $g\colon T \to X$ by $g(T) = a$. Let $\gamma\colon X \to T$ and $\psi\colon T \to X$ be simple G-multi-nets determined by G-maps c and g, respectively. Then $\gamma \circ \psi = \iota^T$. In order to see that X and T are equivariantly shape equivalent, it remains to check that $\psi \circ \gamma \simeq_G \iota^X$, i.e. that $g \circ c \overset{\sigma}{\simeq}_G 1_X$ for every normal cover σ of X.

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Let a $\sigma \in Cov(X)$ be given. Since X is approximately G-contractible, there is a $\tau \in Cov(X)$ such that multi-valued G^τ-functions into X are G^σ-homotopic. But, $g \circ c$ and 1_X are both multi-valued G^τ-functions of X into itself. Hence, $g \circ c \overset{\sigma}{\simeq}_G 1_X$.

($\Leftarrow$). It suffices to observe that every trivial G-space T is approximately G-contractible and that approximate G-contractibility is preserved under equivariant shape dominations and equivariant shape equivalences. ∎

Let us call a G-map $f: X \to Y$ constant provided $f(X)$ is a single (stationary) point of Y. The following notion is a version of path-connectedness for G-spaces. A G-space X is *G-path*-connected provided every two constant G-maps into X are G-homotopic.

THEOREM 14. *If a G-space X has trivial equivariant shape, then every two G-maps from X into a G-path-connected G-space which is both internally G-movable and internally G-calm are G-homotopic.*

Proof. Let Z be a G-path-connected G-space which is both internally G-movable and internally G-calm. Let $a, b: X \to Z$ be G-maps. Let T be a trivial G-space. Since X has trivial equivariant shape, the obvious G-map $c: X \to T$ is an equivariant shape equivalence. It follows from Theorem 7 that there are G-maps $f, g: T \to Z$ with $a \simeq_G f \circ c$ and $b \simeq_G g \circ c$. But, f and g are constant G-maps into Z so that $f \simeq_G g$ because Z is G-path-connected. Hence, $a \simeq_G b$.

THEOREM 15. *Let $\mathscr{P}$ be a class of G-path-connected G-spaces with at least one stationary point. If a G-space X has an inverse $G(\mathscr{P})$-exposition and every two G-maps of X into a member of $\mathscr{P}$ are G-homotopic, then X has trivial equivariant shape.*

Proof. Let $c: X \to T$ be the obvious G-map from X into a trivial G-space T. We shall show that c is left $\mathscr{S}h^G$-invertible. Then X will be equivariantly shape dominated by the approximately G-contractible G-space T and will therefore have trivial equivariant shape by Theorem 13. According to Theorem 11, we must in fact check that the induced function $c^Z_\sharp$ is a bijection for every $Z \in \mathscr{P}$.

In order to see that $c^Z_\sharp$ is a surjection for every $Z \in \mathscr{P}$, let Z be from $\mathscr{P}$ and let $f: X \to Z$ be a G-map. Let $g: T \to Z$ be a constant G-map. Then f and $g \circ c$ are two G-maps of X into Z. By assumption, they are G-homotopic so that $[f]_G = c^Z_\sharp([g]_G)$ and $c^G_\sharp$ is onto.

Finally, in order to see that $c^Z_\sharp$ is an injection for every $Z \in \mathscr{P}$, let Z be from $\mathscr{P}$ and let $f, g: T \to Z$ be G-maps and assume that $c^Z_\sharp([f]_G) = c^Z_\sharp([g]_G)$, i.e. that $f \circ c \simeq_G g \circ c$. But, the G-maps f and g are constant G-maps into Z. Since Z is G-path-connected, we obtain $f \simeq_G g$ so that $c^Z_\sharp$ is indeed one-to-one.

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