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# ON MULTIPLICATIVELY BADLY APPROXIMABLE NUMBERS

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*Abstract.* The Littlewood conjecture states that $\liminf_{q\to\infty} q \cdot \|q\alpha\| \cdot \|q\beta\| = 0$ for all $(\alpha, \beta) \in \mathbb{R}^2$. We show that with the additional factor of $\log q \cdot \log\log q$, the statement is false. Indeed, our main result implies that the set of $(\alpha, \beta)$ for which $\liminf_{q\to\infty} q \cdot \log q \cdot \log\log q \cdot \|q\alpha\| \cdot \|q\beta\| > 0$ is of full dimension.

§1. *Introduction.* The famous Littlewood conjecture (LC) states that for any pair of real numbers $(\alpha, \beta)$,

$$\liminf_{q\to\infty} q \cdot \|q\alpha\| \cdot \|q\beta\| = 0 \tag{1}$$

where $\|\cdot\|$ denotes the distance to the nearest integer. Equivalently, the set

$$\left\{(\alpha, \beta) \in \mathbb{R}^2 : \liminf_{q\to\infty} q \cdot \|q\alpha\| \cdot \|q\beta\| > 0\right\} \tag{2}$$

is empty. This problem was conjectured in the 1930s and is still open. For recent progress concerning this fundamental problem, see [4, 6] and references therein. It is easily seen that (1) holds for all $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ outside the set **Bad** of badly approximable numbers, defined as follows:

$$\mathbf{Bad} := \left\{\alpha \in \mathbb{R} : \liminf_{q\to\infty} q\|q\alpha\| > 0\right\}.$$

In the attempt to understand what should be a proper analogue of badly approximable points in the multiplicative case, several authors have investigated the following set (we will follow the notation introduced in [2]): for $\lambda \geqslant 0$, let

$$\mathbf{Mad}^{\lambda} := \left\{(\alpha, \beta) \in \mathbb{R}^2 : \liminf_{q\to\infty} (\log q)^{\lambda} \cdot q \cdot \|q\alpha\| \cdot \|q\beta\| > 0\right\}.$$

In other words, $\mathbf{Mad}^{\lambda}$ is a modification of the set in (2) such that the corresponding condition is weakened by $(\log q)^{\lambda}$. More generally, given a function $f : \mathbb{N} \to \mathbb{R}^+$, define the set

$$\mathbf{Mad}(f) := \inf\left\{(\alpha, \beta) \in \mathbb{R}^2 : \liminf_{q\to\infty} f(q) \cdot q \cdot \|q\alpha\| \cdot \|q\beta\| > 0\right\}. \tag{3}$$

In [2], the author and Velani conjectured the following.

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CONJECTURE 1 (Badziahin and Velani).

$$\mathbf{Mad}^{\lambda} = \emptyset \quad \textit{for any } \lambda < 1;$$
$$\dim(\mathbf{Mad}^{\lambda}) = 2 \quad \textit{for any } \lambda \geqslant 1$$

*where* $\dim(\cdot)$ *denotes the Hausdorff dimension.*

If true, this conjecture would imply that the proper multiplicative analogue of the set **Bad** is $\mathbf{Mad}^1$. Note that the LC is equivalent to the statement that $\mathbf{Mad}^0$ is empty. Therefore the Badziahin–Velani (BV) conjecture implies the Littlewood conjecture. Regarding the first part of the BV conjecture, all that is known to date is the remarkable result of Einsiedler *et al* [4], which states that $\dim \mathbf{Mad}^0 = 0$. On the other hand, relating to the second part, the best-known result is due to Bugeaud and Moschevitin [3]; it states that $\dim \mathbf{Mad}^2 = 2$. So we have a gap $0 \leqslant \lambda < 2$ where the behavior of $\mathbf{Mad}^{\lambda}$ is completely unknown.

In this paper we will address the second part of the BV conjecture. In particular, we will show that

$$\dim \mathbf{Mad}(f) = 2 \quad \text{if } f(q) = \log q \cdot \log\log q.$$

This will straightforwardly imply that $\dim(\mathbf{Mad}^{\lambda}) = 2$ for any $\lambda > 1$.

It is worth mentioning that the "mixed" analogue of this result was achieved recently by the author and Velani; it was proven that the set

$$\mathbf{Mad}_{\mathcal{D}}(f) := \left\{\alpha \in \mathbb{R} : \liminf_{q\to\infty} f(q) \cdot q \cdot |q|_{\mathcal{D}} \|q\alpha\| > 0\right\}$$

has full Hausdorff dimension. All the details can be found in [2].

1.1. *Simultaneous and dual variants of* **Mad**. It is well known that the Littlewood conjecture has an equivalent formulation in terms of linear forms. In other words, (1) is equivalent to the statement that

$$\liminf_{|AB|\to\infty} |A|^*|B|^* \cdot \|A\alpha - B\beta\| > 0$$

where $|x|^* := \max\{|x|, 1\}$. However, it is not known whether (3) can be reformulated in the same manner. Define the sets

$$\mathbf{Mad}_L(f) := \inf\left\{(\alpha, \beta) \in \mathbb{R}^2 : \liminf_{|AB|\to\infty} f(|A|^*|B|^*) \cdot |A|^*|B|^* \|A\alpha - B\beta\| > 0\right\} \tag{4}$$

and

$$\mathbf{Mad}_L^{\lambda} := \mathbf{Mad}_L(\log^{\lambda} q).$$

Then $\mathbf{Mad}(f)$ and $\mathbf{Mad}_L(f)$ are not necessarily the same. However, as will be shown in the next sections, these sets are closely related to each other. For consistency, in further discussions we will use the notation $\mathbf{Mad}_P^{\lambda}$ and $\mathbf{Mad}_P(f)$ instead of $\mathbf{Mad}^{\lambda}$ and $\mathbf{Mad}(f)$, respectively; this will emphasize the fact that in one case we deal with points and in another case we deal with lines.

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It appears that, rather than investigating $\mathbf{Mad}_P(f)$ and $\mathbf{Mad}_L(f)$ independently, it is easier to deal with them simultaneously. In particular, we prove the following result.

THEOREM 1. *Let* $f(q) = \log q \cdot \log\log q$. *Then*

$$\dim(\mathbf{Mad}_P(f) \cap \mathbf{Mad}_L(f)) = 2.$$

1.2. *Main result.* For convenience, we define the "modified logarithm" function $\log^* : \mathbb{R} \to \mathbb{R}$ by

$$\log^* x := \begin{cases} 1 & \text{for } x < e, \\ \log x & \text{for } x \geqslant e. \end{cases}$$

From now on,

$$f(q) := \log^* q \cdot \log^* \log q.$$

The key to establishing Theorem 1 is to investigate the intersection of the sets $\mathbf{Mad}_P(f)$ and $\mathbf{Mad}_L(f)$ along fixed vertical lines in the $(x, y)$-plane. With this in mind, let $\mathrm{L}_x$ denote the line parallel to the $y$-axis passing through the point $(x, 0)$.

The following constitutes our main theorem.

THEOREM 2. *For any* $\theta \in \mathbf{Bad}$,

$$\dim(\mathbf{Mad}_P(f) \cap \mathbf{Mad}_L(f) \cap \mathrm{L}_\theta) = 1.$$

Since by Jarník (1928) the Hausdorff dimension of **Bad** is one, Theorem 1 can easily be derived from Theorem 2 with the help of the following general result, which relates the dimension of a set to the dimensions of parallel sections, enabling us to establish the complementary lower bound estimate; see [**5**, p. 99].

PROPOSITION. *Let* $F$ *be a subset of* $\mathbb{R}^2$ *and let* $E$ *be a subset of the* $x$*-axis. If* $\dim(F \cap \mathrm{L}_x) \geqslant t$ *for all* $x \in E$, *then* $\dim F \geqslant t + \dim E$.

Indeed, let $F = \mathbf{Mad}_P(f) \cap \mathbf{Mad}_L(f)$ and $E = \mathbf{Bad}$. In view of the fact that $\dim(\mathbf{Bad}) = 1$ and Theorem 2, one gets $\dim \mathbf{Mad}_P(f) \cap \mathbf{Mad}_L(f) \geqslant 2$. Since $\mathbf{Mad}_P(f) \cap \mathbf{Mad}_L(f) \subset \mathbb{R}^2$, the upper bound statement for the dimension is trivial. Therefore the main ingredient in establishing Theorem 1 is Theorem 2.

For the proof of Theorem 2 we will use ideas similar to those in [**2**], which first appeared in joint work of the author, Pollington and Velani [**1**]. However, the technical details in this paper are substantially more complicated than those in [**2**].

§2. *Preliminaries.* Let $S$ be any subset of $\mathbb{R}^2$. We denote by $S_\theta$ its orthogonal projection onto the line $\mathrm{L}_\theta$. Let $P(p, r, q) := (p/q, r/q)$ be a rational point, where $(p, r, q) \in \mathbb{Z}^3$ and $\gcd(p, r, q) = 1$. The height of $P$ is defined to be the value

$$H(P) := q^2|q\theta - p| \geqslant q^2\|q\theta\|.$$

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Consider the following segment on $\mathrm{L}_\theta$:

$$\Delta(P,\,\delta) := \{\theta\} \times \left(\frac{r}{q} - \frac{\delta}{H(P)},\ \frac{r}{q} + \frac{\delta}{H(P)}\right).$$

Note that $|\Delta(P,\,\delta)| = 2\delta H(P)^{-1}$.

Given a line with integer coefficients,

$$L(A,\,B,\,C) := \{(x,\,y) \in \mathbb{R}^2 : Ax - By + C = 0\}$$

where

$$(A,\,B,\,C) \in \mathbb{Z}^3, \quad B \neq 0, \quad \gcd(A,\,B,\,C) = 1, \tag{5}$$

let the height of $L$ be the value

$$H(L) := |A|^* B^2.$$

Consider the following segment on $\mathrm{L}_\theta$:

$$\Delta(L,\,\delta) := \{\theta\} \times \left(\frac{A\theta + C}{B} - \frac{\delta}{H(L)},\ \frac{A\theta + C}{B} + \frac{\delta}{H(L)}\right).$$

Then $|\Delta(L,\,\delta)| = 2\delta H(L)^{-1}$.

Given constants $c > 0$ and $Q > 0$, define the auxiliary sets

$$\begin{aligned}&\mathbf{Mad}_P(f,\,c,\,Q)\\ &\quad := \{(\alpha,\,\beta) \in \mathbb{R}^2 : f(q)\cdot q \cdot \|q\alpha\| \cdot \|q\beta\| > c \ \ \forall q \in \mathbb{N},\ q \geqslant Q\}\end{aligned}$$

and

$$\mathbf{Mad}_L(f,\,c,\,Q) := \left\{(\alpha,\,\beta) \in \mathbb{R}^2 : \begin{array}{c} f(|A|^*|B|^*)\cdot|A|^*|B|^*\|A\alpha - B\beta\| > c \\ \forall (A,\,B) \in \mathbb{Z}^2,\ |A|^*B^2 \geqslant Q \end{array}\right\}.$$

It is easily verified that $\mathbf{Mad}_P(f,\,c,\,Q) \subset \mathbf{Mad}_P(f)$, $\mathbf{Mad}_L(f,\,c,\,Q) \subset \mathbf{Mad}_L(f)$ and

$$\mathbf{Mad}_P(f) \cap \mathbf{Mad}_L(f) = \bigcup_{c>0} (\mathbf{Mad}_P(f,\,c,\,Q) \cap \mathbf{Mad}_L(f,\,c,\,Q)).$$

For convenience, we will omit the parameter $Q$ where it is irrelevant and write $\mathbf{Mad}_P(f,\,c)$ and $\mathbf{Mad}_L(f,\,c)$ for $\mathbf{Mad}_P(f,\,c,\,Q)$ and $\mathbf{Mad}_L(f,\,c,\,Q)$, respectively.

So it suffices to prove that the set $\mathbf{Mad}_P(f,\,c) \cap \mathbf{Mad}_L(f,\,c) \cap \mathrm{L}_\theta$ has full Hausdorff dimension for some positive constant $c$.

Geometrically, the set $\mathbf{Mad}_P(f,\,c)$ consists of points that avoid the "neighborhood" of each rational point $P = (p/q,\,r/q)$ defined by the inequality

$$\left|x - \frac{p}{q}\right| \left|y - \frac{r}{q}\right| < \frac{c}{f(q)q^3}.$$

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This "neighborhood" of $P$ will remove the interval $\Delta(P, cf(q)^{-1})$ from $\mathrm{L}_\theta$. Without loss of generality, we can assume that $|q\theta - p| = \|q\theta\|$; otherwise, we just replace the point $P$ by $P' := (p'/q, r/q)$ such that $|q\theta - p'| = \|q\theta\|$. Then $\Delta(P') \supset \Delta(P)$ and the "neighborhood" of $P$ will not remove more than the one of $P'$.

Similarly, one can show that the set $\mathbf{Mad}_L(f, c)$ consists of points that avoid the "neighborhood" of each line $L(A, B, C)$ defined by

$$|Ax - By + C| < \frac{c}{f(|A|^*|B|^*)|A|^*|B|^*},$$

where the coefficients $A, B, C$ satisfy $(A, B) <> (0, 0)$ and $\gcd(A, B, C) = 1$. For $B = 0$ this leads to the inequality

$$\|Ax\| < \frac{c}{|A|f(|A|)}.$$

Take $c < \inf_{q\in\mathbb{N}} q\|q\theta\|$. Then this inequality is not true for $x = \theta$; in other words, the "neighborhood" of the line does not remove anything from $\mathrm{L}_\theta$. Therefore it is sufficient to consider the lines $L(A, B, C)$ with $B \neq 0$, so the coefficients $(A, B, C)$ will satisfy (5). Then the "neighborhood" of $L(A, B, C)$ will remove the interval $\Delta(L, cf(|A|^*|B|^*)^{-1})$ from $\mathrm{L}_\theta$.

2.1. *Cantor sets.* In the proof we will use the general Cantor framework first introduced in [2]. Here we reproduce the definitions and facts which will be used later. For more details we refer to the paper [2].

Let $I$ be a closed interval in $\mathbb{R}$. Let $\mathbf{R} := (R_n)$ with $n \in \mathbb{Z}_{\geqslant 0}$ be a sequence of natural numbers, and let $\mathbf{r} := (r_{m,n})$ with $m, n \in \mathbb{Z}_{\geqslant 0}$, $m \leqslant n$, be a two-parameter sequence of non-negative real numbers.

*The construction.* We start by subdividing the interval $I$ into $R_0$ closed intervals $I_1$ of equal length, and we denote by $\mathcal{I}_1$ the collection of such intervals. Thus,

$$\#\mathcal{I}_1 = R_0 \quad \text{and} \quad |I_1| = R_0^{-1}|\mathrm{I}|.$$

Next, we remove at most $r_{0,0}$ intervals $I_1$ from $\mathcal{I}_1$. Note that we do not specify which intervals should be removed but just give an upper bound on the number of intervals to be removed. Denote by $\mathcal{J}_1$ the resulting collection. Thus,

$$\#\mathcal{J}_1 \geqslant \#\mathcal{I}_1 - r_{0,0}. \tag{6}$$

For obvious reasons, intervals in $\mathcal{J}_1$ will be referred to as (level-one) survivors. It will be convenient to define $\mathcal{J}_0 := \{I\}$. In general, for $n \geqslant 0$, given a collection $\mathcal{J}_n$ we construct a nested collection $\mathcal{J}_{n+1}$ of closed intervals $J_{n+1}$ using the following two operations.

- *Splitting procedure.* We subdivide each interval $J_n \in \mathcal{J}_n$ into $R_n$ closed sub-intervals $I_{n+1}$ of equal length and denote by $\mathcal{I}_{n+1}$ the collection of such intervals. Thus,

$$\#\mathcal{I}_{n+1} = R_n \times \#\mathcal{J}_n \quad \text{and} \quad |I_{n+1}| = R_n^{-1}\,|J_n|.$$

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- *Removing procedure.* For each interval $J_n \in \mathcal{J}_n$, we remove at most $r_{n,n}$ intervals $I_{n+1} \in \mathcal{I}_{n+1}$ that lie within $J_n$. Note that the number of intervals $I_{n+1}$ removed is allowed to vary amongst the intervals in $\mathcal{J}_n$. Next, for each interval $J_{n-1} \in \mathcal{J}_{n-1}$, we additionally remove at most $r_{n-1,n}$ intervals $I_{n+1} \in \mathcal{I}_{n+1}$ that lie within $J_{n-1}$. In general, for each interval $J_{n-k} \in \mathcal{J}_{n-k}$ $(1 \leqslant k \leqslant n)$, we additionally remove at most $r_{n-k,n}$ intervals $I_{n+1} \in \mathcal{I}_{n+1}$ that lie within $J_{n-k}$. Then the collection $\mathcal{J}_{n+1}$ consists of all intervals $I_{n+1} \in \mathcal{I}_{n+1}$ that survive after all these removing procedures for $k = 1, 2, \ldots, n$. Thus, the total number of survivors is at least

$$\#\mathcal{J}_{n+1} \geqslant R_n \#\mathcal{J}_n - \sum_{k=0}^{n} r_{k,n} \#\mathcal{J}_k. \tag{7}$$

Finally, having constructed the nested collections $\mathcal{J}_n$ of closed intervals, we consider the limit set

$$\mathbf{K}(I, \mathbf{R}, \mathbf{r}) := \bigcap_{n=1}^{\infty} \bigcup_{J \in \mathcal{J}_n} J.$$

Any set $\mathbf{K}(I, \mathbf{R}, \mathbf{r})$ which can be achieved by the procedure described will be referred to as a $(I, \mathbf{R}, \mathbf{r})$ *Cantor set.*

Of course, in general it can happen that for some choice of parameters $\mathbf{R}$ and $\mathbf{r}$ and some choice of removed intervals in the removing procedure, the $(I, \mathbf{R}, \mathbf{r})$ Cantor set becomes empty. However, the next result shows that with some additional conditions on the parameters, the Hausdorff dimension of this set is bounded below.

THEOREM (BV4). *Given a* (I, $\mathbf{R}$, $\mathbf{r}$) *Cantor set* $\mathbf{K}(I, \mathbf{R}, \mathbf{r})$*, suppose that* $R_n \geqslant 4$ *for all* $n \in \mathbb{Z}_{\geqslant 0}$ *and that*

$$\sum_{k=0}^{n} \left( r_{n-k,n} \prod_{i=1}^{k} \left( \frac{4}{R_{n-i}} \right) \right) \leqslant \frac{R_n}{4}. \tag{8}$$

*Then*

$$\dim \mathbf{K}(I, \mathbf{R}, \mathbf{r}) \geqslant \liminf_{n \to \infty} (1 - \log_{R_n} 2).$$

Here we use the convention that the product term in (8) is 1 when $k = 0$ and that, by definition, $\log_{R_n} 2 := \log 2 / \log R_n$. The proof of Theorem BV4 is presented in [2, Theorem 4].

2.2. *Duality between points and lines.* The next two propositions show that there is a "kind" of duality between rational points $P(p, r, q)$ and lines $L(A, B, C)$. It will play a crucial role in our proof.

PROPOSITION 1. *Let* $P_1(p_1, r_1, q_1)$ *and* $P_2(p_2, r_2, q_2)$ *be two different rational points with* $p_1/q_1 \neq p_2/q_2$, $r_1/q_1 \neq r_2/q_2$ *and* $0 < q_1 \|q_2\theta\| \leqslant q_2 \|q_1\theta\|$. *Let* $L(A, B, C)$*, with* $(A, B, C)$ *satisfying (5), be the line passing through* $P_1$ *and* $P_2$. *Assume that* $(P_2)_\theta \in \Delta(P_1, \delta)$. *Then*

$$(P_2)_\theta \in \Delta\left( L, \frac{\delta^2 |B|}{q_2 \|q_1\theta\|} \cdot \frac{H(P_2)}{H(P_1)} \right) \subset \Delta\left( L, 2\delta^2 \frac{H(P_2)}{H(P_1)} \right). \tag{9}$$

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*Moreover,*

$$H(L) \leqslant 4\delta H(P_1)\frac{q_2^3}{q_1^3}. \tag{10}$$

PROPOSITION 2. *Let $L_1(A_1, B_1, C_1)$ and $L_2(A_2, B_2, C_2)$ be two lines with integer coefficients $(A_i, B_i, C_i)$ satisfying (5) and $|A_2B_1| \leqslant |A_1B_2|$. Assume that they intersect at a rational point $P(p, r, q)$ and that $L_2 \cap \mathrm{L}_\theta \in \Delta(L_1, \delta)$. Then*

$$L_2 \cap \mathrm{L}_\theta \in \Delta\left(P, \frac{\delta^2 q}{|B_2A_1|} \cdot \frac{H(L_2)}{H(L_1)}\right) \subset \Delta\left(P, 2\delta^2\frac{H(L_2)}{H(L_1)}\right). \tag{11}$$

*Moreover,*

$$H(P) \leqslant 4\delta H(L_1)\frac{|B_2|^3}{|B_1|^3}. \tag{12}$$

*Proof of Proposition 1.* Since $P_1, P_2 \in L$, we have the following system of equations:

$$\begin{cases} Ap_1 - Br_1 + Cq_1 = 0, \\ Ap_2 - Br_2 + Cq_2 = 0, \\ A\theta - B\omega + C = 0 \end{cases}$$

where $\omega := (A\theta + C)/B$. Since $p_1/q_1 \neq p_2/q_2$ and $r_1/q_1 \neq r_2/q_1$, we get that the coefficients $A$ and $B$ are non-zero. Let $A' := A/d$ and $B' := B/d$ where $d := (A, B)$. Then, since $(A, B, C) = 1$, we get that $q_1 = dq_1'$ and $q_2 = dq_2'$. Then the first two equations of the system lead to

$$A'(p_1q_2' - p_2q_1') = B'(r_1q_2' - r_2q_1').$$

This, together with $(A', B') = 1$, implies that $|p_1q_2' - p_2q_1'| \geqslant |B'|$ and $|r_1q_2' - r_2q_1'| \geqslant |A'|$, or

$$\left|\frac{p_1}{q_1} - \frac{p_2}{q_2}\right| \geqslant \frac{|B|}{q_1q_2} \quad \text{and} \quad \left|\frac{r_1}{q_1} - \frac{r_2}{q_2}\right| \geqslant \frac{|A|}{q_1q_2}.$$

The system also gives us the equalities

$$|A|\left|\frac{p_1}{q_1} - \theta\right| = |B|\left|\frac{r_1}{q_1} - \omega\right| \quad \text{and} \quad |A|\left|\frac{p_2}{q_2} - \theta\right| = |B|\left|\frac{r_2}{q_2} - \omega\right|.$$

The assumption $(P_2)_\theta \in \Delta(P_1, \delta)$ is equivalent to

$$\left|\frac{r_1}{q_1} - \frac{r_2}{q_2}\right| < \frac{\delta}{q_1^2\|q_1\theta\|}.$$

Finally, by the triangle inequality we find that

$$\left|\frac{p_1}{q_1} - \frac{p_2}{q_2}\right| \leqslant 2\max\left\{\left|\frac{p_1}{q_1} - \theta\right|, \left|\frac{p_2}{q_2} - \theta\right|\right\} = 2\max\left\{\frac{\|q_1\theta\|}{q_1}, \frac{q_2\|\theta\|}{q_2}\right\}.$$

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Upon combining all these inequalities we get that

$$|B| \leqslant q_1 q_2 \left| \frac{p_1}{q_1} - \frac{p_2}{q_2} \right| \leqslant 2 \max\{q_2 \|q_1\theta\|, q_1 \|q_2\theta\|\} = 2q_2\|q_1\theta\|,$$
$$|A| \leqslant q_1 q_2 \left| \frac{r_1}{q_1} - \frac{r_2}{q_2} \right| < \frac{\delta q_2}{q_1 \|q_1\theta\|}.$$

Now we are ready to calculate the bound

$$\left| \frac{r_2}{q_2} - \omega \right| = \frac{|A|}{|B|} \frac{\|q_2\theta\|}{q_2} < \frac{1}{|AB|} \cdot \frac{\delta^2 q_2^2 \cdot \|q_2\theta\|}{q_1^2 \|q_1\theta\|^2 \cdot q_2} = \frac{1}{|A|B^2} \cdot \frac{\delta^2 |B| \cdot H(P_1)}{q_2 \|q_1\theta\| \cdot H(P_2)}.$$

Then the first inclusion in (9) follows immediately. For the second, we just use the calculated estimate for $|B|$. Also, by combining the bounds for $|A|$ and $|B|$ we obtain an estimate for the height $H(L)$:

$$H(L) = |A|B^2 \leqslant \frac{4\delta q_2^3 \|q_1\theta\|}{q_1} = 4\delta H(P_1) \frac{q_2^3}{q_1^3}.$$

This completes the proof of Proposition 1. □

Before we start the proof of Proposition 2, let us establish some basic facts regarding the point of intersection of two lines $L_1(A_1, B_1, C_1)$ and $L_2(A_2, B_2, C_2)$ with integer coefficients $(A_i, B_i, C_i) \in \mathbb{Z}^3 \backslash (\{0\}^2 \times \mathbb{Z})$ such that $(A_i, B_i, C_i) = 1$ for $i = 1, 2$. These facts will be of use in further discussions as well. An intersection $L_1 \cap L_2$ is a rational point $P(p, r, q)$ that is the solution of the system of equations

$$\begin{cases} A_1 p - B_1 r + C_1 q = 0, \\ A_2 p - B_2 r + C_2 q = 0, \end{cases}$$

which leads to the following equalities:

$$\frac{p}{q} = \frac{B_1 C_2 - B_2 C_1}{A_1 B_2 - A_2 B_1} \quad \text{and} \quad \frac{r}{q} = \frac{A_1 C_2 - A_2 C_1}{A_1 B_2 - A_2 B_1}.$$

Therefore we get that

$$|B_1 C_2 - B_2 C_1| = dp, \qquad |A_1 C_2 - A_2 C_1| = dr, \qquad |A_1 B_2 - A_2 B_1| = dq \tag{13}$$

where $d := \gcd(A_1 B_2 - A_2 B_1, B_1 C_2 - B_2 C_1) \in \mathbb{Z}$.

Let $i \in \{1, 2\}$. It is easily verified that

$$L_i \cap \mathrm{L}_\theta = \left( \theta, \frac{A_i\theta + C_i}{B_i} \right) = \left( \theta, \frac{r}{q} + \frac{A_i}{B_i} \left( \theta - \frac{p}{q} \right) \right).$$

Therefore

$$|L_1 \cap \mathrm{L}_\theta - L_2 \cap \mathrm{L}_\theta| = \left| \frac{A_1}{B_1} - \frac{A_2}{B_2} \right| \cdot \left| \theta - \frac{p}{q} \right| = \frac{d|q\theta - p|}{|B_1 B_2|}.$$

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Hence

$$|q\theta - p| = d^{-1}|B_1B_2| \cdot |L_1 \cap \mathrm{L}_\theta - L_2 \cap \mathrm{L}_\theta| \tag{14}$$

and

$$|q\omega - r| = \frac{|A_1|}{|B_1|}|q\theta - p| \quad \text{where } \omega := \frac{A_1\theta + C_1}{B_1}. \tag{15}$$

*Proof of Proposition 2.* By (13), an upper bound for $q$ is given by

$$q = d^{-1}|A_1B_2 - A_2B_1| \leqslant 2d^{-1}\max\{|A_1B_2|, |A_2B_1|\} = 2d^{-1}|A_1B_2|.$$

An upper bound for $|q\theta - p|$ can be derived from (14) and the assumption that $L_2 \cap \mathrm{L}_\theta \in \Delta(P, \delta)$:

$$|q\theta - p| < \frac{\delta|B_1B_2|}{d|A_1|B_1^2} = \frac{\delta|B_2|}{d|A_1B_1|}.$$

Finally, we get the required bounds

$$\begin{aligned} |L_2 \cap \mathrm{L}_\theta - P_\theta| &= \frac{|A_2|}{|B_2|} \cdot \frac{|q\theta - p|}{q} \\ &< \frac{|A_2| \cdot \delta^2|B_2|^2}{|B_2| \cdot d^2|A_1B_1|^2 \cdot q|q\theta - p|} \\ &\leqslant \frac{1}{q^2\|q\theta\|} \cdot \frac{\delta^2 q \cdot H(L_2)}{|B_2A_1| \cdot H(L_1)} \end{aligned}$$

and

$$H(P) = q^2|q\theta - p| < 4d^{-2}|A_1B_2|^2 \cdot \frac{\delta|B_2|}{d|A_1B_1|} \leqslant 4\delta H(L_1) \cdot \frac{|B_2|^3}{|B_1|^3}.$$

To get the last inclusion in (11), we just use the calculated bound for $q$. This completes the proof of Proposition 2. □

As we will see, the duality between points and lines will appear throughout this paper.

## §3. *Proof of Theorem 2.*

3.1. *The idea.* By definition, for $\theta \in \mathbf{Bad}$ there exists a quantity $c(\theta) > 0$ such that

$$\inf_{q \in \mathbb{N}} q\|q\theta\| = c(\theta).$$

In other words, for any positive integer $q$, the following inequality is satisfied:

$$q|q\theta - p| \geqslant c(\theta). \tag{16}$$

Let $R \geqslant e^9 c^{-1}(\theta)$ be an integer. Choose constants $c$ and $c_1$ sufficiently small such that they satisfy the inequalities

$$2^{12}c < 1, \qquad 2c < R^2c_1c(\theta), \qquad c < c(\theta) \tag{17}$$

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and

$$2^6 \max\left\{\frac{c}{R^2 c_1 c(\theta)}, 2^{11} c\right\} \frac{(\log R+2)^2 R^4}{(\log 2)^2} + 2^{15} c_1 \frac{R^3(\log R+2)}{\log 2} < 1. \quad (18)$$

Finally, choose the parameter $Q := c(\theta) R^2 F(2)$ where

$$F(n) := \prod_{k=1}^{n} k\,[\log^* k] \text{ for } n \geqslant 1 \quad \text{and} \quad F(n) := 1 \text{ for } n \leqslant 0.$$

The goal is to construct a $(I, \mathbf{R}, \mathbf{r})$ Cantor-type set $\mathbf{K}_c$ with properly chosen parameters $I, \mathbf{R}$ and $\mathbf{r}$ so that $\mathbf{K}_c$ is a subset of $\mathbf{Mad}_P(f, c, Q) \cap \mathbf{Mad}_L(f, c, Q)$. Then we use Theorem BV4 to estimate its Hausdorff dimension. Let $I$ be any interval of length $c_1$ contained within the unit interval $\{\theta\} \times [0, 1] \subset \mathrm{L}_\theta$. Define $\mathcal{J}_0 := \{I\}$. We are going to construct, by induction on $n$, a collection $\mathcal{J}_n$ of closed intervals $J_n$ such that $\mathcal{J}_n$ is nested in $\mathcal{J}_{n-1}$; that is, each interval $J_n$ in $\mathcal{J}_n$ is contained in some interval $J_{n-1}$ in $\mathcal{J}_{n-1}$. The length of an interval $J_n$ will be given by

$$|J_n| := c_1 R^{-n} F^{-1}(n).$$

Moreover, each interval $J_n$ in $\mathcal{J}_n$ will satisfy the conditions

$$J_n \cap \Delta(P, cf^{-1}(q)) = \emptyset \quad \text{for all } P(p, r, q) \in \mathbb{Q}^2 \text{ with } (p, r, q) = 1 \text{ and } Q < H(P) < c(\theta) R^{n-1} F(n-1) \quad (19)$$

and

$$J_n \cap \Delta(L, cf^{-1}(|A|^*|B|^*)) = \emptyset \quad \text{for all } L(A, B, C) \text{ with } (A, B, C) \in \mathbb{Z}^3,\ B \neq 0,\ (A, B, C) = 1 \text{ and } Q < H(L) < c(\theta) R^{n-1} F(n-1). \quad (20)$$

In particular, we put

$$\mathbf{K}_c = \bigcap_{n=1}^{\infty} \bigcup_{J \in \mathcal{J}_n} J.$$

By construction, conditions (19) and (20) ensure that

$$\mathbf{K}_c \subset \mathbf{Mad}_P(f, c) \cap \mathbf{Mad}_L(f, c) \cap \mathrm{L}_\theta.$$

The aim of the rest of the paper is to show that $\mathbf{K}_c$ is in fact a $(I, \mathbf{R}, \mathbf{r})$ Cantor set, with $\mathbf{R} = (R_n)$ given by

$$R_n := R(n+1)[\log^*(n+1)] \quad (21)$$

and $\mathbf{r} = (r_{m,n})$ given by

$$r_{m,n} := \begin{cases} 25R \log R \cdot n^4 (\log^* n)^4 & \text{if } m = n-3, \\ 0 & \text{otherwise.} \end{cases} \quad (22)$$

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Then Theorem 2 will follow from Theorem BV4. Indeed, for $n < 3$ the condition (8) is obviously satisfied. For $n \geqslant 3$ and $R \geqslant 2^7$ we have that

$$\begin{aligned}\text{left-hand side of (8)} &= r_{n-3,n} \cdot \frac{4^3}{R_{n-1}R_{n-2}R_{n-3}} \\ &\leqslant \frac{4^3}{R^3} \cdot \frac{25R \log R \cdot n^4 (\log^* n)^4}{n(n-1)(n-2)\log^* n \log^*(n-1)\log^*(n-2)} \\ &\leqslant \frac{16 \cdot 25 \cdot 4^3 \log R}{R^3} \cdot \frac{R(n+1)[\log^*(n+1)]}{4} \\ &\leqslant \frac{R_n}{4} = \text{ right-hand side of (8).}\end{aligned}$$

Therefore Theorem BV4 implies that

$$\dim \mathbf{K}_c \geqslant \liminf_{n\to\infty}(1 - \log_{R_n} 2) = 1,$$

which completes the proof of Theorem 2.

3.2. *Basic construction: splitting into collections $C_P(n, l, k)$ and $C_L(n, l, k)$.* Now we will describe the procedure of constructing the collections $\mathcal{J}_n$. For $n = 0$, we trivially have that (19) and (20) are satisfied for the sole interval $I \in \mathcal{J}_0$. The point is that, by the choice of $Q$, there are neither points nor lines satisfying the height condition $Q < H(P)$, $H(L) < c(\theta)$. Then we construct $\mathcal{J}_i$, $i = 1, 2, 3$, by just subdividing each $J_{i-1}$ in $\mathcal{J}_{i-1}$ into $R \cdot i[\log^* i]$ closed intervals of equal length. Again, for the same reason, the conditions (19) and (20) are satisfied for any $J_i \in \mathcal{J}_i$, $i = 1, 2, 3$. Note that

$$\#\mathcal{J}_i = R^i F(i), \quad i = 1, 2, 3.$$

In general, given $\mathcal{J}_n$ satisfying (19) and (20), we wish to construct a nested collection $\mathcal{J}_{n+1}$ of intervals $J_{n+1}$ for which (19) and (20) are satisfied with $n$ replaced by $n + 1$. By definition, any interval $J_n$ in $\mathcal{J}_n$ avoids intervals $\Delta(P, cf^{-1}(q))$ and $\Delta(L, cf^{-1}(|A|^*|B|^*))$ arising from points and lines with height bounded above by $c(\theta)R^{n-1}F(n-1)$. Since any "new" interval $J_{n+1}$ is to be nested in some $J_n$, it is enough to show that $J_{n+1}$ avoids intervals $\Delta(P, cf^{-1}(q))$ and $\Delta(L, cf^{-1}(|A|^*|B|^*))$ arising from points and lines with height satisfying

$$c(\theta)R^{n-1}F(n-1) \leqslant H(P), H(L) < c(\theta)R^n F(n). \tag{23}$$

Denote by $C_P(n)$ the collection of all rational points satisfying this height condition. Formally,

$$C_P(n) := \{P(p, r, q) \in \mathbb{Q}^2 : P \text{ satisfies (23)}\},$$

and it is precisely this collection of rationals that comes into play when constructing $\mathcal{J}_{n+1}$ from $\mathcal{J}_n$. By analogy, for "lines" we let

$$C_L(n) := \{L(A, B, C) : L \text{ satisfies (23)}\}.$$

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We now proceed with the construction. Assume that $n \geqslant 3$. We subdivide each $J_n$ in $\mathcal{J}_n$ into $R_n = R(n+1)[\log^*(n+1)]$ closed intervals $I_{n+1}$ of length

$$|I_{n+1}| = c_1 R^{-n-1} F^{-1}(n+1).$$

Denote by $\mathcal{I}_{n+1}$ the collection of such intervals. In view of the nested requirement, the collection $\mathcal{J}_{n+1}$ which we are attempting to construct will be a sub-collection of $\mathcal{I}_{n+1}$. In other words, the intervals $I_{n+1}$ represent possible candidates for $J_{n+1}$. The goal now is simple: it is to remove those "bad" intervals $I_{n+1}$ from $\mathcal{I}_{n+1}$ for which

$$I_{n+1} \cap \Delta(P, cf^{-1}(q)) \neq \emptyset \quad \text{for some } P(p,r,q) \in C_P(n) \tag{24}$$

or

$$I_{n+1} \cap \Delta(L, cf^{-1}(|A|^*|B|^*)) \neq \emptyset \quad \text{for some } L(A,B,C) \in C_L(n). \tag{25}$$

So we define

$$\mathcal{J}_{n+1} := \left\{ J_{n+1} \in \mathcal{I}_{n+1} \;\middle|\; \begin{array}{l} J_{n+1} \cap \Delta(P, cf^{-1}(q)) = \emptyset \\ \quad \text{for any } P \in C_P(n), \\ J_{n+1} \cap \Delta(L, cf^{-1}(|A|^*|B|^*)) = \emptyset \\ \quad \text{for any } L \in C_L(n) \end{array} \right\}.$$

Consider the rational point $P(p,r,q) \in C_P(n)$. Note that since $q^2 \geqslant q^2\|q\theta\| = H(q) \geqslant cR^{n-1}F(n-1)$, we have that

$$f(q) \geqslant \tfrac{1}{2}\log^*(cR^{n-1}F(n-1))\log^*\tfrac{1}{2}\log(c(\theta)R^{n-1}F(n-1)) > \tfrac{1}{2}n(\log^* n)^2 \tag{26}$$

for sufficiently large $R$. We use Stirling's formula to show that for $n \geqslant 3$,

$$c(\theta)R^{n-1}F(n-1) \geqslant c(\theta)R^{n-1}(n-1)! > (8n)^n \quad \text{for } R \geqslant e^9 c^{-1}(\theta).$$

Therefore the left-hand side of (26) is bigger than

$$\tfrac{1}{2}n\log(8n)\cdot\log^*(\tfrac{1}{2}n\log(8n)) > \tfrac{1}{2}n\log^{*2} n.$$

Note that for any line $L(A,B,C) \in C_L(n)$, we have the analogous bound

$$f(|A|^*|B|^*) \geqslant \tfrac{1}{2}n(\log^* n)^2. \tag{27}$$

For $l \in \mathbb{Z}$, we split $C_P(n)$ into sub-collections

$$C_P(n,l) := \left\{ P(p,r,q) \in C_P(n) \;\middle|\; \begin{array}{l} c(\theta)2^l R^{n-1}F(n-1) \leqslant H(P), \\ H(P) < c(\theta)2^{l+1}R^{n-1}F(n-1) \end{array} \right\}. \tag{28}$$

In view of (23), we have that

$$2^l < Rn\log^* n, \tag{29}$$

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and so

$$0 \leqslant l \leqslant [\log_2(Rn \log^* n)] < \log_2 R + 2 \log_2 n < c_3 \log^* n \tag{30}$$

where $c_3 := (\log R + 2)/\log 2$ is an absolute constant independent of $n$ and $l$.

Additionally, with $k \in \mathbb{Z}$, we split the collection $C_P(n, l)$ into sub-collections $C'_P(n, l, k)$ such that

$$C'_P(n, l, k) := \{P(p, r, q) \in C_P(n, l) : c(\theta)2^k \leqslant q\|q\theta\| < c(\theta)2^{k+1}\}. \tag{31}$$

Take any $P(p, r, q) \in C'_P(n, l, k)$. In view of (16), the value $k$ should be non-negative. On the other hand, one can get an upper bound for $k$ by (23):

$$0 \leqslant k \leqslant [\log_2(R^n F(n))] < n \log_2 R + n \log_2 n + n \log_2 \log^* n < c_3 n \log^* n. \tag{32}$$

The upshot is that for fixed $n$ and $l$, the number of classes $C'_P(n, l, k)$ is at most $c_3 n \log^* n$.

Note that within the collection $C'_P(n, l, k)$ we have very sharp control of the height $H(P)$. Then, by (28) and (31), we also have very sharp control over the value $q$, namely

$$2^{l-k-1} R^{n-1} F(n-1) < q < 2^{l-k+1} R^{n-1} F(n-1). \tag{33}$$

Concerning the collection $C_L(n)$, we also partition it into sub-collections. Firstly, we partition it into sub-collections $C_L(n, l)$ such that

$$C_L(n, l) := \left\{ L \in C_L(n) \;\middle|\; \begin{matrix} c(\theta)2^l R^{n-1} F(n-1) \leqslant H(L), \\ H(L) < c(\theta)2^{l+1} R^{n-1} F(n-1) \end{matrix} \right\}. \tag{34}$$

Then we split $C_L(n, l)$ into sub-collections $C'_L(n, l, k)$ such that

$$C'_L(n, l, k) := \{L(A, B, C) \in C_L(n, l) : 2^k \leqslant |B| < 2^{k+1}\}. \tag{35}$$

One can check that $l$ and $k$ satisfy the same conditions (30) and (32) as in the case of points. Note that within each collection we have very good control over all point and line parameters.

The procedure of removing "bad" intervals from $\mathcal{I}_{n+1}$ will be as follows. We will first remove all intervals $I_{n+1} \in \mathcal{I}_{n+1}$ such that there exists a point $P \in C_P(n, 0)$ which satisfies $I_{n+1} \cap \Delta(P, cf^{-1}(q)) \neq \emptyset$ or there exists a line $L \in C_L(n, 0)$ which satisfies $I_{n+1} \cap \Delta(L, cf^{-1}(|A|^*|B|^*)) \neq \emptyset$. Then we repeat this removing procedure for collections

$$C_P(n, 1) \text{ and } C_L(n, 1), \ldots, C_P(n, c_2 \log^* n) \text{ and } C_L(n, c_2 \log^* n)$$

in exactly this order.

We will use lexicographical order for pairs in $\mathbb{Z}^2$; that is, we say that $(a, b) \leqslant (c, d)$ if either $a < c$ or $a = c$ and $b \leqslant d$. Consider the point $P(p, r, q) \in$

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$C'_P(n,l,k)$. If there exists a pair $(n',l') \leqslant (n,l)$ and a line $L(A,B,C) \in C_L(n',l')$ such that

$$H(L) < H(P) \quad \text{and} \quad \Delta(P, cf^{-1}(q)) \subset \Delta(L, cf^{-1}(|A|^*|B|^*)),$$

then such a point will not remove anything more than was removed by a line $L$. Therefore such a point can be ignored. The same is true if there exists a point $P'(p',r',q') \in C_P(n',l')$ such that

$$H(P') < H(P) \quad \text{and} \quad \Delta(P, cf^{-1}(q)) \subset \Delta(P', cf^{-1}(q')).$$

Therefore, instead of collection $C'_P(n,l,k)$, we can work with

$$C_P(n,l,k) := \left\{ P(p,r,q) \in C'_P(n,l,k) \;\middle|\; \begin{array}{l} \forall (n',l') < (n,l), \\ \forall L(A,B,C) \in C_L(n',l') \\ \quad \text{with } H(L) < H(P), \\ \forall P'(p',r',q') \in C_P(n',l') \\ \quad \text{with } H(P') < H(P), \\ \Delta(P, cf^{-1}(q)) \not\subset \Delta(L, cf^{-1}(|A|^*|B|^*)), \\ \Delta(P, cf^{-1}(q)) \not\subset \Delta(P', cf^{-1}(q')) \end{array} \right\}.$$

By the same procedure, we construct the collection $C_L(n,l,k)$ from $C'_L(n,l,k)$. Note that by the construction of $C_P(n,l,k)$, there exists at most one point $P(p,r,q) \in C_P(n,l,k)$ with given second coordinate $r/q$.

3.3. *Blocks of intervals* $\mathbf{B}_P(J)$ *and* $\mathbf{B}_L(J)$. Take the maximal possible constant $c_2 > 0$ such that

$$c_2 \leqslant \frac{1}{2^{10} c(\theta)} \quad \text{and} \quad \frac{R^2 c_1}{c_2} \in \mathbb{Z}. \tag{36}$$

Fix the triple $(n,l,k)$ and consider an arbitrary interval $J \subset \mathrm{L}_\theta$ of length $|J| = c_2 2^{-l} R^{-n+1} F^{-1}(n-1)$. Then for any $P(p,r,q) \in C_P(n,l,k)$ we have $|\Delta(P, cf^{-1}(q))| < |J|$. Indeed, this is true because

$$\begin{aligned} |J| &\geqslant |\Delta(P, cf^{-1}(q))| \\ &\iff \frac{c_2}{2^l R^{n-1} F(n-1)} \geqslant \frac{2c}{f(q) H(P)} \\ &\overset{(28)}{\impliedby} \frac{c_2}{2^l R^{n-1} F(n-1)} \geqslant \frac{2c}{c(\theta) 2^l R^{n-1} F(n-1) \cdot f(q)}. \end{aligned}$$

The last inequality is true provided that $c_2 c(\theta) \geqslant 2c$, which in turn is true by the second inequality of (17) and (36). One can easily check that the same fact is true for any $\Delta(L, cf^{-1}(|A|^*|B|^*))$ where $L(A,B,C) \in C_L(n,l,k)$.

LEMMA 1. *Let $J$ be an interval on $\mathrm{L}_\theta$ of length $|J| = c_2 2^{-l} R^{-n+1} F^{-1}(n-1)$. Then all rational points $P(p,r,q) \in C_P(n,l,k)$ such that $\Delta(P, cf^{-1}(q)) \cap J \neq \emptyset$ lie on a single line.*

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*Proof.* Consider an arbitrary point $P(p, r, q) \in C_P(n, l, k)$. Then

$$\left|\theta - \frac{p}{q}\right| = \frac{H(P)}{q^3} \overset{(28),(33)}{<} \frac{c(\theta)}{2^{2l-3k-4} R^{2(n-1)} F^2(n-1)}. \tag{37}$$

Suppose we have three points $P_i(p_i, r_i, q_i) \in C_P(n, l, k)$, $i = 1, 2, 3$, such that $\Delta(P_i, cf^{-1}(q_i)) \cap J \neq \emptyset$ and they do not lie on a single line. Then they form a triangle which has an area of

$$\mathbf{area}(\triangle P_1 P_2 P_3) \geqslant \frac{1}{2q_1q_2q_3} \overset{(33)}{\geqslant} \frac{1}{2^{3l-3k+4} R^{3(n-1)} F^3(n-1)}.$$

On the other hand, the first coordinates $p_i/q_i$ of the points $P_i$ should satisfy (37), and their second coordinates $r_i/q_i$ should lie within the interval of length $|J| + |\Delta(P_i, cf^{-1}(q_i))| \leqslant 2|J|$. Therefore we have the following upper bound for the area of the triangle $\triangle P_1 P_2 P_3$:

$$\begin{aligned}\mathbf{area}(\triangle P_1 P_2 P_3) &< \frac{2c_2 2^{-l} R^{-n+1} F^{-1}(n-1) \cdot 2c(\theta)}{2^{2l-3k-4} R^{2(n-1)} F^2(n-1)} \\ &\leqslant 2^{10} c_2 c(\theta) \cdot \frac{1}{2^{3l-3k+4} R^{3(n-1)} F^3(n-1)}.\end{aligned}$$

Finally, by (36) we get that the last value is bounded above by

$$\frac{1}{2^{3l-3k+4} R^{3(n-1)} F^3(n-1)} \leqslant \mathbf{area}(\triangle P_1 P_2 P_3),$$

which is impossible. So we get a contradiction. □

So, given an interval $J$ of length $c_2 2^{-l} R^{n-1} F^{-1}(n-1)$, if we have at least two points $P \in C_P(n, l, k)$ as in Lemma 1, then all points with this property will lie on a single line $L$. We denote this line by $L_J$. If there is at most one point $P \in C_p(n, l, k)$ as in Lemma 1, then we just say that $L_J$ is undefined.

Note that $L_J$ cannot be horizontal, because by the construction of $C_P(n, l, k)$ there is only one point $P(p, r, q) \in C_P(n, l, k)$ with the given second coordinate $r/q$. Also, $L_J$ cannot be vertical either; otherwise its equation would be expressable as $x = C/A$ where $\gcd(A, C) = 1$. Then, by the construction of $\theta$, we have that

$$\left|\theta - \frac{p}{q}\right| = \left|\theta - \frac{C}{A}\right| \geqslant \frac{c(\theta)}{A^2},$$

which together with (37) gives us

$$|A| \geqslant 2^{l-3/2k-2} R^{n-1} F(n-1).$$

Then, by the definition of $L_J$, there exist two points $P_1(p_1, r_1, q_1)$ and $P_2(p_2, r_2, q_2)$ with $|r_1/q_1 - r_2/q_2| < 2|J|$. However,

$$\left|\frac{r_1}{q_1} - \frac{r_2}{q_2}\right| \geqslant \frac{|A|}{q_1q_2} \overset{(33)}{\geqslant} 2^{-l+k/2-4} R^{-n+1} F^{-1}(n-1) > 2|J|.$$

So we get a contradiction.

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The statement of Lemma 1 can be strengthened if we have more than two points $P \in C_P(n, l, k)$ such that $\Delta(P, cf^{-1}(q)) \cap J \neq \emptyset$.

LEMMA 2. *Let $J$ be an interval on $\mathrm{L}_\theta$ of length $|J| = c_2 2^{-l} R^{-n+1} F^{-1}(n-1)$. Assume that there exists a line $L_J$. Consider the sequence of consecutive intervals $M_i \subset \mathrm{L}_\theta$, $i \in \mathbb{N}$, such that $|M_i| = |J|$, $M_1 := J$ and the bottom end of $M_i$ coincides with the top end of $M_{i+1}$. Define the set*

$$\mathcal{P}(J, m) := \left\{ P \in C_P(n, l, k) : P \in L_J \text{ and } \Delta(P, cf^{-1}(q)) \cap \left( \bigcup_{i=1}^{m} M_i \right) \neq \emptyset \right\}$$

*and the value*

$$m_P(J) := \max\{m \in \mathbb{N} : \#\mathcal{P}(J, m) \geqslant m + 1\}.$$

*Then all rational points $P \in C_P(n, l, k)$ such that*

$$\Delta(P, cf^{-1}(q)) \cap \left( \bigcup_{i=1}^{m_P(J)} M_i \right) \neq \emptyset$$

*lie on a line $L_J$.*

*Remark* 1. Since the number of points $P \in C_P(n, l, k)$, $P \in L_J$, is finite, the value $m_P(J)$ is correctly defined. Indeed, since by assumption $\#\mathcal{P}(J, 1) \geqslant 2$, $m + 1 \to \infty$ and $\#\mathcal{P}(J, m)$ is bounded, we have that $m(J)$ exists and is finite.

*Remark* 2. We define the block of intervals

$$\mathbf{B}_P(J) := \bigcup_{i=1}^{m(J)} M_i.$$

We will work with it as with one unit. If for some interval $J$ the line $L_J$ is undefined, then we define $m(J) := 1$ and $\mathbf{B}_P(J) := J$. So now $m(J)$ and $\mathbf{B}_P(J)$ are well-defined for all intervals $J$ of length $c_2 2^{-l} R^{-n+1} F^{-1}(n-1)$.

*Proof.* This is similar to the proof of Lemma 1. Let

$$\mathcal{P}(J, m(J)) = (P_i(p_i, r_i, q_i))_{1 \leqslant i \leqslant m(J)+1}$$

where the sequence $r_i/q_i$ is arranged in ascending order. Assume that there is a point $P(p, r, q) \in C_P(n, l, k)$ such that $P \notin L_J$ and $\Delta(P, cf^{-1}(q)) \cap \mathbf{B}_P(J) \neq \emptyset$. Then the triangle $\Delta(P P_1 P_{m(J)+1})$ splits into $m_P(J)$ disjoint triangles

$$\Delta(P P_i P_{i+1}), \quad 1 \leqslant i \leqslant m_P(J),$$

each of which has area

$$\mathbf{area}(\Delta(P P_i P_{i+1})) \geqslant \frac{1}{2 q q_i q_{i+1}}.$$

On the other hand, the first coordinates of the points $P_1, \ldots, P_{m_P(J)+1}$ and $P$ satisfy (37), and their second coordinates lie within the interval of length at most $(m_P(J) + 1)|J|$. Therefore we have the following estimate for the area of the

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triangle:

$$\frac{m_P(J)}{2^{3l-3k+4}R^{3(n-1)}F^3(n-1)} \leqslant \mathbf{area}(\triangle P P_1 P_{m_P(J)+1}) \leqslant \frac{2^9(m_P(J)+1)c_2c(\theta)}{2^{3l-3k+4}R^{3(n-1)}F^3(n-1)}.$$

But this is impossible since the left-hand side of the inequality is bigger than its right-hand side. □

Lemmas 1 and 2 have full analogues for lines $L \in C_L(n, l, k)$, which we will formulate in the next two lemmas. However, the proofs are slightly different.

LEMMA 3. *Let $J$ be an interval on $\mathrm{L}_\theta$ of length $|J| = c_2 2^{-l} R^{-n+1} F^{-1}(n-1)$. Then all lines $L(A, B, C) \in C_L(n, l, k)$ such that $\Delta(L, cf^{-1}(|A|^*|B|^*)) \cap J \neq \emptyset$ pass through a single rational point $P$.*

*Proof.* We will use the following well-known fact. Consider three planar lines $L_i(A_i, B_i, C_i)$, $i = 1, 2, 3$, defined by equations $A_i x - B_i y + C_i = 0$. Then they intersect in one point (possibly at infinity) if and only if

$$\det \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} = 0.$$

Suppose that there are three lines $L_1, L_2, L_3 \in C_L(n, l, k)$ which do not intersect at one point but are such that their thickenings intersect $J$. Then

$$\left| \det \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} \right| \geqslant 1.$$

On the other hand, we can make vertical shifts of $L_1$, $L_2$ and $L_3$ by distances $\delta_i < |J| + |\Delta(L_i)| < 2|J|$, $i = 1, 2, 3$, such that they will intersect at one point on $J$. Upon vertically shifting a line by the distance $\epsilon$ we change its $C$-coordinate by the value $B\epsilon$. Therefore we have

$$\det \begin{pmatrix} A_1 & B_1 & C_1 + B_1\delta_1 \\ A_2 & B_2 & C_2 + B_2\delta_2 \\ A_3 & B_3 & C_3 + B_3\delta_3 \end{pmatrix} = 0 \quad \Rightarrow \quad \left| \det \begin{pmatrix} A_1 & B_1 & B_1\delta_1 \\ A_2 & B_2 & B_2\delta_2 \\ A_3 & B_3 & B_3\delta_3 \end{pmatrix} \right| \geqslant 1.$$

However, the latter determinant is bounded above by

$$2|J|(|B_1(A_2B_3 - A_3B_2)| + |B_2(A_1B_3 - A_3B_1)| + |B_3(A_1B_2 - A_2B_1)|) \overset{(34),(35)}{\leqslant} 2c_2c(\theta)2^{-l}R^{-n+1}F^{-1}(n-1) \cdot 6 \cdot 2^{l+3}R^{n-1}F(n-1) \overset{(36)}{<} 1.$$

Hence we get a contradiction. □

So, given an interval $J$ of length $c_2 2^{-l} R^{n-1} F^{-1}(n-1)$, if we have at least two lines from $C_L(n, l, k)$ as in Lemma 3, then all lines with this property will

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intersect at one rational point $P$. We denote this point by $P_J$. If there is at most one line from $C_L(n, l, k)$ as in Lemma 3, then we just say that $P_J$ is undefined.

The next lemma is a "line" analogue of Lemma 2.

LEMMA 4. *Let $J$ be an interval on $\mathrm{L}_\theta$ of length $|J| = c_2 2^{-l} R^{-n+1} F^{-1}(n-1)$. Assume that there exists a point $P_J$. Consider the sequence of consecutive intervals $M_i \subset \mathrm{L}_\theta$, $i \in \mathbb{N}$, such that $|M_i| = |J|$, $M_1 := J$ and the bottom end of $M_i$ coincides with the top end of $M_{i+1}$. Define the set*

$$\mathcal{L}(J, m) := \left\{ L \in C_L(n, l, k) : P_J \in L,\ \Delta(L, cf^{-1}(|A|^*|B|^*)) \cap \left( \bigcup_{i=1}^{m} M_i \right) \neq \emptyset \right\}$$

*and the value*

$$m_L(J) := \max\{m \in \mathbb{N} : \#\mathcal{L}(J, m) \geqslant m + 1\}.$$

*Then all lines $L \in C_L(n, l, k)$ such that*

$$\Delta(L, cf^{-1}(|A|^*|B|^*)) \cap \left( \bigcup_{i=1}^{m_L(J)} M_i \right) \neq \emptyset$$

*intersect at a point $P_J$.*

By analogy with Remark 1, the value $m_L(J)$ is correctly defined. We define the block of intervals

$$\mathbf{B}_L(J) := \bigcup_{i=1}^{m_L(J)} M_i.$$

We will work with it as with one unit. As in Remark 2, if for some interval $J$ the point $P_J$ is not defined, then we define $m_L(J) := 1$ and $\mathbf{B}_L(J) := J$.

*Proof.* If $m_L(J) = 1$, then this is simply the statement of Lemma 3. Now assume that $m_L(J) > 1$. Let

$$\mathcal{L}(J, m_L(J)) = (L_i(A_i, B_i, C_i))_{1 \leqslant i \leqslant m_L(J)+1}.$$

Write

$$\omega_i := \frac{A_i\theta + C_i}{B_i} \quad \text{for } 1 \leqslant i \leqslant m_L(J) + 1.$$

Then all the triples $(A_i, B_i, C_i)$ lie inside the figure $F$ defined by the inequalities

$$|A_i| = \frac{H(L_i)}{|B_i|^2} \overset{(34),(35)}{<} c(\theta) 2^{l-2k+1} R^{n-1} F(n-1),$$

$$|B_i| \overset{(35)}{<} 2^{k+1}$$

and

$$|A_i\theta - B_i\omega_1 + C_i| < |B_i| \cdot |\omega_1 - \omega_i| < c_2 m_L(J) 2^{k+2-l} R^{-n+1} F^{-1}(n-1).$$

The volume of this figure is $16 c_2 c(\theta) m_L(J)$, which in view of (36) is smaller than $m_L(J)/6$. All points $(A_i, B_i, C_i)$, together with $(0, 0, 0)$, lie on the

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plane defined by $A_i p - B_i r + C_i q = 0$. And since $\gcd(A_i, B_i, C_i) = 1$, their convex body contains at least $m_L(J)$ disjoint triangles with vertices at points $(A_i, B_i, C_i)$ and $(0, 0, 0)$.

Now suppose that there is a line $L(A, B, C) \in C_L(n, l, k)$ such that $P_J \notin L$ and $\Delta(L, cf^{-1}(|A|^*|B|^*)) \cap \mathbf{B}_L(J) \neq \emptyset$. Then $(A, B, C) \in F$, but now this point does not lie on the same plane as the points $(A_i, B_i, C_i)$ and $(0, 0, 0)$. Thus it forms at least $m_L(J)$ disjoint tetrahedrons with them, each of which has a volume of at least $1/6$. Therefore the volume of $F$ is bounded by

$$\tfrac{1}{6} m_L(J) \leqslant \mathbf{vol}(F) < \tfrac{1}{6} m_L(J).$$

But the last inequality is impossible. Therefore the line $L$ has to pass through the point $P_J$. □

3.4. *Properties of blocks* $\mathbf{B}_P(J)$, $\mathbf{B}_L(j)$ *and quantities* $m_P, m_L$. Take an arbitrary interval $M$ of length $c_2 2^{-l} R^{n-1} F(n-1)$, and consider the collection $\mathcal{P}_M$ of points $P \in C_P(n, l, k)$ such that $\Delta(P, cF^{-1}(q)) \cap \mathbf{B}_P(M) \neq \emptyset$. Then one of the following cases should be true.

*Case 1P.* For any interval $J \subset \mathbf{B}_P(M)$ such that $|J| = |M|$,

$$\#\mathcal{S}_J := \#\big\{P \in \mathcal{P}_M : \Delta(P, cF^{-1}(q)) \cap J \neq \emptyset\big\} \leqslant 2^2.$$

*Case 2P.* There exists $J \subset \mathbf{B}_P(M)$ with $|J| = |M|$ such that $\#\mathcal{S}_J > 2^2$. Then the line $L_J$ is correctly defined and therefore $L_M = L_M(A, B, C)$ is correctly defined as well. Let the coefficient $B$ satisfy the condition

$$|B| < \frac{c(\theta) 2^{k+6}}{1/2cn(\log^* n)^2}. \tag{38}$$

*Case 3P.* There exists $J \subset \mathbf{B}_P(M)$ with $|J| = |M|$ such that $\#\mathcal{S}_J > 2^2$ and

$$|B| \geqslant \frac{c(\theta) 2^{k+6}}{1/2cn(\log^* n)^2}. \tag{39}$$

Consider cases 2P and 3P. Since for any $P \in \mathcal{S}_J$ all numbers $P_\theta$ lie inside an interval of length at most $2|J|$, there are at least two points $P_1(p_1, r_1, q_1)$ and $P_2(p_2, r_2, q_2)$ from $\mathcal{S}_J$ such that

$$\left|\frac{r_1}{q_1} - \frac{r_2}{q_2}\right| < 2^{-1}|J|.$$

Without loss of generality, assume that $q_2\|q_1\theta\| > q_1\|q_2\theta\|$. Then

$$(P_2)_\theta \in \Delta(P_1, 2^{-1}|J|H(P_1)) \overset{(28)}{\subset} \Delta(P_1, c(\theta)c_2) \overset{(36)}{\subset} \Delta(P_1, 2^{-10}). \tag{40}$$

Since $L_M$ is neither vertical nor horizontal, Proposition 1 is applicable for $\delta = 2^{-10}$. It states that

$$(P_2)_\theta \in \Delta\left(L_M, \frac{2^{-20}|B|}{q_2\|q_1\theta\|} \cdot \frac{H(P_2)}{H(P_1)}\right)$$

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and

$$H(L_M) \leqslant 2^{-8} H(P_1) \frac{q_2^3}{q_1^3} \overset{(33)}{\leqslant} \frac{1}{2} H(P_1).$$

This shows that $L_M$ belongs to the class which, within the basic construction, had been considered before considering the points from $\mathcal{P}_M$.

Now let us look at case 2P. By (38), (28), (31) and (33), the inclusion (40) implies that

$$(P_2)_\theta \in \Delta\left(L_M, \frac{2^{-11}}{1/2cn(\log^* n)^2}\right) \overset{(27)}{\subset} \Delta\left(L_M, \frac{1}{8cf(|A|^*|B|^*)}\right).$$

Now, since for any $P(p', r', q') \in C_P(n, l, k)$ the distance $|\theta - p'/q'|$ can differ from $|\theta - p/q|$ by a factor of at most 4, the same is true of the value $|\omega - r'/q'|$. An implication of this is that for all $P \in \mathcal{P}_M$,

$$P_\theta \in \Delta(L_M, 1/2cf^{-1}(|A|^*|B|^*)),$$

whence

$$\bigcup_{P(p,r,q)\in\mathcal{P}_M} \Delta(P, cf^{-1}(q)) \subset \Delta(L_M, cf^{-1}(|A|^*|B|^*)).$$

However, by the construction of the collection $C_P(n, l, k)$, for all $P \in C_P(n, l, k)$ the intervals $\Delta(P, cf^{-1}(q))$ are not contained in any interval $\Delta$ previously considered. Therefore, since $\mathcal{P}_M \subset C_P(n, l, k)$, the set $\mathcal{P}_M$ in case 2P should be empty, which is a contradiction. Thus case 2P is impossible.

Turning to the last case, 3P, let us order all the points in $\mathcal{P}_M = (P_i(p_i, r_i, q_i))_{1\leqslant i\leqslant m_L(J)+1}$ in such a way that the sequence $p_i/q_i$ is increasing. Then we have

$$\left|\frac{p_1}{q_1} - \frac{p_{m_P(M)+1}}{q_{m_P(M)+1}}\right| \leqslant \left|\theta - \frac{p_1}{q_1}\right| + \left|\theta - \frac{p_{m_P(M)+1}}{q_{m_P(M)+1}}\right| \overset{(37)}{\leqslant} \frac{c(\theta)}{2^{2l-3k-5} R^{2(n-1)} F^2(n-1)}.$$

On the other hand, the smallest possible difference between consecutive numbers $p_i/q_i$ and $p_{i+1}/q_{i+1}$ is bounded below by

$$\frac{p_{i+1}}{q_{i+1}} - \frac{p_i}{q_i} \geqslant \frac{|B|}{q_i q_{i+1}},$$

and therefore

$$\left|\frac{p_1}{q_1} - \frac{p_{m_P(M)+1}}{q_{m_P(M)+1}}\right| \overset{(33)}{\geqslant} \frac{|B| m_P(M)}{2^{2l-2k+2} R^{2(n-1)} F^2(n-1)}.$$

By combining the preceding two inequalities and (39), we finally get an estimate

$$m_P(M) \leqslant cn(\log^* n)^2. \tag{41}$$

Now, for the same interval $M$, define the collection $\mathcal{L}_M$ of lines $L(A, B, C) \in C_L(n, l, k)$ such that $\Delta(L, cF^{-1}(|A|^*|B|^*)) \cap \mathbf{B}_L(M) \neq \emptyset$. Consider the following three cases, which are full analogues to cases 1P, 2P and 3P above.

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*Case 1L.* For any interval $J \subset \mathbf{B}_L(M)$ such that $|J| = |M|$,

$$\#\mathcal{S}_J := \#\left\{L(A, B, C) \in \mathcal{L}_M : \Delta(L, cF^{-1}(|A|^*|B|^*)) \cap J \neq \emptyset\right\} \leqslant 2^2.$$

*Case 2L.* There exists $J \subset \mathbf{B}_L(M)$ with $|J| = |M|$ such that $\#\mathcal{S}_J > 2^2$. Then the point $P_J$ is correctly defined and therefore $P_M = P_M(p, r, q)$ is correctly defined as well. Let the coefficient $q$ satisfy the condition

$$q < \frac{c(\theta)2^{l-k+3}R^{n-1}F(n-1)}{1/2cn(\log^* n)^2}. \tag{42}$$

*Case 3L.* There exists $J \subset \mathbf{B}_L(M)$ with $|J| = |M|$ such that $\#\mathcal{S} > 2^2$ and

$$q \geqslant \frac{c(\theta)2^{l-k+3}R^{n-1}F(n-1)}{1/2cn(\log^* n)^2}. \tag{43}$$

Consider cases 2L and 3L. The arguments will be essentially the same as those for cases 2P and 3P, so one can get that there are at least two lines $L_1(A_1, B_1, C_1)$ and $L_2(A_2, B_2, C_2)$ from $\mathcal{S}$ such that

$$|L_1 \cap \mathrm{L}_\theta - L_2 \cap \mathrm{L}_\theta| < 2^{-1}|J|.$$

Without loss of generality, suppose that $|A_2B_1| < |A_1B_2|$. Then

$$L_2 \cap \mathrm{L}_\theta \in \Delta(L_1, 2^{-10})$$

and Proposition 2 is applicable with $\delta = 2^{-10}$. Therefore arguments analogous to those used in cases 2P and 3P give us

$$L_2 \cap \mathrm{L}_\theta \in \left(P_M, \frac{2^{-19}q}{|B_2A_1|}\right)$$

and

$$H(P_M) \leqslant \tfrac{1}{2}H(L_1).$$

Therefore the point $P_M$ is from the class which has already been considered before considering lines from $\mathcal{L}_M$.

First, look at case 2L. By (42), (34) and (35) we have that

$$L_2 \cap \mathrm{L}_\theta \in \Delta\left(P_M, \frac{2^{-14}}{1/2cn(\log^* n)^2}\right) \overset{(26)}{\subset} \Delta\left(P_M, \frac{1}{32cf(q)}\right).$$

Note that for any line $L(A, B, C) \in C_L(n, l, k)$ which goes through $P_M(p, r, q)$, the distance

$$\left|\frac{A\theta + C}{B} - \frac{r}{q}\right| = \frac{|A|}{|B|}\left|\theta - \frac{p}{q}\right|$$

can differ by a factor of at most 16 from the same distance for line $L_2$. Therefore, for all $L \in \mathcal{L}_M$,

$$L \cap \mathrm{L}_\theta \in \Delta(P_M, 1/2cf^{-1}(q)),$$

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whence

$$\bigcup_{L(A,B,C)\in\mathcal{L}_M} \Delta(L, cf^{-1}(|A|^*|B|^*)) \subset \Delta(P_M, cf^{-1}(q)).$$

However, since $\mathcal{L}_M \subset C_L(n, l, k)$, we get by the construction of $C_L(n, l, k)$ that $\mathcal{L}_M$ has to be empty; this is a contradiction. Therefore case 2L is impossible.

Now consider case 3L. Let us order all the lines in $\mathcal{L}_M = (L_i(A_i, B_i, C_i))_{1\leqslant i\leqslant m_L(J)+1}$ in such a way that the sequence of the second coordinates of $L_i \cap \mathrm{L}_\theta$ is increasing. Then we have

$$\begin{aligned}|L_1 \cap \mathrm{L}_\theta - L_{m_L(M)+1} \cap \mathrm{L}_\theta| &\leqslant \left|L_1 \cap \mathrm{L}_\theta - \frac{r}{q}\right| + \left|L_{m_L(J)+1} \cap \mathrm{L}_\theta - \frac{r}{q}\right| \\ &= \left(\frac{|A_1|}{|B_1|} + \frac{|A_{m_L(M)+1}|}{|B_{m_L(M)+1}|}\right) \cdot \frac{|q\theta - p|}{q} \\ &\overset{(34),(35)}{<} c(\theta) 2^{l-3k+2} R^{n-1} F(n-1) \cdot \frac{|q\theta - p|}{q}.\end{aligned}$$

On the other hand, by (14) and (35) the smallest difference between two consecutive $L_i \cap \mathrm{L}_\theta$ and $L_{i+1} \cap \mathrm{L}_\theta$ is at least

$$\frac{|q\theta - p|}{|B_i B_{i+1}|} > 2^{-2k-2}|q\theta - p|,$$

and therefore

$$|L_1 \cap \mathrm{L}_\theta - L_{m_L(M)+1} \cap \mathrm{L}_\theta| > m_L(M) 2^{-2k-2}|q\theta - p|.$$

By combining the upper and lower bounds for $|L_1 \cap \mathrm{L}_\theta - L_{m_L(M)+1} \cap \mathrm{L}_\theta|$ with (43), we finally obtain the estimate

$$m_L(M) \leqslant cn(\log^* n)^2. \tag{44}$$

3.5. *Final step of the proof.* Let $n \geqslant 3$. Fix an interval $J_{n-3} \in \mathcal{J}_{n-3}$. We will first estimate the quantity

$$\#\{P(p, r, q) \in C_P(n, l, k) : \Delta(P, cf^{-1}(q)) \cap J_{n-3} \neq \emptyset\}.$$

Split $J_{n-3}$ into

$$K := c_1/c_2 \cdot 2^l R^2 (n-1)(n-2)[\log^*(n-1)][\log^*(n-2)]$$

subintervals $M_1, \ldots, M_K$ of equal length $c_2 2^{-l} R^{-n+1} F^{-1}(n-1)$ such that the bottom endpoint of $M_i$ coincides with the top endpoint of $M_{i+1}$ $(1 \leqslant i \leqslant K-1)$.

We start by constructing blocks from intervals $M_1, \ldots, M_K$. Define $B_1 := \mathbf{B}_P(M_{n_1})$, $B_2 := \mathbf{B}_P(M_{n_2}), \ldots, B_t := \mathbf{B}_P(M_{n_t})$ in such a way that $n_1 := 1$ and the bottom endpoint of $\mathbf{B}_P(M_{n_i})$ coincides with the top endpoint of $\mathbf{B}_P(M_{n_{i+1}})$. By Lemma 2, for any $1 \leqslant i < t$ we have

$$\begin{aligned}&\#\{P(p, r, q) \in C_P(n, l, k) : \Delta(P, cf^{-1}(q)) \cap B_i \neq \emptyset 1\} \\ &\quad \leqslant m_P(M_{n_i}) + 1 \leqslant 2m_P(M_{n_i}).\end{aligned} \tag{45}$$

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Now let us consider the last block $B_t$. The problem is that this block is not necessarily included in $J_{n-3}$, so we need to treat it independently. As was discussed in §3.4, we have two possible cases. In case 1P we have that for any $i \geqslant n_t$,

$$\#\{P(p, r, q) \in C_P(n, l, k) : \Delta(P, cf^{-1}(q)) \cap M_i \neq \emptyset\} \leqslant 2^2.$$

Upon combining this with (45) we get that

$$\#\{P(p, r, q) \in C_P(n, l, k) : \Delta(P, cf^{-1}(q)) \cap J_{n-3} \neq \emptyset\} \leqslant 4K. \tag{46}$$

In case 3P we have

$$\begin{aligned}&\#\{P(p, r, q) \in C_P(n, l, k) : \Delta(P, cf^{-1}(q)) \cap B_t \neq \emptyset\}\\ &\overset{(41)}{\leqslant} cn(\log^* n)^2 + 1 \overset{(17),(36)}{<} K.\end{aligned}$$

Upon combining this estimate with (45) we get that

$$\#\{P(p, r, q) \in C_P(n, l, k) : \Delta(P, cf^{-1}(q)) \cap J_{n-3} \neq \emptyset\} \leqslant 3K < 4K.$$

Now estimate the number of intervals $I_{n+1} \in \mathcal{I}_{n+1}$ which are removed by $\Delta(P, cf^{-1}(q))$ where $P$ is some interval from $C_P(n, l, k)$:

$$\begin{aligned}&\#\{I_{n+1} \in \mathcal{I}_{n+1} : I_{n+1} \cap \Delta(P, cf^{-1}(q)) \neq \emptyset\}\\ &\leqslant \frac{|\Delta(P, cf^{-1}(q))|}{|I_{n+1}|} + 2\\ &= \frac{2cR^{n+1}F(n+1)}{c_1 f(q)H(q)} + 2\\ &\leqslant \frac{2cR^2 n(n+1)[\log^* n][\log^*(n+1)]}{c_1 c(\theta) f(q) 2^l} + 2\\ &\overset{(26)}{<} \frac{8cR^2(n+1)}{c_1 c(\theta) 2^l} + 2.\end{aligned} \tag{47}$$

The upshot of the cardinality estimates (46) and (47) is that

$$\begin{aligned}&\#\left\{I_{n+1} \in \mathcal{I}_{n+1} : J_{n-3} \cap I_{n+1} \cap \Delta(P, cf^{-1}(q)) \neq \emptyset \text{ for some } P \in C_P(n, l)\right\}\\ &\overset{(32)}{\leqslant} c_3 n \log^* n \cdot \#\left\{I_{n+1} \in \mathcal{I}_{n+1} \;\middle|\; \begin{array}{c} J_{n-3} \cap I_{n+1} \cap \Delta(P, cf^{-1}(q)) \neq \emptyset \\ \text{for some } P \in C_P(n, l, k)\end{array}\right\}\\ &\leqslant (c_3 n \log^* n \cdot 4K)\left(2 + \frac{8cR^2(n+1)}{c_1 c(\theta) 2^l}\right)\\ &\leqslant 8R^2 c_3 \frac{c_1}{c_2} \cdot 2^l n^3 (\log^* n)^3 + 2^5 \frac{c_3 c}{c_2 c(\theta)} R^4 \cdot n^4 (\log^* n)^3.\end{aligned}$$

By analogy we get the same estimate for

$$\#\left\{I_{n+1} \in \mathcal{I}_{n+1} \;\middle|\; \begin{array}{c} J_{n-3} \cap I_{n+1} \cap \Delta(L, cf^{-1}(|A|^*|B|^*)) \neq \emptyset \\ \text{for some } L \in C_L(n, l)\end{array}\right\}.$$

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By taking lines and points together and summing over $l$ satisfying (30), we find that

$$\#\left\{ I_{n+1}\in\mathcal{I}_{n+1} \;\middle|\; \begin{array}{l} J_{n-3}\cap I_{n+1}\cap\Delta(P,\,cf^{-1}(q))\neq\emptyset \\ \quad\text{for some } P\in C_P(n) \text{ or} \\ J_{n-3}\cap I_{n+1}\cap\Delta(L,\,cf^{-1}(|A|^*|B|^*))\neq\emptyset \\ \quad\text{for some } L\in C_L(n) \end{array} \right\}$$
$$\leqslant 16R^2c_3\frac{c_1}{c_2}n^3(\log^*n)^3\sum_{2^l<Rn\log^*n}2^l+2^6\frac{c_3^2c}{c_2c(\theta)}R^4n^4(\log^*n)^4.$$

If $(2^{10}c(\theta))^{-1}>R^2c_1$, then in view of (36) we have that $c_2=R^2c_1$. Otherwise we have that $c_2\geqslant(2^{11}c(\theta))^{-1}$ and

$$\frac{c}{c_2c(\theta)}\leqslant 2^{11}c,\qquad \frac{c_1}{c_2}\leqslant 2^{11}c_1.$$

In any case, the last expression is bounded above by

$$c_4n^4(\log^*n)^4$$

where

$$c_4:=2^6\max\left\{\frac{c}{R^2c_1c(\theta)},\,2^{11}c\right\}\frac{(\log R+2)^2R^4}{(\log 2)^2}$$
$$+16\max\{R^{-2},\,2^{11}c_1\}\frac{R^3(\log R+2)}{\log 2}$$

(recall that $c_3=(\log R+2)/\log 2$). In view of (18), the right-hand side of this inequality is bounded by

$$25R\log R\cdot n^4(\log^*n)^4=r_{n-3,n}.$$

The upshot is that for any interval $J_{n-3}\in\mathcal{J}_{n-3}$, the number of "bad" intervals $I_{n+1}\in\mathcal{I}_{n+1}$ which are to be removed is bounded by $r_{n-3,n}$. Therefore the desired set $\mathbf{K}_c$ is indeed a $(I,\mathbf{R},\mathbf{r})$ Cantor-type set. The proof of Theorem 2 is thus complete.

§4. *Final remark.* In the proof of Theorem 2 we showed that $\mathbf{Mad}_P(f,c)\cap\mathbf{Mad}_L(f,c)\cap\mathrm{L}_\theta$ contains a $(I,\mathbf{R},\mathbf{r})$ Cantor-type set; this allows us to use [**2**, Theorem 5].

THEOREM (BV5). *For each integer $1\leqslant i\leqslant k$, suppose we are given a Cantor set $\mathbf{K}(I,\mathbf{R},\mathbf{r}_i)$. Then*

$$\bigcap_{i=1}^{k}\mathbf{K}(I,\mathbf{R},\mathbf{r}_i)$$

*is a $(I,\mathbf{R},\mathbf{r})$ Cantor set where*

$$\mathbf{r}:=(r_{m,n})\quad\text{with } r_{m,n}:=\sum_{i=1}^{k}r_{m,n}^{(i)}.$$

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For sets of the form $\mathbf{Mad}_P(f) \cap \mathbf{Mad}_L(f) \cap \mathrm{L}_\theta$, Theorem BV5 enables us to show that for any finite family $\theta_1, \ldots, \theta_n$ of badly approximable numbers one can find $\alpha \in \mathbb{R}$ such that the following inclusion holds simultaneously for all $1 \leqslant i \leqslant n$:

$$(\alpha, \theta_i) \in \mathbf{Mad}_P(f) \cap \mathbf{Mad}_L(f).$$

Moreover, the set of such numbers $\alpha$ is of full Hausdorff dimension. The proof is based on intersecting the corresponding Cantor-type sets $K_c(i)$ associated with each set $\mathbf{Mad}_P(f, c) \cap \mathbf{Mad}_L(f, c) \cap \mathrm{L}_{\theta_i}$ for $c$ sufficiently small and then applying Theorem BV4 to the intersection. We will leave the details to the interested reader.

We also believe that the same fact will be true for a countable collection $\{\theta_i\}$ of badly approximable numbers. However, this cannot be proven with existing techniques.

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