Evolution of the Euler-Maclaurin sum formula

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The correspondence between the discrete and the continuous is a fascinating theme in mathematics. The Euler-Maclaurin sum formula, discovered independently and almost contemporaneously by Leonhard Euler (1707–1783) and Colin Maclaurin (1698–1746), in the early 1730s, relates the sum of the values of a function at the integers in the interval $[a, b]$ with its integral over $[a, b]$. It thus equates a discrete sum with a continuous sum (integral) of a related function.

Rota underlined the importance of the Euler-MacLaurin sum formula when he said: 'The Euler-MacLaurin summation formula is one of the most remarkable formulas of mathematics.' [1, p. 11] The formula has proved useful for about 280 years and has not lost its importance even in this age of high-speed computing. There is an extensive literature on the topic that still continues to attract the attention of researchers.

1. *Euler's motivation and beginnings*

The sum formula seems to have originated in Euler's efforts to evaluate the partial sums of the harmonic series and the sum of the infinite series of reciprocal squares. In §14 of his paper E20 (presented to the St Petersburg Academy on 5 March 1731 and originally published in Commentarii academiae scientiarum Petropolitanae 5, 1738, pp. 91-105), Euler explains a method of getting an integral corresponding to a progression with general term $\frac{1}{a+(n-1)b}$. He notes in §17 that the sum of the the series

$$\frac{a}{a\cdot\alpha a}+\frac{a}{(a+b)(\alpha a+b)}+\frac{a}{(a+2b)(\alpha a+2b)}+\cdots$$

to infinity will be $\int y^{\alpha-2}dy\int\frac{dy}{a\left(1-y^{\frac{b}{a}}\right)}$. Putting $\alpha=1$ and $a=b=1$, he arrives at the relation

$$\int_0^1\frac{dy}{y}\int_0^y\frac{dx}{1-x}=\sum_{n=0}^{\infty}\frac{1}{2^n(n+1)^2}+(\ln 2)^2=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

in §22 and obtains an approximate value 1.644934 for the integral and therefore the series. The identity amounts to Euler's evaluation of $\mathrm{Li}_2\left(\frac{1}{2}\right)$ in §199, Ch. IV of his *Institutiones Calculi Integralis* (1768).

The summation formula was enunciated by Euler in a pioneering paper E25 (presented to the St Petersburg Academy on 20 June 1732) [2]. Euler begins by stating (in E20) that in the past year he had proposed a 'synthetic method' of summing innumerable progressions, and says that he will communicate an 'analytical method' having wide application. He then introduces his formula [1, §2] for the sum s (as a function of n) from the first to the t (n)-th term, given by the equation

$$t=\frac{ds}{dn}-\frac{d^2s}{2!\,dn^2}+\frac{d^3s}{3!\,dn^3}-\frac{d^4s}{4!\,dn^4}+\text{etc.},$$

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which can be transmuted into

$$s = \sum_{n=0}^{m} t(n) = \int t\,dn + \alpha t + \beta\frac{dt}{dn} + \gamma\frac{d^2t}{dn^2} + \delta\frac{d^3t}{dn^3} + \ldots .$$

The coefficients are determined recursively using the following relations:

$$\alpha = \frac{1}{2}, \beta = \frac{\alpha}{2!} - \frac{1}{3!}, \gamma = \frac{\beta}{2!} - \frac{\alpha}{3!} + \frac{1}{4!}, \delta = \frac{\gamma}{2!} - \frac{\beta}{3!} + \frac{\alpha}{4!} - \frac{1}{5!},$$

$$\varepsilon = \frac{\delta}{2!} - \frac{\gamma}{3!} + \frac{\beta}{4!} - \frac{\alpha}{5!} + \frac{1}{6!}, \text{ etc.}$$

Then the equation becomes

$$s = \int t\,dn + \frac{t}{2} + \frac{dt}{12\,dn} - \frac{d^3}{720\,dn^3} + \frac{d^5t}{30420\,dn^5} + \text{etc.} \qquad (1)$$

He illustrates it with $t = n^2 + 2n$; then $dt = 2n\,dn + 2\,dn$, $d^2t = 2\,dn^2$, $d^3t = 0$.

$$s = \int (n^2 + 2n)\,dn + \frac{n^2 + 2n}{2} + \frac{2n + 2}{12} = \frac{2n^3 + 9n^2 + 7n}{6}.$$

Since $s = 0$ at $n = 0$, the constant term is left out.

2. *Geometrical treatment*

Euler's method evolved over a period of time. He took up the subject again in his paper E46 [3] (presented to the St. Petersburg Academy on 9 June 1735). He notes: 'Hac autem ratione serierum summationem ad integrationem reducebam.' It translates as: *By this method the summation of series is reduced to integration*. He explains:

> Suppose n to be the summation index of the terms which sum to s. The sum s will get augmented with each unit of n. Let the increase be x; the quantity x becomes very small as n becomes very large. Consider now 1 and x as the instantaneous increment in n and s so that $dn : ds = 1 : x$. Then, $ds = x\,dn$ and $s = \int x\,dn$. In this, the constant of integration has to be added; if the sum s is continued to infinity, the constant of integration is left out of calculation. Thus, by this method we obtain a sum from the term x up to infinity. When we add the sum s to the (previously computed) finite sum, we obtain the sum of the proposed series.

As he remarks, this method always sums the series with some small error.

Euler elucidates his method geometrically, investigated analytically earlier. He forms lower and upper bounds for the sum.

He notes that the series $a + b + c + d + e +$ etc. is the sum of the areas of the parallelograms Aa, Bb, Cc, Dd etc. shown in Figure 1 where AP has been partitioned into AB, BC, CD, DE etc. each of unit length. So to get the sum of this series, we must determine an approximation to the area of these parallelograms. He argues to establish that $a + b + c + d + \ldots + x > \int y\,dn$. Therefore we have discovered a lower limit, such that the sum of the series

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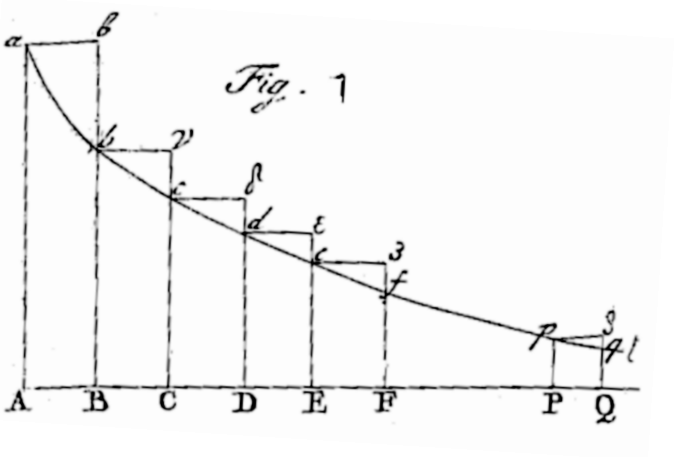


FIGURE 1

is larger, and we now find the upper limit by a similar method as in Figure 2 getting $a + b + c + d + \ldots + x < \int y\,dn$. Therefore $\int x\,dn$ and $\int y\,dn$ are the upper and lower limits of the series.

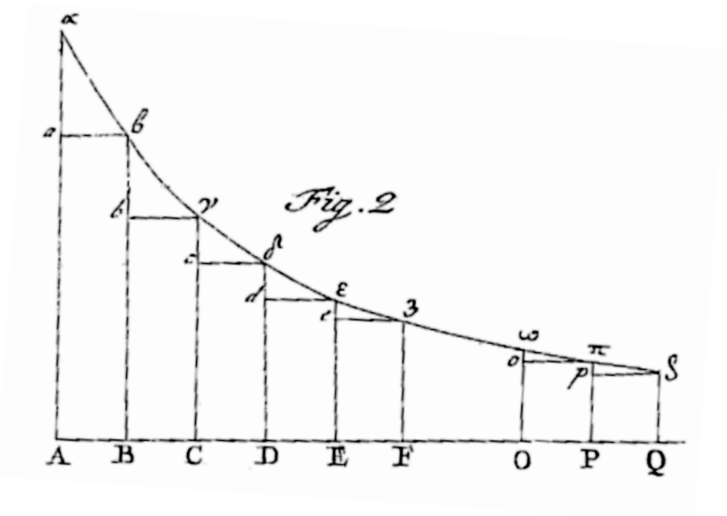


FIGURE 2

We have obtained close approximations to the sums of these series by neglecting the small curvilinear triangles in each figure. In the first figure we must add the area of these curvilinear triangles $\int y\,dn$, to obtain the sum of the series. The area of each curvilinear triangle is the sum of a rectilinear triangle and a curvilinear segment. The sum of all the rectilinear triangles

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equals $(Aa - Qq)AB/2 = \frac{1}{2}(a - y)$. But even if $\frac{1}{2}(a - y)$ is added to $\int y\,dn$ it will not be sufficient to complete the area. This way Euler obtains an improved lower bound $a + b + c + d + \ldots + x > \int y\,dn + \frac{1}{2}(a - y)$.

In the second figure, from the area $A\alpha\pi P$, which equals $\int x\,dn$ one has to subtract the area of the curvilinear triangles (which is less than the area of the rectilinear triangles) to obtain the true sum of the rectangles. Euler subtracts $\frac{1}{2}(a - y)$ to get the inequality $a + b + c + d + \ldots + x < \int y\,dn - \frac{1}{2}(a - y)$.

Euler states in §10 that the value of the former of these two new limits is closer than the latter. To improve the approximation we must add the areas of the curvilinear segments bounded by the arcs *ab*, *bc*, etc. and the corresponding chords *ab*, *bc*, etc. the first two of which are shown in his Figure 3.

Referring to Fig.3, Euler writes:

> A true approximation of these segments is obtained by extending the chord *cb* leftward to *n*. Bisect the straight line *an* in *m*, the line *bm* is an approximation to the tangent of the curve at *b*. *The area of the segment aba is nearly one third the area of the triangle abm and consequently one sixth the area of the triangle abn* (emphasis added).

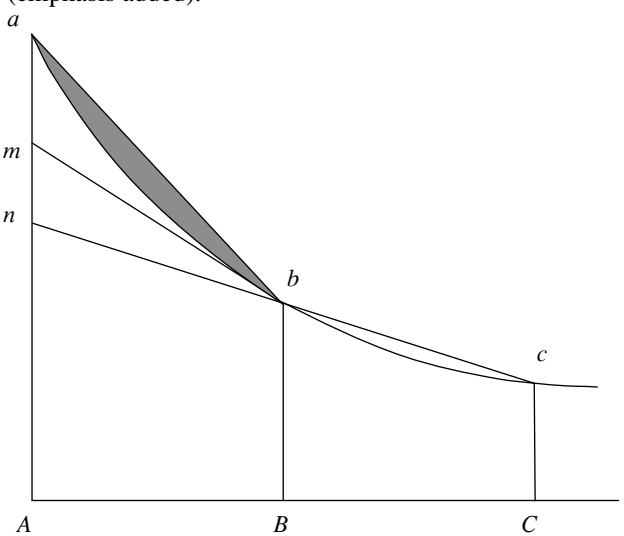


FIGURE 3

Thus if $Aa = a$, $Bb = b$, $Cc = c$, $an = a - 2b + c$, and since $AB = 1$, the triangle $abn = \frac{1}{2}(a - 2b + c)$. So this sixth part equalling $abn = \frac{1}{12}(a - 2b + c)$ is the first curvilinear segment *aba*. Similarly the second segment equals $\frac{1}{12}(b - 2c + d)$ and the last equals $\frac{1}{12}(x - 2y + z)$, denoting by z the term indexed by $n + 2$. Therefore the result of the sum of

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all of the segments equals $\frac{1}{12}\big((a-b)-(y-z)\big)$. Euler thus obtains a formula in §11 which may be written in this form

$$\sum_{k=n}^{N} \approx \int_{n}^{N} f(x)\,dx + \frac{7f(n) - f(n+1) + 5f(N) - f(N+1)}{12}.$$

He observes in §12 that if the series converges, this formula gives approximate sums as close to the true value as you wish. The greater the number of initial terms added exactly, the better the approximation we can obtain. The formula then assumes a simpler form:

$$\sum_{k=0}^{\infty} \approx \int_{n}^{\infty} f(x)\,dx + \frac{7f(n) - f(n+1)}{12}.$$

Euler determines the sum of the first million terms of the harmonic series to be 14.392669 (correct to 3 decimal places) by first computing the sum of the first ten terms and then adding an approximate value given by his formula. The formula also yields this:

$$\begin{aligned}\zeta(2) &\approx \sum_{k=1}^{10} \frac{1}{k^2} + \int_{11}^{\infty} \frac{dx}{x^2} + \frac{7\cdot 121^{-1} - 144^{-1}}{12}\\ &\approx 1.5497677311 + 0{\cdot}0909090909 + 0.004242232\\ &\approx 1.644919055.\end{aligned}$$

2.1. *Verification of Euler's observation in a particular case*

The discussion is based on the Notes on Section 10 in [4]. We assume the curve to be a parabola $y = x^2$ and consider that portion of the curve which lies on the left of the y-axis. In Figure 3, the trapezium $nACc$ (where Bb bisects AC) shows that

$$\begin{aligned}Bb &= \tfrac{1}{2}\big[An + Cc\big] \Rightarrow An = 2Bb - Cc = 2f(\alpha) - 2f(\alpha+1)\\ &= 2\alpha^2 - (\alpha+1)^2,\end{aligned}$$

i.e. $A_n = \alpha^2 - 2\alpha - 1$. Hence

$$an = aA - An = (\alpha-1)^2 - (\alpha^2 - 2\alpha - 1) \text{ and } am = 1.$$

Therefore, the area of $\Delta anb = \frac{2\cdot 1}{2} = 1$ and the area of $\Delta nmb = \frac{1}{2}$.

The area of the trapezium $aABb$ equals arithmetic mean of Aa and Bb multiplied by height 1. So its area is $\frac{1}{2}\big((\alpha-1)^2 + \alpha^2\big)$.

Here α is the x-coordinate of B, so that $\alpha - 1$ and $\alpha + 1$ are the x-coordinates of A and C, respectively.

The area of the curvilinear segment bab is given by the area of the trapezium $aABb$ minus the area under the parabola from a to b. This is given by

$$\frac{(\alpha-1)^2+\alpha^2}{2} - \int_{\alpha-1}^{\alpha} x^2\,dx = \frac{2\alpha^2 - 2\alpha + 1}{2} - \frac{\alpha^3}{3} + \frac{(\alpha-1)^3}{3} = \frac{1}{6}.$$

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3. *Euler's proof of the sum formula*

In the next paper E47 [5] (presented to the St Petersburg Academy on October 13, 1735) Euler explains how he discovered the formula. He states that if $y = f(x)$, then $f(x+dx) = y + dy$, $f(x + 2dx) = y + 2dy + d^2y$, $f(x + 3dx) = y + 3dy + 3d^2y + d^3y$, where the coefficients are the same as those of the powers of a binomial. So if $x + mdx$ is put in place of x, then y will take on this form

$$y + \frac{m}{1}dy + \frac{m(m-1)}{1\cdot 2}d^2y + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}d^3y + \ldots,$$

and letting m be an infinitely large number such that mdx represents a finite quantity he changes it into

$$y + \frac{m}{1}dy + \frac{m^2}{1\cdot 2}d^2y + \frac{m^3}{1\cdot 2\cdot 3}d^3y + \frac{m^4}{1\cdot 2\cdot 3\cdot 4}d^4y + \ldots.$$

Then putting $mdx = a$ or $m = \dfrac{a}{dx}$, he writes it as

$$y + \frac{a\,dy}{1\,dx} + \frac{a^2\,d^2y}{1\cdot 2\,dx^2} + \frac{a^3\,d^3y}{1\cdot 2\cdot 3\,dx^3} + \ldots,$$

in which all the terms are of finite magnitude.

Euler then mentions Taylor series in this connection. Putting $a = -x$ so that y vanishes, he obtains

$$y = \frac{x\,dy}{1\,dx} - \frac{x^2\,d^2y}{1\cdot 2\,dx^2} + \frac{x^3\,d^3y}{1\cdot 2\cdot 3\,dx^3} - \ldots,$$

Now, setting $z = \dfrac{dy}{dx}$,

$$\int z\,dz = \frac{xz}{1} - \frac{x^2\,dz}{1\cdot 2\,dx} + \frac{x^3\,d^2z}{1\cdot 2\cdot 3\,dx^2} - \ldots,$$

in which equation the integral of $z\,dx$ is expressed by an infinite series. He then remarks: 'And this is the general quadrature of curves which the most insightful Johann Bernoulli gave in the *Acta eruditorum* of Leipzig; however, he did not attach the analysis which led to this series.'

He then returns to his series in §12 and writes: Let us have some series $A + B + C + D + \ldots + X$, in which A denotes the first term, B the second, and X the term with index x, so that X is the general term of the given series. Let us also put the sum of this progression to be

$$A + B + C + D + \ldots + X = S;$$

S will be the summatory term, and if the series is determined it will be composed from x and fixed as much as X is. He obtains the following equation in the next section:

$$X = \frac{dS}{1!\,dx} - \frac{d^2S}{2!\,dx^2} + \frac{d^3S}{3!\,dx^3} - \frac{d^4S}{4!\,dx^4} + \ldots$$

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and remarks that this series can be inverted by a known method, so that from the general term X the summatory term S can be determined, which is what is required.

In §16 he calculates the coefficients α, β, etc. that we encountered earlier; this time he begins correctly with $\alpha = 1$. In §19 he derives the formula

$$S = \int X\,dx + \frac{X}{2!} + \frac{dX}{3!\,2\,dx} - \frac{d^3X}{5!\,6\,dx^3} + \frac{d^5X}{7!\,6\,dx^5} - \frac{3\,d^7X}{9!\,10\,dx^7} + \frac{5\,d^9X}{11!\,6\,dx^9} - \frac{691\,d^{11}X}{13!\,210\,dx^{11}} + \frac{35\,d^{13}X}{15!\,2\,dx^{13}} - \frac{3617\,d^{15}X}{17!\,30\,dx^{15}} + \ldots. \tag{2}$$

Euler had not yet discerned the connection between the coefficients occurring in the formula and the Bernoulli numbers B_m which first appeared in *Ars Conjectandi* in connection with the evaluation of sums of powers of consecutive integers. B_{2k} is the coefficient of $\frac{x^{2k}}{(2k)!}$ in the expansion of $\frac{x}{e^x - 1}$. He himself evaluates the power sums $\sum_{m=1}^{n} m^p$ for $p = 1, 2, \ldots, 16$ in §23.

Euler employs his method to approximate the partial harmonic sums (H_n) and $\zeta(2)$ for a second time and obtains amazing approximations. In §27 he gives H_{10^6} correct to 16 decimal places. In §32, he gives an approximated value of $\zeta(2)$ correct to 20 decimal places! He computes it in this way: $\int X\,dx = C - \frac{1}{x}$ and $\frac{d^{2k-1}X}{dx^{2k-1}} = -\frac{(2k)!}{x^{2k+1}}$. First he calculated

$$\sum_{n=1}^{10} \frac{1}{n^2} = \frac{1968329}{1270080} = 1.549767731166540\ldots.$$

Then he used a few terms of the divergent sum

$$S = \frac{1}{10} - \frac{1}{2\cdot 10^2} + \frac{1}{6\cdot 10^3} - \frac{1}{3\cdot 10^6} + \frac{1}{42\cdot 10^7} - \frac{1}{3\cdot 10^{10}} + \frac{1}{132\cdot 10^{10}} - \frac{691}{273\cdot 10^{13}} + \frac{7}{6\cdot 10^{15}} - \ldots.$$

The stated terms yield

$$S = \frac{571569012104204987}{6006000000000000000} = 0.0951663356816858\overline{1202131}$$

leading to an approximate value 1.64493 40668 48226 50 correct to 15 decimal places. Euler does not provide details of his computations for the 20-place accuracy. He must have used more terms. We find a few typographical errors in the paper. In §19, the 8th term on the right-hand side with coefficient 691 has + instead of the − sign. Then in §32, the term with 691 has 13 zeros in the denominator instead of 14.

The sum formula is further elaborated in Euler's paper E55 [6]. In §4 the formula (2) occurs with an additional term having the coefficient $\frac{43867}{19!\,42}$.

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The stated aim of the paper is to get a formula for an infinite sum.

Euler unified the material relating to the summation formula and its applications in chapters 5 and 6 of part II of [7]. The coefficients α, β, γ, … are investigated in depth in articles 112–120 of chapter 5. As the letters are determined from the preceding by a constant law, they form a recurrent series. Writing $V = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \ldots$, he obtains the following generating function:

$$V = \frac{1}{\sum_{k=0}^{\infty} (-1)^k \frac{u^k}{k!}} = \frac{u}{1 - e^{-u}}.$$

In art. 122 Euler recognises the connection between these coefficients and the Bernoulli numbers. Dividing α, β, γ, etc. by odd numbers 3, 5, 7, etc., Euler defines $\frac{\alpha}{3} = \frac{1}{6} = \mathfrak{A}$, $\frac{\beta}{5} = \frac{1}{30} = \mathfrak{B}$, $\frac{\gamma}{7} = \frac{1}{42} = \mathfrak{C}$, $\frac{\delta}{9} = \frac{1}{30} = \mathfrak{D}$, $\frac{\varepsilon}{11} = \frac{5}{66} = \mathfrak{E}$, and so on. We find that $\mathfrak{A} = B_2$, $\mathfrak{B} = -B_4$, $\mathfrak{C} = B_6$ and so on. These numbers arose in finding the sums $\sum_{m=1}^{n} m^p$ and now they pop up surprisingly in a totally different situation, namely, in the sum formula for

$$\sum_{m=1}^{\infty} \frac{1}{m^{2k}} = \frac{(-1)^{k-1}}{2} \frac{B_{2k}}{(2k)!} (2\pi)^{2k}.$$

The summation formula is used in Chapter 6 to obtain approximations even when it diverges. Euler takes up the harmonic series and finds the value of his constant γ using his own notation in art. 143: $C = 0.5772156649015325$ and notes that this constant expresses the sum of the series

$$\frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{D}}{8} + \frac{\mathfrak{E}}{10} - \ldots.$$

4. *Maclaurin and the sum formula*

Colin Maclaurin (1698-1746) forms, with Gregory and Stirling, the trio of notable Scottish mathematicians who developed and extended Newton's work in calculus. Maclaurin entered the University of Glasgow at the tender age of 11 and graduated in his fifteenth year. At 19, he was appointed professor of mathematics at Marischal College, Aberdeen. Two years later he was elected a fellow of the Royal Society of London. He soon became acquainted with Newton on whose recommendation Maclaurin was selected in November 1725 to assist Professor James Gregory (1666-1742), a nephew of the famous James Gregory (1638-1675), at Edinburgh university. Maclaurin eventually succeeded Gregory as Professor of Mathematics in 1742 and occupied the Chair until his death.

4.1. *The background of Maclaurin's fluxions*

In 1734, George Berkeley, an Irish philosopher and bishop, assailed Newton's doctrine of fluxions in his *The analyst; or, A discourse addressed to an infidel mathematician*.* The Bishop maintained that it was not possible

* The 'infidel mathematician' was Edmond Halley (1656-1742) whose 1684 visit put Newton onto *De motu corporum in gyrum*; Halley played a leading role in Royal Society's endorsement of the *Principia* and got it published at his own expense.

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to imagine a finite ratio between two evanescent quantities. 'And what are these fluxions?' he asked adding sarcastically: 'The velocities of evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?' (XXXV) Maclaurin staunchly defended Newton's methods in his two-volume *A treatise of fluxions* (1742) that aims at putting calculus on a rigorous geometric basis.

Maclaurin begins his *Preface* by referring to Berkeley's pamphlet:

> A Letter published in the Year 1734 under the Title of The Analyst, first gave Occasion to the ensuing Treatise, and several Reasons concurred to induce me to write on this Subject at so great length. The Author of that Piece had represented the Method of Fluxions as founded on false Reasoning, and full of Mysteries. His Objections seemed to have been occasioned by the concise Manner in which the Elements of this Method have been usually described, and their having been so much misunderstood by a Person of his Abilities appeared to me to be a sufficient Proof that a fuller Account of the Grounds of them was required.

Among several other things, Maclaurin obtained in his book the celebrated Maclaurin series for which he is best remembered. It is a special case of Taylor's theorem and was in fact given earlier by Stirling in his *Methodus Differentialis* published in 1730. In Part II, Proposition 19, Stirling expands any general ordinate T in terms of of the equidistant ordinates and finally in terms of the first ordinate and its fluxions

$$T = A + \dot{A}z + \tfrac{1}{2}\ddot{A}z^2 + \tfrac{1}{6}\dddot{A}z^3 + \tfrac{1}{24}\ddddot{A}z^4 + \ldots .$$

Stirling writes: 'And Mr. Taylor was the first to discover this in his *Methodus Incrementorum*, and afterwards Herman gave it in the appendix to his *Phoronomia*.' (In 1772, J.-L. Lagrange proclaimed Taylor's theorem 'the basic principle of differential calculus.') Maclaurin mentions Taylor's book in *Fluxions*, Vol. II, Ch. II, art. 751, p. 611.

4.2. *The sum formula*

We now come to the summation formula that occurs at two places in Maclaurin's *Fluxions*. His treatment of the summation formula is akin to Euler's geometrical treatment in E46. In Chapter X (Volume I) of *Treatise of Fluxions* [8], he discusses areas and the sums of progressions.

Referring to Figure 4, he writes in art. 350 (pp. 289–290):

> Let the terms of any progression be represented by the perpendiculars *AF*, *BE*, *CK*, *HL*, &c. that stand upon the base *AD* at equal distances; and let *PN* be any ordinate of the curve *FNe* that passes through the extremities of those perpendiculars. Suppose *AP* to be produced; and according as the area *APNF* has a limit which it never amounts to, or may be

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produced till it exceed any give space, there is a limit which the sum of the progression never amounts to, or it may be continued till its sum exceed any given number. For let the rectangles *FB*, *EC*, *KH*, *LI*, &c. be completed, and, the area *APNF* being continued over the same base, it is always less than the sum of all those rectangles, but greater than the sum of all the rectangles after the first. Therefore the area *APNF* and the sum of those rectangles either both have limits, or both have none; and it is obvious, that the same is to be said of the sum of the ordinates *AF*, *BE*, *CK*, *HL*, &c. and of the sum of the terms of the progression that are represented by them.

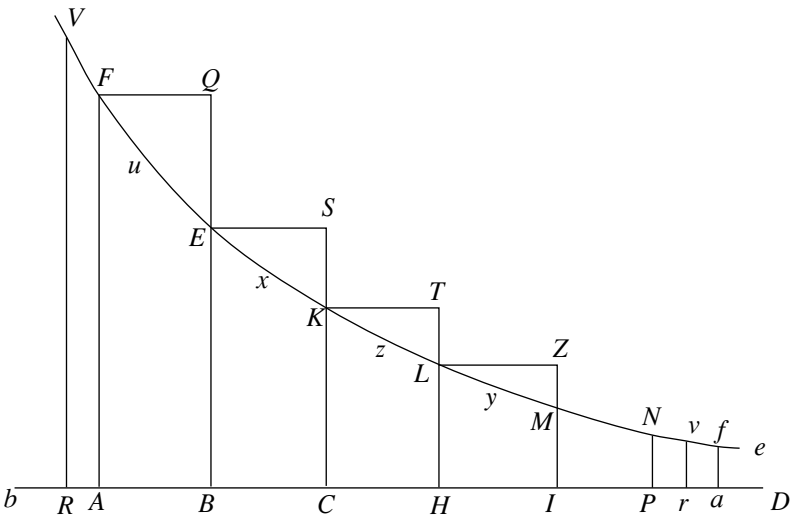


FIGURE 4

He then argues in art. 352 (pp. 291-292):

When the area *APNF* has a limit, we not only conclude from this, that the sum of the progression represented by the ordinates has a limit; but when the former limit is known, we may by it approximate to the value of the latter. Let *AB* represent unit; then the sum of the rectangles *FB*, *EC*, *KH*, *LI*, &c. and the sum of the ordinates *AF*, *BE*, *CK*, *HL*, &c. shall be expressed by the same number But because the curve is convex towards the asymptote, the sum of those rectangles always exceeds the curvilineal area over the same base, by the triangles *FEQ*, *EKS*, *KLT*, *LMZ*, &c. and by the curvilineal spaces *FuE*, *ExK*, *KzL*, *LyM*, &c. The sum of those triangles approaches to one half of the rectangle *FB*, as its limit; therefore the sum of the rectangles *FB*, *EC*, *KH*, *LI*, &c. is greater, than the curvilineal area *AFEKMLI* that is over the

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same base added to one half of the rectangle *FB*, the excess being the sum of the spaces *FuE*, *ExK*, *KzL*, *LyM*, &c. that are bounded by the arches *FuE*, *ExK*, *KzL*, &c. and their chords *FE*, *EK*, *KL*, &c. Hence, when *AF* is a term at a great distance from the beginning of a progression of this kind, the number that expresses the limit of the curvilineal area *APNF* added to one half of the term *AF* gives nearly the value of *AF* and the subsequent terms *BE*, *CK*, &c. the spaces *FuE*, *ExK*, &c. being neglected. ... Let the limit of the area *APNF* be expressed by *A*, the ordinate *AF* by *a*, the first fluxion of *AF* (the fluxion of the base being measured by *AB*, or unit) by *b*, the third, fifth and the subsequent fluxions of *AF* taken alternately, and always positively, by *d*, *f*, &c. Then the sum of the progression represented by *AF*, *BE*, *CK*, *HL*, &c. shall be found nearly by computing $A + \frac{1}{2}a + \frac{1}{12}b - \frac{1}{720}d + \frac{1}{30240}f +$ &c. Or if *AR* be taken towards *b* equal to one half of *AB*, the ordinate at *R* meet the curve in *V*, the limit of the area *RPNV* (or its value when *RP* is supposed to be produced infinitely) be now expressed by *A*, the first, third, fifth and subsequent fluxions of *RV* taken alternately, and always positively, by *b*, *d*, *f*, &c. then the sum of the terms *AE*, *BE*, *CK*, *HL*, &c. may be found by computing $A - \frac{1}{24}b + \frac{7}{5760}d - \frac{31}{967680}f +$ &c.

The formula also occurs in [8, vol.II, Ch.IV, art. 828, p. 672] where Maclaurin explains how the coefficients are determined. He uses the cumbersome dot notation for the derivatives.

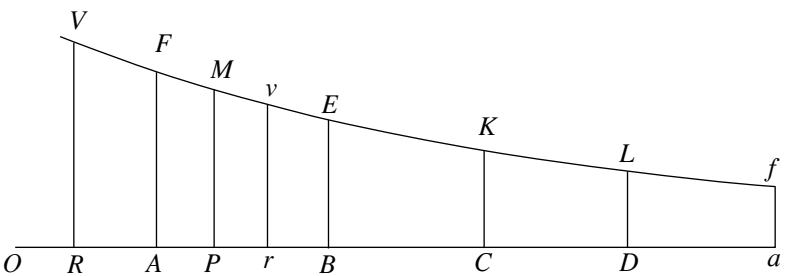


FIGURE 5

Referring to Figure 5, he writes:

Suppose the base $AP = z$, the ordinate $PM = y$, and, the base being supposed to flow uniformly, let $\dot{z} = 1$. Let the first ordinate *AF* be represented by *a*, *AB* = 1, and the area *ABEF* = *A*. As *A* is the area generated by the ordinate *y*, so let *B*, *C*, *D*, *E*, *F*, &c. represent the areas upon the same base *AB* generated by the respective ordinates $\dot{y}$, $\ddot{y}$, $\dddot{y}$, $\ddddot{y}$, &c. Then

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$AF = a = A - \frac{B}{2} + \frac{C}{12} - \frac{E}{720} + \frac{G}{30240} -$ &c. For by art. 752, $A = a + \frac{\dot{a}}{2} + \frac{\ddot{a}}{6} + \frac{\dddot{a}}{24} + \frac{\ddddot{a}}{120} +$ &c. whence we have the equation

$$a = A - \frac{\dot{a}}{2} - \frac{\ddot{a}}{6} - \frac{\dddot{a}}{24} - \frac{\ddddot{a}}{120} - \text{\&c}. \qquad (Q)$$

In like manner, $\dot{a} = B - \frac{\ddot{a}}{2} - \frac{\dddot{a}}{6} - \frac{\ddddot{a}}{24} -$ &c. $\ddot{a} = C - \frac{\dddot{a}}{2} - \frac{\ddddot{a}}{6} -$ &c. $\dddot{a} = D - \frac{\ddddot{a}}{2} -$ &c. $\ddddot{a} = E -$ &c. by which latter equations, if we exterminate $\dot{a}$, $\ddot{a}$, $\dddot{a}$, $\ddddot{a}$, &c. from the value of a in the equation Q, we shall find that $a = A - \frac{B}{2} + \frac{C}{12} - \frac{E}{720} +$ &c. The coefficients are continued thus: let k, ℓ, m, n, &c. denote the respective coefficients of $\dot{a}$, $\ddot{a}$, $\dddot{a}$, $\ddddot{a}$, &c. in the equation Q; that is, let $k = \frac{1}{2}$, $\ell = \frac{1}{6}$, $m = \frac{1}{120}$, &c.; suppose $K = k = \frac{1}{2}$, $L = kK - l = \frac{1}{12}$, $M = kL - \ell K + m = 0$, $N = kM - \ell L + mK - n = -\frac{1}{720}$, and so on; then $a = A - KB + LC - MD + NE -$ &c. where the coefficients of the alternate areas D, F, H &c. vanish.

He further states in art. 829, p. 673:

> As A is the fluent of $y\dot{z}$, so B is the fluent of $\dot{y}\dot{z}$, C of $\ddot{y}\dot{z}$, E of $\ddddot{y}\dot{z}$, &c. Therefore, since $\dot{z} = 1$, and these areas are generated while the ordinate PM moves from AF to BE, the area B will be express'd by the excess of the last ordinate BE above the first AF, C by the difference of the fluxions of the ordinates (having due regard to the signs of these fluxions), E by the difference of their third fluxions, and the other areas G, I, &c. by the respective differences of their fluxions of the corresponding higher orders. Therefore if α represent $BE - AF$ the difference of the ordinates, and β, δ, ζ, &c. the differences of their fluxions of the first, third, fifth orders, &c. then
>
> $$a = A - \frac{\alpha}{2} + \frac{\beta}{12} - \frac{\delta}{720} + \frac{\zeta}{30240} + \text{\&c}.$$

Maclaurin concludes his discussion in art.830, pp. 673-674:

> Supposing now the base Aa to be divided into the equal successive parts AB, BC, CK, &c. and each part equal to unit, let the sum of the equidistant ordinates AF, BE, CK, &c. exclusive of the last ordinate af be represented by S, the total area $AFfa$ upon the base Aa by A, the excess of af above AF by α, the respective excesses of their first, third, fifth fluxions, &c., by β, δ, ζ &c. the fluxion of the base being supposed equal to 1, then it follows from the last article that

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$S=A-\frac{\alpha}{2}+\frac{\beta}{12}-\frac{\delta}{720}+\frac{\zeta}{30240}+$ &c. and $A=S+\frac{\alpha}{2}-\frac{\beta}{12}+\frac{\delta}{720}-\frac{\zeta}{30240}$ +&c. which give two of the theorems mentioned in art. 352. and 353. where we had hyperbolic figures chiefly in view. This proposition more generally express'd, without supposing z or $\dot{z}$ equal to unit, is that

$$S = \frac{A}{z} - \frac{\alpha}{2} + \frac{z\beta}{12\dot{z}} - \frac{z^3\delta}{720\dot{z}^3} + \frac{z^5\zeta}{30240\dot{z}^5} - \frac{z^7\theta}{30240\dot{z}^7} + \&c.$$

5. *Nomenclature*

We may note here that while discussing interpolation of intermediate terms of a series in vol.II, art. 853, p. 691, Maclaurin states in a footnote:

> I take this opportunity to mention, that having occasionally shown in 1737, the 292, 293 pages of this treatise (after they were printed) to Mr. Stirling, he took notice that a theorem similar to the first of these described in art. 352. had been communicated to him by Mr. Euler.

Euler had communicated his formula in his letter (labelled OO2621 at http://eulerarchive.maa.org/) to James Stirling (1692-1770) on 8 June 1736 [9, p. 496]. The letter remained unanswered for about two years. Then in 1737, Stirling received from Maclaurin the galley proofs of certain portions of *Treatise of Fluxions* containing the Euler-Maclaurin formula. Explaining his belated reply dated 16 April 1738 (labelled OO2622 at http://eulerarchive.maa.org/, rough draft in [10, pp.178–181]) Stirling wrote:

> During these last two years I have been involved in a great many business matters which have required me to go frequently to Scotland, and then return to London. And it was on account of these affairs that first of all your letter came late into my hands and then that, even to this very day, there is scarcely time available for reading through your letter with the attention which it deserves. For after deliberations have been interrupted, not to say neglected, for a long time, patience is required before the mind can be brought to think about the same things once again.

Stirling acknowledged the importance of Euler's Theorem, and remarked that his own theorem for summing logarithms was only a particular case. He further told Euler that Maclaurin had, in the proof-sheets of *Treatise of Fluxions* to appear shortly, two theorems for summing series by means of derivatives of the terms, one of which was identical to the result that Euler had sent. Stirling also informed Euler about his communications with Maclaurin on Euler's formula. He also told Euler about Maclaurin's promise to acknowledge Euler's work in his *Treatise*.

We may reproduce here what Tweedie wrote in this regard:

> Maclaurin asked for, and obtained, Stirling's help and criticisms, while his *Treatise of Fluxions* was in proof.

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> This had an interesting sequel. About 1736, Euler wrote to Stirling a letter (now lost) in which he communicated his *Summation Formula*. Stirling in his reply recognised its importance, and that it included his own theorem on log *n* as a particular case, but "warned" Euler that he had read the *identical* theorem in the *Fluxions*.
>
> Euler in a long reply, full of interesting mathematical information, waived his claims to priority before Maclaurin.
>
> [11, p.304]

Tweedie reiterated his view in his book [10, p. 211-212] and noted: 'With characteristic modesty and absolute freedom from jealousy, Euler in his reply waives his claim to priority over Maclaurin.' We give here an English translation of the relevant excerpt from Euler's letter (in Latin) dated 27 July 1738 (labelled OO2623 with wrong month at http://eulerarchive.maa.org/) from [10, p. 181-182]:

> In this regard, I have the least desire for anything that would detract from the admiration of the celebrated Maclaurin; since he happened to have discovered the same theorem for summing series before me, hence he deserves to be named as its first Inventor. I found this theorem about four years ago at which time I also gave its demonstration and use before our Academy and provided a detailed exposition in my dissertation, and by means of which I composed the powers of the periphery of the circle in our *Commentariis* brought out annually in the public domain.

Though sounding magnanimous, Euler put the record straight at the same time. However, he did not mention Maclaurin's work in his book *Institutiones calculi differentialis cum eiususu in analysi finitorum ac doctrina serierum* (1755) where he discussed his sum formula in detail.

As stated earlier, Euler's paper E20 presented to the St Petersburg Academy on 5 March 1731 is referred to in E25 which contains the first version of the sum formula. From the information gathered from http://eulerarchive.maa.org/, E24 was written in 1731 and E26 in 1732. E25 was presented to the St Petersburg Academy on 20 June 1732. So Euler must have found the formula sometime in the second half of 1731 or the first half of 1732. Most probably he found the formula in the first quarter of 1732. Though we have no definite information as to when Maclaurin began writing his *Fluxions*, we may infer that it originated after Berkeley's pamphlet of 1734. Stirling received its proofs in 1737. The circumstantial evidence indicates that Euler in fact found the formula before Maclaurin. But Poisson and Jacobi as well as Darboux called it *Maclaurin's formula*.

As Knopp observes: 'For long it was known as *Maclaurin's* formula, or the *Euler-Maclaurin* formula; it is only recently that *Euler*'s undoubted priority has been established.' [12, p. 521, fn.3] R. Reiff (1855–1908), a German mathematics historian, discussed the issue in his *Geschichte der Unendlichen Reihen* (1889) and suggested on page 87 that 'it may not be

https://doi.org/10.1017/mag.2022.116 Published online by Cambridge University Press

inappropriate to combine both names and to use the name Euler-Maclaurin's sum formula.' His suggestion has found wide acceptance among mathematicians.

Acknowledgement

I would like to thank the anonymous referee for useful suggestions.

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https://doi.org/10.1017/mag.2022.116 Published online by Cambridge University Press