Ergod. Th. & Dynam. Sys. (1997), **17**, 793–819
Printed in the United Kingdom © *1997 Cambridge University Press*

Hypercyclic and chaotic semigroups of linear operators*

WOLFGANG DESCH†, WILHELM SCHAPPACHER† and GLENN F. WEBB‡

† *Universität Graz, Institut für Mathematik, Heinrichstraße 36, A-8010 Graz, Austria*
‡ *Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA*

(*Received* 6 *March* 1995 *and revised* 21 *February* 1996)

Abstract. We investigate hypercyclic and chaotic behavior of linear strongly continuous semigroups. We give necessary and sufficient conditions on the semigroup to be hypercyclic, and sufficient conditions on the spectrum of an operator to generate a hypercyclic semigroup. A variety of examples is provided.

1. *Introduction*

Chaos is usually viewed as a nonlinear phenomenon, but the idea of chaotic behavior also arises in discrete and continuous linear dynamical systems. In the discrete case the concepts of hypercyclicity and chaos have been studied in the theory of bounded linear operators in Banach spaces. A bounded linear operator T in a Banach space X is hypercyclic provided that there exists $x \in X$ such that $\{T^n x \mid n = 1, 2, 3 \ldots\}$ is dense in X. T is chaotic provided that it is hypercyclic and the set of periodic points is dense in X. (A point is called periodic if there exists some $n > 1$ such that $T^n x = x$.) This definition of chaos is consistent with the definition given by Devaney [**7**, p. 50] for an arbitrary continuous mapping f on a topological space, which requires that f is topological transitive and the periodic points of f are dense. (These two conditions imply sensitive dependence on initial conditions [**2**].) The theory of hypercyclic and chaotic bounded linear operators has been extensively developed in connection with the invariant subspace problem of Hilbert spaces (see [**3**], [**10**], [**11**], [**21**]).

Our objective in this paper is to investigate hypercyclicity and chaos in the context of strongly continuous semigroups of bounded linear operators in Banach spaces. Hypercyclic and chaotic behavior in strongly continuous linear semigroups can be viewed as the continuous time analog of the discrete time case mentioned above. A strongly continuous semigroup of bounded linear operators $\{T(t) \mid t \geq 0\}$ is hypercyclic provided that there exists $x \in X$ such that $\{T(t)x \mid t \geq 0\}$ is dense in X and chaotic if the set

* Work by G. W. supported by the National Science Foundation under Grant No. DMS-9202550. Work by W. D. done within the Spezialforschungsbereich F 003, Optimierung und Kontrolle. Work by W. D. and W. S. supported in part by a SCIENCE project of the European Community.

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

$\{x \in X \mid \exists t > 0 : T(t)x = x\}$ is also dense in X. Chaos in linear semigroups can occur when the infinitesimal generator is bounded or unbounded, but only in infinite-dimensional spaces. The case of unbounded infinitesimal generators makes the linear semigroup formulation of chaos applicable to linear partial differential equations. Chaotic behavior of strongly continuous linear semigroups has been studied in [**15**], [**16**], and [**23**]. In [**15**] and [**16**] chaotic linear semigroups are associated with the idea of exactness and applied to first-order partial differential equations ([**15**, p. 297] and [**16**, p. 339]). In [**15**] and [**16**] exact linear semigroups are applied to nonlinear continuous semidynamical systems via the Frobenius–Perron operators and the Koopman operators. In [**4**], [**5**], [**14**], [**19**], and [**20**] chaotic behavior is analyzed in semi-linear first-order partial differential equations with logistic nonlinearities. In these investigations the essential idea is the construction of densely wandering orbits. Some more remarks on the existing literature will be in order when we state our results in detail.

In this paper we investigate hypercyclicity and chaos in linear semigroups in a general context. In §2 we establish the equivalence of hypercyclicity and topological transitivity and some consequences, which are quite similar to the discrete case and serve as the main technical tools in the subsequent sections. In §3 we relate hypercyclicity to the spectrum of the infinitesimal generator. Section 4 deals with translation semigroups, which are associated with transport processes. It is particularly easy to characterize those translation semigroups that are hypercyclic, so that this class is well suited to generate examples and counterexamples concerning the general theory of hypercyclic semigroups. We add an example of a linear partial differential equation with both diffusion and transport terms. In §5 we consider a class of semigroups generated by discrete shift operators in l^1-spaces. The most striking feature of these examples is that they can lead to hypercyclic semigroups generated by bounded operators. Finally, §6 shows an example of a globally stable nonlinear system, whose linearization is yet a hypercyclic linear semigroup.

2. *Hypercyclic and chaotic semigroups*

Let X be a Banach space and let $\{T(t) \mid t \geq 0\}$ be a strongly continuous semigroup of linear operators in X. By A we denote its infinitesimal generator. We will consider the following subsets of X:

X_0 is the set of all $x \in X$ such that $\lim_{t\to\infty} T(t)x = 0$.
X_∞ is the set of all $x \in X$ such that for each $\epsilon > 0$ there exist some $w \in X$ and some $t > 0$ with $\|w\| < \epsilon$ and $\|T(t)w - x\| < \epsilon$.
X_p is the set of periodic points, i.e. the set of all $x \in X$ such that there exists some $t > 0$ with $T(t)x = x$.

Definition 2.1. The semigroup $\{T(t) \mid t \geq 0\}$ is called hypercyclic provided that there exists $x \in X$ such that $\{T(t)x \mid t \geq 0\}$ is dense in X. If in addition X_p is dense in X, then the semigroup $\{T(t) \mid t \geq 0\}$ is called chaotic.

Examples 4.10 and 4.11 show hypercyclic semigroups which are not chaotic. Although hypercyclic semigroups seem to be very irregular, there are even bounded linear operators generating hypercyclic semigroups (see §5).

This section gives the topological characterization of hypercyclic C_0-semigroups. There is no significant difference to the theory of discrete hypercyclic semigroups,

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

neither with respect to the assertions of the theorems nor to the methods of the proofs. Theorem 2.2 below has its discrete analog in [**13**, Theorem 2.1] and [**10**, 1.2 Theorem]. [**10**, 1.4 Corollary] also gives a discrete analog to Theorem 2.3. Related theorems are also found in [**9**, 2.2 Theorem] and [**13**, Theorem 1.4], to mention just a few.

We give a necessary and sufficient condition for a semigroup to be hypercyclic. $T(t)$ is called topologically transitive on X if for every pair of open subsets $U, V \subset X$ there exists some $t > 0$ such that $T(t)U \cap V \neq \emptyset$. The following theorem states that $\{T(t)\}$ is hypercyclic if and only if it is topologically transitive. Notice that the existence of an hypercyclic semigroup on X obviously implies that X is separable.

THEOREM 2.2. *Let $\{T(t) \mid t \geq 0\}$ be a strongly continuous linear semigroup on a separable Banach space X. Then the following assertions are equivalent:*

(i) *$\{T(t)\}$ is hypercyclic;*

(ii) *for all $y \in X$, $z \in X$ and all $\epsilon > 0$ there exist some $v \in X$ and some $t > 0$ such that $\|y - v\| < \epsilon$ and $\|z - T(t)v\| < \epsilon$;*

(iii) *for all $\epsilon > 0$ there exists a dense subset $D \subset X$ such that for all $z \in D$ there exists a dense subset $D' \subset X$ such that for all $y \in D'$ there exist $v \in X$ and $t > 0$ such that $\|y - v\| < \epsilon$ and $\|z - T(t)v\| < \epsilon$.*

Proof. (i) implies (ii). Let $\{T(t)x \mid t \geq 0\}$ be dense in X. Obviously for each $s > 0$, $\{T(t)x \mid t > s\}$ is also dense. Given $y, z \in X$ there exists some $s > 0$ such that $\|y - T(s)x\| < \epsilon$, and there exists some $u > s$ such that $\|z - T(u)x\| < \epsilon$. Now put $v = T(s)x$ and $t = u - s$.

(ii) implies (i). Let $\{z_1, z_2, z_3, \ldots\}$ be a dense sequence in X. We construct sequences $\{y_1, y_2, y_3, \ldots\} \subset X$ and $\{t_1, t_2, t_3, \ldots\} \subset [0, \infty)$ inductively:

- put $y_1 = z_1$, $t_1 = 0$;
- for $n > 1$ find y_n, t_n such that

$$\|y_n - y_{n-1}\| \leq \frac{2^{-n}}{\sup\{\|T(t_j)\| \mid j < n\}}, \tag{2.1}$$

$$\|z_n - T(t_n)y_n\| \leq 2^{-n}. \tag{2.2}$$

In particular, (2.1) implies $\|y_n - y_{n-1}\| \leq 2^{-n}$, so that the sequence y_n has a limit x. Applying (2.2) and once again (2.1), we infer that

$$\begin{aligned} \|z_n - T(t_n)x\| &\leq \|z_n - T(t_n)y_n\| + \|T(t_n)\|\|y_n - x\| \\ &\leq \|z_n - T(t_n)y_n\| + \sum_{i=n+1}^{\infty} \|T(t_n)\|\|y_i - y_{i-1}\| \\ &\leq 2^{-n} + \sum_{i=n+1}^{\infty} 2^{-i} = 2^{-n+1}. \end{aligned}$$

Given $z \in X$ and $\epsilon > 0$ there are arbitrarily large n such that $\|z_n - z\| < \epsilon/2$. Choosing n large enough such that $2^{-n+1} < \epsilon/2$, we obtain

$$\|T(t_n)x - z\| \leq \|z - z_n\| + \|z_n - T(t_n)x\| < \epsilon.$$

Therefore, $\{T(t)x \mid t \geq 0\}$ is dense.

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

(ii) implies (iii). Put $D = D' = X$.

(iii) implies (ii). Let $\epsilon > 0$ and $z \in X$. Pick $\tilde{z} \in D$ such that $\|z - \tilde{z}\| < \epsilon/2$. Then specify D' according to (iii) with $\tilde{z}$ instead of z and $\epsilon/2$ instead of ϵ. For $y \in X$ pick $\tilde{y} \in D'$ with $\|\tilde{y} - y\| < \epsilon/2$. Finally, we choose t and v according to (iii) with $\epsilon/2$, $\tilde{y}$, $\tilde{z}$ instead of ϵ, y, z and obtain

$$\|T(t)v - z\| \leq \|T(t)v - \tilde{z}\| + \|\tilde{z} - z\| < \epsilon,$$
$$\|v - y\| \leq \|v - \tilde{y}\| + \|\tilde{y} - y\| < \epsilon. \qquad \square$$

As a corollary we obtain a sufficient condition for hypercyclicity.

THEOREM 2.3. *Let $\{T(t) \mid t \geq 0\}$ be a strongly continuous semigroup in a separable Banach space X. If both X_∞ and X_0 are dense subsets, then $\{T(t)\}$ is hypercyclic.*

Proof. We utilize Theorem 2.2(iii) with $D = X_\infty$ (independent of ϵ) and $D' = X_0$ (independent of ϵ and z). Let $z \in X_\infty$ and $y \in X_0$. Then for each $\epsilon > 0$ there are arbitrarily large $t > 0$ and $w \in X$ such that

$$\|T(t)w - z\| < \frac{\epsilon}{2} \quad \text{and} \quad \|w\| < \epsilon.$$

Since $y \in X_0$, for sufficiently large t we have $\|T(t)y\| < \epsilon/2$. We put $v = y + w$ and infer

$$\|z - T(t)v\| \leq \|z - T(t)w\| + \|T(t)y\| < \epsilon,$$
$$\|y - v\| = \|w\| < \epsilon. \qquad \square$$

Remark 2.4. Theorem 2.3 is only a sufficient condition: if $\{T(t)\}$ is hypercyclic, then $X_\infty = X$, but there are hypercyclic semigroups with $X_0 = \{0\}$.

Proof. Theorem 2.2(ii) with $y = 0$ states precisely that $X = X_\infty$. Example 4.10 shows a hypercyclic group with $X_0 = \{0\}$. $\square$

If the semigroup is a group, one might think of hypercyclicity in terms of the forward and the backward trajectory. The following theorem states that both notions of hypercyclicity are equivalent.

THEOREM 2.5. *If $\{T(t) \mid t \in \mathbb{R}\}$ is a strongly continuous group of linear operators on X, then the following assertions are equivalent:*

(i) *the semigroup $\{T(t) \mid t \geq 0\}$ is hypercyclic;*
(ii) *the semigroup $\{T(-t) \mid t \geq 0\}$ is hypercyclic;*
(iii) *there exists some $x \in X$ such that both sets $\{T(t)x \mid t \geq 0\}$ and $\{T(-t)x \mid t \geq 0\}$ are dense.*

Proof. (i) implies (ii). Let $y, z \in X$ and $\epsilon > 0$ be arbitrary. If $\{T(t) \mid t \geq 0\}$ is hypercyclic, then by Theorem 2.2(ii), there exist $v \in X$ and $t > 0$ such that

$$\|v - y\| < \epsilon \quad \text{and} \quad \|T(t)v - z\| < \epsilon.$$

We put $w = T(t)v$ and obtain

$$\|w - z\| < \epsilon \quad \text{and} \quad \|T(-t)w - y\| < \epsilon.$$

Again using Theorem 2.2 we infer that $\{T(-t) \mid t \geq 0\}$ is hypercyclic.

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

(ii) implies (i). By symmetry.

(i) and (ii) imply (iii). We modify the proof of Theorem 2.2. Let $\{\tilde{z}_i\}$ be a dense sequence. We define

$$z_{2i} = \tilde{z}_i, \quad z_{2i+1} = \tilde{z}_i.$$

Now we construct sequences $\{y_i\}$, $\{t_i\}$ exactly as in the proof of Theorem 2.2, satisfying (2.1) and (2.2), with the additional assumption that

$$t_j \begin{cases} > 0 & \text{if } j = 2i, \\ < 0 & \text{if } j = 2i+1. \end{cases}$$

This is possible since assertion (ii) of Theorem 2.2 holds for $\{T(t)\}$ as well as $\{T(-t)\}$. One can show again that the limit x of $\{y_j\}$ has dense forward and backward trajectories. □

3. *Spectral conditions*

In this section we give sufficient conditions on the spectrum of the generator so that a semigroup is hypercyclic. The results are based on the hypothesis that there is an abundance of eigenvectors. These conditions are only sufficient: there are hypercyclic semigroups whose generators have empty point spectrum (see Example 4.11).

Although some of our results follow known patterns from the discrete case very closely, extreme care should be taken at the attempt to generalize spectral results from discrete semigroups to continuous semigroups by 'taking logarithms'. It is well known that there is not necessarily a one-to-one correspondence between the spectrum of the operator $T(t)$ belonging to a C_0-semigroup, and the spectrum of its generator A.

THEOREM 3.1. *Let X be a separable Banach space and let A be the infinitesimal generator of a strongly continuous semigroup $\{T(t) \mid t \geq 0\}$ on X. Let U be an open subset of the point spectrum of A, which intersects the imaginary axis, and for each $\lambda \in U$ let x_λ be a nonzero eigenvector, i.e. $Ax_\lambda = \lambda x_\lambda$. For each $\phi \in X^*$ we define a function $F_\phi : U \to \mathbb{C}$ by $F_\phi(\lambda) = \langle \phi, x_\lambda \rangle$. Assume that for each $\phi \in X^*$ the function F_ϕ is analytic and that F_ϕ does not vanish identically on U unless $\phi = 0$. Then $\{T(t)\}$ is chaotic.*

Example 4.12 shows an application of Theorem 3.1. (The following proof is essentially an abstract reformulation of the techniques applied to multipliers in spaces of holomorphic functions in [**10**, Proposition 4.2, Theorem 4.5].)

Proof. We start with an abstract observation. Let $V \subset U$ be an arbitrary subset which admits a cluster point in U. We prove that $Y_V = \operatorname{span}\{x_\lambda \mid \lambda \in V\}$ is dense in X. Suppose the contrary. By the Hahn–Banach theorem there exists some $\phi \in X^*$ such that $\phi \neq 0$ and $\langle \phi, x \rangle = 0$ for all $x \in Y_V$. Thus for all $\lambda \in V$, $F_\phi(\lambda) = \langle \phi, x_\lambda \rangle = 0$. Since V admits a cluster point in U and F_ϕ is analytic on U, this implies that F_ϕ vanishes identically. This contradicts our assumptions.

We will now show that the subspaces X_0 and X_∞ are dense, so that $\{T(t)\}$ is hypercyclic by Theorem 2.3. Moreover, we show that X_p is dense so that the semigroup is chaotic.

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

X_0 is dense. Choose V to be a subset of $\{\lambda \in U \mid \Re(\lambda) < 0\}$ which admits a cluster point in U. Evidently for $\lambda \in V$ the eigenvector x_λ is contained in X_0. As X_0 is a subspace, Y_V is contained in X_0, hence X_0 is dense.

X_∞ is dense. This time let V be a subset of $\{\lambda \in U \mid \Re(\lambda) > 0\}$, which admits a cluster point in U. Since

$$\sum_{j=1}^{n} \alpha_j x_{\lambda_j} = T(t)\left(\sum_{j=1}^{n} \alpha_j e^{-\lambda_j t} x_{\lambda_j}\right),$$

each x in the dense subspace Y_V is contained in X_∞.

X_p is dense. Let $V = \{\lambda_1, \lambda_2, \lambda_3, \ldots\}$, where each λ_j is purely imaginary with a rational imaginary part, and such that the sequence $\{\lambda_j\}$ admits a limit. Then each x_{λ_j} is periodic with a period which is a rational multiple of π. This also implies that each linear combination $\alpha_1 x_{\lambda_1} + \cdots + \alpha_n x_{\lambda_n}$ is periodic. Therefore the dense subspace Y_V is contained in X_p, so that X_p is dense. □

The next theorem does not require the assumption that X_0 is dense in order to show hypercyclicity. We require the following hypotheses on the point spectrum of A.

(A.1) There exist two positive numbers $\gamma_1 < \gamma_2$ such that $\gamma_m + i\pi\mu$ is contained in the point spectrum of A for all rational $\mu \in \mathbb{R}$ and $m \in \{1, 2\}$.

(A.2) Let $x_{\gamma_m + \mu i}$ be an eigenvector of A with the eigenvalue $\gamma_m + \mu i$. By $X_{m,k}$ we denote

$$X_{m,k} = \operatorname{span}\{x_{\gamma_m + j\pi i/k} \mid j = 0, \pm 1, \pm 2, \ldots\}.$$

We assume that for each $y \in X$ and each $\epsilon > 0$ there exists some positive integer k and some $x \in X_{1,k}$ with $\|x - y\| < \epsilon$.

(A.3) Let k_1 be a positive integer, let $x_1 \in X_{1,k_1}$ and $\epsilon > 0$. Then there exist arbitrarily large multiples k_2 of k_1 and corresponding vectors $x_2 \in X_{2,k_2}$ with

$$\|x_1 - x_2\| < \epsilon e^{-2k_2\gamma_1}.$$

These conditions are weaker than the Hypotheses (H.1)–(H.3) in [**23**, p. 6]. Therefore the following theorem implies immediately [**23**, Proposition 4.2].

THEOREM 3.2. *Let $\{T(t)\}$ be a strongly continuous semigroup in a separable Banach space X. Let its generator A satisfy the hypotheses (A.1)–(A.3) above. Then $\{T(t)\}$ is hypercyclic.*

Proof. We utilize Theorem 2.2(ii). Let $\epsilon > 0$ and $y, z \in X$. By Hypothesis (A.2) there exists a positive integer k_1 and vectors $x_1, z_1 \in X_{1,k_1}$ such that

$$\|y - x_1\| \le \frac{\epsilon}{4}, \quad \|z - z_1\| \le \frac{3\epsilon}{4}.$$

By Hypothesis (A.3) there exist arbitrarily large k_2 which are multiples of k_1 and corresponding vectors $x_2 \in X_{2,k_2}$ such that

$$\|x_1 - x_2\| \le \frac{\epsilon}{4} e^{-2k_2\gamma_1}.$$

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

We choose k_2 sufficiently large such that

$$e^{-2k_2\gamma_1}\|z_1\| \le \frac{\epsilon}{4}, \quad e^{2k_2(\gamma_1-\gamma_2)}\|x_1\| \le \frac{\epsilon}{4},$$

and put

$$v = x_1 + e^{-2k_2\gamma_1}z_1 - e^{2k_2(\gamma_1-\gamma_2)}x_2.$$

Then we have

$$\begin{aligned}\|y - v\| &= \|y - x_1 - e^{-2k_2\gamma_1}z_1 + e^{2k_2(\gamma_1-\gamma_2)}x_2\| \\ &\le \|y - x_1\| + e^{-2k_2\gamma_1}\|z_1\| + e^{2k_2(\gamma_1-\gamma_2)}\|x_2 - x_1\| + e^{2k_2(\gamma_1-\gamma_2)}\|x_1\| \\ &\le \frac{\epsilon}{4} + \frac{\epsilon}{4} + e^{-2k_2\gamma_2}\frac{\epsilon}{4} + \frac{\epsilon}{4} \le \epsilon.\end{aligned}$$

Now notice that for any $x \in X_{m,k}$ and any multiple k_2 of k we have $T(2k_2)x = e^{2k_2\gamma_m}x$. (Here $m \in \{1, 2\}$.) Therefore

$$\begin{aligned}\|z - T(2k_2)v\| &= \|z - e^{2k_2\gamma_1}x_1 - z_1 + e^{2k_2\gamma_1}x_2\| \\ &\le \|z - z_1\| + e^{2k_2\gamma_1}\|x_1 - x_2\| \\ &\le \frac{3\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.\end{aligned}$$

We have found v and $t = 2k_2$ to satisfy assertion (ii) of Theorem 2.2. Consequently $\{T(t)\}$ is hypercyclic. □

We close this section with some necessary conditions on the dual semigroup.

THEOREM 3.3. *Let $\{T(t) \mid t \ge 0\}$ be a hypercyclic semigroup generated by A in a Banach space X. Then the adjoint A^* of A and the dual semigroup $\{T^*(t)\}$ have the following properties:*

(i) *if $\phi \in X^*$, $\phi \ne 0$, then the orbit $\{T^*(t)\phi \mid t \ge 0\}$ is unbounded;*

(ii) *the point spectrum of A^* is empty.*

(This is, in fact, a very elementary observation. Compare [**13**, Corollary 2.4], also published in [**6**, Proposition 4.1] for the discrete case. As an anonymous referee tells us, it has also been proved in unpublished work by P. Bourdon, and P. Bourdon and J. H. Shapiro.)

Proof. Let $x \in X$ be such that $\{T(t)x \mid t \ge 0\}$ is dense in X. Assume that for some $\phi \in X^*$, $\phi \ne 0$ the orbit is bounded, i.e. $\|T^*(\phi)\| \le M$ for all $t \ge 0$. Let $y \in X$ be such that $|\langle\phi, y\rangle| > 3M\|x\|$, and choose $t > 0$ such that $\|T(t)x - y\| \le M\|x\|/\|\phi\|$. Then we have the estimate

$$\begin{aligned}3M\|x\| &< |\langle\phi, y\rangle| \le |\langle\phi, T(t)x\rangle| + |\langle\phi, T(t)x - y\rangle| \\ &= |\langle T^*(t)\phi, x\rangle| + |\langle\phi, T(t)x - y\rangle| \le M\|x\| + \|\phi\|\frac{M\|x\|}{\|\phi\|} = 2M\|x\|\end{aligned}$$

which is a contradiction. Thus (i) is proved.

To prove (ii), assume that λ is in the point spectrum of A^*, i.e. $A^*\phi = \lambda\phi$ for some $\phi \ne 0$. Then $T^*(t)\phi = e^{\lambda t}\phi$. Because of (i) we infer $\Re(\lambda) > 0$. The equation

$$\langle\phi, T(t)x\rangle = e^{\lambda t}\langle\phi, x\rangle$$

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

implies that either ϕ annihilates the whole orbit of x or

$$|\langle \phi, T(t)x\rangle| \geq |\langle \phi, x\rangle|$$

for all $t \geq 0$. Since the orbit of x is dense, the first case would imply that $\phi = 0$. On the other hand let $y \in X$ be such that $\langle \phi, y\rangle = 0$ and choose t such that

$$\|T(t)x - y\| \leq \frac{|\langle \phi, x\rangle|}{2\|\phi\|}.$$

Then

$$|\langle \phi, T(t)x\rangle| \leq |\langle \phi, y\rangle| + \|\phi\|\|T(t)x - y\| < \tfrac{1}{2}|\langle \phi, x\rangle|$$

in contradiction to the second case. □

Theorem 3.3 can be used to rule out important classes of semigroups from being hypercyclic. These classes include differentiable (in particular, analytic) solution semigroups for partial differential equations on compact space domains. Notice, however, that analyticity by itself is not sufficient to exclude hypercyclicity. Counter-examples are provided by the semigroups in §5, which are generated by bounded operators.

COROLLARY 3.4. *Let $\{T(t)\}$ be a strongly continuous semigroup generated by A in a Banach space X. Assume that $\{T(t)\}$ is eventually uniformly continuous in the sense that with respect to the operator norm $\lim_{h\to 0+} \|T(t+h) - T(t)\| = 0$, if t is sufficiently large. Let X_A be the domain of A endowed with the graph norm of A. Assume that the embedding of X_A into X is compact. Then $\{T(t)\}$ is not hypercyclic.*

Proof. Assume that $\{T(t)\}$ is hypercyclic. Let ω_1 be the spectral bound of $\{T(t)\}$,

$$\omega_1 = \sup\{\Re(\lambda) \mid \lambda \text{ in the spectrum of } A\}.$$

Being eventually uniformly continuous, the semigroup $\{T(t)\}$ has spectrum determined growth: for each $\omega > \omega_1$ there exists a constant M such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ (see, for example, [**1**, p. 87]). Since $T^*(t)$ has unbounded trajectories, this estimate cannot hold for negative ω, consequently the spectrum of A is not empty. Therefore the spectrum of A^* is not empty either. The compact embedding of X_A implies that the spectrum of A^* is a pure point spectrum, since the resolvent $(\lambda - A)^{-1}$ and its adjoint $(\overline{\lambda} - A^*)^{-1}$ are compact. Therefore A^* has nonempty point spectrum in contradiction to Theorem 3.3. □

4. *Hypercyclic translation semigroups*

Semigroups of translations on weighted function spaces on the real line or the positive half-line are particularly well suited to serve as examples for understanding hypercyclicity since it is easy to determine the cases when they are hypercyclic. Rolewicz [**18**] in his 1969 paper initializing the study of hypercyclicity had already observed that the translation semigroup in $\mathbf{L}^p_\rho(\mathbb{R}, \mathbb{R})$ is hypercyclic if $\rho(t) = a^{-|t|}$ with $a > 1$ serving as a weight function. We start out with the characterization of the weight functions that make the translation semigroup hypercyclic and proceed to some examples to round off the theory presented in the previous sections.

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

We begin this section with a disclaimer. Hypercyclicity of translation semigroups in spaces of entire functions has been a focus of study for some years (see, for example, [**6**], [**9**], [**10**]). In these spaces, norms or seminorms may be based on weights attributed to the Taylor coefficients or the derivatives of a function. Nevertheless, the reader should not be confused when we talk about admissible weights. In this section we put weights on the value of the function, evaluated at different points of their domain. In such spaces, mostly Lebesgue $\mathbf{L}^p$ spaces, semigroups of translations have a completely different flavor than in spaces of holomorphic functions. The study of translation semigroups in these spaces arises naturally in the context of partial differential equations and semigroup settings for delay equations.

Definition 4.1. Let I be one of the intervals $(-\infty, \infty)$ or $[0, \infty)$. By an admissible weight function on I we mean a measurable function $\rho : I \to \mathbb{R}$ satisfying the following conditions:

(i) $\rho(\tau) > 0$ for all $\tau \in I$;

(ii) there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(\tau) \leq Me^{\omega t}\rho(t + \tau)$ for all $\tau \in I$ and all $t > 0$.

LEMMA 4.2. *Let I be one of the intervals $(-\infty, \infty)$ or $[0, \infty)$ and let ρ be an admissible weight function on I. For each $l > 0$ there are constants $0 < m_1 \leq M_1$ (depending on ρ and l only) such that for each $\sigma \in I$ and each $\tau \in [\sigma, \sigma + l]$*

$$m_1\rho(\sigma) \leq \rho(\tau) \leq M_1\rho(\sigma + l).$$

In particular, on compact intervals ρ is bounded and bounded away from 0.

Proof. Due to condition (ii) in Definition 4.1 we have for $\tau \in [\sigma, \sigma + l]$

$$\rho(\tau) \leq Me^{\omega(\sigma+l-\tau)}\rho(\sigma + l),$$
$$\rho(\sigma) \leq Me^{\omega(\tau-\sigma)}\rho(\tau).$$

Therefore we can pick $M_1 = Me^{\omega l}$, $m_1 = (1/M)e^{-\omega l}$. □

Definition 4.3. For an admissible weight function ρ on $I = (-\infty, \infty)$ or $I = [0, \infty)$ we consider the following function spaces:

$$\mathbf{L}^p_\rho(I, \mathbb{C}) = \left\{u : I \to \mathbb{C} \mid u \text{ measurable}, \int_I |u(\tau)|^p\rho(\tau)\,d\tau < \infty\right\}$$
$$\text{with } \|u\|_p = \left(\int_I |u(\tau)|^p\rho(\tau)\,d\tau\right)^{1/p},$$

$$\mathcal{C}_{0,\rho}(I, \mathbb{C}) = \left\{u : I \to \mathbb{C} \mid u \text{ continuous}, \lim_{\tau\to\pm\infty} \rho(\tau)u(\tau) = 0\right\}$$
$$\text{with } \|u\|_\infty = \sup_{\tau\in I} |u(\tau)|\rho(\tau).$$

Similar spaces are considered for real-valued functions.

Remark 4.4. Notice that we have only specified separable spaces. In particular, the condition $\lim_{t\to\pm\infty} \rho(\tau)u(\tau) = 0$ cannot be omitted from the definition of $\mathcal{C}_{0,\rho}(I, X)$ without losing separability. All spaces are constructed so that there are dense subsets of smooth functions with compact support.

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

Definition 4.5. Let $I = (-\infty, \infty)$ or $I = [0, \infty)$, and let X be one of the spaces $\mathbf{L}^p_\rho(I, \mathbb{C})$ or $\mathcal{C}_{0,\rho}(I, \mathbb{C})$ with an admissible weight function ρ. For $t \geq 0$ and $u \in X$ we define $T(t)u$ by

$$[T(t)u](\tau) = u(\tau + t).$$

We call $\{T(t) \mid t \geq 0\}$ the (forward) translation semigroup on X.

LEMMA 4.6. *Let $I = (-\infty, \infty)$ or $I = [0, \infty)$, and let X be one of the spaces $\mathbf{L}^p_\rho(I, \mathbb{C})$ or $\mathcal{C}_{0,\rho}(I, \mathbb{C})$ with an admissible weight function ρ. The translation semigroup $\{T(t)\}$ on X is a strongly continuous semigroup of linear operators. Its generator A is given by*

$$\mathrm{dom}(A) = \{u \in X \mid u \text{ absolutely continuous, } u' \in X\}, \quad Au = u'.$$

Proof. This can be viewed as a folk theorem, therefore we will not check the details of the proof. (For constant or exponential weights see [**12**, §§19.2, 19.3].)

We mention the main ideas. It is evident that $T(t)$ is linear and that the semigroup property holds. Condition (ii) of Definition 4.1 is designed to assure that for $u \in X$ also $T(t)u \in X$ with

$$\|T(t)u\| \leq \begin{cases} Me^{\omega t}\|u\| & \text{if } X = \mathcal{C}_{0,\rho}(I, \mathbb{C}), \\ M^{1/p}e^{\omega t/p}\|u\| & \text{if } X = \mathbf{L}^p_\rho(I, \mathbb{C}). \end{cases}$$

The strong continuity of $T(t)u$ is proved easily for smooth u with compact support. Since these u form a dense set, the semigroup is strongly continuous on the whole space X.

The computation of the infinitesimal generator can be performed directly by investigating $\lim_{t\to 0+}(T(t)u - u)$ or by the resolvent

$$[(\lambda - A)^{-1}u](\tau) = \left[\int_0^\infty e^{-\lambda t}T(t)u\,dt\right](\tau) = \int_0^\infty e^{-\lambda t}u(t + \tau)\,dt. \qquad \square$$

THEOREM 4.7. *Let $I = [0, \infty)$ and let X be one of the spaces $\mathbf{L}^p_\rho(I, \mathbb{C})$ or $\mathcal{C}_{0,\rho}(I, \mathbb{C})$ with an admissible weight function ρ. The translation semigroup $\{T(t)\}$ on X is hypercyclic if and only if*

$$\liminf_{t\to\infty} \rho(t) = 0.$$

Proof. For each function u with compact support there is some $t > 0$ such that $T(t)u = 0$. Therefore X_0 contains all functions of compact support, so that X_0 is dense. Due to Theorem 2.3 and Remark 2.4 the semigroup $\{T(t)\}$ is hypercyclic if and only if X_∞ is dense, which is again equivalent to $X_\infty = X$.

First suppose that $X_\infty = X$. For some fixed interval length l consider a function $u \neq 0$ whose support is contained in $[0, l]$. Then for arbitrarily small $\epsilon > 0$ there exists a function $v_\epsilon \in X$ and some $t_\epsilon > 0$ such that

$$\|v_\epsilon\| \leq \epsilon, \quad \|T(t_\epsilon)v_\epsilon - u\| < \epsilon,$$

which implies that the restriction w_ϵ of v_ϵ to the interval $[t_\epsilon, l + t_\epsilon]$ satisfies

$$\|T(t_\epsilon)w_\epsilon\| > \|u\| - \epsilon.$$

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

We use the constant p for $X = \mathbf{L}^p_\rho(I, \mathbb{C})$ and $p = 1$ for $X = \mathcal{C}_{0,\rho}(I, \mathbb{C})$. From Lemma 4.2 we infer that in any case

$$\|T(t_\epsilon)w_\epsilon\|^p \leq \frac{M_1\rho(l)}{m_1\rho(t_\epsilon)}\|v_\epsilon\|^p.$$

If ϵ tends to 0, then $\|T(t_\epsilon)w_\epsilon\| \to \|u\|$ while $\|v_\epsilon\| \to 0$. Therefore the inequality above implies that $\rho(t_\epsilon) \to 0$.

Assume on the contrary that $\liminf_{t\to\infty} \rho(t) = 0$. Let u be an arbitrary function of compact support, say $[0, l]$. We define

$$v_t(\tau) = \begin{cases} 0 & \text{if } \tau < t, \\ u(\tau - t) & \text{if } \tau \geq t. \end{cases}$$

Due to Lemma 4.2 we have

$$\|v_t\|^p \leq \frac{M_1\rho(t + l)}{m_1\rho(0)}\|u\|^p.$$

Now choose a sequence $\{t_j\}$ such that $\rho(t_j + l) \to 0$. Then $\|v_{t_j}\| \to 0$ while $T(t_j)v_{t_j} = u$. Consequently $u \in X_\infty$. Since u has been picked from a dense subset, X_∞ is dense. □

THEOREM 4.8. *Let $I = (-\infty, \infty)$ and let X be one of the spaces $\mathbf{L}^p_\rho(I, \mathbb{C})$ or $\mathcal{C}_{0,\rho}(I, \mathbb{C})$ with an admissible weight function ρ. The translation semigroup $\{T(t)\}$ on X is hypercyclic if and only if for each $\theta \in \mathbb{R}$ there exists a sequence $\{t_j\}_{j=1}^\infty$ of positive real numbers such that*

$$\lim_{j\to\infty} \rho(t_j + \theta) = \lim_{j\to\infty} \rho(-t_j + \theta) = 0.$$

(Notice that Theorem 4.8 is very closely related to a discrete analog in [**22**, Theorem 2.1].)

Proof. The condition is necessary. Suppose that $\{T(t)\}$ is hypercyclic. Let y and $z \in X$ be functions whose support is contained in the interval $[\theta, \theta + l]$ with some $l > 0$. We may pick y and z such that $y \geq 0$ and $z \leq 0$ on the whole real line, and $\|y\| = \|z\| = 1$. By Theorem 2.2 for each $\epsilon > 0$ there exist some $v_\epsilon \in X$ and $t_\epsilon > 0$ such that

$$\|T(t_\epsilon)v_\epsilon - z\| < \epsilon \quad \text{and} \quad \|v_\epsilon - y\| < \epsilon.$$

Let v_ϵ^+ and v_ϵ^- denote the positive and negative parts of v_ϵ. Let w_1 be the restriction of v_ϵ^+ to $[\theta, \theta + l]$ and w_2 the restriction of v_ϵ^- to $[\theta + t_\epsilon, \theta + l + t_\epsilon]$. (If $X = \mathcal{C}_{0,\rho}(I, \mathbb{C})$, then the functions w_i may not belong to X. However, the following computations make sense if the definition of the norm is extended from continuous to piecewise continuous functions.) Notice that the restrictions have been defined to be the parts of $v_\epsilon^\pm$ whose supports satisfy

$$w_1 \geq 0, \quad \operatorname{supp}(w_1) \subset [\theta, \theta + l], \quad \operatorname{supp}(T(t_\epsilon)w_1) \subset [\theta - t_\epsilon, \theta + l - t_\epsilon],$$
$$w_2 \leq 0, \quad \operatorname{supp}(w_2) \subset [\theta + t_\epsilon, \theta + l + t_\epsilon], \quad \operatorname{supp}(T(t_\epsilon)w_2) \subset [\theta, \theta + l].$$

From the negativity of z and $\|T(t_\epsilon)v_\epsilon - z\| < \epsilon$ we obtain $\|T(t_\epsilon)w_1\| < \epsilon$. On the other hand y is positive and has its domain in $[\theta, \theta + l]$. Since $\|v_\epsilon - y\| < \epsilon$, we infer that $\|w_1\| > \|y\| - \epsilon$.

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

We proceed with our estimates for $X = \mathbf{L}^p_\rho(I, \mathbb{C})$. (In the case $X = \mathcal{C}_{0,\rho}$ the same estimates work with $p = 1$ and all integrals replaced by suprema.) By Lemma 4.2

$$\|T(t_\epsilon)w_1\|^p = \int_{\theta - t_\epsilon}^{\theta + l - t_\epsilon} \rho(\tau)|w_1(\tau + t_\epsilon)|^p \, d\tau \geq \int_\theta^{\theta + l} m_1 \rho(\theta - t_\epsilon)|w_1(\tau)|^p \, d\tau,$$

$$\|w_1\|^p = \int_\theta^{\theta + l} \rho(\tau)|w_1(\tau)|^p \, d\tau \leq \int_\theta^{\theta + l} M_1 \rho(\theta + l)|w_1(\tau)|^p \, d\tau.$$

Thus

$$\epsilon^p \geq \|T(t_\epsilon)w_1\|^p \geq \frac{m_1 \rho(\theta - t_\epsilon)}{M_1 \rho(\theta + l)} \|w_1\|^p \geq \frac{m_1 \rho(\theta - t_\epsilon)}{M_1 \rho(\theta + l)} (1 - \epsilon)^p.$$

This implies that $\rho(\theta - t_\epsilon) \to 0$ as $\epsilon \to 0$.

Similarly, from the positivity of y and $\|v_\epsilon - y\| < \epsilon$ we obtain $\|w_2\| < \epsilon$. On the other hand z is negative and has its domain in $[\theta, \theta + l]$. Since $\|T(t_\epsilon)v_\epsilon - z\| < \epsilon$, we infer that $\|w_2\| > \|z\| - \epsilon$. We use now Lemma 4.2 to estimate

$$\epsilon^p \geq \|w_2\|^p \geq \frac{m_1 \rho(\theta + t_\epsilon)}{M_1 \rho(\theta + l)} \|T(t_\epsilon)w_2\|^p \geq \frac{m_1 \rho(\theta + t_\epsilon)}{M_1 \rho(\theta + l)} (1 - \epsilon)^p.$$

Thus $\rho(\theta + t_\epsilon) \to 0$ as $\epsilon \to 0$.

The condition is sufficient. Assume now that for each $\theta \in \mathbb{R}$ there exists a sequence $\{t_j\} \to \infty$ such that $\lim_{j \to \infty} \rho(\pm t_j + \theta) = 0$. Let y and z be two functions in X with compact support, say $[\theta - l, \theta]$. We will show that condition (iii) of Theorem 2.2 holds with y and z. Since these functions are taken from a dense set, this implies that $\{T(t)\}$ is hypercyclic.

We take $t = t_j$ sufficiently large, so that the intervals in the following definition do not overlap, and define

$$v_j(\tau) = \begin{cases} y(\tau) & \text{if } \tau \in [\theta - l, \theta], \\ z(\tau - t_j + l) & \text{if } \tau \in [t_j + \theta - l, t_j + \theta], \\ 0 & \text{else.} \end{cases}$$

Notice that by Lemma 4.2

$$\rho(\tau) \begin{cases} \leq M_1 \rho(t_j + \theta) & \text{if } \tau \in [t_j + \theta - l, t_j + \theta], \\ \leq M_1 \rho(-t_j + \theta) & \text{if } \tau \in [-t_j + \theta - l, -t_j + \theta], \\ \geq m_1 \rho(\theta - l) & \text{if } \tau \in [\theta - l, \theta]. \end{cases}$$

The support of $y - v_j$ is contained in $[t_j + \theta - l, t_j + \theta]$, while the support of $z - T(t)v_j$ is contained in $[-t_j + \theta - l, -t_j + \theta]$. We infer easily that

$$\|v_j - y\|^p \leq \frac{M_1 \rho(t_j + \theta)}{m_1 \rho(\theta - l)} \|z\|^p,$$

$$\|T(t)v_j - z\|^p \leq \frac{M_1 \rho(-t_j + \theta)}{m_1 \rho(\theta - l)} \|y\|^p.$$

Therefore $\|T(t_j)v_j - z\|$ and $\|v_j - y\|$ converge to 0 as $j \to \infty$. □

THEOREM 4.9. *Let $I = (-\infty, \infty)$ and let X be one of the spaces $\mathbf{L}^p_\rho(I, \mathbb{C})$ or $\mathcal{C}_{0,\rho}(I, \mathbb{C})$ with an admissible weight function ρ. Moreover, let $\rho_1(\tau) = \rho(-\tau)$ be an admissible weight function. Then the translation semigroup $\{T(t) \mid t \geq 0\}$ can be extended to a group $\{T(t) \mid t \in \mathbb{R}\}$ on X. Moreover, the following assertions are equivalent:*

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

(i) *the forward translation semigroup* $\{T(t) \mid t \geq 0\}$ *is hypercyclic;*
(ii) *the backward translation semigroup* $\{T(-t) \mid t \geq 0\}$ *is hypercyclic;*
(iii) *there exists a sequence* $\{t_j\}_{j=1}^{\infty}$ *of positive real numbers such that*

$$\lim_{j\to\infty} \rho(t_j) = \lim_{j\to\infty} \rho(-t_j) = 0.$$

Proof. Let $X_1 = \mathbf{L}^p_{\rho_1}$ or $X_1 = \mathcal{C}_{0,\rho_1}$, respectively. The semigroup $\{T(-t) \mid t \geq 0\}$ is isomorphic to the forward translation semigroup on the space X_1, which is a strongly continuous semigroup since ρ_1 is also admissible.

Lemma 4.2, applied to ρ and ρ_1, states that for each $l > 0$ there are constants m_1, m_2, M_1, M_2 depending only on l such that

$$m_1\rho(\tau - l) \leq \rho(\tau) \leq M_1\rho(\tau + l),$$
$$m_2\rho(\tau + l) \leq \rho(\tau) \leq M_2\rho(\tau - l).$$

For each $\theta \in \mathbb{R}$ we may pick $l = |\theta|$ and infer

$$\lim_{j\to\infty} \rho(\pm t_j + \theta) = 0 \quad \text{iff} \quad \lim_{j\to\infty} \rho(\pm t_j) = 0.$$

From Theorem 4.8 we infer that the forward translation semigroup is hypercyclic iff (iii) of Theorem 4.9 holds. The same conditions can be rewritten equivalently in terms of ρ_1:

$$\lim_{j\to\infty} \rho(\pm t_j) = 0 \quad \text{iff} \quad \lim_{j\to\infty} \rho_1(\mp t_j) = 0.$$

Therefore condition (iii) is also necessary and sufficient for the hypercyclicity of the backward translation semigroup. □

Example 4.10. There exists an admissible weight function ρ such that $\rho_1(t) = \rho(-t)$ is also admissible, and the translation groups $\{T(t) \mid t \in \mathbb{R}\}$ on $X = \mathbf{L}^p_\rho((-\infty, \infty), \mathbb{C})$ or $X = \mathcal{C}_{0,\rho}((-\infty, \infty), \mathbb{C})$ have the following properties:
(i) $\{T(t)\}$ is hypercyclic;
(ii) there exist no bounded trajectories $\{T(t)x \mid t \geq 0\}$ or $\{T(-t)x \mid t \geq 0\}$ except the zero trajectory. In particular, $X_0 = \{0\}$, the semigroup $\{T(t)\}$ admits no periodic orbits, and it is not chaotic.

Proof. Choose an even real-valued function $\phi : \mathbb{R} \to \mathbb{R}$ such that ϕ is continuously differentiable and its derivative is bounded by some constant ω on $\mathbb{R}$, and

$$\limsup_{\tau\to\infty} \phi(\tau) = \limsup_{\tau\to-\infty} \phi(\tau) = \infty,$$
$$\liminf_{\tau\to\infty} \phi(\tau) = \liminf_{\tau\to-\infty} \phi(\tau) = -\infty.$$

Put $\rho(\tau) = e^{\phi(\tau)}$. Then evidently

$$\frac{\rho(\tau \pm t)}{\rho(\tau)} = e^{\phi(\tau\pm t)-\phi(\tau)} \leq \exp\left[\left|\int_\tau^{\tau+t} \phi'(\sigma)\, d\sigma\right|\right] \leq e^{\omega|t|}.$$

Hence, both ρ and ρ_1 are admissible. Since $\liminf_{\tau\to\infty} \phi(\tau) = -\infty$ and ϕ is even, assertion (iii) of Theorem 4.9 holds. Therefore $\{T(t)\}$ is hypercyclic.

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

We finish the proof for $X = \mathbf{L}^p_\rho(\mathbb{R}, \mathbb{C})$. (The estimates for $X = \mathcal{C}_{0,\rho}(\mathbb{R}, \mathbb{C})$ are done similarly.) Let $x \in \mathbf{L}^p_\rho(\mathbb{R}, \mathbb{C})$, $x \neq 0$, and take l large enough such that $\eta = \int_{-l}^{l} |x(\tau)|^p \, d\tau > 0$. Since $\limsup_{\tau \to -\infty} \phi(\tau) = \infty$, for any $M > 0$ there exists some $t_M < 0$ such that $\rho(-t_M - l) > M$. Using Lemma 4.2 for an interval length of $2l$, we estimate

$$\|T(t_M)x\|^p \geq \int_{-t_M - l}^{-t_M + l} |x(t_M + \tau)|^p \rho(\tau) \, d\tau \geq \int_{-l}^{l} |x(\tau)|^p m_1 \rho(-t_M - l) \, d\tau \geq M\eta.$$

Therefore $\{T(t)x \mid t \geq 0\}$ is unbounded. The same argument shows that $\{T(-t)x \mid t \geq 0\}$ is unbounded.

It is an immediate consequence that $X_0 = X_p = \{0\}$. □

Example 4.11. There exists an admissible weight function ρ on $[0, \infty)$ such that the translation semigroup $\{T(t) \mid t \geq 0\}$ on $X = \mathbf{L}^1_\rho([0, \infty), \mathbb{C})$ is hypercyclic, and its infinitesimal generator has empty point spectrum.

Proof. We will construct an admissible function ρ on $I = [0, \infty)$ such that

$$\liminf_{\tau \to \infty} \rho(\tau) = 0.$$

Then by Theorem 4.7 the translation semigroup $\{T(t) \mid t \geq 0\}$ on $X = \mathbf{L}^1_\rho$ is hypercyclic. Moreover, we will construct ρ such that

$$\int_0^\infty e^{-\kappa s} \rho(s) \, ds = \infty \quad \text{for all } \kappa > 0. \tag{4.1}$$

Then, if x is an eigenvector of the generator of $T(t)$, then $T(t)x = e^{\lambda t}x$ implies that

$$x(s) = x(0)e^{-\lambda s},$$

which is a contradiction to $x \in X$ by (4.1). Therefore the generator has empty point spectrum.

To construct ρ we start with increasing sequences

$$0 = s_0 < s_1 < s_2 < \cdots \to \infty, \quad 1 < r_1 < r_2 < r_3 \cdots \to \infty,$$

with the following properties:

$$\limsup_{i \to \infty} \frac{\ln(r_i)}{s_i} = \infty, \quad \liminf_{i \to \infty} \frac{\ln(r_i)}{s_i} = 0.$$

With some arbitrary $\omega > 0$ we define

$$\rho(0) = 1,$$
$$\rho(s) = r_i e^{-\omega s} \quad \text{for } s \in (s_{i-1}, s_i].$$

Since the sequence $\{r_i\}$ is increasing, we infer easily that for all $s, t \geq 0$

$$\frac{\rho(s)}{\rho(s + t)} \leq e^{\omega t}.$$

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

Therefore ρ is admissible. Given $\epsilon > 0$ we pick i such that

$$\frac{\ln(r_i)}{s_i} < \frac{\omega}{2}.$$

Then

$$\rho(s_i) = r_i e^{-\omega s_i} < e^{-(\omega/2)s_i}.$$

Thus $\liminf_{\tau\to\infty} \rho(\tau) = 0$.

Given $\kappa > 0$ and arbitrary large $M > 1/(\kappa + \omega)$ we pick i such that $s_i > 1$ and

$$\frac{\ln(r_i)}{s_i} > \eta := \omega + \kappa + \ln[M(\kappa + \omega)].$$

Then

$$\rho(s_i) = r_i e^{-\omega s_i} > e^{(\eta-\omega)s_i}.$$

For $s \geq s_i$,

$$\rho(s) \geq e^{-\omega(s-s_i)}\rho(s_i).$$

Thus

$$\begin{aligned}\int_{s_i}^{\infty} e^{-\kappa s}\rho(s)\,ds &\geq \int_{s_i}^{\infty} e^{-\kappa s}e^{-\omega(s-s_i)}\rho(s_i)\,ds = \int_{s_i}^{\infty} e^{-(\kappa+\omega)(s-s_i)}e^{-\kappa s_i}\rho(s_i)\,ds \\ &> \frac{1}{\kappa+\omega}e^{(-\kappa+\eta-\omega)s_i} = \frac{[M(\kappa+\omega)]^{s_i}}{\kappa+\omega} > M.\end{aligned}$$

Since M can be chosen arbitrary large, we infer (4.1). □

The last example of this section is not a translation semigroup, but it contains a diffusion part. It shows that (in spite of Corollary 3.4) analyticity does not rule out that a semigroup is hypercyclic.

Example 4.12. In $X = \mathbf{L}^2([0,\infty), \mathbb{C})$ we consider the partial differential equation

$$\begin{aligned}u_t(x,t) &= au_{xx}(x,t) + bu_x(x,t) + cu(x,t), \\ u(0,t) &= 0 \quad \text{for } t \geq 0, \\ u(x,0) &= f(x) \quad \text{for } x \geq 0 \text{ with some } f \in X.\end{aligned}$$

Let $a, b, c > 0$ and $c < b^2/(2a) < 1$. Then the solution semigroup to this partial differential equation is chaotic.

Proof. We define the operators

$$\begin{aligned}A_1 f = f'' \quad \text{on } \operatorname{dom}(A_1) &= \{f \in \mathbf{W}^{2,2}([0,\infty), \mathbb{C}) \mid f(0) = 0\}, \\ B_1 f = f' \quad \text{on } \operatorname{dom}(B_1) &= \mathbf{W}^{1,2}([0,\infty), \mathbb{C}), \\ A &= aA_1 + bB_1 + cI.\end{aligned}$$

A_1 is the infinitesimal generator of a contraction semigroup, B_1 is dissipative, and $\operatorname{dom}(A_1) \subset \operatorname{dom}(B_1)$. For all $f \in \operatorname{dom}(A_1)$,

$$\|bB_1 f\| \leq \frac{b}{\sqrt{2a}}(\|aA_1 f\| + \|f\|).$$

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

Since $b/\sqrt{2a} < 1$, [**17**, Corollary 3.3, p. 82] implies that $aA_1 + bB_1$ generates a contraction semigroup. Consequently A is the generator of a strongly continuous semigroup.

To show that this semigroup is hypercyclic we verify the conditions of Theorem 3.1. Let U be the slotted disk

$$U = \left\{ \lambda \in \mathbb{C} \mid \left| \lambda - \left(c - \frac{b^2}{4a} \right) \right| \leq \frac{b^2}{4a}, \Im(\lambda) \neq 0 \text{ if } \Re(\lambda) \leq c - \frac{b^2}{4a} \right\}.$$

Since $c < b^2/(2a)$, the domain U intersects the imaginary axis.

For $\lambda \in U$ we define

$$f_\lambda(x) = e^{-(b/2a)x} \sin\left(x \sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}} \right).$$

Observe that

$$af_\lambda'' + bf_\lambda' + cf_\lambda = \lambda f_\lambda \quad \text{and} \quad f_\lambda(0) = 0.$$

Moreover, we have the estimate

$$\begin{aligned} |f_\lambda(x)| &\leq e^{-(b/2a)x} \exp\left[\left| \sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a^2}} \right| x \right] \\ &\leq \exp\left[\frac{x}{\sqrt{a}} \left[-\sqrt{\frac{b^2}{4a}} + \sqrt{\left| \lambda - \left(c - \frac{b^2}{4a} \right) \right|} \right] \right]. \end{aligned}$$

Since for $\lambda \in U$ the exponent in the last term is negative, we infer that $f_\lambda \in X$. A similar argument shows that $f_\lambda'' \in X$, so that $f_\lambda \in \mathrm{dom}(A)$. The vectors f_λ are therefore eigenvectors of A with the eigenvalues λ.

For $\phi \in X^* = X$ and $\lambda \in U$, we now define $F_\phi(\lambda) = \langle \phi, f_\lambda \rangle$. Put

$$\psi(x) = \begin{cases} e^{-(b/2a)x} \phi(x) & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Then F_ϕ can be written in terms of the Fourier transform $\tilde{\psi}$ of ψ:

$$F_\phi(\lambda) = \frac{1}{2i} \left[\tilde{\psi}\left(-\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a}} \right) - \tilde{\psi}\left(\sqrt{\frac{c-\lambda}{a} - \frac{b^2}{4a}} \right) \right].$$

Since the integrals involved converge absolutely and the square root is analytic for $\lambda \in U$, one can show easily that $F_\phi(\lambda)$ depends analytically on $\lambda \in U$. If F_ϕ vanishes on the whole set U, we infer by analyticity that

$$\tilde{\psi}(\mu) = \tilde{\psi}(-\mu) \quad \text{for all } \mu \in \mathbb{R}.$$

This implies that ψ is an even function, and since $\psi = 0$ on the negative half-line, ψ vanishes everywhere, which implies that $\phi = 0$.

We have verified the conditions of Theorem 3.1 and infer that the semigroup generated by A is chaotic. □

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

5. *Semigroups generated by discrete shifts*

In this section we consider the weighted l^1-space

$$X = l^1_r = \left\{ \{x_i\}_{i=1}^{\infty} \mid x_i \in \mathbb{C}, \sum_{i=1}^{\infty} r_i |x_i| < \infty \right\}$$

normed by

$$\|\{x_i\}_{i=1}^{\infty}\| = \sum_{i=1}^{\infty} r_i |x_i|.$$

Here $\{r_i\}_{i=1}^{\infty}$ is a sequence of weights $r_i > 0$, such that for some constant M

$$\frac{r_i}{r_{i+1}} \leq M \quad \text{for all } i. \tag{5.1}$$

(Similar results can also be proved for weighted l^p-spaces with $1 \leq p < \infty$.) In these spaces we define the unlateral backward shift A by

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ \vdots \end{pmatrix}.$$

Weighted shifts are a good example of bounded generators for continuous hypercyclic semigroups. Another source of such examples are translation semigroups in spaces of holomorphic functions (see the remarks at the beginning of §4).

Salas [**22**] has proved that $\mathrm{id} + A$ is hypercyclic. It is no surprise then, that the continuous semigroup generated by A is also hypercyclic, and the technicalities of the proof are, in fact, very closely related to the proof of Salas.

LEMMA 5.1. *If the weights r_i satisfy the condition (5.1), then the unilateral backward shift A is a bounded linear operator on l^1_r. Moreover, the linear semigroup $\{T(t)\}$ generated by A is given by*

$$T(t) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ \vdots \end{pmatrix} \quad \text{with } u_i(t) = \sum_{j=i}^{\infty} \frac{t^{j-i}}{(j-i)!} x_j.$$

Proof. One checks easily that the norm of A is the least constant M such that (5.1) holds. For a bounded operator, the semigroup generated by A is given by the power series

$$T(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

which yields immediately the formula given above. □

THEOREM 5.2. *Let the weights r_i satisfy (5.1), let A be the left shift and $\{T(t)\}$ be the semigroup generated by A in $X = l^1_r$ as in Lemma 5.1. Then $\{T(t)\}$ is hypercyclic.*

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

Proof. We prove assertion (iii) from Theorem 2.2 for y, z in the dense subset

$$D = \{\{x_i\}_{i=1}^{\infty} \in l_r^1 \mid \exists L : \forall i > L : x_i = 0\}.$$

Thus, let $y = \{y_i\}_{i=1}^{\infty}$ and $z = \{z_i\}_{i=1}^{\infty}$ be sequences in D with

$$y_i = z_i = 0 \quad \text{for } i > L.$$

Let $\epsilon > 0$ be arbitrary. We will construct a vector $v = \{v_i\}_{i=1}^{\infty} \in l_r^1$ and some large $t > 0$ such that

$$\|v - y\| \le \epsilon \quad \text{and} \quad \|T(t)v - z\| \le \epsilon.$$

Starting out with some t (to be determined later) we define the corresponding vector $v(t)$ by the identities

$$v_i(t) = y_i \quad \text{for } i = 1, \dots, L,$$
$$v_i(t) = 0 \quad \text{for } i = L+1, \dots, 2L \text{ and for } i > 3L,$$
$$u_i(t) = z_i \quad \text{for } i = 1, \dots, L, \text{ where } u_i(t) = \sum_{j=i}^{3L} \frac{t^{j-i}}{(j-i)!} v_j(t).$$

The coefficients $v_{2L+1}(t), \dots, v_{3L}(t)$ are then the solutions of the following system of linear equations:

$$\begin{pmatrix} \frac{t^{2L}}{(2L)!} & \frac{t^{2L+1}}{(2L+1)!} & \cdots & \frac{t^{3L-1}}{(3L-1)!} \\ \frac{t^{2L-1}}{(2L-1)!} & \frac{t^{2L}}{(2L)!} & \cdots & \frac{t^{3L-2}}{(3L-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{t^{L+1}}{(L+1)!} & \frac{t^{L+2}}{(L+2)!} & \cdots & \frac{t^{2L}}{(2L)!} \end{pmatrix} \begin{pmatrix} v_{2L+1}(t) \\ v_{2L+2}(t) \\ \vdots \\ v_{3L}(t) \end{pmatrix}$$
$$+ \begin{pmatrix} 1 & \frac{t}{1!} & \cdots & \frac{t^{L-1}}{(L-1)!} \\ 0 & 1 & \cdots & \frac{t^{L-2}}{(L-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_L \end{pmatrix}.$$

Notice that the coefficient matrix of the system can be written

$$\begin{pmatrix} \frac{t^{2L}}{(2L)!} & \frac{t^{2L+1}}{(2L+1)!} & \cdots & \frac{t^{3L-1}}{(3L-1)!} \\ \frac{t^{2L-1}}{(2L-1)!} & \frac{t^{2L}}{(2L)!} & \cdots & \frac{t^{3L-2}}{(3L-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{t^{L+1}}{(L+1)!} & \frac{t^{L+2}}{(L+2)!} & \cdots & \frac{t^{2L}}{(2L)!} \end{pmatrix}$$
$$= \begin{pmatrix} t^{2L} & 0 & \dots & 0 \\ 0 & t^{2L-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{L+1} \end{pmatrix} W \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & t & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{L-1} \end{pmatrix}$$

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

with a constant regular matrix

$$W = \begin{pmatrix} \frac{1}{(2L)!} & \frac{1}{(2L+1)!} & \cdots & \frac{1}{(3L-1)!} \\ \frac{1}{(2L-1)!} & \frac{1}{(2L)!} & \cdots & \frac{1}{(3L-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(L+1)!} & \frac{1}{(L+2)!} & \cdots & \frac{1}{(2L)!} \end{pmatrix}.$$

Therefore the solution of the system is given by

$$\begin{pmatrix} v_{2L+1}(t) \\ v_{2L+2}(t) \\ \vdots \\ v_{3L}(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & t^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{-L+1} \end{pmatrix} W^{-1} \begin{pmatrix} t^{-2L} & 0 & \dots & 0 \\ 0 & t^{-2L+1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{-L-1} \end{pmatrix}$$
$$\cdot \left[\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_L \end{pmatrix} - \begin{pmatrix} 1 & \frac{t}{1!} & \cdots & \frac{t^{L-1}}{(L-1)!} \\ 0 & 1 & \dots & \frac{t^{L-2}}{(L-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{pmatrix} \right].$$

It is easily seen that there is some constant C_1 such that all coefficients of

$$\begin{pmatrix} t^{-2L} & 0 & \dots & 0 \\ 0 & t^{-2L+1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{-L-1} \end{pmatrix} \cdot \left[\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_L \end{pmatrix} - \begin{pmatrix} 1 & \frac{t}{1!} & \cdots & \frac{t^{L-1}}{(L-1)!} \\ 0 & 1 & \dots & \frac{t^{L-2}}{(L-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{pmatrix} \right]$$

are bounded by $C_1 t^{-L-1}$. Since W^{-1} is independent of t, there exists some constant C_2 such that for $i = 2L+1, \dots, 3L$,

$$|v_i(t)| \le C_2 t^{-(i-2L-1)} t^{-L-1} = C_2 t^{L-i}.$$

This will be the crucial estimate in the following consideration.

We use first that $v_i(t) = y_i$ for $i \le 2L$ and estimate

$$\|v(t) - y\| = \sum_{i=2L+1}^{3L} r_i |v_i(t)| \le \sum_{i=2L+1}^{3L} r_i C_2 t^{L-i}.$$

Notice that all powers of t in this inequality are negative, so that

$$\|v(t) - y\| \le \epsilon \quad \text{for sufficiently large } t > 0.$$

Next we observe that $u_i(t)$ is the ith coefficient of $T(t)v(t)$, and that it equals z_i for $i \le L$. Therefore

$$\|T(t)v(t) - z\| = \sum_{i=L+1}^{3L} r_i \left| \sum_{j=i}^{3L} \frac{t^{j-i}}{(j-i)!} v_j(t) \right| \le \sum_{i=L+1}^{3L} r_i \sum_{j=i}^{3L} \frac{t^{j-i}}{(j-i)!} C_2 t^{L-j}$$

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

$$\leq \quad C_2 \sum_{i=L+1}^{3L} r_i t^{L-i} \sum_{j=i}^{3L} \frac{1}{(j-i)!} \leq C_2 e \sum_{i=L+1}^{3L} r_i t^{L-i}.$$

Since again all powers of t are negative, we obtain

$$\|T(t)v(t) - z\| \leq \epsilon \quad \text{for sufficiently large } t > 0.$$

Therefore assertion (iii) of Theorem 2.2 holds for y and z in the dense subset D. □

Notice that unlike the case of translation semigroups, Theorem 5.2 does not require any assumptions on the weight functions except (5.1) which is necessary in order that $\{T(t)\}$ is a semigroup of bounded operators on l^1_r. However, the properties of the weights determine whether the semigroup is chaotic:

THEOREM 5.3. *Let the weights r_i satisfy (5.1), let A be the left shift and $\{T(t)\}$ be the semigroup generated by A in $X = l^1_r$ as in Lemma 5.1. Let R denote the radius of convergence for the power series $\sum_{j=1}^{\infty} r_j x^j$ and let $\omega \in \mathbb{C}$. Then the semigroup $\{e^{\omega t} T(t)\}$ is chaotic if and only if $|\Re(\omega)| < R$.*

(By (5.1) R is always finite. However, R may be 0, so that $\{T(t)\}$ is not chaotic.)

Proof. Necessity. If $\{e^{\omega t} T(t)\}$ is chaotic, then the point spectrum of $\omega + A$ contains a number μ with $\Re(\mu) = 0$ and $\Im(\mu) \neq \Im(\omega)$. Thus the point spectrum of A contains a nonreal number $\lambda = \mu - \omega$ with $\Re(\lambda) = -\Re(\omega)$. In particular, $|\lambda| > |\Re(\omega)|$. Let $v = \{v_j\}_{j=1}^{\infty}$ be a corresponding eigenvector. From $Av = \lambda v$ we infer $v_j = \lambda^{j-1} v_1$, whence

$$\|v\| = \sum_{j=1}^{\infty} r_j |\lambda^{j-1} v_1| = |v_1| \sum_{j=1}^{\infty} r_j |\lambda|^{j-1}.$$

Since $\|v\| < \infty$, we infer that $|\Re(\omega)| < |\lambda| \leq R$.

Sufficiency. We will use Theorem 3.1. Let $|\Re(\omega)| < R$. Let U be the open disk with radius R and center ω. Since $|\Re(\omega)| < R$, the disk U intersects the imaginary axis.

For $\lambda \in U$ we define $x_\lambda = \{x_j(\lambda)\}_{j=1}^{\infty}$ by $x_j(\lambda) = (\lambda - \omega)^j$. From $|\lambda - \omega| < R$ we infer $x_\lambda \in l^r_1$. An easy computation shows that $(\omega + A)x_\lambda = \lambda x_\lambda$. Now let ϕ be in the dual space of l^1_r, i.e. ϕ is given by

$$\langle \phi, y \rangle = \sum_{j=1}^{\infty} \phi_j y_j,$$

where the sequence $\{r_j^{-1} |\phi_j|\}$ is bounded. Then

$$\langle \phi, x_\lambda \rangle = \sum_{j=1}^{\infty} r_j^{-1} \phi_j r_j (\lambda - \omega)^j.$$

This is an analytic function of $\lambda \in U$. If $\langle \phi, x_\lambda \rangle = 0$ for all $\lambda \in U$, then all coefficients of the power series vanish, which implies that $\phi = 0$.

We have therefore verified the assumptions of Theorem 3.1 and infer that $\{e^{\omega t} T(t)\}$ is chaotic. □

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

6. *Hypercyclic linearization of a stable system*

One of the most important features of the stability theory for linear systems is that it can be carried over to yield local results for nonlinear systems by linearization. It is known that a nonlinear semigroup whose Fréchet derivative is an exponentially stable linear semigroup is locally asymptotically stable. Additional hypotheses are required to prove that a semigroup with an unstable linearization is unstable (cf. [**8**]). Although hypercyclicity seems to be a property of a very unstable semigroup, it can also occur in the critically unstable case, when trajectories grow slower than exponentially. From this viewpoint, it is no surprise that there are globally stable nonlinear systems with hypercyclic linearizations, as the following example shows.

In this section we require an admissible weight function ρ on $[0,\infty)$ which is continuous and satisfies $\liminf_{\tau\to\infty}\rho(\tau)=0$. We know that for such ρ the translation semigroup $\{T(t)\mid T\geq 0\}$ on $X=\mathcal{C}_{0,\rho}([0,\infty),\mathbb{R})$ is hypercyclic.

Moreover, let $\phi:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ be a continuously differentiable function satisfying

$$\phi(t,0)=0 \quad \text{for all } t>0, \tag{6.1}$$

$$\phi(t,\cdot) \text{ is bijective with inverse } \phi^{-}(t,\cdot), \tag{6.2}$$

$$\frac{\partial}{\partial x}\phi(t,0)=\frac{1}{\rho(t)} \quad \text{for all } t>0, \tag{6.3}$$

$$\frac{\partial}{\partial x}\phi(t,x)>0 \quad \text{for all } t>0,\ x\in\mathbb{R}. \tag{6.4}$$

Assume that for all $t>0$, the following limits converge uniformly for x in compact subsets in $\mathbb{R}$:

$$\lim_{\tau\to\infty}|\phi(\tau,x)-x|=0, \tag{6.5}$$

$$\lim_{\tau\to\infty}|\phi^{-}(\tau,x)-x|=0, \tag{6.6}$$

$$\lim_{\tau\to\infty}\left|\frac{\partial}{\partial x}\phi^{-}(\tau,x)-\frac{\partial}{\partial x}\phi^{-}(\tau+t,x)\right|=0. \tag{6.7}$$

Example 6.1. Let ρ be a continuous admissible weight function on $[0,\infty)$ such that $\liminf_{\tau\to\infty}\rho(\tau)=0$, and let $\phi:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ be a continuously differentiable function satisfying the hypotheses (6.1)–(6.7). We define a nonlinear semigroup $\{S(t)\mid t\geq 0\}$ on $X=\mathcal{C}_{0,\rho}([0,\infty),\mathbb{R})$ by

$$[S(t)u](\tau)=\frac{1}{\rho(\tau)}\phi^{-}(\tau,\phi(t+\tau,\rho(t+\tau)u(t+\tau))).$$

Then the zero trajectory $S(t)(0)=0$ is globally asymptotically stable in the sense that for each $u\in X$ we have $\lim_{t\to\infty}S(t)u=0$.

However, for each fixed $t>0$, the Fréchet derivative of $S(t)$ is the translation operator $T(t)$ on X (which forms a hypercyclic semigroup).

The rest of this section consists of several lemmas proving Example 6.1.

LEMMA 6.2. *The following limit holds uniformly for $t\geq 0$ and u in bounded subsets of $\mathcal{C}_{0,\rho}([0,\infty),\mathbb{R})$:*

$$\lim_{\tau\to\infty}|\rho(\tau)[S(t)u](\tau)-\rho(t+\tau)u(t+\tau)|=0.$$

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

Proof.

$$\begin{aligned}
&\lim_{\tau\to\infty} |\rho(\tau)[S(t)u](\tau) - \rho(t+\tau)u(t+\tau)| \\
&\quad = \lim_{\tau\to\infty} \left|\rho(\tau)\frac{1}{\rho(\tau)}\phi^-(\tau, \phi(t+\tau, \rho(t+\tau)u(t+\tau))) - \rho(t+\tau)u(t+\tau)\right| \\
&\quad \le \lim_{\tau\to\infty} |\phi^-(\tau, \phi(t+\tau, \rho(t+\tau)u(t+\tau))) - \phi(t+\tau, \rho(t+\tau)u(t+\tau))| \\
&\qquad + \lim_{\tau\to\infty} |\phi(t+\tau, \rho(t+\tau)u(t+\tau)) - \rho(t+\tau)u(t+\tau)|.
\end{aligned}$$

If $\|u\| \le M$, then $\rho(t+\tau)|u(t+\tau)| \le M$ is uniformly bounded. Therefore (6.5) implies that the second limit is zero. This also implies that $\phi(t+\tau, \rho(t+\tau)u(t+\tau))$ is bounded, therefore (6.6) implies that the first limit is 0. □

LEMMA 6.3. *$\{S(t) \mid t \ge 0\}$ is a strongly continuous nonlinear semigroup on $X = \mathcal{C}_{0,\rho}([0,\infty), \mathbb{R})$ in the following sense:*

- *for all $t \ge 0$, the operator $S(t)$ is a continuous nonlinear map from X into X;*
- *$S(0)$ is the identity;*
- *for all $s, t \ge 0$ and all $u \in X$, the semigroup identity holds: $S(s)(S(t)u) = S(s+t)u$;*
- *for all $u \in X$, the function $t \mapsto S(t)u$ is continuous in $t \in [0,\infty)$;*
- *moreover, $S(t)0 = 0$ for all $t \ge 0$.*

Proof. We show $S(t)u \in X$. Let $u \in X$ and $t \ge 0$ be fixed. Since all ingredients are continuous, the function

$$\tau \mapsto \frac{1}{\rho(\tau)}\phi^-(\tau, \phi(t+\tau, \rho(t+\tau)u(t+\tau)))$$

is continuous. Moreover,

$$\lim_{\tau\to\infty} \rho(\tau)|[S(t)u](\tau)| \le \lim_{\tau\to\infty} |\rho(\tau)[S(t)u](\tau) - \rho(t+\tau)u(t+\tau)| + \lim_{\tau\to\infty} |\rho(t+\tau)u(t+\tau)|.$$

Since $u \in X$, the second limit is 0. Lemma 6.2 states that the first limit is 0. Hence $S(t)u \in X$.

$S(t) : X \to X$ is a continuous map. Let $u_n \to u_0$ be a convergent sequence in X and let $t \ge 0$ be fixed. Our assumptions imply that ϕ and ϕ^- are uniformly continuous on compact sets in $[0,\infty) \times \mathbb{R}$. Also, the set $\{\rho(t+\tau)u_n(t+\tau) \mid \tau \ge 0, n \in \mathbb{N}\}$ is bounded, since u_n is a bounded sequence. Using the definition of $S(t)$ we can conclude that

$$\lim_{n\to\infty} \rho(\tau)|[S(t)u_n](\tau) - [S(t)u_0](\tau)| = 0$$

uniformly for τ in compact intervals. Moreover, by Lemma 6.2

$$\lim_{\tau\to\infty} |\rho(\tau)[S(t)u_n](\tau) - \rho(t+\tau)u_n(t+\tau)| = 0$$

uniformly for all n. Given $\epsilon > 0$ choose $\tau_0 > 0$ such that

$$|\rho(\tau)[S(t)u_n](\tau) - \rho(t+\tau)u_n(t+\tau)| < \frac{\epsilon}{3} \quad \text{for all } \tau \ge \tau_0, n = 0, 1, 2, \ldots.$$

Then choose n_0 large enough such that

$$\rho(\tau)|[S(t)u_n](\tau) - [S(t)u_0](\tau)| < \epsilon \quad \text{for all } \tau \le \tau_0, n \ge n_0,$$

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

and such that

$$\rho(\tau)|u_n(\tau) - u_0(\tau)| < \frac{\epsilon}{3} \quad \text{for all } \tau \geq 0, n \geq n_0.$$

Combining all estimates we obtain for any $\tau \geq 0$:

$$\begin{aligned}\rho(\tau)|[S(t)u_n](\tau) - [S(t)u_0](\tau)| &\leq \epsilon \quad \text{if } \tau \leq \tau_0 \\ &\leq |\rho(\tau)[S(t)u_n](\tau) - \rho(t+\tau)u_n(t+\tau)| \\ &\quad +\rho(t+\tau)|u_n(t+\tau) - u_0(t+\tau)| \\ &\quad +|\rho(t+\tau)u_0(t+\tau) - \rho(\tau)[S(t)u_0](\tau)| \\ &\leq \epsilon \quad \text{if } \tau \geq \tau_0.\end{aligned}$$

Thus $\|S(t)u_n - S(t)u_0\| \leq \epsilon$.

The algebraic identities. $S(0)u = u$ and $S(t)(S(s)u) = S(t+s)u$ follow by direct computation, exploiting that $\phi^-(t,\cdot)$ is the inverse of $\phi(t,\cdot)$. Also, $S(t)0 = 0$ is shown by direct computation.

Strong continuity in t. Let $u \in X$ be fixed. We estimate

$$\begin{aligned}\lim_{t\to 0+} \rho(\tau)|[S(t)u](\tau) - u(\tau)| &\leq \lim_{t\to 0+} |\rho(\tau)[S(t)u](\tau) - \rho(t+\tau)u(t+\tau)| \\ &\quad + \lim_{t\to 0+} |\rho(t+\tau)u(t+\tau) - \rho(\tau)u(\tau)|.\end{aligned}$$

Using again the uniform continuity of ϕ and ϕ^- on compact subsets of $[0,\infty)\times\mathbb{R}$ we infer that the first limit is 0 uniformly for all $\tau \geq 0$. Since $\rho(\tau)u(\tau) \to 0$ for $\tau \to \infty$ and both ρ and u are uniformly continuous on bounded intervals, we infer that the second limit is 0 uniformly for all τ. Thus $\|S(t)u - u\| \to 0$ as $t \to 0+$. □

LEMMA 6.4. *For $t \geq 0$ and $u \in X$ we define the function*

$$g_{t,u}(\tau) = \frac{\partial}{\partial x}\phi^-(\tau, \phi(t+\tau, \rho(t+\tau)u(t+\tau))) \cdot \frac{\partial}{\partial x}\phi(t+\tau, \rho(t+\tau)u(t+\tau)).$$

Fix some $t \geq 0$. Then $g_{t,u}$ has the following properties:

(i) $\lim_{\tau\to\infty} g_{t,u}(\tau) = 1$ *uniformly for u in bounded sets;*

(ii) $g_{t,u}(\tau)$ *is uniformly bounded for u in bounded sets and $\tau \geq 0$;*

(iii) *for each convergent sequence $u_n \to u_0$ in X the following limit converges uniformly for all $\tau \geq 0$:* $\lim_{n\to\infty} g_{t,u_n}(\tau) = g_{t,u_0}(\tau)$.

Proof. For simplicity we abbreviate the partial derivatives: $\phi_x = (\partial/\partial x)\phi$, $\phi_x^- = (\partial/\partial x)\phi^-$.

If $\|u\| \leq M$, then $\rho(t+\tau)|u(t+\tau)| \leq M$. From (6.5) we infer that $|\phi(t+\tau, \rho(t+\tau)u(t+\tau))|$ is uniformly bounded. Therefore (6.5)–(6.7) and the continuity of ϕ, ϕ^-, ϕ_x, and ϕ_x^- hold uniformly for $x = \rho(t+\tau)u(t+\tau)$ or $x = \phi(t+\tau, \rho(t+\tau)u(t+\tau))$ if u is taken from bounded sets. In particular, we infer that $g_{t,u}(\tau)$ depends continuously on t and u, uniformly for τ in compact intervals.

We use the fact that $\phi^-(\tau,\cdot)$ is the inverse of $\phi(\tau,\cdot)$ to obtain

$$\phi_x^-(t+\tau, \phi(t+\tau, w)) \cdot \phi_x(t+\tau, w) = 1.$$

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

Thus

$$\begin{aligned}&|g_{t,u}(\tau)-1|\\ &\quad= |\phi_x^-(\tau,\phi(t+\tau,\rho(t+\tau)u(t+\tau)))\cdot\phi_x(t+\tau,\rho(t+\tau)u(t+\tau))\\ &\qquad -\phi_x^-(t+\tau,\phi(t+\tau,\rho(t+\tau)u(t+\tau)))\cdot\phi_x(t+\tau,\rho(t+\tau)u(t+\tau))|.\end{aligned}$$

By (6.7) this expression converges to 0 as $\tau\to\infty$, uniformly for u in bounded sets. This proves (i).

Assertions (ii) and (iii) are now easy, since we have uniform continuity for τ in compact sets, and a uniform limit 1 for $\tau\to\infty$. □

LEMMA 6.5. *For each fixed $t>0$, the operator $S(t):X\to X$ is Fréchet differentiable. The Fréchet derivative of $S(t)$ at a point $u\in X$ is the operator $W_{t,u}:X\to X$:*

$$[W_{t,u}v](\tau)=\frac{\rho(t+\tau)}{\rho(\tau)}g_{t,u}(\tau)v(t+\tau)$$

with $g_{t,u}$ defined as in Lemma 6.4.

- *For fixed $t\geq 0$, the operator $W_{t,u}$ depends continuously on $u\in X$.*
- *For fixed $v\in X$, $W_{t,u}v$ depends continuously on t and u.*
- *$W_{t,0}$ is the translation operator $T(t)$ given by $[T(t)u](\tau)=u(t+\tau)$.*

Proof. The Fréchet derivative. For any $u,v\in X$ and any $t,\tau\geq 0$ we obtain by differentiation

$$\begin{aligned}&\frac{\partial}{\partial h}[S(t)(u+hv)](\tau)\\ &\quad= \frac{1}{\rho(\tau)}\phi_x^-(\tau,\phi(t+\tau,\rho(t+\tau)[u(t+\tau)+hv(t+\tau)]))\\ &\qquad\cdot\phi_x(t+\tau,\rho(t+\tau)[u(t+\tau)+hv(t+\tau)])\cdot\rho(t+\tau)v(t+\tau)\\ &\quad= \frac{\rho(t+\tau)}{\rho(\tau)}g_{t,u+hv}(\tau)v(t+\tau).\end{aligned}$$

We can now estimate

$$\begin{aligned}&\rho(\tau)|[S(t)(u+v)](\tau)-[S(t)u](\tau)-[W_{t,u}v](\tau)|\\ &\quad= \left|\int_0^1\rho(t+\tau)g_{t,u+hv}(\tau)v(t+\tau)\,dh-\rho(t+\tau)g_{t,u}(\tau)v(t+\tau)\right|\\ &\quad\leq \rho(t+\tau)|v(t+\tau)|\int_0^1|g_{t,u+hv}(\tau)-g_{t,u}(\tau)|\,dh\\ &\quad\leq \|v\|\int_0^1|g_{t,u+hv}(\tau)-g_{t,u}(\tau)|\,dh.\end{aligned}$$

From Lemma 6.4(iii) we infer that the integral converges to 0 uniformly for $\tau\geq 0$ if $v\to 0$. Therefore $W_{t,u}$ as defined above is the Fréchet derivative of $S(t)$. Notice that Lemma 6.4(ii), implies that $W_{t,u}$ is a bounded linear operator.

Continuous dependence on u. Let $u_n\to u_0$ be a convergent sequence in X. For any $v\in X$ we have

$$\rho(\tau)|[W_{t,u_n}v](\tau)-[W_{t,u_0}v](\tau)|=|g_{t,u_n}(\tau)-g_{t,u_0}(\tau)|\rho(t+\tau)|v(t+\tau)|.$$

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

Lemma 6.4(iii) implies that this expression converges to 0 uniformly for $\tau \geq 0$ and v in bounded sets. Therefore $W_{t,u}$ depends continuously on u with respect to the operator norm.

Strongly continuous dependence on t. This is done in a similar manner to the continuous dependence on u. We use the fact that for given $v \in X$ the function $\rho(t+\tau)v(t+\tau)$ depends uniformly continuously on t and again treat the behavior on compact τ-intervals and the asymptotics for $\tau \to \infty$ separately.

Derivative at the zero trajectory. From (6.3) we infer

$$\frac{\rho(t+\tau)}{\rho(\tau)} g_{t,0}(\tau) = \frac{\rho(t+\tau)}{\rho(\tau)} \phi_x^-(\tau, 0) \cdot \phi_x(\tau, 0) = \frac{\rho(t+\tau)}{\rho(\tau)} \rho(\tau) \frac{1}{\rho(t+\tau)} = 1.$$

Thus $W_{t,0} = T(t)$. □

LEMMA 6.6. *The zero trajectory of $S(t)$ is locally stable in the sense that for all $\epsilon > 0$ there exists some $\delta > 0$ such that $\|S(t)u\| \leq \epsilon$ for all $u \in X$ with $\|u\| \leq \delta$ and all $t \geq 0$.*

The zero trajectory of $S(t)$ is globally asymptotically stable in the following sense: $\lim_{t\to\infty} S(t)u = 0$ *for each $u \in X$.*

Proof. Local Stability. Let $\epsilon > 0$. By (6.6) we can pick $T_1 > 0$ such that

$$|\phi^-(\tau, x) - x| \leq \frac{\epsilon}{2} \quad \text{for } \tau \geq T_1, |x| \leq \epsilon.$$

By the uniform continuity of ϕ^- on compact sets we can now choose some $\delta_1 \in (0, \epsilon/2)$ such that

$$|\phi^-(\tau, x)| \leq \epsilon \quad \text{for } \tau \in [0, T_1], |x| \leq \delta_1.$$

By (6.5) choose $T_2 > 0$ such that

$$|\phi(\tau, x) - x| \leq \frac{\delta_1}{2} \quad \text{for } \tau \geq T_2, |x| \leq \epsilon,$$

and choose $\delta \in (0, \delta_1/2)$ such that

$$|\phi(\tau, x)| \leq \delta_1 \quad \text{for } \tau \in [0, T_2], |x| \leq \delta.$$

Now let $u \in X$ with $\|u\| \leq \delta$, hence $\rho(t+\tau)|u(t+\tau)| \leq \delta \leq \epsilon$. Then

$$\begin{aligned} &|\phi(t+\tau, \rho(t+\tau)u(t+\tau))| \\ &\quad \leq \quad \delta_1 \quad \text{if } t+\tau \leq T_2 \\ &\quad \leq \quad |\phi(t+\tau, \rho(t+\tau)u(t+\tau)) - \rho(t+\tau)u(t+\tau)| + |\rho(t+\tau)u(t+\tau)| \\ &\quad \leq \quad \frac{\delta_1}{2} + \delta \leq \delta_1 \quad \text{if } t+\tau \geq T_2. \end{aligned}$$

Therefore in any case

$$|\phi(t+\tau, \rho(t+\tau)u(t+\tau))| \leq \delta_1.$$

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

This implies

$$\begin{aligned}
&|\phi^-(\tau, \phi(t+\tau, \rho(t+\tau)u(t+\tau)))| \\
&\quad \leq \quad \epsilon \quad \text{if } \tau \leq T_1 \\
&\quad \leq \quad |\phi^-(\tau, \phi(t+\tau, \rho(t+\tau)u(t+\tau))) - \phi(t+\tau, \rho(t+\tau)u(t+\tau))| \\
&\qquad +|\phi(t+\tau, \rho(t+\tau)u(t+\tau))| \\
&\quad \leq \quad \frac{\epsilon}{2} + \delta_1 \leq \epsilon \quad \text{if } \tau \geq T_1.
\end{aligned}$$

In any case

$$\rho(\tau)|[S(t)u](\tau)| \leq \epsilon, \quad \text{i.e. } \|S(t)u\| \leq \epsilon.$$

Global stability. Let $\epsilon > 0$ and $u \in X$. Choose δ such that $\|S(t)v\| < \epsilon$ for all $t \geq 0$ and all $v \in X$ with $\|v\| \leq \delta$. Since $u \in X$, we have $\lim_{\tau\to\infty} \rho(\tau)u(\tau) = 0$. We can therefore change u on a compact interval $[0, T]$ to obtain $v \in X$ with

$$u(\tau) = v(\tau) \quad \text{for } \tau > T, \text{ and } \|v\| < \delta.$$

Then we have for $t > T$,

$$\|S(t)u\| = \|S(t)v\| < \epsilon. \qquad \square$$

Acknowledgements. The authors wish to thank Charles Batty, Michael Jung and Gieri Simonett for helpful discussions. Moreover, we thank an anonymous referee for valuable references, in particular for a copy of C. Kitai's unpublished thesis.

REFERENCES

[1] W. Arendt *et al*. *One Parameter Semigroups of Positive Operators (Lecture Notes in Mathematics 1184)*. Ed R. Nagel. Springer, Berlin, Heidelberg, New York, Tokyo, 1986.

[2] J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey. On Devaney's defininition of chaos. *Amer. Math. Monthly* **99** (1992), 332–339.

[3] B. Beauzamy. *Introduction to Operator Theory and Invariant Subspaces*. North-Holland Mathematical Library, North-Holland, Amsterdam, 1988.

[4] P. Brunovsky. Notes on chaos in the cell population partial differential equation. *Nonlinear Anal.* **7** (1983), 167–176.

[5] P. Brunovsky and J. Komornik. Ergodicity and exactness of the shift on $C[0, \infty)$ and the semiflow of a first order partial differential equation. *J. Math. Anal. Appl.* **104** (1984), 235–245.

[6] K. C. Chan and J. H. Shapiro. The cyclic behavior of translation operators on Hilbert spaces of entire functions. *Indiana University Math. J.* **40** (1991), 1421–1449.

[7] R. L. Devaney. *An Introduction to Chaotic Dynamical Systems*, 2nd edn., Addison-Wesley, New York, 1989.

[8] W. Desch and W. Schappacher. Linearized stability for nonlinear semigroups. *Differential Equations in Banach Spaces (Bologna 1985) (Lecture Notes in Mathematics 1223)*. Eds A. Favini and E. Obrecht. Springer, 1986, pp. 61–73.

[9] R. M. Gethner and J. H. Shapiro. Universal vectors for operators on spaces of holomorphic functions. *Proc. Amer. Math. Soc.* **100** (1987), 281–288.

[10] G. Godefroy and J. H. Shapiro. Operators with dense, invariant, cyclic vector manifolds. *J. Functional Analysis* **98** (1991), 229–269.

[11] D. A. Herrero. Hypercyclic operators and chaos. *J. Operator Theory* **28** (1992), 93–103.

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press

[12] E. Hille and R. S. Phillips. *Functional Analysis and Semi-Groups (Amer. Math. Soc. Colloquium Publications 31)*, revised edn. American Mathematical Society, Providence, 1957.
[13] C. Kitai. Invariant closed sets for linear operators. *Thesis*, submitted to the University of Toronto, 1982.
[14] A. Lasota. Stable and chaotic solutions of a first order partial differential equation. *Nonlinear Anal.* **5** (1981), 1183–1193.
[15] A. Lasota and M. C. Mackey. *Probabilistic Properties of Deterministic Systems*, Cambridge University Press, Cambridge, 1985.
[16] A. Lasota and M. C. Mackey. *Chaos, Fractals, and Noise, Stochastic Aspects of Dynamics (Applied Mathematical Science Series 97)*. Springer, New York, 1994.
[17] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations (Applied Mathematical Series 44)*. Springer, New York, 1983.
[18] S. Rolewicz. On orbits of elements. *Studia Math.* **32** (1969), 17–22.
[19] A. O. Rey and M. C. Mackey. Multistability and boundary layer development in a transport equation with delayed arguments. *Canad. Appl. Math. Quar.* **1** (1993), 61–81.
[20] R. Rudnicki. Strong ergodic properties of a first-order partial differential equation. *J. Math. Anal. Appl.* **133** (1988), 14–26.
[21] H. R. Salas. A hypercyclic operator whose adjoint is also hypercyclic. *Proc. Amer. Math. Soc.* **112** (1991), 765–770.
[22] H. R. Salas. Hypercyclic weighted shifts. *Trans. Amer. Math. Soc.* **347** (1995), 993–1004.
[23] G. F. Webb. Periodic and chaotic behavior in structured models of cell population dynamics. *Recent Developments in Evolution Equations (Pitman Research Notes in Mathematics 324)*, 1995 pp. 40–49.

https://doi.org/10.1017/S0143385797084976 Published online by Cambridge University Press