Ergod. Th. & Dynam. Sys. (2016), **36**, 1892–1921 © *Cambridge University Press, 2015*
doi:10.1017/etds.2014.144

Toeplitz flows and their ordered K-theory

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(*Received* 29 *March* 2014 *and accepted in revised form* 21 *November* 2014)

Abstract. To a Toeplitz flow (X, T) we associate an ordered K^0-group, denoted $K^0(X, T)$, which is order isomorphic to the K^0-group of the associated (non-commutative) C^*-crossed product $C(X) \rtimes_T \mathbb{Z}$. However, $K^0(X, T)$ can be defined in purely dynamical terms, and it turns out to be a complete invariant for (strong) orbit equivalence. We characterize the K^0-groups that arise from Toeplitz flows (X, T) as exactly those simple dimension groups (G, G^+) that contain a non-cyclic subgroup H of rank one that intersects G^+ non-trivially. Furthermore, the Bratteli diagram realization of (G, G^+) can be chosen to have the ERS property, i.e. the incidence matrices of the Bratteli diagram have equal row sums. We also prove that for any Choquet simplex K there exists an uncountable family of pairwise non-orbit equivalent Toeplitz flows (X, T) such that the set of T-invariant probability measures $M(X, T)$ is affinely homeomorphic to K, where the entropy $h(T)$ may be prescribed beforehand. Furthermore, the analogous result is true if we substitute strong orbit equivalence for orbit equivalence, but in that case we can actually prescribe both the entropy and the maximal equicontinuous factor of (X, T). Finally, we present some interesting concrete examples of dimension groups associated to Toeplitz flows.

1. *Introduction*

Toeplitz flows have been extensively studied, as both topological and measure-theoretic dynamical systems, since they were first introduced by Jacobs and Keane in 1969 [**JK**]. (Note: in spite of the word 'flow', which would indicate an $\mathbb{R}$-action, Toeplitz flows are dynamical systems with a $\mathbb{Z}$-action generated by a single homeomorphism.) In this paper we will exclusively study Toeplitz flows as topological dynamical systems. A Toeplitz flow is a special minimal subshift on a finite alphabet Λ (in particular, it is a *symbolic system*) defined in terms of a so-called *Toeplitz sequence* on Λ, the latter being associated to certain arithmetic progressions. (Incidentally, the reason the name 'Toeplitz' has been attached to these systems is that Toeplitz in 1928 [**T**] gave an explicit construction of almost periodic functions (in the Bohr sense) on $\mathbb{R}$, where he as a device used arithmetic

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progressions.) Toeplitz flows are closely related to the well-understood odometer systems, and may in a certain sense be seen as the simplest Cantor minimal systems beyond the odometer systems. Yet, surprisingly, Toeplitz flows exhibit a richness of properties that the odometer systems do not have. As examples of this we mention that any Choquet simplex can be realized as the set of invariant probability measures for some Toeplitz flow. Furthermore, for every $0 \leq t < \infty$ there exists a Toeplitz flow (in fact, an uncountable family of pairwise non-isomorphic Toeplitz flows) whose topological entropy is equal to t.

An entirely new approach to study Toeplitz flows and, for that matter, Cantor minimal systems in general—thereby providing powerful new tools—came from an unsuspected source: non-commutative C^*-algebras. In fact, by studying the so-called C^*-crossed product associated to a Cantor minimal system a complete isomorphism invariant turns out to be a special ordered abelian group, a so-called dimension group. This group is the K^0-group of the crossed product and it comes with a natural ordering. It turns out that the K^0-group can be defined purely in dynamical terms, and that it is a complete invariant for (strong) orbit equivalence. This invariant is completely independent of other invariants traditionally used to study dynamical systems, like spectral invariants and entropy. Furthermore, the invariant is in many cases effectively computable. We mention two papers that illustrate this. The first is the paper by Durand *et al* [**DHS**], where substitutional dynamical systems were studied using the K^0-approach. The other is the paper by Gjerde and Johansen [**GJ**], where Toeplitz flows were treated. In the latter paper a 'clean' conceptual proof—as seen from the theory of dimension groups—is given of the Choquet simplex realization result alluded to above. (The original proof of that result is due to Downarowicz [**D**], after preliminary results had been obtained by Williams [**W**].) Furthermore, the dimension groups (i.e. the K^0-groups) associated to Toeplitz flows are described in their paper in terms of Bratteli diagrams with what they call the EPN property (which is the same as the ERS property—the term we prefer to use). In this paper we will extend their results considerably and we will give a more satisfactory description of the K^0-groups. In fact, the main thrust of this paper is to give the definite characterization of the K^0-groups associated to Toeplitz flows. This is done in an intrinsic way, meaning that we give a characterization in terms of the group itself, while the Bratteli diagram realizations of the groups in question play an important, but auxiliary, role. By our characterization we can easily exhibit concrete examples of such groups; some of these will be described in §5. Furthermore, our results underscore in a striking manner that the K^0-group of a Toeplitz flow (X, T) does not 'see' the entropy $h(T)$ of T; these two entities are independent. (See the Remark after Theorem 2.4.)

We remark that Toeplitz flows, in contrast to substitution minimal systems, are 'unstable'. By that we mean that whereas an induced system of a substitution system is again a substitution system, and a Cantor factor of a substitution system is again either a substitution system or an odometer, this it not true for Toeplitz flows in general (cf. [**DD**]). There is a 'tiny' overlap between Toeplitz flows and substitution minimal systems. In fact, some—but not all—substitution minimal systems associated to (primitive, aperiodic) substitutions of constant length are Toeplitz flows (cf. [**M**, Theorem 6.03]).

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2. *Main results*

We formulate our main results, referring to §3 for definitions of some of the terms occurring in the statements of the theorems below.

THEOREM 2.1. *Let $0 \le t < \infty$. The following two sets are equal (up to order isomorphisms):*

(i) $\{(K^0(X, T), K^0(X, T)^+) \mid (X, T)$ *Toeplitz flow,* $h(T) = t\}$;

(ii) $\{(G, G^+) \mid G$ *simple dimension group containing a non-cyclic subgroup* H *of rank one such that* $H \cap G^+ \neq \{0\}\}$.

THEOREM 2.2. *Let (G, G^+, u) be a simple dimension group with order unit u, and assume that the rational (and hence rank-one) subgroup $\mathbb{Q}(G, u)$ is non-cyclic. Let $0 \le t < \infty$. There exists a Toeplitz flow (X, T) such that:*

(i) $(G, G^+, u) \cong (K^0(X, T), K^0(X, T)^+, \mathbb{1})$;

(ii) *the entropy $h(T)$ of T equals t;*

(iii) *the set of (continuous) eigenvalues of T is $\{e^{2\pi i s} \mid s \in \mathbb{Q}(G, u)\}$.*

THEOREM 2.3. *Let $0 \le t < \infty$. The following three sets of simple dimension groups with order units are equal (up to order isomorphisms preserving order units):*

$$\mathcal{T}_t = \{(K^0(X, T), K^0(X, T)^+, \mathbb{1}) \mid (X, T) \text{ Toeplitz flow, } h(T) = t\},$$

$$\mathcal{G} = \{(G, G^+, u) \mid G \text{ simple dimension group with } \mathbb{Q}(G, u) \text{ non-cyclic}\},$$

$$\mathcal{B} = \left\{ (K_0(V, E), K_0(V, E)^+, [\mathbf{1}]) \;\middle|\; \begin{array}{r}(V, E) \text{ simple Bratteli diagram} \\ \text{with the ERS property}\end{array} \right\}.$$

THEOREM 2.4. *Let $0 \le t < \infty$. For any Choquet simplex K there exists an uncountable family of pairwise non-orbit equivalent Toeplitz flows $\mathcal{A} = \{(X, T)\}$ such that for each $(X, T) \in \mathcal{A}$:*

(i) $K \cong M(X, T)$, *i.e. K is affinely homeomorphic to the T-invariant probability measures on X;*

(ii) $h(T) = t$.

Now let (Y, S) be any odometer and let K be any Choquet simplex. There exists an uncountable family of pairwise non-strong orbit equivalent Toeplitz flows $\widetilde{\mathcal{A}} = (\widetilde{X}, \widetilde{T})$ such that for each $(\widetilde{X}, \widetilde{T}) \in \widetilde{\mathcal{A}}$:

(i) $K \cong M(\widetilde{X}, \widetilde{T})$;

(ii) $h(\widetilde{T}) = t$;

(iii) *the maximal equicontinuous factor of $(\widetilde{X}, \widetilde{T})$ is (Y, S).*

Remark. As we alluded to above, Theorem 2.4 yields in a striking way as a corollary that for Toeplitz flows orbit structure and entropy are independent entities.

Our result, Theorem 2.2(i), has an interesting consequence, which we will briefly describe. In [**GMPS**], the following remarkable result is proved (cf. [**GMPS**, Theorem 2.5]).

Let d be a natural number. A minimal $\mathbb{Z}^d$-action on the Cantor set is orbit equivalent to a minimal $\mathbb{Z}$-action (i.e. to a Cantor minimal system (X, T)). A complete invariant

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for orbit equivalence is a simple dimension group with order unit (G, G^+, u) such that $\mathrm{Inf}G = \{0\}$. Specifically, G is given by

$$G = \mathcal{C}(X, \mathbb{Z})/\left\{ f \in \mathcal{C}(X, \mathbb{Z}) \;\middle|\; \int_X f\,d\mu = 0 \text{ for all } \mu \in M(X, \mathbb{Z}^d) \right\}.$$

(Here $M(X, \mathbb{Z}^d)$ denotes the set of invariant probability measures under the $\mathbb{Z}^d$-action.) G^+ is given by the induced ordering from $\mathcal{C}(X, \mathbb{Z})$ and u corresponds to the constant function $1 \in \mathcal{C}(X, \mathbb{Z})$.

Remark. An open problem raised in [**GMPS**] is: what is the range of the invariant (G, G^+, u) when $d > 1$? (For $d = 1$, it is known that the range is all simple dimension groups with order unit (G, G^+, u) such that $\mathrm{Inf}G = \{0\}$, cf. [**HPS**, Theorem 5.4].)

In [**CP**], the following theorem is proved (cf. [**CP**, Theorem B]).

Let (X, T) be a Toeplitz flow and let $d \geq 1$. There exists a Toeplitz $\mathbb{Z}^d$-subshift (and so, in particular, a minimal $\mathbb{Z}^d$-action on the Cantor set) which is orbit equivalent to (X, T).

Combining Theorem 2.2(i) with the two results cited above, we get the following interesting result.

PROPOSITION 2.5. *Let d be a natural number. The range of the orbit invariant for minimal $\mathbb{Z}^d$-actions on the Cantor set contains all simple dimension groups with order unit (G, G^+, u), where $\mathrm{Inf}G = \{0\}$, such that $\mathbb{Q}(G, u)$ is non-cyclic.*

3. *Basic concepts and definitions and key background results*

3.1. *Dynamical systems.* Throughout this paper we will use the term dynamical system to mean a compact metric space X together with a homeomorphism $T : X \to X$, and we will denote this by (X, T). This induces in a natural way a $\mathbb{Z}$-action on X. The orbit of $x \in X$ under this action is $\{T^n x \mid n \in \mathbb{Z}\}$ and will be denoted by $\mathrm{orbit}_T(x)$. If all the orbits are dense in X, we say that (X, T) is a *minimal system*. It is a simple observation that (X, T) is minimal if and only if $TA = A$ for some closed $A \subseteq X$ implies that $A = X$ or $\varnothing$.

(We will denote the natural numbers $\{1, 2, 3, \ldots\}$ by $\mathbb{N}$, the integers by $\mathbb{Z}$, the rational numbers by $\mathbb{Q}$ and the real numbers by $\mathbb{R}$. Also, let $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$, $\mathbb{Q}^+ = \{r \in \mathbb{Q} \mid r \geq 0\}$, $\mathbb{R}^+ = \{t \in \mathbb{R} \mid t \geq 0\}$.)

Definition 3.1. We say that a dynamical system (Y, S) is a *factor* of (X, T) and that (X, T) is an *extension* of (Y, S) if there exists a continuous surjection $\pi : X \to Y$ which satisfies $S(\pi(x)) = \pi(Tx)$ for all $x \in X$. We call π a *factor map*. If π is a bijection, then we say that (X, T) and (Y, S) are *conjugate*, and we write $(X, T) \cong (Y, S)$. We say that (X, T) is *flip conjugate* to (Y, S) if $(X, T) \cong (Y, S)$ or $(X, T) \cong (Y, S^{-1})$.

Definition 3.2. The dynamical systems (X, T) and (Y, S) are *orbit equivalent* if there exists a homeomorphism $F : X \to Y$ such that $F(\mathrm{orbit}_T(x)) = \mathrm{orbit}_S(F(x))$ for all $x \in X$. We call F an *orbit map*.

Remark. Clearly, flip conjugacy implies orbit equivalence. One can show that if X (and hence Y) is a connected space, then orbit equivalence between (X, T) and (Y, S) implies flip conjugacy. (This follows by a simple argument using a result of Sierpiński, cf. [**K**, Theorem 6, Ch. V, §47, III].) This has the consequence that the study of orbit equivalence

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is only interesting as it pertains to *Cantor minimal systems* (X, T), i.e. X is a Cantor set on which T acts minimally. The K-theoretic invariant we are going to introduce is an invariant for orbit equivalence, and so we will assume henceforth that our dynamical systems are Cantor minimal, even though some of the subsequent definitions apply to more general systems.

Let (X, T), (Y, S) and F be as in Definition 3.2, both (X, T) and (Y, S) being Cantor minimal systems. For each $x \in X$, there exists a unique integer $n(x)$ (respectively, $m(x)$) such that $F(Tx) = S^{n(x)}(F(x))$, $F(T^{m(x)}x) = S(F(x))$. We call $m, n : X \to \mathbb{Z}$ the orbit cocycles associated to the orbit map F.

Definition 3.3. We say that (X, T) and (Y, S) are strong orbit equivalent if there exists an orbit map $F : X \to Y$ such that each of the two associated orbit cocycles $m, n : X \to \mathbb{Z}$ has at most one point of discontinuity.

Remark. Boyle (cf. [**BT**]) proved that if the orbit cocycles are continuous for all $x \in X$, then (X, T) and (Y, S) are flip conjugate. So, strong orbit equivalence is in a sense the mildest weakening possible of flip conjugacy.

We shall need the concept of *induced transformation* and *Kakutani equivalence*.

Definition 3.4. Let (X, T) be a Cantor minimal system and let A be a clopen subset of X (hence A is again a Cantor set). Let $T_A : A \to A$ be the *first return map*, i.e. $T_A(x) = T^{r_A(x)}x$, where $r_A(x) = \min\{n \in \mathbb{N} \mid T_A^n x \in A\}$. We say that (A, T_A), which is again Cantor minimal, is the induced system of (X, T) with respect to A.

Definition 3.5. The Cantor minimal systems (X, T) and (Y, S) are *Kakutani equivalent* if (up to conjugacy) they have a common induced system. (One can show that this is an equivalence relation on the family of Cantor minimal systems.)

Definition 3.6. Let (X, T) and (Y, S) be (Cantor) minimal systems, (Y, S) being a factor of (X, T) by a map $\pi : X \to Y$. If there exists a point $x \in X$ such that $\pi^{-1}(\pi(x)) = \{x\}$, we say that π is almost one-to-one and we say that (X, T) is an almost one-to-one extension of (Y, S).

Remark. One can show that the subset B of X consisting of the points x in X satisfying the above condition is a dense $\mathcal{G}_\delta$-set in X. In fact, define $A(x, n) = \{y \in Y \mid \pi^{-1}(y) \subseteq B(x, 2^{-n})\}$, where $B(x, 2^{-n})$ is the open ball around x with radius 2^{-n}. One shows that $A(x, n)$ is an open subset of Y. Define $A_n = \bigcup_{v \in X} A(v, n)$. Then A_n is an open subset of Y. A simple argument yields that the subset B in question is equal to $\pi^{-1}\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} \pi^{-1}(A_n)$, which is a dense $\mathcal{G}_\delta$-set.

3.2. *Toeplitz flows.*

Definition 3.7. (X, T) is *expansive* if there exists $\delta > 0$ such that if $x \neq y$ then $\sup_n \mathrm{d}(T^n x, T^n y) > \delta$, where d is a metric that gives the topology of X. (Expansiveness is independent of the metric as long as the metric gives the topology of X.)

Let $\Lambda = \{a_1, a_2, \ldots, a_n\}$, $n \geq 2$, be a finite alphabet and let $Z = \Lambda^{\mathbb{Z}}$ be the set of all bi-infinite sequences of symbols from Λ with Z given the product topology—thus Z is a

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Cantor set. Let $S : Z \to Z$ denote the shift map, $S : (x_n) \to (x_{n+1})$. If X is a closed subset of Z such that $S(X) = X$, we say that (X, S) is a *subshift*, where we denote the restriction of S to X again by S. Subshifts are easily seen to be expansive. We state the following well-known fact as a proposition (cf. [**Wa**, Theorem 5.24]).

PROPOSITION 3.8. *Let (X, T) be a Cantor minimal system. Then (X, T) is conjugate to a minimal subshift on a finite alphabet if and only if (X, T) is expansive.*

As a general reference on Toeplitz flows, we refer to [**W**] (cf. also [**D**]).

Definition 3.9. Let $\eta = (\eta(n))_{n\in\mathbb{Z}} \in \Lambda^{\mathbb{Z}}$, where Λ is a finite alphabet. Then we define for $\sigma \in \Lambda$, $p \in \mathbb{N}$

$$\mathrm{Per}_p(\eta, \sigma) = \{n \in \mathbb{Z} \mid \eta(n + mp) = \sigma \text{ for all } m \in \mathbb{Z}\}.$$

Let

$$\mathrm{Per}_p(\eta) = \bigcup_{\sigma\in\Lambda} \mathrm{Per}_p(\eta, \sigma).$$

We say that η is a *Toeplitz sequence* if $\bigcup_{p\in\mathbb{N}} \mathrm{Per}_p(\eta) = \mathbb{Z}$.

By the *p-skeleton* of η we will mean the part of η which is periodic with period p; more precisely, we define the p-skeleton to be the sequence obtained from η by replacing $\eta(n)$ with a new symbol $*$ for all $n \notin \mathrm{Per}_p(\eta)$. We say that p is an *essential period* of η if the p-skeleton of η is not periodic with any smaller period. The least common multiple of two essential periods is again an essential period, a fact which is easily verified.

Definition 3.10. Assume that a Toeplitz sequence η is non-periodic. A *periodic structure* for η is a strictly increasing sequence $(p_i)_{i\in\mathbb{N}}$ such that p_i is an essential period of η for all i, $p_i | p_{i+1}$ and $\bigcup_{i=1}^{\infty} \mathrm{Per}_{p_i}(\eta) = \mathbb{Z}$. (A periodic structure always exists for a (non-periodic) Toeplitz sequence, cf. [**W**].)

Definition 3.11. Let $\eta \in \Lambda^{\mathbb{Z}}$ be a Toeplitz sequence. The dynamical system $(\overline{\mathcal{O}(\eta)}, S)$ is called a *Toeplitz flow*, where $\mathcal{O}(\eta) = \mathrm{orbit}_S(\eta)$ and S is the shift map.

Periodic sequences are Toeplitz sequences. Every Toeplitz sequence is almost periodic, i.e. every word occurring in η appears with bounded gap between successive occurrences. Hence, $(\overline{\mathcal{O}(\eta)}, S)$ is minimal. In what follows, we will only consider non-periodic Toeplitz flows, and so Toeplitz flows are expansive Cantor minimal systems.

Let $(p_i)_{i\in\mathbb{N}}$ be a periodic structure for the (non-periodic) Toeplitz sequence η, and denote the associated Toeplitz flow by (Y, S). Let $(G_{\mathfrak{a}}, \rho_{\widehat{1}})$ denote the *odometer* (also called *adding machine*) associated to the $\mathfrak{a}$-adic group

$$G_{\mathfrak{a}} = \prod_{i=1}^{\infty} \left\{0, 1, \ldots, \frac{p_i}{p_{i-1}} - 1\right\},$$

where $\mathfrak{a} = \{p_i/p_{i-1}\}_{i\in\mathbb{N}}$ (we set $p_0 = 1$) and where $\rho_{\widehat{1}}(x) = x + \widehat{1}$, where $\widehat{1} = (1, 0, 0, \ldots)$. We note that $G_{\mathfrak{a}}$ is naturally isomorphic to the inverse limit group

$$\mathbb{Z}/p_1\mathbb{Z} \xleftarrow{\phi_1} \mathbb{Z}/p_2\mathbb{Z} \xleftarrow{\phi_2} \mathbb{Z}/p_3\mathbb{Z} \xleftarrow{\phi_3} \cdots,$$

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where $\phi_i(n)$ is the residue of n modulo p_i. It is a fact that the family consisting of compact groups G that are both monothetic (i.e. contains a dense copy of $\mathbb{Z}$, which of course implies that G is abelian) and Cantor (as a topological space) coincides with the family of $\mathfrak{a}$-adic groups. It is also noteworthy that all minimal rotations (in particular rotations by $\widehat{1}$) on such groups are conjugate. This is a consequence of the fact that the dual group of an $\mathfrak{a}$-adic group is a torsion group. If $\mathfrak{a} = \{p\}_{i\in\mathbb{N}}$, where p is a prime, then $G_\mathfrak{a}$ is the p-adic integers. (We refer to [**HR**, Vol 1] for background information on $\mathfrak{a}$-adic groups.)

Recall that every (minimal) dynamical system has a *maximal equicontinuous factor*, the latter being, by a well-known theorem, conjugate to a minimal rotation on a compact abelian group G, and so, in particular, is uniquely ergodic. Specifically, let (G, ρ_g) be the maximal equicontinuous factor of (X, T), where $\rho_g : G \to G$ is rotation by g, i.e. $\rho_g(x) = x + g$ for $x \in G$, and $\Phi : X \to G$ is the factor map. Let (H, ρ_h) be another minimal group rotation factor of (X, T), where $\Psi : X \to H$ is the factor map. Then (H, ρ_h) is a factor of (G, ρ_g) by a factor map $\pi : G \to H$ such that $\Psi = \pi \circ \Phi$.

One can detect the maximal equicontinuous factor (G, ρ_g) of (X, T) by determining the (continuous) eigenvalues Γ of T, i.e.

$$\Gamma = \{\lambda \in \mathbb{T} \mid f \circ T = \lambda f \text{ for some } 0 \neq f \in \mathcal{C}(X)\},$$

where $\mathbb{T}$ is the unit circle. Now Γ is a countable and discrete subgroup of the discrete circle group $\mathbb{T}_d$, and $G = \widehat{\Gamma}$ is the compact dual group with $g \in G$ being the character on Γ defined by $g(\gamma) = \gamma$ for all $\gamma \in \Gamma$. Incidentally, the dual group $\widehat{G_\mathfrak{a}}$ of the odometer group $G_\mathfrak{a}$ has only torsion elements and is equal to the following subgroup of $\mathbb{T}_d$, namely

$$\widehat{G_\mathfrak{a}} = \{e^{(2\pi i l)/(a_1 a_2 \cdots a_k)} \mid \mathfrak{a} = \{a_i\}_{i\in\mathbb{N}}, k \in \mathbb{N}, l \in \mathbb{Z}\}.$$

If (X, T) is a Toeplitz flow with a periodic structure $(p_i)_{i\in\mathbb{N}}$, then the maximal equicontinuous factor is the odometer system associated to $\mathfrak{a} = (p_i/p_{i-1})_{i\in\mathbb{N}}$. Furthermore, the factor map $\pi : X \to G_\mathfrak{a}$ is an almost one-to-one map. The converse is also true. We state two theorems that establish this fact.

THEOREM 3.12. [**P**] *Let (X, T) be a minimal almost one-to-one extension of a (minimal) group rotation system (G, ρ_g). Then (G, ρ_g) is the maximal equicontinuous factor of (X, T).*

THEOREM 3.13. [**MP**] *An expansive Cantor minimal system (X, T) is a Toeplitz flow if and only if (X, T) is an almost one-to-one extension of an odometer system $(G_\mathfrak{a}, \rho_{\widehat{1}})$. Furthermore, $(G_\mathfrak{a}, \rho_{\widehat{1}})$ is the maximal equicontinuous factor of (X, T).*

A more detailed analysis of the situation is found in [**W**]. We summarize by listing some salient points from [**W**] which are relevant for this paper.

Let η be a (non-periodic) Toeplitz sequence and let $(\overline{\mathcal{O}(\eta)}, S)$ be the associated Toeplitz flow. Let $(p_i)_{i\in\mathbb{N}}$ be a periodic structure for η. Define $A_n^i = \{S^m\eta \mid m \equiv n \pmod{p_i}\}$.

(1) $\overline{A_n^i}$ is the set of all $\omega \in \overline{\mathcal{O}(\eta)}$ with the same p_i-skeleton as $S^n(\eta)$.

(2) $\{\overline{A_n^i} \mid n \in \mathbb{Z}/p_i\mathbb{Z}\}$ is a partition of $\overline{\mathcal{O}(\eta)}$ into clopen sets.

(3) $\overline{A_n^i} \supset \overline{A_m^j}$ for $i < j$ and $m \equiv n \pmod{p_i}$.

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(4) $S\overline{A_n^i} = \overline{A_{n+1}^i}$.

(5) $\omega, \theta \in \bigcap_{i=1}^{\infty} \overline{A_{n_i}}$ (where $n_i \equiv n_j \pmod{p_i}$ for $j \geq i$) if and only if ω and θ have the same p_i-skeleton for all $i \in \mathbb{N}$. In particular, $\bigcap_{i=1}^{\infty} \overline{A_{n_i}} = \{\omega\}$ if and only if ω is a Toeplitz sequence. This implies that $\pi^{-1}(\pi(\omega)) = \{\omega\}$ if and only if ω is a Toeplitz sequence. Here $\pi : \overline{\mathcal{O}(\eta)} \to G_{\mathfrak{a}}$ is the factor map, $(G_{\mathfrak{a}}, \rho_{\widehat{1}})$ being the maximal equicontinuous factor of $(\overline{\mathcal{O}(\eta)}, S)$.

Since $\mathrm{Per}_{p_i}(\eta)$ is periodic, it has a density in $\mathbb{Z}$ given by

$$d_i = \frac{1}{p_i} |\{n \in \mathbb{Z}/p_i\mathbb{Z} \mid n \in \mathrm{Per}_{p_i}(\eta)\}|,$$

where $|E|$ denotes the number of elements in the set E. The d_i are increasing, and we set $d = \lim_{i \to \infty} d_i$.

Definition 3.14. The Toeplitz sequence η is *regular* if $d = 1$.

Notation. Let (X, T) be a dynamical system. We denote by $M(X, T)$ the set of T-invariant probability measures on X. We say that (X, T) is *uniquely ergodic* if $M(X, T)$ is a singleton.

Fact. If (X, T) is a minimal dynamical system, then $M(X, T)$ is a Choquet simplex (where $M(X, T)$ is given the w^*-topology).

THEOREM 3.15. [**JK**] *If $(\overline{\mathcal{O}(\eta)}, S)$ is a regular Toeplitz flow (i.e. η is a regular Toeplitz sequence), then it is uniquely ergodic and the (topological) entropy $h(S)$ is zero.*

We will return to Toeplitz flows and their properties after we have introduced the (ordered) K-theory associated to such systems.

3.3. *Bratteli diagrams and dimension groups.* (As general references for the material in this section, we refer to [**E**, **HPS**, **GPS1**].)

3.3.1. *Bratteli diagrams.* A Bratteli diagram (V, E) consists of a set of vertices $V = \bigsqcup_{n=0}^{\infty} V_n$ and a set of edges $E = \bigsqcup_{n=1}^{\infty} E_n$, where the V_n and the E_n are finite disjoint sets and where $V_0 = \{v_0\}$ is a one-point set. The edges in E_n connect vertices in V_{n-1} with vertices in V_n. If e connects $v \in V_{n-1}$ with $u \in V_n$, we write $s(e) = v$ and $r(e) = u$, where $s : E_n \to V_{n-1}$ and $r : E_n \to V_n$ are the source and range maps, respectively. We will assume that $s^{-1}(v) \neq \varnothing$ for all $v \in V$ and that $r^{-1}(v) \neq \varnothing$ for all $v \in V \setminus V_0$. A Bratteli diagram can be given a diagrammatic presentation with V_n the vertices at level n and E_n the edges between V_{n-1} and V_n. If $|V_{n-1}| = t_{n-1}$ and $|V_n| = t_n$, then the edge set E_n is described by a $t_n \times t_{n-1}$ incidence matrix $M_n = (m_{ij}^n)$, where m_{ij}^n is the number of edges connecting $v_i^n \in V_n$ with $v_j^{n-1} \in V_{n-1}$ (see Figure 1). Let $k, l \in \mathbb{Z}^+$ with $k < l$ and let $E_{k+1} \circ E_k \circ \cdots \circ E_l$ denote all the paths from V_k to V_l. Specifically, $E_{k+1} \circ E_k \circ \cdots \circ E_l = \{(e_{k+1}, \ldots, e_l) \mid e_i \in E_i, i = k+1, \ldots, l; r(e_i) = s(e_{i+1}), i = k+1, \ldots, l-1\}$. We define $r((e_{k+1}, \ldots, e_l)) = r(e_l)$ and $s((e_{k+1}, \ldots, e_l)) = s(e_{k+1})$. Notice that the corresponding incidence matrix is the product $M_l M_{l-1} \cdots M_{k+1}$ of the incidence matrices.

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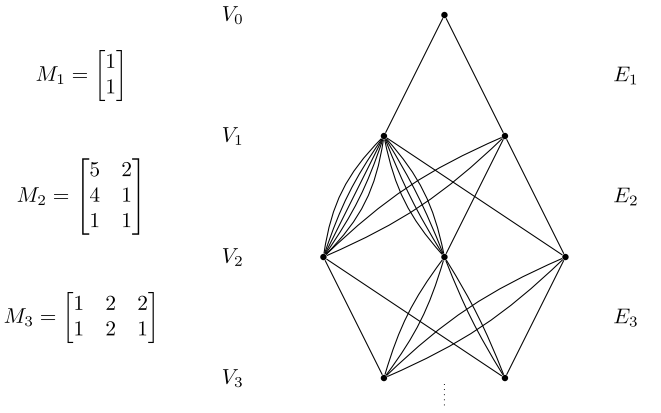


FIGURE 1. An example of a Bratteli diagram.

Definition 3.16. The Bratteli diagram (V, E) with incidence matrices $(M_n)_{n=1}^{\infty}$ has the ERS property (ERS = equal row sum) if the row sums of the incidence matrices are constant. Let the constant row sum of M_n be r_n. We associate the supernatural number $\prod_{n=1}^{\infty} r_n$ to (V, E). (See the comments after Definition 3.23.)

Definition 3.17. Given a Bratteli diagram (V, E) and a sequence $0 = m_0 < m_1 < m_2 < \cdots$ in $\mathbb{Z}^+$, we define the *telescoping* of (V, E) to $\{m_n\}$ as (V', E'), where $V'_n = V_{m_n}$ and $E'_n = E_{m_{n-1}+1} \circ \cdots \circ E_{m_n}$, and the source and the range maps are as above.

Remark. Observe that the ERS property is preserved under telescoping, and that is also the case for the associated supernatural number.

Definition 3.18. We say that the Bratteli diagram (V, E) is *simple* if there exists a telescoping of (V, E) such that the resulting Bratteli diagram (V', E') has full connection between all consecutive levels, i.e. the entries of all the incidence matrices are non-zero.

Given a Bratteli diagram (V, E), we define the infinite path space associated to (V, E), namely

$$X_{(V,E)} = \{(e_1, e_2, \ldots) \mid e_i \in E_i, r(e_i) = s(e_{i+1}) \text{ for all } i \geq 1\}.$$

Clearly, $X_{(V,E)} \subseteq \prod_{n=1}^{\infty} E_n$, and we give $X_{(V,E)}$ the relative topology, $\prod_{n=1}^{\infty} E_n$ having the product topology. Loosely speaking, this means that two paths in $X_{(V,E)}$ are close if the initial parts of the two paths agree on a long initial stretch. Also, $X_{(V,E)}$ is a closed subset of $\prod_{n=1}^{\infty} E_n$, and is compact.

Let $p = (e_1, e_2, \ldots, e_n) \in E_1 \circ \cdots \circ E_n$ be a finite path starting at $v_0 \in V_0$. We define the *cylinder set* $U(p) = \{(f_1, f_2, \ldots) \in X_{(V,E)} \mid f_i = e_i, i = 1, 2, \ldots, n\}$. The collection of cylinder sets is a basis for the topology on $X_{(V,E)}$. The cylinder sets are clopen sets, and so $X_{(V,E)}$ is a compact, totally disconnected metric space—metric because the collection of cylinder sets is countable. If (V, E) is simple, then $X_{(V,E)}$ has no isolated points, and so $X_{(V,E)}$ is a Cantor set. (We will in the following disregard the trivial case where $|X_{(V,E)}|$ is finite.)

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Let $P_n = E_1 \circ \cdots \circ E_n$ be the set of finite paths of length n (starting at the top vertex). We define the truncation map $\tau_n : X_{(V,E)} \to P_n$ by $\tau_n((e_1, e_2, \ldots)) = (e_1, e_2, \ldots, e_n)$. If $m \geq n$, we have the obvious truncation map $\tau_{m,n} : P_m \to P_n$.

There is an obvious notion of isomorphism between Bratteli diagrams (V, E) and (V', E'); namely, there exists a pair of bijections between V and V' preserving the gradings and intertwining the respective source and range maps. Let $\sim$ denote the equivalence relation on Bratteli diagrams generated by isomorphism and telescoping. One can show that $(V, E) \sim (V', E')$ if and only if there exists a Bratteli diagram (W, F) such that telescoping (W, F) to odd levels $0 < 1 < 3 < \cdots$ yields a diagram isomorphic to some telescoping of (V, E), and telescoping (W, F) to even levels $0 < 2 < 4 < \cdots$ yields a diagram isomorphic to some telescoping of (V', E').

3.3.2. *Dimension groups.* By an ordered group we shall mean a countable abelian group G together with a subset G^+, called the positive cone, such that:

(1) $G^+ - G^+ = G$;
(2) $G^+ \cap (-G^+) = \{0\}$;
(3) $G^+ + G^+ \subset G^+$.

We shall write $a \leq b$ if $b - a \in G^+$. We say that an ordered group is *unperforated* if $a \in G$ and $na \in G^+$ for some $a \in G$ and $n \in \mathbb{N}$ implies that $a \in G^+$. Observe that an unperforated group is torsion free. By an *order unit* for (G, G^+) we mean an element $u \in G^+$ such that for every $a \in G$, $a \leq nu$ for some $n \in \mathbb{N}$.

Definition 3.19. A *dimension group* (G, G^+, u) with distinguished order unit u is an unperforated ordered group (G, G^+) satisfying the *Riesz interpolation property*, i.e. given $a_1, a_2, b_1, b_2 \in G$ with $a_i \leq b_j$ $(i, j = 1, 2)$, there exists $c \in G$ with $a_i \leq c \leq b_j$ $(i, j = 1, 2)$.

We write $(G_1, G_1^+, u_1) \cong (G_2, G_2^+, u_2)$ if there exists an order isomorphism $\phi : G_1 \to G_2$, i.e. ϕ is a group isomorphism such that $\phi(G_1^+) = G_2^+$, and $\phi(u_1) = u_2$.

To a Bratteli diagram (V, E) we can associate an ordered group, which we will denote by $K_0(V, E)$ (because of its connection to (ordered) K-theory). Let $V = \bigsqcup_{n=0}^{\infty} V_n$ and let $(M_n)_{n=1}^{\infty}$ be the incidence matrices. Then we have a system of simplicially ordered groups and positive maps

$$(\mathbb{Z} =)\ \mathbb{Z}^{|V_0|} \xrightarrow{M_1} \mathbb{Z}^{|V_1|} \xrightarrow{M_2} \mathbb{Z}^{|V_2|} \longrightarrow \cdots,$$

where the positive homomorphism $M_n : \mathbb{Z}^{|V_{n-1}|} \to \mathbb{Z}^{|V_n|}$ is given by matrix multiplication with the incidence matrix M_n. ($\mathbb{Z}^{|V_n|}$ is a column vector, and an element in $\mathbb{Z}^{|V_n|}$ is positive if all its entries are non-negative.) By definition, $K_0(V, E)$ is the inductive limit of the system above, and $K_0(V, E)$ is given the induced order. We denote the positive cone by $K_0(V, E)^+$. $K_0(V, E)$ has a distinguished *order unit*, namely the element $[\mathbf{1}]$ in $K_0(V, E)^+$ corresponding to the element $1 \in \mathbb{Z}^{|V_0|} = \mathbb{Z}$. The triple $(K_0(V, E), K_0(V, E)^+, [\mathbf{1}])$ denotes the countable, ordered abelian group $K_0(V, E)$ with positive cone $K_0(V, E)^+$ and distinguished order unit $[\mathbf{1}]$. We will sometimes in what follows for short only write $K_0(V, E)$, the ordering and the order unit being implicitly understood.

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One can show that $(V, E) \cong (V', E')$ if and only if $K_0(V, E)$ is order isomorphic to $K_0(V', E')$ by a map sending the distinguished order unit of $K_0(V, E)$ to the distinguished order unit of $K_0(V', E')$.

It is straightforward to verify that $(K_0(V, E), K_0(V, E)^+)$ is a dimension group. The converse, however, is not obvious and we state it as a theorem formulated in such a way that it suits our purpose.

THEOREM 3.20. [**EHS**] *Let (G, G^+, u) be a dimension group with distinguished order unit u. Then there exists a Bratteli diagram (V, E) such that $(G, G^+, u) \cong (K_0(V, E), K_0(V, E)^+, [\mathbf{1}])$.*

A dimension group (G, G^+) is *simple* if it contains no non-trivial order ideals. An order ideal is a subgroup J such that $J = J^+ - J^+$ (where $J^+ = J \cap G^+$) and $0 \le a \le b \in J$ implies that $a \in J$. It is easily seen that (G, G^+) is a simple dimension group if and only if every $a \in G^+ \backslash \{0\}$ is an order unit. In what follows, we will exclusively work with non-cyclic (i.e. $G \ncong \mathbb{Z}$) simple dimension groups (G, G^+).

Let (G, G^+, u) be a simple dimension group with distinguished order unit u. We say that a homomorphism $p : G \to \mathbb{R}$ is a *state* if p is positive (i.e. $p(G^+) \ge 0$) and $p(u) = 1$. Denote the collection of all states on (G, G^+, u) by $S_u(G)$. Now $S_u(G)$ is a convex compact subset of the locally convex space $\mathbb{R}^G$ with the product topology. In fact, one can show that $S_u(G)$ is a Choquet simplex. It is a fact that $S_u(G)$ determines the order on G. In fact,

$$G^+ = \{a \in G \mid p(a) > 0 \text{ for all } p \in S_u(G)\} \cup \{0\}.$$

Definition 3.21. Let (G, G^+) be a simple dimension group and let $u \in G^+ \backslash \{0\}$. We say that $a \in G$ is *infinitesimal* if $-\epsilon u \le a \le \epsilon u$ for all $0 < \epsilon \in \mathbb{Q}^+$. (If $\epsilon = p/q$, $p, q \in \mathbb{N}$, then $a \le \epsilon u$ means that $qa \le pu$. It is easy to see that the definition does not depend upon the particular order unit u.) An equivalent definition is: $a \in G$ is infinitesimal if $p(a) = 0$ for all $p \in S_u(G)$. The collection of infinitesimal elements of G forms a subgroup, *the infinitesimal subgroup of* G, which we denote by $\mathrm{Inf}(G)$.

Remark. The quotient group $G/\mathrm{Inf}(G)$ is again a simple dimension group in the induced order, and the infinitesimal subgroup of $G/\mathrm{Inf}(G)$ is trivial. Also, an order unit for G maps to an order unit for $G/\mathrm{Inf}(G)$.

The following theorem summarizes some facts that are relevant for our situation; the proof can be found in [**E**].

THEOREM 3.22. *Let (G, G^+, u) be a simple (non-cyclic) dimension group with distinguished order unit u. The map $\theta : G \to \mathrm{Aff}(S_u(G))$ from G to the additive group of continuous affine functions on the Choquet simplex $S_u(G)$ defined by $\theta(g)(p) = p(g)$ is a strict order preserving map (i.e. $\theta(g)(p) > 0$ for all $p \in S_u(G)$ implies that $g > 0$). Furthermore,* $\mathrm{Im}(\theta)$ *is dense in* $\mathrm{Aff}(S_u(G))$ *in the uniform norm and contains the constant function* 1, *and* $\ker(\theta) = \mathrm{Inf}(G)$.

Conversely, suppose that K is a Choquet simplex and H is a countable dense subgroup of $\mathrm{Aff}(K)$, *and that $\theta : G \to H$ is a homomorphism of a torsion-free countable abelian*

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group G onto H. Then, letting

$$G^+ = \{g \in G \mid \theta(g)(p) > 0 \text{ all } p \in K\} \cup \{0\},$$

we get that (G, G^+) is a simple dimension group such that $\operatorname{Inf} G = \ker(\theta)$. *In particular, if $G = H$ (with θ the identity map) and G contains the constant function* 1, *then* $\operatorname{Inf}(G) = \{0\}$ *and $S_1(G)$ is affinely homeomorphic to K by the map sending $k \in K$ to $\widehat{k} : G \to \mathbb{R}$, where $\widehat{k}(g) = g(k)$, $g \in G$.*

Definition 3.23. By a *rational group H* we shall mean an (additive) subgroup of $\mathbb{Q}$ that contains $\mathbb{Z}$. We say that H is a *non-cyclic* rational group if H is not isomorphic to $\mathbb{Z}$. Clearly, $(H, H^+, 1)$ is a simple dimension group with distinguished order unit 1, where $H^+ = H \cap \mathbb{Q}^+$.

Since rational dimension groups are going to play such an important role in this paper, we will give a short description of them (cf. [**F**, Ch. XIII, §85]). First of all, they are exactly the countable torsion-free groups of rank one. Let $n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots$ be a *supernatural number*, where $p_1, p_2, p_3, \ldots$ are the primes $2, 3, 5, \ldots$ listed in increasing order, and $0 \le k_i \le \infty$ for each i. (If $(r_n)_{n=1}^{\infty}$ is a sequence of natural numbers, then we get a supernatural number $\prod_{n=1}^{\infty} r_n$ in an obvious way by factoring each r_n into a product of primes.) Clearly, $n \in \mathbb{N}$ if and only if $k_i < \infty$ for all i, and $k_i = 0$ for all but finitely many i. If $m = p_1^{l_1} p_2^{l_2} p_3^{l_3} \cdots$ is another supernatural number, we multiply n and m as $nm = p_1^{k_1+l_1} p_2^{k_2+l_2} p_3^{k_3+l_3} \cdots$, where $k_i + l_i = \infty$ if either k_i or l_i is equal to ∞. We say that m divides n (notation $m|n$) if $l_i \le k_i$ for all i. For $n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots$, let

$$G(n) = \left\{ \frac{a}{b} \;\middle|\; a \in \mathbb{Z}, b \in \mathbb{N}, b|n \right\}.$$

Then $G(n)$ is a rational group and all rational groups are of this form. Furthermore, $G(n)$ is isomorphic to $G(m)$ if and only if there exist $a, b \in \mathbb{N}$ such that $an = bm$. In particular, all groups $G(n)$, where $n \in \mathbb{N}$, are isomorphic to $G(1) = \mathbb{Z}$. We note that $G(n)$ is p-divisible (i.e. for every $a \in G$, there exists $x \in E$ such that $px = a$) for some prime p if and only if p occurs with infinite multiplicity in the factorization of n. The group $G(n)$ can be made into a dimension group in exactly two ways, namely by letting the positive cone $G(n)^+$ be $G(n) \cap \mathbb{Q}^+$ or $-G(n) \cap \mathbb{Q}^+$, respectively. If $\mathfrak{a} = (a_1, a_2, \ldots)$ is a sequence of natural numbers with $a_i \ge 2$ for all i, we associate the non-cyclic rational group $\{m/(a_1 a_2 \cdots a_k) | m \in \mathbb{Z}, k \in \mathbb{N}\}$. Clearly, this is the same as the group $G(n)$, where n is the supernatural number $\prod_{i=1}^{\infty} a_i$. (Here the factorization of $\prod_{i=1}^{\infty} a_i$ into products of primes is obviously understood.)

Definition 3.24. Let (G, G^+, u) be a simple dimension group with order unit u. We define the rational subgroup of G, denoted $\mathbb{Q}(G, G^+, u)$ (or $\mathbb{Q}(G, u)$ for short), to be

$$\mathbb{Q}(G, u) = \{g \in G \mid ng = mu \text{ for some } n \in \mathbb{N}, m \in \mathbb{Z}\}.$$

PROPOSITION 3.25. *Let $\mathbb{Q}(G, u)$ be as in Definition 3.24. The map $\Gamma : \mathbb{Q}(G, u) \to \mathbb{Q}$ defined by $\Gamma(g) = m/n$ if $ng = mu$ is an injective order homomorphism sending u to $1 \in \mathbb{Z}$, where $\mathbb{Q}(G, u)^+ = \mathbb{Q}(G, u) \cap G^+$. Thus, $\mathbb{Q}(G, u)$ is order isomorphic to a rational group. Furthermore, $G/\mathbb{Q}(G, u)$, as an abstract group, is torsion free.*

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Proof. The map Γ is well defined. In fact, if $ng = mu$ and $n'g = m'u$, then we get by multiplying with n' and n, respectively, $nn'g = n'mu$ and $nn'g = nm'u$. Subtracting, we get $(n'm - nm')u = 0$. Since G is torsion free, we get $n'm - nm' = 0$, and so $m/n = m'/n'$. Similarly, we show that Γ is a group homomorphism sending u to 1. If $\Gamma(g) = 0$, then $ng = 0$ for some $n \in \mathbb{N}$, and so $g = 0$ by torsion freeness of G. So, Γ is injective. It is straightforward to show that $\Gamma(g) \geq 0 \Leftrightarrow g \geq 0$, and so Γ is an order isomorphism onto its image. Hence, $\mathbb{Q}(G, u)$ is order isomorphic to a rational group.

To show that $G/\mathbb{Q}(G, u)$ is torsion free, let $k\overline{g} = 0$ for some $k \in \mathbb{N}$, where $\overline{g}$ is the image of $g \in G$ under the quotient map. This implies that $kg \in \mathbb{Q}(G, u)$, and so there exist $n \in \mathbb{N}$, $m \in \mathbb{Z}$ such that $nkg = mu$. Hence, $g \in \mathbb{Q}(G, u)$, and so $\overline{g} = 0$, thus proving that $G/\mathbb{Q}(G, u)$ is torsion free. □

Remark. Let $(G_1, G_1^+, u_1) \cong (G_2, G_2^+, u_2)$ by a map $\phi : G_1 \to G_2$. Then it is easily seen that $\phi(\mathbb{Q}(G_1, u_1)) = \mathbb{Q}(G_2, u_2)$. So, loosely speaking, we have that isomorphic dimension groups with distinguished order units have the same rational subgroups.

The notion of a rational subgroup of a dimension group with distinguished order unit depends heavily upon the choice of the order unit, as the following example shows.

Example 3.26. Let H be a non-cyclic subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$. Let $G = H \oplus \mathbb{Z}$ with $G^+ = \{(h, k) \mid h > 0, k \in \mathbb{Z}\} \cup \{(0, 0)\}$. Then (G, G^+) is a simple dimension group with $\mathrm{Inf}(G) = 0 \oplus \mathbb{Z}$. If we choose the order unit $u = (1, 0)$ for G, then one shows easily that $\mathbb{Q}(G, u) = H \oplus 0 \cong H$. However, if we choose the order unit $\widetilde{u} = (1, 1)$ for G, then $\mathbb{Q}(G, \widetilde{u}) = \{(k, k) \mid k \in \mathbb{Z}\} \cong \mathbb{Z}$.

3.3.3. *Ordered Bratteli diagram and the Bratteli–Vershik model.* An *ordered Bratteli diagram* $(V, E, \geq)$ is a Bratteli diagram (V, E) together with a partial order $\geq$ in E so that edges $e, e' \in E$ are comparable if and only if $r(e) = r(e')$. In other words, we have a linear order on each set $r^{-1}(v)$, $v \in V \setminus V_0$. We let $E_{\min}$ and $E_{\max}$, respectively, denote the minimal and maximal edges of the partially ordered set E.

Note that if $(V, E, \geq)$ is an ordered Bratteli diagram and $k < l$ in $\mathbb{Z}^+$, then the set $E_{k+1} \circ E_{k+2} \circ \cdots \circ E_l$ of paths from V_k to V_l with the same range can be given an induced (lexicographic) order as follows:

$$(e_{k+1} \circ e_{k+2} \circ \cdots \circ e_l) > (f_{k+1} \circ f_{k+2} \circ \cdots \circ f_l)$$

if, for some i with $k + 1 \leq i \leq l$, $e_j = f_j$ for $i < j \leq l$ and $e_i > f_i$. If (V', E') is a telescoping of (V, E), then, with this induced order from $(V, E, \geq)$, we get again an ordered Bratteli diagram $(V', E', \geq)$.

Definition 3.27. We say that the ordered Bratteli diagram $(V, E, \geq)$, where (V, E) is a simple Bratteli diagram, is *properly ordered* if there exist a unique min path $x_{\min} = (e_1, e_2, \ldots)$ and a unique max path $x_{\max} = (f_1, f_2, \ldots)$ in $X_{(V,E)}$. (That is, $e_i \in E_{\min}$ and $f_i \in E_{\max}$ for all $i = 1, 2, \ldots$.)

Let (V, E) be a properly ordered Bratteli diagram, and let $X_{(V,E)}$ be the path space associated to (V, E). Then $X_{(V,E)}$ is a Cantor set. Let $T_{(V,E)}$ be the *lexicographic map* on $X_{(V,E)}$, i.e. if $x = (e_1, e_2, \ldots) \in X_{(V,E)}$ and $x \neq x_{\max}$, then $T_{(V,E)}x$ is the successor of x

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in the lexicographic ordering. Specifically, let k be the smallest natural number so that $e_k \notin E_{\max}$. Let f_k be the successor of e_k (and so $r(e_k) = r(f_k)$). Let $(f_1, f_2, \ldots, f_{k-1})$ be the unique least element in $E_1 \circ E_2 \circ \cdots \circ E_{k-1}$ from $s(f_k) \in V_{k-1}$ to the top vertex $v_0 \in V_0$. Then $T_{(V,E)}((e_1, e_2, \ldots)) = (f_1, f_2, \ldots, f_k, e_{k+1}, e_{k+2}, \ldots)$. We define $T_{(V,E)}x_{\max} = x_{\min}$. Then it is easy to check that $T_{(V,E)}$ is a minimal homeomorphism on $X_{(V,E)}$. We note that if $x \neq x_{\max}$, then x and $T_{(V,E)}x$ are *cofinal*, i.e. the edges making up x and $T_{(V,E)}x$, respectively, agree from a certain level on. We will call the Cantor minimal system $(X_{(V,E)}, T_{(V,E)})$ a *Bratteli–Vershik system*. There is an obvious way to telescope a properly ordered Bratteli diagram, getting another properly ordered Bratteli diagram, such that the associated Bratteli–Vershik systems are conjugate—the map implementing the conjugacy is the obvious one. By telescoping, we may assume without loss of generality that the properly ordered Bratteli diagram has the property that at each level all the minimal edges (respectively the maximal edges) have the same source, cf. [**HPS**, Proposition 2.8].

THEOREM 3.28. [**HPS**] *Let (X, T) be a Cantor minimal system. Then there exists a properly ordered Bratteli diagram $(V, E, \geq)$ such that the associated Bratteli–Vershik system $(X_{(V,E)}, T_{(V,E)})$ is conjugate to (X, T).*

Proof sketch. Let $x_0 \in X$ and let $\{U_n\}_{n\in\mathbb{Z}_+}$ be a decreasing sequence of clopen sets of X such that $U_0 = X$ and $U_n \searrow \{x_0\}$. For each U_n, we construct a finite number of towers 'built' over U_n. These are determined by the map $\lambda_n : U_n \to \mathbb{N}$, $\lambda_n(y) = \inf\{m \in \mathbb{N} \mid T^m y \in U_n\}$. If $\lambda_n(U_n) = \{m_1, m_2, \ldots, m_{k_n}\}$, then we get at first k_n towers of height $m_1, m_2, \ldots, m_{k_n}$, respectively. These may be vertically subdivided, giving rise to more towers (some of them of the same height), such that we obtain the following scenario: the clopen partitions $\{\mathcal{P}_n\}_{n\in\mathbb{Z}^+}$ of X that the towers associated to the various U_n generate are nested, $\mathcal{P}_0 \prec \mathcal{P}_1 \prec \mathcal{P}_2 \prec \cdots$, and the union of the $\mathcal{P}_n$ is a basis for the topology of X. We build the properly ordered Bratteli diagram $(V, E, \geq)$ by letting the vertices V_n at level n correspond to the various towers built over U_n. The ordering of the edges between levels $n-1$ and n is determined by the order in which the towers at level n traverse the towers at level $n-1$.

Remark. The simplest Bratteli–Vershik model $(V, E, \geq)$ for the odometer $(G_{\mathfrak{a}}, T)$ associated to $\mathfrak{a} = (a_i)_{i\in\mathbb{N}}$ is obtained by letting $V_n = 1$ for all n, and the number of edges between V_{n-1} and V_n be a_n.

Definition 3.29. Let (X, T) be a Cantor minimal system. Define the (additive) *coboundary map* $\partial_T : \mathcal{C}(X, \mathbb{Z}) \to \mathcal{C}(X, \mathbb{Z})$ by $\partial_T f = f - f \circ T^{-1}$. Define

$$\mathrm{K}^0(X, T) = \mathcal{C}(X, \mathbb{Z})/\partial_T(\mathcal{C}(X, \mathbb{Z}))$$

and give $\mathrm{K}^0(X, T)$ the induced order, i.e.

$$\mathrm{K}^0(X, T)^+ = \{[f] \in \mathrm{K}^0(X, T) \mid [f] = [g] \text{ for some } g \geq 0\},$$

where $[f]$ denotes the class of $f \in \mathcal{C}(X, \mathbb{Z})$. Let $\mathbb{1}(= [1])$ denote the distinguished order unit corresponding to the constant function 1 on X. We will in what follows sometimes for short only write $K^0(X, T)$, the ordering and the order unit being implicitly understood.

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Remark. Let $(G_{\mathfrak{a}}, \rho_{\widehat{\mathbb{1}}})$ be the odometer associated to $\mathfrak{a} = (a_i)_{i\in\mathbb{N}}$. Then $K^0(G_{\mathfrak{a}}, \rho_{\widehat{\mathbb{1}}})$ is order isomorphic to the rational group associated to $\mathfrak{a}$, namely

$$\left\{ \frac{m}{a_1 a_2 \cdots a_n} \;\middle|\; m \in \mathbb{Z},\, n \in \mathbb{N} \right\},$$

and so, in particular, $\mathbb{Q}(K^0(X, T), \mathbb{1}) \cong \mathbb{Q}(K_0(V, E), [\mathbf{1}])$ by a map preserving the canonical order units.

The following result, which is implicit in [**GPS1**, §2], is highly relevant for this paper and we present a proof due to B. Host (cf. [**O**, Theorem 2.2]).

PROPOSITION 3.30. *Let (X, T) be a Cantor minimal system and let p be a natural number greater than one. The following are equivalent:*

(i) $1/p \in \mathbb{Q}(K^0(X, T), \mathbb{1})$;

(ii) $(2\pi i)/p$ *is a continuous eigenvalue for T, i.e. there exists $f \in \mathcal{C}(X)$, $f \neq 0$, such that* $f \circ T = e^{(2\pi i)/p} f$.

Proof. (i) $\Rightarrow$ (ii): For any set E, let 1_E denote the characteristic function of E. By condition (i), there exist continuous functions $f, g : X \to \mathbb{Z}$ such that $pf - 1_X = g - g \circ T^{-1}$, where f is non-zero (and hence g is non-zero). Let $n \in \mathrm{Im}(g)$ and let $A = g^{-1}(\{n\})$. Then A is a non-empty clopen set. For $x \in A$, let $r(x)$ be the first return time of x to A, i.e. $r(x)$ is the smallest positive integer such that $T^{r(x)}x \in A$. Since T is minimal, the function $r : A \to \mathbb{Z}$ is uniformly bounded and $\bigcup_{x\in A} \bigcup_{j=1}^{r(x)} T^j x = X$. We get

$$\begin{aligned} p\left(\sum_{j=1}^{r(x)} f(T^j x)\right) - r(x) &= \sum_{j=1}^{r(x)} (pf(T^j x) - 1_X(T^j x)) = \sum_{j=1}^{r(x)} (g(T^j x) - g(T^{j-1}x)) \\ &= g(T^{r(x)}x) - g(x). \end{aligned}$$

If $x \in A$, the right-hand side is zero, and so the left-hand side yields that $r(x)$ must be a (positive) multiple of p. Let $\widetilde{A} = \bigcup_{l\geq 0} T^{lp}(A)$, and so, in particular, A is an open set. Then for all $x \in A$ the first return time to $\widetilde{A}$ is p. Then $X = \bigsqcup_{j=0}^{p-1} T^j(\widetilde{A})$ (disjoint union), and so, in particular, $\widetilde{A}$, $T(\widetilde{A})$, $\ldots$, $T^{p-1}(\widetilde{A})$ are clopen sets. Define $f \in \mathcal{C}(X)$ by

$$f(x) = e^{(j2\pi i)/p} \quad \text{if } x \in T^j(\widetilde{A}).$$

Then one easily sees that $f \circ T = e^{(2\pi i)/p} f$, and so $(2\pi i)/p$ is a continuous eigenvalue for T.

(ii) $\Rightarrow$ (i): Let $(2\pi i)/p$ be a continuous eigenvalue for T, and let $f \in \mathcal{C}(X)$ be a non-zero eigenfunction, i.e. $f \circ T = e^{(2\pi i)/p} f$. We may assume that $|f(x)| = 1$ for all $x \in X$ and that $A = f^{-1}(\{1\})$ is non-empty. Now, clearly, A, $T(A)$, $\ldots$, $T^{p-1}(A)$ are disjoint closed sets, and $T\big(\bigcup_{j=0}^{p-1} T^j(A)\big) = \bigcup_{j=0}^{p-1} T^j(A)$, and so, by minimality of T, we have $\bigcup_{j=0}^{p-1} T^j(A) = X$. Hence, A is clopen, and so $1_A \in \mathcal{C}(X)$. Now

$$\begin{aligned} p1_A - 1_X &= (1_A - 1_A) + (1_A - 1_{T(A)}) + \cdots + (1_A - 1_{T^{p-1}(A)}) \\ &= 0 + (1_A - 1_A \circ T^{-1}) + \cdots + (1_A - 1_A \circ T^{-(p-1)}). \end{aligned}$$

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Also,

$$1_A - 1_A \circ T^{-j} = (1_A - 1_A \circ T^{-1}) + (1_A \circ T^{-1} - (1_A \circ T^{-1}) \circ T^{-1}) + \cdots + (1_A \circ T^{-(j-1)} - (1_A \circ T^{-(j-1)}) \circ T^{-1})$$

is a coboundary, and so $p1_A - 1_X$ is a coboundary. This shows that $1/p \in \mathbb{Q}(K^0(X, T), \mathbb{1})$. □

Combining Proposition 3.30 with Theorems 3.12 and 3.13 (and the remarks just preceding these theorems), we get the following result.

PROPOSITION 3.31. *Let (X, T) be a Cantor minimal system. Then $\mathbb{Q}(K^0(X, T), \mathbb{1})$ completely determines the maximal equicontinuous* Cantor *factor of (X, T). Specifically, the maximal equicontinuous Cantor factor is the odometer $(G_\mathfrak{a}, \rho_{\widehat{\mathbb{1}}})$ associated to the $\mathfrak{a}$-adic number $\mathfrak{a} = (a_1, a_2, \ldots)$, where $\mathbb{Q}(K^0(X, T), \mathbb{1}) \cong K^0(G_\mathfrak{a}, \rho_{\widehat{\mathbb{1}}})$. (The latter group is described in the Remark preceding Proposition 3.30.)*

The following theorem is a fairly straightforward corollary of Theorem 3.28.

THEOREM 3.32. **[HPS]** *Let (X, T) be a Cantor minimal system. Then $(\mathrm{K}^0(X, T), \mathrm{K}^0(X, T)^+)$ is a simple dimension group, and all (non-cyclic) simple dimension groups (G, G^+, u) with distinguished order unit u are order isomorphic to $(\mathrm{K}^0(X, T), \mathrm{K}^0(X, T)^+, \mathbb{1})$ for some Cantor minimal system (X, T). In particular, if $(V, E, \geq)$ is a properly ordered Bratteli diagram such that (X, T) is conjugate to $(X_{(V,E)}, T_{(V,E)})$, then $K^0(X, T) \cong K_0(V, E)$ (as ordered groups with canonical order units), and so, in particular, $\mathbb{Q}(K^0(X, T), \mathbb{1}) \cong \mathbb{Q}(K_0(V, E), [\mathbf{1}])$. Furthermore, the state space $S_{\mathbb{1}}(K^0(X, T))$ may be identified in an obvious way with the Choquet simplex $M(X, T)$ of T-invariant probability measures; in fact, these two Choquet simplices are affinely homeomorphic.*

Remark. Changing the order unit corresponds dynamically to considering induced systems (cf. Definitions 3.4 and 3.5). In fact, let $(V, E, \geq)$ be a simply ordered Bratteli diagram, and let $(V', E', \geq)$ be the resulting simply ordered Bratteli diagram after we have made a finite change to $(V, E, \geq)$. (By a finite change, we mean adding and/or removing a finite number of edges and then making arbitrary choices of linear orderings of the edges meeting at the same vertex for a finite number of vertices.) Then $(X_{(V,E)}, T_{(V,E)})$ is Kakutani equivalent to $(X_{(V',E')}, T_{(V',E')})$, which can be seen as an immediate consequence of how the Vershik map is defined. Furthermore, if (X, T) is Cantor minimal with associated dimension group $K^0(X, T)$ with distinguished order unit, then, choosing a new order unit, say u, there exists a Cantor minimal system (Y, S) which is Kakutani equivalent to (X, T) such that $(K^0(Y, S), K^0(Y, S)^+, \mathbb{1}) \cong (K^0(X, T), K^0(X, T)^+, u)$. In fact, (Y, S) is obtained from (X, T) by making a finite change to the Bratteli–Vershik model for (X, T).

THEOREM 3.33. **[GPS1]** *The Cantor minimal systems (X, T) and (Y, S) are* strong orbit equivalent *if and only if $(K^0(X, T), K^0(X, T)^+, \mathbb{1}) \cong (K^0(Y, S), K^0(Y, S)^+, \mathbb{1})$.*

Remark. The idea behind introducing $K^0(X, T)$ with an ordering for a Cantor minimal system (X, T) comes from (non-commutative) operator algebra theory. In fact, one can

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show that $K^0(X, T)$ as defined above is abstractly isomorphic to the K_0-group of the associated C^*-crossed product $\mathcal{C}(X) \rtimes_T \mathbb{Z}$. The latter K_0-group comes with a natural order, which translates to the order we introduced on $K^0(X, T)$. It turns out that the ordered K_0-group of $\mathcal{C}(X) \rtimes_T \mathbb{Z}$, with its natural scaling corresponding to the unit element, is a complete isomorphism invariant, and so it follows that (X, T) is strong orbit equivalent to (Y, S) if and only if $\mathcal{C}(X) \rtimes_T \mathbb{Z}$ is $*$-isomorphic to $\mathcal{C}(Y) \rtimes_S \mathbb{Z}$ [**GPS1**].

THEOREM 3.34. [**GPS1**] *The Cantor minimal systems (X, T) and (Y, S) are* orbit equivalent *if and only if $(\widetilde{K^0(X, T)}, \widetilde{K^0(X, T)}^+, \widetilde{\mathbb{1}}) \cong (\widetilde{K^0(Y, S)}, \widetilde{K^0(Y, S)}^+, \widetilde{\mathbb{1}})$, where $\widetilde{K^0(X, T)} = K^0(X, T)/\mathrm{Inf}(K^0(X, T))$, $\widetilde{K^0(Y, S)} = K^0(Y, S)/\mathrm{Inf}(K^0(Y, S))$, and the positive cones and order units are induced by the quotient maps.*

Let $(V, E, \geq)$ be a properly ordered Bratteli diagram, and let $(X_{(V,E)}, T_{(V,E)})$ be the associated Bratteli–Vershik system. For each $k \in \mathbb{N}$, let P_k as above denote the paths from V_0 to V_k, i.e. the paths from $v_0 \in V_0$ to some $v \in V_k$. For $x \in X_{(V,E)}$, we associate the bi-infinite sequence $\pi_k(x) = (\tau_k(T^n_{(V,E)}x))_{n=-\infty}^{\infty} \in P_k^{\mathbb{Z}}$ over the finite alphabet P_k, where $\tau_k : X_{(V,E)} \to P_k$ is the truncation map. Let S_k denote the shift map on $P_k^{\mathbb{Z}}$. Then the following diagram commutes:

$$\begin{array}{ccc} X_{(V,E)} & \xrightarrow{T_{(V,E)}} & X_{(V,E)} \\ \Big\downarrow{\scriptstyle \pi_k} & & \Big\downarrow{\scriptstyle \pi_k} \\ X_k & \xrightarrow{S_k} & X_k \end{array}$$

where $X_k = \pi_k(X_{(V,E)})$. One observes that π_k is a continuous map, and so X_k is a compact shift-invariant subset of $P_k^{\mathbb{Z}}$. So, (X_k, S_k) is a factor of $(X_{(V,E)}, T_{(V,E)})$. For $k > l$, there is an obvious factor map $\pi_{k,l} : X_k \to X_l$, and one can show that $(X_{(V,E)}, T_{(V,E)})$ is the inverse limit of the system $\{(X_k, S_k)\}_{k\in\mathbb{N}}$. All the systems (X_k, S_k) are clearly expansive. One has the following result, which will be important for us. Even though the result is well known by people familiar with (ordered) Bratteli diagrams, no proof has been written down as far as we know. So, we will present a proof here.

PROPOSITION 3.35. *Assume that $(X_{(V,E)}, T_{(V,E)})$ is expansive. Then there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $(X_{(V,E)}, T_{(V,E)})$ is conjugate to (X_k, S_k) by the map $\pi_k : X_{(V,E)} \to X_k$.*

Proof. Since the π_k are factor maps, all we need to show is that there exists k_0 such that π_k is injective for all $k \geq k_0$. Recall that $(X_{(V,E)}, T_{(V,E)})$ being expansive means that there exists $\delta > 0$ such that given $x \neq y$ there exists $n_0 \in \mathbb{Z}$ such that $d(T^{n_0}_{(V,E)}x, T^{n_0}_{(V,E)}y) > \delta$, where d is some metric on $X_{(V,E)}$ compatible with the topology. Choose k_0 such that $d(x, y) < \delta$ if x and y agree (at least) on the k_0 first edges. Now assume that $\pi_k(x) = \pi_k(y)$ for some $k \geq k_0$. By the definition of π_k, this means that, for all $n \in \mathbb{Z}$, $\tau_k(T^n_{(V,E)}x) = \tau_k(T^n_{(V,E)}y)$, and so $d(T^n_{(V,E)}x, T^n_{(V,E)}y) < \delta$ for all $n \in \mathbb{Z}$ because of our choice of k_0. This contradicts that $d(T^{n_0}_{(V,E)}x, T^{n_0}_{(V,E)}y) > \delta$. Hence, π_k is injective for all $k \geq k_0$, proving the proposition. □

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3.4. *Order embeddings of dimension groups associated to factor maps.* We have seen that Toeplitz flows can be characterized by being (expansive) almost one-to-one Cantor minimal extensions of odometer systems (cf. Theorem 3.13). The following theorem will therefore be important for us, since it relates the K^0-groups of the extension and the factor, respectively, of Cantor minimal systems.

THEOREM 3.36. [**GW**, Proposition 3.1] *Let (X, T) and (Y, S) be Cantor minimal systems such that (X, T) is an extension of (Y, S) by the factor map $\pi : X \to Y$. Then $\pi^* : K^0(Y, S) \to K^0(X, T)$ defined by $\pi^*([h]) = [h \circ \pi]$ is an order embedding, i.e. $[h] \geq 0$ if and only if $\pi^*([h]) \geq 0$ for $h \in \mathcal{C}(Y, \mathbb{Z})$. (Here $[h]$ and $[h \circ \pi]$ denote the class of h and $h \circ \pi$ in $K^0(Y, S)$ and $K^0(X, T)$, respectively.)*

Remark. The proof of Theorem 3.36 has two ingredients. The first is the use of the Gottschalk–Hedlund lemma [**GH**], which in our context says that $g \in \mathcal{C}(X, \mathbb{Z})$ is a coboundary, i.e. $g = f - f \circ T^{-1}$ for some $f \in \mathcal{C}(X, \mathbb{Z})$, if and only if $\sup_n \left|\sum_{i=0}^{n-1} g(T^i x_0)\right| < \infty$ for some $x_0 \in X$. This will establish that π^* is well defined. The second ingredient, which gives the order embedding, is applying Theorem 3.22 (cf. also Theorem 3.32) together with the fact that π induces a surjective map of $M(X, T)$ onto $M(Y, S)$.

4. *Proofs of the main results*

The proofs of Theorems 2.1–2.3 stated in §2 will rest heavily upon results obtained earlier by Gjerde and Johansen [**GJ**] and Sugisaki [**S1**, **S2**, **S3**], where the paper [**S3**] is extending and being inspired by, so to speak, an analogous result proved by Giordano *et al* [**GPS2**]. The proofs in [**S1**, **S2**, **S3**] are rather technical, involving very clever manipulations of Bratteli diagrams. Looking carefully at crucial steps in the proofs of the main theorems in [**S2**, **S3**], in particular, we could deduce more specific properties of the Bratteli diagrams that appear, starting with our basic setting. This, combined with the Bratteli–Vershik model for Toeplitz flows established in [**GJ**] and the embedding result of [**GW**], will, loosely speaking, give the proofs of our three first theorems. (The proof of the fourth theorem, Theorem 2.4, requires a somewhat different approach.) However, this should not be construed as saying that our theorems are simply corollaries of these earlier results. Rather, one could perhaps say that the results we obtain represent the culmination of the study of (topological) Toeplitz flows from the perspective of orbit equivalence and/or K^0-groups.

LEMMA 4.1. *The Bratteli–Vershik model associated to a properly ordered Bratteli diagram $(V, E, \geq)$ with the ERS property has the odometer corresponding to the supernatural number of (V, E) (cf. Definition 3.16) as a factor. The factor map π is almost one-to-one. If the Bratteli–Vershik system $(X_{(V,E)}, T_{(V,E)})$ is expansive, then, for some k, we get that $\pi_k(x_{\min})$ is a Toeplitz sequence and $\pi_k : X_{(V,E)} \to X_k$ is a conjugacy map between $(X_{(V,E)}, T_{(V,E)})$ and (X_k, S_k) (cf. Proposition 3.35). In particular, $(X_{(V,E)}, T_{(V,E)})$ is a Toeplitz flow.*

Proof. We will give the proof by appealing to Figure 2, where we have drawn on the left a particular Bratteli diagram (V, E) with the ERS property. Let the incidence matrices be $(M_n)_{n=1}^{\infty}$, say, with r_n equal to the (constant) row sum of M_n. We 'collapse' (V, E) in an

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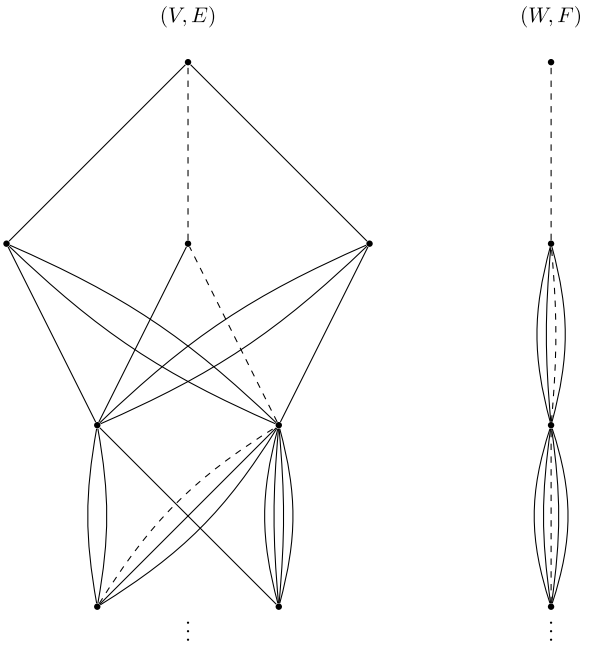


FIGURE 2. Assuming that the edges in both (V, E) and (W, F) are ordered left to right, the factor map π will take the dashed path in (V, E) to the dashed path in (W, F).

obvious way to create the diagram (W, F) on the right in Figure 2, where $|W_n| = 1$ for all n and the number of edges between W_{n-1} and W_n is r_n. Let them be linearly ordered as $f_1^{(n)} < f_2^{(n)} < \cdots < f_{r_n}^{(n)}$. The factor map $\pi : X_{(V,E)} \to X_{(W,F)}$ is defined by reading off the labels, so to say. That is, let $x = (e_n)_{n=1}^{\infty} \in X_{(V,E)}$ and let e_n be the (k_n)th edge in the linear ordering of $r^{-1}(r(e_n))$. We then map e_n to $f_{k_n}^{(n)}$, thus getting $\pi(x) \in X_{(W,F)}$. It is easy to see that π is a continuous map that intertwines $T_{(V,E)}$ and $T_{(W,F)}$. Also, $\pi^{-1}\{\pi(x_{\min})\} = \{x_{\min}\}$, where $x_{\min}$ is the unique minimal path of $X_{(V,E)}$, and so π is almost one-to-one. The map π is onto, since the image of π is a compact, hence closed, subset of $X_{(W,F)}$, and π clearly maps $\text{orbit}_{T_{(V,E)}}(x_{\min})$ to a dense subset of $X_{(W,F)}$. The factor $(X_{(W,F)}, T_{(W,F)})$ is by construction an odometer with the properties stated. If $(X_{(V,E)}, T_{(V,E)})$ is expansive, then, for some $k \in \mathbb{N}$, $\pi_k : X_{(V,E)} \to X_k$ is a conjugacy map by Proposition 3.35. As pointed out in §3.3.3, we may assume that at each level all the minimal edges (respectively the maximal edges) have the same source. This scenario is obtained by telescoping the original diagram, and all essential properties are preserved by this operation. The ERS property of (V, E) implies that $\pi_k(x_{\min})$ is a Toeplitz sequence, a fact that is easily shown; we omit the details. So, $(X_{(V,E)}, T_{(V,E)})$ is a Toeplitz flow (which, incidentally, also is a consequence of Theorem 3.13). □

Remark. Lemma 4.1 is the easy part of [**GJ**, Theorem 8]. Conversely, if one starts with a Toeplitz flow (X, T) one can construct a properly ordered Bratteli diagram $(V, E, \geq)$ such that (V, E) has the ERS property and $(X, T) \cong (X_{(V,E)}, T_{(V,E)})$. This is achieved by

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merging the structure of Toeplitz flows as described in §3.2 with the construction described in the proof sketch of Theorem 3.28. (See [**GJ**, Theorem 8].) In [**S1**], it is proved that for any ordered Bratteli diagram $(V, E, \geq)$ such that (V, E) has the ERS property, there exists another properly ordered Bratteli diagram $(V', E', \geq)$ such that (V', E') has the ERS property, with $K_0(V', E') \cong K_0(V, E)$ (as ordered groups with distinguished order units) and $(X_{(V',E')}, T_{(V',E')})$ is Toeplitz. By Theorem 3.32, $K^0(X, T) \cong K^0(X_{(V',E')}, T_{(V',E')})$, and so (X, T) is strong orbit equivalent to $(X_{(V',E')}, T_{(V',E')})$ by Theorem 3.33.

LEMMA 4.2. *Let $\pi : (X, T) \to (Y, S)$ be an almost one-to-one factor map between Cantor minimal systems. Then*

$$K^0(X, T)/\pi^*(K^0(Y, S))$$

is torsion free.

Proof. We will give a heuristic argument in order to highlight the basic idea behind the proof, avoiding the somewhat messy details that tend to obscure the understanding. (For a more detailed proof, see [**S3**, Theorem 3.1].) Since π is a factor map, we know from Theorem 3.36 that $\pi^* : K^0(Y, S) \to K^0(X, T)$ is an order embedding, where $\pi^*([h]) = [h \circ \pi]$. (Here $h \in \mathcal{C}(Y, \mathbb{Z})$ and $[h]$ and $[h \circ \pi]$ denote the classes of h and $h \circ \pi$ in $K^0(Y, S)$ and $K^0(X, T)$, respectively.) We will construct specific Bratteli–Vershik models for (X, T) and (Y, S), respectively. Let $x_0 \in X$ be such that $\pi^{-1}(\pi(x_0)) = x_0$. Let $\{U_n\}_{n\in\mathbb{Z}^+}$ be a decreasing sequence of clopen sets in Y such that $U_0 = Y$ and $U_n \searrow \{y_0\}$, where $y_0 = \pi(x_0)$. We now proceed as described in the proof sketch of Theorem 3.28. Let $\{\mathcal{P}_n\}_{n\in\mathbb{Z}^+}$ be the nested sequence of clopen partitions of Y associated to the tower constructions built over the various U_n, such that the union of the $\mathcal{P}_n$ is a basis for the topology of Y. Let $(W, F, \geq)$ be the resulting properly ordered Bratteli diagram, from which we get a Bratteli–Vershik model for (Y, S). Consider the clopen sets $\widetilde{U_n} = \pi^{-1}(U_n)$, $n \in \mathbb{Z}^+$. Clearly, $\{\widetilde{U_n}\}_{n\in\mathbb{Z}^+}$ is a descending sequence of clopen subsets of X such that $\widetilde{U_0} = X$, $\widetilde{U_n} \searrow \{x_0\}$. Proceeding again as described in the proof sketch of Theorem 3.28, we get a simply ordered Bratteli diagram $(V, E, \geq)$, from which we get a Bratteli–Vershik model for (X, T). Now $K^0(X, T) \cong K_0(V, E)$ and $K^0(Y, S) \cong K_0(W, F)$ as ordered groups with canonical order units. We note that the functions $\lambda_n : U_n \to \mathbb{N}$ and $\widetilde{\lambda}_n : \widetilde{U}_n \to \mathbb{N}$ defined by $\lambda_n(u) = \inf\{m \in \mathbb{N} \mid S^m u \in U_n\}$ and $\widetilde{\lambda}_n(\widetilde{u}) = \inf\{m \in \mathbb{N} \mid T^m \widetilde{u} \in \widetilde{U}_n\}$, respectively, are related by $\lambda_n(u) = \widetilde{\lambda}_n(\widetilde{u})$ if $\widetilde{u} \in \pi^{-1}(u)$. This has implications for how $K_0(W, F)$ is embedded in $K_0(V, E)$. Loosely speaking, each $w \in W_n$ (which corresponds to a tower T_w over U_n) is split into a finite number of vertices $v_1, v_2, \ldots, v_l$ in V_n (this corresponds to the tower T_w being subdivided into l towers $T_{v_1}, T_{v_2}, \ldots, T_{v_l}$ over $\widetilde{U}_n$, each of the same height as T_w). The factor map $\pi : X_{(V,E)} \to X_{(W,F)}$ is a kind of 'collapsing' map similarly to the one exhibited in Figure 2. While the scenario exhibited in Figure 2 is very simple, it does illustrate the essential point. In fact, the single vertices at levels 1, 2 and 3 of (W, F) in Figure 2 split into three, two and two vertices, respectively, at levels 1, 2 and 3 of (V, E). The image of the group element of $K_0(W, F)$ that is -7, say, at level 2 of (W, F) in Figure 2 is represented by $(-7, -7)$ at level 2 of (V, E). In general, a group element in $K_0(W, F)$ which is represented as being $b \in \mathbb{Z}$ at $w \in W_n$, and zero at the other vertices in W_n, is mapped to the group element in $K_0(V, E)$

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that is represented by being b at each of the vertices in V_n associated to w, and zero elsewhere. This extends by linearity in an obvious way to any element in $K_0(W, F)$ that is represented as a vector at level n. This 'locally constancy' property, which is preserved at the higher levels under the canonical mappings of the Bratteli diagram, is what completely characterizes the embedding of $K_0(W, F)$ into $K_0(V, E)$. This clearly implies that $K_0(V, E)/\pi^*(K_0(W, F))$ is torsion free. If namely $g \in K_0(V, E)$ is such that $kg \in \pi^*(K_0(W, F))$ for some $k \in \mathbb{N}$, then kg is represented by some 'locally constant' vector at level n of (V, E), say. But then clearly g is also represented by a 'locally constant' vector (at some higher level of (V, E) than n, perhaps), and hence g lies in $\pi^*(K_0(W, F))$. □

Remark. The converse of Lemma 4.2 is not true. In a recent paper by Glasner and Host, they constructed Cantor minimal systems (Y, S) and (X, T) such that (X, T) is an extension of (Y, S) by a map $\pi : (X, T) \to (Y, S)$ which is not an almost one-to-one extension, and $K^0(X, T)/\pi^*(K^0(Y, S))$ is non-zero and torsion free [**GHo**, Appendix C]. In fact, their example can be adjusted to make (Y, S) an odometer and (X, T) to be expansive.

LEMMA 4.3. *Let $(V, E, \geq)$ be a simple Bratteli diagram with the ERS property. Let M_n be the incidence matrix between levels $n-1$ and n, and let r_n be the (constant) row sum of M_n. Let m be the supernatural number $\prod_{n=1}^{\infty} r_n$. Let $H = \mathbb{Q}(K_0(V, E), [\mathbf{1}])$, where $[\mathbf{1}]$ is the canonical order unit of $K_0(V, E)$. Then H is order isomorphic to the rational group $G(m)$ by a map sending $[\mathbf{1}]$ to $1 \in G(m)$. Furthermore, H is represented in an obvious way by constant vectors at each level of (V, E), i.e. vectors of the form $(a, a, a, \ldots, a)^{\mathrm{tr}} \in \mathbb{Z}^{|V_n|}$ for each n (*tr *denotes the transpose).*

Proof. The proof is an immediate consequence of the fact that

$$M_n(1, 1, \ldots, 1)^{\mathrm{tr}} = (r_n, r_n, \ldots, r_n)^{\mathrm{tr}},$$

where $(1, 1, \ldots, 1)^{\mathrm{tr}} \in \mathbb{Z}^{|V_{n-1}|}$, $(r_n, r_n, \ldots, r_n)^{\mathrm{tr}} \in \mathbb{Z}^{|V_n|}$, which yields

$$M_n M_{n-1} \cdots M_1(1) = \left(\prod_{k=1}^{n} r_k, \prod_{k=1}^{n} r_k, \ldots, \prod_{k=1}^{n} r_k\right)^{\mathrm{tr}} \in \mathbb{Z}^{|V_n|}.$$

□

Remark. We will say that (V, E) is an ERS realization of $G \cong K_0(V, E)$ with respect to a subdimension group $H \subseteq G$ if H is embedded in $K_0(V, E)$ as in Lemma 4.3.

LEMMA 4.4. *Let $(J, J^+, 1)$ be a non-cyclic rational group (cf. Definition 3.23), and let (Y, S) be an odometer such that $K^0(Y, S) \cong J$ (as ordered groups with distinguished order units). Let $\iota : J \to G$ be an order embedding of J into a simple dimension group (G, G^+, u) preserving the order units, such that $G/\iota(J)$ is torsion free. There exists a properly ordered Bratteli diagram $(V, E, \geq)$ such that:*

(i) *(V, E) has the ERS property;*
(ii) *$K^0(X_{(V,E)}, T_{(V,E)}) \cong G$ (as ordered groups with distinguished order units);*
(iii) *$(X_{(V,E)}, T_{(V,E)})$ is an almost one-to-one extension of (Y, S);*
(iv) *(Y, S) is the maximal equicontinuous factor of (X, T);*
(v) *$(J, J^+, 1) \cong \mathbb{Q}(K_0(V, E), [\mathbf{1}])$.*

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Proof. Assertions (ii) and (iii) are the main result of [**S3**], namely Theorem 1.1 (see also Corollary 1.2). Since the properly ordered Bratteli diagram associated to (Y, S) is very special—having a single vertex at each level—the almost one-to-one extension of (Y, S) constructed in the proof of Theorem 1.1 in [**S3**], which is obtained by constructing a properly ordered Bratteli diagram $(V, E, \geq)$ based on the one associated to (Y, S), will have property (i). (For details, cf. [**S3**, Remark 3.2 and Proposition 3.3].) The assertion (iv) is a consequence of Theorem 3.12. The assertion (v) follows from (iii) and Lemma 4.3. In fact, $K^0(Y, S)$ embeds into $K_0(V, E)$ as constant vectors at each level of (V, E) (cf. the proof of Lemma 4.2, keeping in mind that in our case the properly ordered Bratteli diagram $(W, F, \geq)$ appearing there and being associated to (Y, S) has one vertex at each level). □

LEMMA 4.5. *Let $(V, E, \geq)$ be a properly ordered Bratteli diagram such that (V, E) has the ERS property. Let $0 \leq t < \infty$. There exists a properly ordered Bratteli diagram $(\widetilde{V}, \widetilde{E}, \geq)$ such that:*

(i) *$(\widetilde{V}, \widetilde{E})$ has the ERS property;*
(ii) *$K_0(\widetilde{V}, \widetilde{E}) \cong K_0(V, E)$ (as ordered groups with distinguished order units);*
(iii) *$(X_{(\widetilde{V}, \widetilde{E})}, T_{(\widetilde{V}, \widetilde{E})})$ is expansive;*
(iv) *$h(T_{(\widetilde{V}, \widetilde{E})}) = t$.*

Proof. The assertions (ii), (iii) and (iv) are the main result (Theorem 1.1) of [**S2**]. (Note that by Theorems 3.32, 3.33 and Proposition 3.8, respectively, strong orbit equivalence is related to K_0-groups, and expansiveness is related to subshifts, respectively.) Now, in the proof of Theorem 1.1 of [**S2**], various simple Bratteli diagrams are constructed, modifying the given Bratteli diagram (V, E). However, each modification preserves the ERS property of the original Bratteli diagram (V, E). (For details, cf. Propositions 4.2 and 4.4 and §§5.1, 5.3 and 5.4 of [**S2**].) So, the properly ordered Bratteli diagram $(\widetilde{V}, \widetilde{E}, \geq)$ that eventually arises in the proof of Theorem 1.1 of [**S2**] will have all the properties listed in Lemma 4.5. □

Proof of Theorem 2.1. Let (X, T) be a Toeplitz flow (with $h(T) = t$). By Theorem 3.13, (X, T) is an almost one-to-one extension of an odometer (Y, S), the factor map being $\pi : X \to Y$. By Theorem 3.36, $\pi^* : K^0(Y, S) \to K^0(X, T)$ is an order embedding sending the distinguished order unit of $K^0(Y, S)$ to the one in $K^0(X, T)$. Set $G = K^0(X, T)$, $H = \pi^*(K^0(Y, S))$. Then H is a non-cyclic subgroup of G of rank one such that $H \cap G^+ \neq \{0\}$.

Conversely, assume that (G, G^+) is a simple dimension group containing the non-cyclic subgroup H such that $H \cap G^+ \neq \{0\}$. Let $u \in H \cap G^+$ be any non-zero element. We consider the simple dimension group (G, G^+, u) with distinguished order unit u. Now H is a subgroup of the rational subgroup $\mathbb{Q}(G, u)$ of G. In fact, if $h \in H$, then there exist $m, n \in \mathbb{Z}$ such that $nh = mu$, since H is of rank one. (We may assume without loss of generality that $n \in \mathbb{N}$.) In particular, we get that $\mathbb{Q}(G, u)$ is non-cyclic. Now we apply Lemma 4.4 with $J = \mathbb{Q}(G, u)$ and $\iota : J \to G$ the inclusion map. We keep the notation of Lemma 4.4. Apply Lemma 4.5 to the properly ordered Bratteli diagram $(V, E, \geq)$ constructed in Lemma 4.4 to get $(\widetilde{V}, \widetilde{E}, \geq)$ with the properties listed in Lemma 4.5.

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Set $X = X_{(\widetilde{V},\widetilde{E})}$ and $T = T_{(\widetilde{V},\widetilde{E})}$. Then $K^0(X, T) \cong K_0(\widetilde{V}, \widetilde{E}) \cong K_0(V, E) \cong G$ as ordered groups with distinguished order units. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Set $J = \mathbb{Q}(G, u)$. By assumption, J is a non-cyclic rational group (hence of rank one). The inclusion map $\iota : J \to G$ is an order embedding and, by Proposition 3.25, G/J is torsion free. Let (Y, S) be an odometer system such that $K^0(Y, S) \cong J$ (as ordered groups with distinguished order units). Applying Lemma 4.4, we get a properly ordered Bratteli diagram $(V, E, \geq)$ satisfying the properties listed in Lemma 4.4. Now apply Lemma 4.5 to $(V, E, \geq)$ to get $(\widetilde{V}, \widetilde{E}, \geq)$ satisfying the properties listed in Lemma 4.5. Let $(X, T) = (X_{(\widetilde{V},\widetilde{E})}, T_{(\widetilde{V},\widetilde{E})})$. By Lemma 4.1, we get that (X, T) is a Toeplitz flow. Finally, invoking Proposition 3.30, we get that (X, T) satisfies all the properties listed in Theorem 2.2. (Recall that by the Remark to Proposition 3.25 we have $(\mathbb{Q}(G, u) \cong) \mathbb{Q}(K_0(V, E), [\mathbf{1}]) \cong \mathbb{Q}(K_0(\widetilde{V}, \widetilde{E}), [\mathbf{1}])$.) □

Proof of Theorem 2.3. By Theorem 2.2, we get the inclusion $\mathcal{G} \subseteq \mathcal{T}_t$. The inclusion $\mathcal{T}_t \subseteq \mathcal{G}$ is a consequence of Theorem 3.13 and Proposition 3.31. Hence, we have $\mathcal{G} = \mathcal{T}_t$.

The inclusion $\mathcal{B} \subseteq \mathcal{G}$ follows from Lemma 4.3. The inclusion $\mathcal{T}_t \subseteq \mathcal{B}$ follows from the main result, [**GJ**, Theorem 8]. Altogether, this implies that $\mathcal{T}_t = \mathcal{G} = \mathcal{B}$.

Proof of Theorem 2.4. We prove the first part: let H be a dense rational subgroup of $\mathbb{Q}$. Let x_0 be any point in K, and let $\mathcal{E}$ be a countable set in $\mathrm{Aff}(K)$, the continuous, real-valued affine functions on K, such that $\mathcal{E}$ is a dense (in the uniform topology) subset of $\{a \in \mathrm{Aff}(K) \mid a(x_0) = 0\}$. Let G be the (countable) additive subgroup of $\mathrm{Aff}(K)$ generated by H and $\mathcal{E}$, where we identify every element in H with a constant affine function. Then $(G, G^+, 1)$ is a simple dimension group with order unit the constant function 1, and G^+ is the obvious positive cone; furthermore, $\mathrm{Inf}(G) = \{0\}$, and the state space $S_1(G)$ is affinely homeomorphic with K (cf. Theorem 3.22). It is a simple matter to check that $\mathbb{Q}(G, 1) = H$. By Theorem 2.2, there exists a Toeplitz flow (X, T) such that $(G, G^+, 1) \cong (K^0(X, T), K^0(X, T)^+, \mathbb{1})$ and $h(T) = t$. By Theorem 3.32, we get that $M(X, T)$ is affinely homeomorphic to $S_1(G)$, and so to K. Now, if $(G_1, G_1^+, 1) \cong (G_2, G_2^+, 1)$, where G_1 and G_2 are constructed as G above, then $\mathbb{Q}(G_1, 1) \cong \mathbb{Q}(G_2, 1)$ as ordered groups by a positive map sending 1 to 1. Clearly, one can choose an uncountable family of non-isomorphic groups H of the type described above. Hence, there exists an uncountable family of non-isomorphic dimension groups $(G, G^+, 1)$ of the type constructed above. The corresponding uncountable family of Toeplitz flows (X, T) are then pairwise non-orbit equivalent by Theorem 3.34.

We prove the second part: let (Y, S) be associated to the $\mathfrak{a}$-adic group $G_{\mathfrak{a}}$, where $\mathfrak{a} = (a_1, a_2, \ldots)$. Choose H to be the rational group associated to $\mathfrak{a}$, i.e. $H = \{m/(a_1 a_2 \cdots a_n) \mid m \in \mathbb{Z},\ n \in \mathbb{N}\}$. As above, we identify H with constant affine functions on K in an obvious way. Let $\mathcal{E}$ be as above, and let G be generated by H and $\mathcal{E}$ as above. Let N be any countable torsion-free abelian group, and let $\widetilde{G} = G \oplus N$, with $\widetilde{G}^+ = G^+ \oplus 0$ and order unit $u = 1 \oplus 0$. It is easily seen that $\mathbb{Q}(\widetilde{G}, u) \cong \mathbb{Q}(G, 1) \cong H$ as ordered groups with distinguished order units. Clearly, $\mathrm{Inf}(\widetilde{G}) = 0 \oplus N \cong N$. Also, $S_u(\widetilde{G}) \cong S_1(G) \cong K$. By Theorem 2.2, there exists a Toeplitz flow $(\widetilde{X}, \widetilde{T})$ such that $K^0(\widetilde{X}, \widetilde{T}) \cong \widetilde{G}$ (as ordered groups with distinguished order units) and $h(\widetilde{T}) = t$. By Proposition 3.31,

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we get that the maximal equicontinuous factor of $(\widetilde{X}, \widetilde{T})$ is (Y, S), and by Theorem 3.32 we get that $M(\widetilde{X}, \widetilde{T}) \cong K$. Clearly, there exists an uncountable family of pairwise non-isomorphic groups N of the above type. The associated simple dimension groups $\widetilde{G} = G \oplus N$ are then pairwise non-isomorphic. (Note that if $(G_1, G_1^+, u_1) \cong (G_2, G_2^+, u_2)$, then $\mathrm{Inf}G_1 \cong \mathrm{Inf}G_2$.) The corresponding Toeplitz flows $(\widetilde{X}, \widetilde{T})$ are then pairwise non-strong orbit equivalent by Theorem 3.33. This finishes the proof of Theorem 2.4. □

5. *Examples*

We will first state some results that—combined with our theorems—will provide a rich source of examples of Toeplitz flows. We first need a definition.

Definition 5.1. Let (V, E) be a simple Bratteli diagram such that $|V_n| \leq l < \infty$ for all n. We say that (V, E) is of finite rank. If $|V_n| = k$ for all $n = 1, 2, \ldots$, we say that (V, E) is of rank k.

Remark. If (V, E) is of finite rank, we may telescope (V, E) to get a new Bratteli diagram of rank some k. It is easily seen that if (V, E) is of rank k, then the dimension group $K_0(V, E)$ has rank $\leq k$. Furthermore, the state space $S_{[\mathbf{1}]}(K_0(V, E))$ of $K_0(V, E)$ has at most k extreme points, and so is a (finite-dimensional) m-simplex for some $m \leq k$.

THEOREM 5.2. [**DM**, **Ho**] *Let $(V, E, \geq)$ be a properly ordered Bratteli diagram such that (V, E) is of finite rank. Then $(X_{(V,E)}, T_{(V,E)})$ is either expansive or it is an odometer.*

THEOREM 5.3. *Let $(V, E, \geq)$ be a properly ordered Bratteli diagram such that (V, E) has finite rank. Then the entropy of $(X_{(V,E)}, T_{(V,E)})$ is zero.*

We will prove Theorem 5.3 at the end of this section. Combining Theorems 5.2 and 5.3 with Lemma 4.1, we get the following result.

THEOREM 5.4. *Let $(V, E, \geq)$ be a properly ordered Bratteli diagram with the following two properties:*

(i) *(V, E) is of finite rank (and so $K_0(V, E)$ is of finite rank);*

(ii) *(V, E) has the ERS property (and so $\mathbb{Q}(K_0(V, E), [\mathbf{1}])$ is non-cyclic).*

Then $(X_{(V,E)}, T_{(V,E)})$ is either an odometer or a Toeplitz flow of entropy zero. In particular, if $K_0(V, E)$ is not a (non-cyclic) rational group, then $(X_{(V,E)}, T_{(V,E)})$ is a Toeplitz flow.

There is a partial converse to Theorem 5.4 due to Handelman [**H**, Theorem 8.5]. (Incidentally, the paper [**H**] treats a more general situation under the assumption that the dimension groups in question have a unique state.) We formulate his result using our terminology and notation.

THEOREM 5.5. *Let (G, G^+, u) be a simple dimension group with order unit u. Assume that $\mathbb{Q}(G, u)$ is non-cyclic and that $S_u(G)$ is a one-point set. (We say that G has a unique state, the order unit being understood.) Assume that* $\mathrm{rank}(G) = k$. *There exists a simple Bratteli diagram (V, E) of rank at most $k + 1$ with the ERS property, such that $(G, G^+, u) \cong (K_0(V, E), K_0(V, E)^+, [\mathbf{1}])$ and (V, E) is an ERS realization of G with respect to $\mathbb{Q}(G, u)$ (cf. the Remark after Lemma 4.3). Giving (V, E) a proper ordering,*

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the associated Bratteli–Vershik system is either an odometer or a uniquely ergodic Toeplitz flow of zero entropy.

Remark. In particular, the scenario described in Theorem 5.5 occurs when $G = H \oplus \mathbb{Z}^m$, where H is a non-cyclic rational group, and $G^+ = H^+ \oplus 0$, with $u = 1 \oplus 0$. Clearly, $\mathrm{Inf}(G) = 0 \oplus \mathbb{Z}^m \cong \mathbb{Z}^m$, and $\mathbb{Q}(G, u) \cong H$ (as ordered groups with order units u and 1, respectively). Clearly, G has a unique state and $\mathrm{rank}(G) = m + 1$. So, by Theorem 5.5, there exists an ERS realization (V, E) of G with respect to H such that (V, E) has rank at most $m + 2$.

Another example where Theorem 5.5 applies is $G = \mathbb{Q} + \mathbb{Q}\alpha \subseteq \mathbb{R}$, where α is an irrational number, and G inherits the ordering from $\mathbb{R}$, i.e. $G^+ = G \cap \mathbb{R}^+$, the order unit being 1. Then $\mathbb{Q}(G, 1) = \mathbb{Q}$ and $\mathrm{rank}(G) = 2$. So, by Theorem 5.5, G has an ERS realization (V, E) with respect to $\mathbb{Q}$, with (V, E) of rank at most three. (In fact, in this particular case it suffices with a rank equal to two, cf. Example 5.6.)

We will give explicit examples below illustrating some of these scenarios.

Example 5.6. Let $G = \mathbb{Q} + \mathbb{Q}\alpha \subseteq \mathbb{R}$ with G^+ and order unit 1 as above, and so $\mathbb{Q}(G, 1) = \mathbb{Q}$. We may assume without loss of generality that $0 < \alpha < 1/2$. Let

$$\alpha = \frac{1}{a_0 + \dfrac{1}{a_1 + \dfrac{1}{a_2 + \cdots}}}$$

be the continued fraction expansion of α. Then $\mathbb{Q} + \mathbb{Q}\alpha$ is order isomorphic to the inductive limit

$$\mathbb{Q} \xrightarrow{A_1} \mathbb{Q}^2 \xrightarrow{A_2} \mathbb{Q}^2 \xrightarrow{A_3} \mathbb{Q}^2 \xrightarrow{A_4} \cdots, \tag{*}$$

where the incidence matrices are

$$A_1 = s \begin{bmatrix} a_0 \\ 1 \end{bmatrix}, \quad A_i = \begin{bmatrix} a_{i-1} & 1 \\ 1 & 0 \end{bmatrix}, \quad i \geq 2,$$

and the order unit is the canonical one, i.e. represented by $1 \in \mathbb{Q}$. This follows from the fact that the dimension group $\mathbb{Z} + \mathbb{Z}\alpha \subseteq \mathbb{R}$ is represented by (*) with the $\mathbb{Q}$ replaced by $\mathbb{Z}$ (cf. [**Sk**, 3.3, Example (ii)]). We will assume that $a_0 = 1$ and so $A_1 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. (We make this assumption just for convenience; nothing essential is changed by this, but the ensuing construction becomes more streamlined.) We will show that the inductive limit in (*) is order isomorphic to the inductive limit

$$\mathbb{Z} \xrightarrow{B_1} \mathbb{Z}^2 \xrightarrow{B_2} \mathbb{Z}^2 \xrightarrow{B_3} \mathbb{Z}^2 \xrightarrow{B_4} \cdots, \tag{**}$$

where the incidence matrices B_n have equal row sums—hence the associated simple Bratteli diagram has the ERS property. The B_n are obtained from the A_n by the following procedure, where we have adapted the construction in the proof of [**GJ**, Theorem 11] to our setting: let $A_2 \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \alpha_1 \\ \alpha_2 \end{smallmatrix}\right]$. Then $J_2' A_2 \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, where J_2' is the diagonal 2×2 matrix $\mathrm{diag}(\alpha_1^{-1}, \alpha_2^{-1})$. Let m_2 be the least common multiple of the denominators of the entries of $J_2' A_2$, and let $k_2 = 2m_2$. Set $J_2 = k_2 J_2'$. Then all entries of $J_2 A_2$ are in $\mathbb{N}$ and are divisible by two. Furthermore, $J_2 A_2 \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} k_2 \\ k_2 \end{smallmatrix}\right]$. Assume that we have constructed $J_2, \ldots, J_{n-1}$. Let J_n' be the appropriate diagonal matrix over $\mathbb{Q}^+$ such that $J_n' A_n J_{n-1}^{-1} \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. Let m_n be the least common multiple of the denominators of the entries of $J_n' A_n J_{n-1}^{-1}$, and

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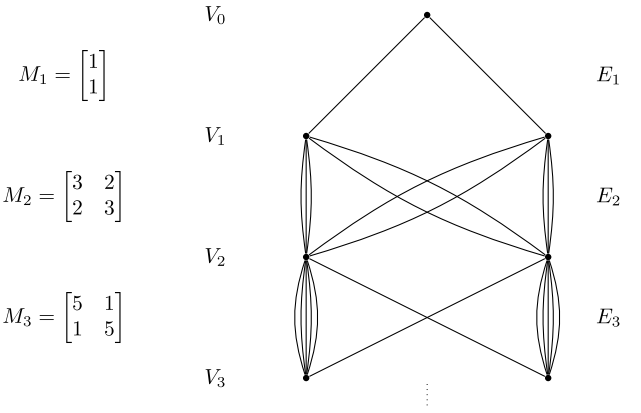


FIGURE 3. A 2-symmetric Bratteli diagram.

let $k_n = nm_n$. Set $J_n = k_n J'_n$. Then all entries of $J_n A_n J_{n-1}^{-1}$ are in $\mathbb{N}$ and are divisible by n. Furthermore, $J_n A_n J_{n-1}^{-1}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] = \left[\begin{smallmatrix}k_n\\k_n\end{smallmatrix}\right]$, where we observe that n is a divisor of k_n. Set $B_n = J_n A_n J_{n-1}^{-1}$. Setting $J_0 = \mathrm{id}$, $J_1 = \mathrm{id}$, we get the commutative diagram

$$\begin{array}{ccccccccccc}
G: & \mathbb{Q} & \xrightarrow{A_1} & \mathbb{Q}^2 & \xrightarrow{A_2} & \mathbb{Q}^2 & \xrightarrow{A_3} & \mathbb{Q}^2 & \xrightarrow{A_4} & \cdots \\
 & \downarrow{\scriptstyle J_0=\mathrm{id}} & & \downarrow{\scriptstyle J_1=\mathrm{id}} & & \downarrow{\scriptstyle J_2} & & \downarrow{\scriptstyle J_3} & & \\
H: & \mathbb{Q} & \xrightarrow{B_1} & \mathbb{Q}^2 & \xrightarrow{B_2} & \mathbb{Q}^2 & \xrightarrow{B_3} & \mathbb{Q}^2 & \xrightarrow{B_4} & \cdots
\end{array}$$

which establishes an order isomorphism between the two associated inductive limits G and H (respecting the distinguished order units). Now H is order isomorphic to the dimension group associated to (**), since the latter is a divisible group (because n is a divisor of each entry of B_n). We conclude that $G = \mathbb{Q} + \mathbb{Q}\alpha \cong K_0(V, E)$ (as ordered groups with distinguished order units), where (V, E) is the Bratteli diagram associated to (**). Since $B_n\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] = \left[\begin{smallmatrix}k_n\\k_n\end{smallmatrix}\right]$, we see that (V, E) has the ERS property. Furthermore, since n divides k_n, we observe that (V, E) is a rank-two ERS realization of G with respect to $\mathbb{Q}(G, 1)$ $(= \mathbb{Q})$.

Finally, any proper ordering of (V, E) will yield a Bratteli–Vershik (BV) system which is a uniquely ergodic Toeplitz system of entropy zero. An intriguing question is if by considering all proper orderings there arises an uncountable family of pairwise non-conjugate BV systems? These BV systems are of course all strong orbit equivalent, having the same K^0-group $\mathbb{Q} + \mathbb{Q}\alpha$.

Example 5.7. (2-symmetric Bratteli diagrams (V, E) and their associated simple dimension groups) Let $|V_n| = 2$ for all $n \geq 1$ and let the incidence matrix M_n between levels $n-1$ and n be 2-symmetric, i.e. of the form

$$M_n = \begin{bmatrix} l_n & k_n \\ k_n & l_n \end{bmatrix} \quad \text{where } 1 \leq k_n < l_n,$$

for all $n \geq 2$, and $M_1 = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ (see Figure 3). So, $\mathrm{rank}(V, E) = 2$ and (V, E) has the ERS property (as well as the ECS property, i.e. equal column sums). (The 2-symmetric Bratteli

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diagrams were studied in [**FM**] in connection with classifying the symmetries of UHF C*-algebras; cf. also [**B**, Ch. III, §7.7.4].) It will be convenient to write for $n \geq 2$,

$$l_n = \frac{q_n + r_n}{2}, \quad k_n = \frac{q_n - r_n}{2} \quad \text{where } q_n \text{ and } r_n \text{ have the same parity,}$$

and so $q_n = l_n + k_n$, $r_n = l_n - k_n$. By a simple computation, we get that

$$M_n M_{n-1} \cdots M_2 = \frac{1}{2}\begin{bmatrix} s_n + t_n & s_n - t_n \\ s_n - t_n & s_n + t_n \end{bmatrix} \quad \text{where } s_n = \prod_{i=2}^{n} q_i, \ t_n = \prod_{i=2}^{n} r_i.$$

We note that s_n and t_n have the same parity and that

$$\frac{s_n}{t_n} = \prod_{i=2}^{n} \frac{q_i}{r_i} \nearrow \prod_{i=2}^{\infty} \frac{q_i}{r_i} = \alpha.$$

Two scenarios can occur. Firstly, if $\alpha < \infty$, then the state space $S_{[\mathbf{1}]}(K_0(V, E))$ of $K_0(V, E)$ has two extreme points. Specifically,

$$K_0(V, E) = \left\{ \left(\frac{k}{2s_n}, \frac{l}{2t_n} \right) \,\middle|\, k, l \in \mathbb{Z} \text{ with the same parity}, n \in \mathbb{N} \right\} \subseteq \mathbb{Q}^2 \subseteq \mathbb{R}^2.$$

$K_0(V, E)^+$ consists of those elements in $K_0(V, E)$ that lie inside the cone determined by the two half-lines with slopes α and $-\alpha$, respectively. The canonical order unit [**1**] of $K_0(V, E)$ is represented by $(1, 0)$ $(= (2/2, 0))$ and the two extreme states are determined by projecting orthogonally to the two lines with slopes α and $-\alpha$, respectively. We note that $K_0(V, E)$ does not split as a direct sum of two (non-cyclic) rational groups, although it 'almost' does. (There is more on that when describing the unique state case below.) Furthermore, $\mathbb{Q}(K_0(V, E), [\mathbf{1}])$ is the subgroup of $K_0(V, E)$ represented by

$$\left\{ \left(\frac{k}{2s_n}, 0 \right) \,\middle|\, k \in \mathbb{Z} \text{ even}, n \in \mathbb{N} \right\}$$

and so is order isomorphic to the non-cyclic rational group $G(m)$, where $m = \prod_{i=1}^{\infty} q_i$.

The second scenario occurs when $\alpha = \infty$. Then $S_{[\mathbf{1}]}(K_0(V, E)) = \{\tau\}$ is a one-point set. The unique state τ is intimately related to the Perron–Frobenius (P–F) eigenvalues and eigenvectors of the incidence matrices $\{M_n\}_{n=2}^{\infty}$. In fact, q_n is the P–F eigenvalue of M_n with left (right) eigenvector $(1, 1)$ $((1, 1)^{\text{tr}})$. The other eigenvalue is r_n with left (right) eigenvector $(1, -1)$ $((1, -1)^{\text{tr}})$. The state τ is determined by

$$\tau(v_n) = \frac{1}{q_1 q_2 \cdots q_n}(1, 1)\,\vec{v_n},$$

where v_n denotes the element in $K_0(V, E)$ represented by the column vector $\vec{v_n} \in \mathbb{Z}^2$ at level n of the Bratteli diagram, and where we for normalization purposes have set $q_1 = 2$ so that $\tau([\mathbf{1}]) = 1$. In particular, we get that

$$\tau(K_0(V, E)) = \left\{ \frac{m}{q_1 q_2 \cdots q_n} \,\middle|\, m \in \mathbb{Z}, n \in \mathbb{N} \right\} = Q \ (\subseteq \mathbb{Q}).$$

Furthermore, $\tau(H) = \{(m/(q_1 q_2 \cdots q_n)) \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$, where $H = \mathbb{Q}(K_0(V, E), [\mathbf{1}])$, since H is represented as 'constant' vectors at each level of (V, E), cf. Lemma 4.3. We

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notice that $\tau(H) = 2\tau(K_0(V, E)) = 2Q$. Now we focus on the interesting special case where $r_n = 1$ for all $n \geq 2$. Hence, $l_n = k_n + 1$, and q_n is odd for all $n \geq 2$. In particular, $Q = \tau(K_0(V, E))$ is not 2-divisible. Now $\ker(\tau) = \mathrm{Inf}(K_0(V, E))$ and $K_0(V, E)^+ = \tau^{-1}(\mathbb{R}^+ - \{0\})$. We observe that $\ker(\tau) \cong \mathbb{Z}$ since $\tau(v_n) = 0$ for all $n \geq 1$, where $\vec{v_n} = (1, -1)^{\mathrm{tr}}$ (cf. the notation above), and $M_n(1, -1)^{\mathrm{tr}} = (1, -1)^{\mathrm{tr}}$ for all $n \geq 2$. Thus, we get the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow K_0(V, E) \xrightarrow{\tau} Q \longrightarrow 0.$$

However, there is no splitting $K_0(V, E) = Q \oplus \mathbb{Z}$, with τ the projection onto Q, the reason being that Q is not 2-divisible. In fact, if such a splitting occurred then clearly Q would be equal to the rational subgroup H. This would imply that $\tau(H) = Q$ and, combining this with $\tau(H) = 2Q$, we would get that Q is 2-divisible, which is a contradiction. (For more details, cf. [**H**, §8].)

Remark. By our results we know that all proper orderings of a 2-symmetric Bratteli diagram yield Bratteli–Vershik maps that are Toeplitz flows. One can show that when $r_n = 1$ ($n \geq 1$) and (V, E) is given the left–right ordering (meaning that we order the edges ranging at the same vertex from left to right), then the Toeplitz sequence that corresponds to the (unique) minimal path is regular (cf. Definition 3.14).

Proof of Theorem 5.3. By telescoping, we may assume that $k = |V_1| = |V_2| = \cdots (= \mathrm{rank}(V, E))$, where $V = \{V_0, V_1, V_2, \ldots\}$, $E = \{E_1, E_2, \ldots\}$. By Theorem 5.2, we may assume that $(X_{(V,E)}, T_{(V,E)})$ is expansive, since if not it would be an odometer which has zero entropy. This implies in particular that we may assume that $k \geq 2$. By telescoping the initial part of (V, E), we may assume that $(X_{(V,E)}, T_{(V,E)})$ is conjugate to (X_1, S_1) (and hence conjugate to (X_n, S_n) for all $n \geq 1$) by the map $\pi_1 : X_{(V,E)} \to X_1$ $(= \pi_1(X_{(V,E)}) \subseteq E_1^{\mathbb{Z}} (= P_1^{\mathbb{Z}}))$ defined by $\pi_1(x) = (\tau_1(T^n_{(V,E)}x))_{n=-\infty}^{\infty}$, where $\tau_1 : X_{(V,E)} \to E_1$ $(= P_1)$ is the truncation map (cf. Proposition 3.35 and the description of key constructions associated to $(V, E, \geq)$, as well as notation and terminology preceding the proposition). For $v \in V_n$, we consider all the paths, say $\{p_1, p_2, \ldots, p_l\}$, from $v_0 \in V_0$ to v, which will be a subset of P_n. Let those paths have the ordering $p_1 < p_2 < \cdots < p_l$, say, in the induced lexicographic ordering. Then $w(v) = \tau_1(p_1)\tau_1(p_2) \cdots \tau_1(p_l)$ will be a word over E_1. Let $W_n = \{w(v) \mid v \in V_n\}$. Notice that $|W_n| = k$ for all n. Define (X_{W_n}, S_{W_n}) to be the subshift of the full shift on $E_1^{\mathbb{Z}}$, where X_{W_n} is the set of all bisequences formed by concatenation of words in W_n. Now we observe that $E_1^{\mathbb{Z}} \supseteq X_{W_1} \supseteq X_{W_2} \supseteq \cdots \supseteq X_1$. In fact, this is an immediate consequence of how the Vershik map is defined.

Recall that the entropy of a subshift (X, T) is $h(T) = \lim_{q\to\infty} 1/q \log |B_q(X)|$, where $B_q(X)$ is the set of words of length q occurring in X (cf. [**Wa**, Theorem 7.13]). Clearly, $h(S_1) \leq h(S_{W_n})$ for all n, and so it suffices to show that $h(S_{W_n}) \to 0$ as $n \to \infty$. Let l_n be the length of the shortest word in W_n. (Clearly, $l_n \to \infty$ as $n \to \infty$.) Assume that $w \in X_{W_n}$ is a subword of length m of a concatenation of s words from W_n. Then we easily see that

$$s \leq \left\lceil \frac{m}{l_n} \right\rceil + 1,$$

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where $\lceil x \rceil$ is the smallest integer which is larger than or equal to x. There are at most k^s different ways to concatenate s words in W_n, and so we get

$$|B_m(W_n)| \leq \sum_{s=0}^{\lceil m/l_n \rceil + 1} k^s = \frac{k^{\lceil m/l_n \rceil + 2} - 1}{k - 1} \leq k^{\lceil m/l_n \rceil + 2} \leq k^{(m/l_n)+3}.$$

This implies that $1/n \log |B_m(W_n)| \leq (m/l_n) + 3/n \log k \to 0$ as $n \to \infty$. Hence, $h(S_{W_n}) \to 0$ as $n \to \infty$, thus completing the proof.

Acknowledgements. I would like to thank David Handelman and Fumiaki Sugisaki for helpful private communications. Furthermore, some of the results in this paper depend upon earlier results obtained by them. I thank the referee for pointing out the relevance of our Theorem 2.2(i) to a realization result of K^0-groups occurring in minimal $\mathbb{Z}^d$-actions on the Cantor set as stated in Proposition 2.5. Finally, I would like to extend my sincere gratitude to Christian Skau for suggesting the basic problems considered in this paper and for his constant encouragement and valuable advice.

References

[**B**] B. Blackadar. *K-theory for Operator Algebras (Mathematical Sciences Research Institute Publications, 5)*. Springer, New York, 1986.

[**BT**] M. Boyle and J. Tomiyama. Bounded topological orbit equivalence and C*-algebras. *J. Math. Soc. Japan* **50**(2) (1998), 317–329.

[**CP**] M. I. Cortez and S. Petite. Invariant measures and orbit equivalence for generalized Toeplitz subshifts. *Groups Geom. Dyn.*, accepted for publication. *Preprint*, arXiv:1106.4280.

[**D**] T. Downarowicz. Survey of odometers and Toeplitz flows. *Algebraic and Topological Dynamics (Contemporary Mathematics, 385)*. Eds. S. Kolyada, Y. Manin and T. Ward. American Mathematical Society, Providence, RI, 2005, pp. 7–38.

[**DD**] T. Downarowicz and F. Durand. Factors of Toeplitz flows and other almost 1–1 extensions over group rotations. *Math. Scand.* **90** (2002), 57–72.

[**DHS**] F. Durand, B. Host and C. F. Skau. Substitutional dynamical systems, Bratteli diagrams and dimension groups. *Ergod. Th. & Dynam. Sys.* **19** (1999), 995–1035.

[**DM**] T. Downarowicz and A. Maass. Finite-rank Bratteli–Vershik diagrams are expansive. *Ergod. Th. & Dynam. Sys.* **28** (2008), 739–747.

[**E**] E. Effros. *Dimensions and C*-Algebras (CBMS Regional Conference Series in Mathematics, 46)*. American Mathematical Society, Providence, RI, 1981.

[**EHS**] E. Effros, D. Handelman and C. Shen. Dimension groups and their affine representations. *Amer. J. Math.* **102** (1980), 385–407.

[**F**] L. Fuchs. *Infinite Abelian Groups.* Vol. II. Academic Press, New York, 1973.

[**FM**] T. Fack and O. Marechal. Sur la classification des symétries des C*-algébres UHF. *Canad. J. Math.* **31** (1979), 496–523.

[**GH**] W. H. Gottschalk and G. A. Hedlund. *Topological Dynamics (American Mathematical Society Colloquium Publications, 36)*. American Mathematical Society, Providence, RI, 1955.

[**GHo**] E. Glasner and B. Host. Extensions of Cantor minimal systems and dimension groups. *J. Reine Angew. Math.* **682** (2013), 207–243.

[**GJ**] R. Gjerde and Ø. Johansen. Bratteli–Vershik models for Cantor minimal systems: application to Toeplitz flows. *Ergod. Th. & Dynam. Sys.* **20** (2000), 1687–1710.

[**GMPS**] T. Giordano, H. Matui, I. F. Putnam and C. Skau. Orbit equivalence for Cantor minimal $\mathbb{Z}^d$-systems. *Invent. Math.* **179**(1) (2010), 119–158.

[**GPS1**] T. Giordano, I. F. Putnam and C. Skau. Topological orbit equivalence and C*-crossed products. *J. Reine Angew. Math.* **469** (1995), 51–111.

https://doi.org/10.1017/etds.2014.144 Published online by Cambridge University Press

[**GPS2**] T. Giordano, I. F. Putnam and C. Skau. K-theory and asymptotic index for certain almost one-to-one factors. *Math. Scand.* **89** (2001), 297–319.
[**GW**] E. Glasner and B. Weiss. Weak orbit equivalence of Cantor minimal systems. *Int. J. Math.* **6** (1995), 569–579.
[**H**] D. Handelman. Equal column sum and equal row sum dimension group realizations. *Preprint*, arXiv:1301.2799.
[**Ho**] S.-M. Høynes. Finite-rank Bratteli–Vershik diagrams are expansive – a new proof. *Preprint*, 2014, arXiv:1411.3371.
[**HPS**] R. H. Herman, I. F. Putnam and C. F. Skau. Ordered Bratteli diagrams, dimension groups, and topological dynamics. *Int. J. Math.* **3** (1992), 827–864.
[**HR**] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis I (Grundlehren der Mathematische Wissenschaften, 115)*. Springer, Berlin, 1963.
[**JK**] K. Jacobs and M. Keane. 0–1-sequences of Toeplitz type. *Z. Wahrscheinlichkeitsth. verw. Geb.* **13** (1969), 123–131.
[**K**] K. Kuratowski. *Topology.* Vol. II. Academic Press, New York, 1968.
[**M**] J. C. Martin. Substitution minimal flows. *Amer. J. Math.* **93** (1971), 503–526.
[**MP**] N. G. Markley and M. E. Paul. Almost automorphic symbolic minimal sets without unique ergodicity. *Israel J. Math.* **34** (1979), 259–272.
[**O**] N. S. Ormes. Strong orbit realization for minimal homeomorphisms. *J. Anal. Math.* **71** (1997), 103–133.
[**P**] M. E. Paul. Construction of almost automorphic symbolic minimal flows. *Appl. Gen. Topol.* **6** (1976), 45–56.
[**Sk**] C. F. Skau. Ordered K-theory and minimal symbolic dynamical systems. *Colloq. Math.* **84–85** (2000), 203–227.
[**S1**] F. Sugisaki. Toeplitz flows, ordered Bratteli diagrams and strong orbit equivalence. *Ergod. Th. & Dynam. Sys.* **21** (2001), 1867–1881.
[**S2**] F. Sugisaki. On the subshift within a strong orbit equivalence class for minimal homeomorphisms. *Ergod. Th. & Dynam. Sys.* **27** (2007), 971–990.
[**S3**] F. Sugisaki. Almost one-to-one extensions of Cantor minimal systems and order embeddings of simple dimension groups. *Münster J. Math.* **4** (2011), 141–169.
[**T**] O. Toeplitz. Beispiele zur Theorie der fastperiodischen Funktionen. *Math. Ann.* **98** (1928), 281–295.
[**W**] S. Williams. Toeplitz minimal flows which are not uniquely ergodic. *Z. Wahrscheinlichkeitsth. verw. Geb.* **67** (1984), 95–107.
[**Wa**] P. Walters. *An Introduction to Ergodic Theory (Graduate Texts in Mathematics, 79)*. Springer, New York, 1982.

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