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On K-automorphisms, Bernoulli shifts and Markov random fields

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Abstract. We show that for translation invariant Markov random fields: (1) the K-property implies a trivial full tail; (2) the Bernoulli property implies Følner independence. The existence of bilaterally deterministic Bernoulli shifts tells us that neither result is true without the Markov assumption (even in one dimension). We also show that for general translation invariant random fields: (3) Følner independence implies a trivial full tail.

1. *Introduction*

Whereas one-dimensional Markov processes are well understood, their structure in higher dimensions is less fully explored. The classification of ergodic properties for Markov random fields is an area that is still under investigation. This paper makes a small contribution to this classification by proving the three results stated in the abstract. In §1 we give a quick review of some necessary basic concepts from ergodic theory. In §2 we formulate our theorems and in §3 we give their proofs.

1.1. *Basic notation and definitions.*

Throughout this paper we shall consider stationary stochastic processes $X = \{X_x\}_{x \in \mathbf{Z}^d}$ taking values in a *finite* set F. We shall also view X as a probability measure μ on $\Omega = F^{\mathbf{Z}^d}$ that is invariant under the natural $\mathbf{Z}^d$-action. We shall continually go back and forth between these two viewpoints.

We write $B_n = [-n, n]^d \cap \mathbf{Z}^d$ to denote the n-box in $\mathbf{Z}^d$ (where d will be understood from the context) and $\partial(U) = \{x \notin U : \exists y \in U \text{ with } \|x - y\| = 1\}$ to denote the outer boundary of $U \subseteq \mathbf{Z}^d$ (where $\|x\| = |x_1| + |x_2| + \cdots + |x_d|$). If Σ is a measurable space, then we let $P(\Sigma)$ denote the set of probability measures on Σ.

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Definition 1.1. If $\mu_1, \mu_2 \in P(F^N)$ with N finite, then

$$\bar{d}(\mu_1, \mu_2) = \inf_{\nu \in P(F^N \times F^N): \nu_1 = \mu_1, \nu_2 = \mu_2} \left\{ E^\nu \left(\frac{1}{N} \sum_{i=1}^{N} 1_{\{\eta_i \neq \xi_i\}} \right) \right\},$$

where ν_1 and ν_2 are the first and second marginal of ν, respectively, E^ν denotes expectation with respect to ν, and a typical element of $F^N \times F^N$ is denoted by $(\{\eta_i\}_{i=1}^N, \{\xi_i\}_{i=1}^N)$.

The inf runs over all couplings (or joinings) ν of μ_1 and μ_2. The r.h.s. without the inf measures the expected percentage of errors under the coupling ν.

Definition 1.2. An *abstract dynamical system* is a quadruple $(\Omega, \mathcal{B}, \mu, \pi_G)$, where Ω is a set, $\mathcal{B}$ is a σ-field of subsets of Ω, μ is a probability measure on $\mathcal{B}$, and π_G is a group action of G on Ω by bijective bimeasurable measure-preserving transformations.

Definition 1.3. Two dynamical systems $(\Omega, \mathcal{B}, \mu, \pi_G)$ and $(\Omega', \mathcal{B}', \mu', \psi_G)$ are *isomorphic* if there are G-invariant measurable sets $B \subseteq \Omega$ and $B' \subseteq \Omega'$ with $\mu(B) = \mu'(B') = 1$ such that π_g and ψ_g are bijective for all g when restricted to B and B', respectively, and such that there exists a bimeasurable measure-preserving mapping $f : B \to B'$ satisfying $f(\pi_g(x)) = \psi_g(f(x))$ for all $g \in G$ and $x \in B$.

A stationary process with index set $\mathbf{Z}^d$ and finite state space F is an example of a dynamical system, namely one where $\Omega = F^{\mathbf{Z}^d}$, $\mathcal{B}$ is the collection of Borel subsets of Ω, μ is the measure on Ω corresponding to the joint distribution of the process, and $\pi_{\mathbf{Z}^d}$ is the canonical action of $\mathbf{Z}^d$ on Ω. (The stationarity of the process is equivalent to this action being measure-preserving.) These are the only dynamical systems that will be considered in this paper. (It is important to keep in mind that the group acting on a stationary process, viewed as a dynamical system, is the index set for the stationary process.)

1.2. *Ergodic, mixing, Bernoulli and very weak Bernoulli.* In order to distinguish between dynamical systems, there are a number of important ergodic theoretic properties, all of which are isomorphism invariants. Some examples, in increasing order of strength, now follow.

Definition 1.4. A dynamical system $(\Omega, \mathcal{B}, \mu, \pi_{\mathbf{Z}^d})$ is *ergodic* if $\mu(A) = 0$ or 1 for all $A \in \mathcal{B}$ that are $\mathbf{Z}^d$-invariant, i.e. $\pi_g A = A$ for all $g \in \mathbf{Z}^d$.

Definition 1.5. A dynamical system $(\Omega, \mathcal{B}, \mu, \pi_{\mathbf{Z}^d})$ is *mixing* if $\lim_{|g| \to \infty} \mu(\pi_g(A) \cap B) = \mu(A)\mu(B)$ for all $A, B \in \mathcal{B}$.

Definition 1.6. A dynamical system $(\Omega, \mathcal{B}, \mu, \pi_{\mathbf{Z}^d})$ is *Bernoulli* if it is isomorphic to $((W, \mathcal{F}, p)^{\mathbf{Z}^d}, \pi_{\mathbf{Z}^d})$ for some Lebesgue space $(W, \mathcal{F}, p)$.

To define very weak Bernoulli, we need to place a linear order $\prec^d$ on $\mathbf{Z}^d$. This ordering will be defined by induction on d and can be thought of as a backwards dictionary ordering. For $d = 1$, we use the natural linear order $<$. For $d \geq 2$, we define $\prec^d$ by induction via

$$(x_1, \ldots, x_d) \prec^d (y_1, \ldots, y_d)$$

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if and only if

$$x_d < y_d, \text{ or}$$
$$x_d = y_d \text{ and } (x_1, \ldots, x_{d-1}) \prec^{d-1} (y_1, \ldots, y_{d-1}).$$

For $x \in \mathbf{Z}^d$, let

$$\text{Past}(x) = \{y \in \mathbf{Z}^d : y \prec^d x\}.$$

We would like to define the past also for a box B_n. This is done as follows. We tile the lattice $\mathbf{Z}^d$ by B_n and disjoint translates of B_n. There is then a natural mapping f from $\mathbf{Z}^d$ onto itself which maps each translate of B_n to a single point and sends B_n to the origin (i.e. f collapses each translate of B_n to a point so that the relative position of two translates corresponds to the relative position of their images under f). Past(B_n) is now defined to be f^{-1}(Past(0)), where Past(0) is defined above.

If $S \subseteq \mathbf{Z}^d$ is finite, then we let $\sigma(S)$ denote the sub σ-field on $\Omega = F^{\mathbf{Z}^d}$ generated by the atoms $\{\omega \in \Omega : \omega(x) = \omega_x, x \in S\}$ $(\omega_1, \ldots, \omega_{|S|} \in F)$ and we let Atom(S) denote the collection of these $|F|^{|S|}$ atoms.

Definition 1.7. A $\mathbf{Z}^d$-invariant $\mu \in P(F^{\mathbf{Z}^d})$ is *very weak Bernoulli* (VWB) if for all $\epsilon > 0$ there exists an N such that: if $n \geq N$ and $S \subseteq \text{Past}(B_n)$ with S finite, then

$$\bar{d}(\mu|_{B_n}, \mu|_{B_n}/A) < \epsilon$$

for all $A \in \text{Atom}(S)$, except for an ϵ-portion as measured by μ.

Here $\mu|_{B_n}$ denotes the measure on $F^{|B_n|}$ obtained by projecting μ onto B_n, $\mu|_{B_n}/A$ means $\mu|_{B_n}$ conditioned on A, and the above proviso means that the union of the atoms A where the above inequality fails has μ-measure $\leq \epsilon$.

We now mention an important theorem. For $d = 1$ this theorem is due to [**O**] and [**OW3**]. For general d, we refer to [**C**], [**K**], [**KW**] and [**T**].

THEOREM 1.8. *If $\mu \in P(F^{\mathbf{Z}^d})$ is $\mathbf{Z}^d$-invariant and ergodic, then Bernoulli and VWB are equivalent.*

(Actually, the version of VWB given in [**K**] is formally weaker than the one in Definition 1.7 but easily shown to be equivalent.)

1.3. *Entropy.* We next need to discuss the notion of entropy, which is also an isomorphism invariant. Although we shall give the necessary definitions here, we refer the reader to [**W**] for a complete discussion.

Definition 1.9. Let $\mathcal{P} = \{P_1, \ldots, P_s\}$ and $\mathcal{Q} = \{Q_1, \ldots, Q_t\}$ be two finite partitions of a measure space $(\Omega, \mathcal{B}, \mu)$. Then the *conditional entropy of $\mathcal{P}$ given $\mathcal{Q}$* is given by

$$H(\mathcal{P}|\mathcal{Q}) = -\sum_{i,j} \mu(P_i \cap Q_j) \log \frac{\mu(P_i \cap Q_j)}{\mu(Q_j)}.$$

Definition 1.10. If $\mu \in P(F^{\mathbf{Z}^d})$ is $\mathbf{Z}^d$-invariant (or equivalently, a stationary process with state space F and index set $\mathbf{Z}^d$), then the *entropy* of μ is given by

$$H(\mu) = \lim_{m \to \infty} H(\mathcal{P}|\mathcal{Q}_m),$$

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where $\mathcal{P}$ is the canonical partition of $F^{\mathbf{Z}^d}$ into $|F|$ sets corresponding to the coordinate x_0 and $\mathcal{Q}_m$ is the partition of $F^{\mathbf{Z}^d}$ into $|F|^m$ sets corresponding to the coordinates $\{x_1, \ldots, x_m\}$ with Past(0) enumerated as $\{x_i\}_{i=1}^{\infty}$ (in any order).

It can be shown that the above definition is independent of the enumeration of Past(0). It is not immediate that entropy is an isomorphism invariant, but, in fact, it is (see [**W**] for the case $d = 1$).

The following theorem is the celebrated isomorphism theorem due to [**O**]. For general d, we refer to [**C**], [**KW**], [**OW1**] and [**T**].

THEOREM 1.11. *If two Bernoulli shifts have the same entropy, then they are isomorphic.*

1.4. *Følner independence.* A strengthening of the VWB property was introduced in [**A**] under the name Følner independence (with the aim of extending the results of [**OW4**] to an amenable group). It is this property that was verified in [**OW4**], although no name was attached to it.

Definition 1.12. A $\mathbf{Z}^d$-invariant $\mu \in P(F^{\mathbf{Z}^d})$ is *Følner independent* (FI) if for all $\epsilon > 0$ there exists an N such that: if $n \geq N$ and $S \subseteq B_n^c$ with S finite, then

$$\bar{d}(\mu|_{B_n}, \mu|_{B_n}/A) < \epsilon$$

for all $A \in \text{Atom}(S)$, except for an ϵ-portion as measured by μ.

It is clear from Definitions 1.7 and 1.12 that FI implies VWB. On the other hand, the construction in [**OW2**] shows that VWB does not imply FI. The last statement implies that, while VWB is an isomorphism invariant (in view of Definition 1.6 and Theorem 1.8), FI is not. Indeed, the example in [**OW2**] is VWB and therefore isomorphic to an i.i.d. process. However, it is not FI and therefore we have two isomorphic processes one of which is FI and the other which is not.

1.5. *K-Automorphisms and trivial full tail.* The next important set of processes we introduce are the so-called K-Automorphisms, named after Kolmogorov. If we are given a translation invariant probability measure μ on $\Omega = F^{\mathbf{Z}^d}$ and a finite partition $\mathcal{Q}$ of Ω, then this induces a translation invariant probability measure $\nu_{\mathcal{Q}}$ on $\mathcal{Q}^{\mathbf{Z}^d}$ in a natural way, which is called a factor process and which in terms of stationary stochastic processes is defined as follows. If $\{X_x\}_{x \in \mathbf{Z}^d}$ corresponds to μ, then let $\{Y_y\}_{y \in \mathbf{Z}^d}$ be defined as a function of $\{X_x\}_{x \in \mathbf{Z}^d}$ by putting Y_y equal to the partition element of $\mathcal{Q}$ that $\tau_y(\{X_x\}_{x \in \mathbf{Z}^d})$ lies in (where τ_y is the translation by y).

Definition 1.13. A $\mathbf{Z}^d$-invariant $\mu \in P(F^{\mathbf{Z}^d})$ is called a *K-Automorphism* (K) if $\nu_{\mathcal{Q}}$ has positive entropy for every nontrivial partition $\mathcal{Q}$ of Ω.

It is easy to see that K is an isomorphism invariant. For $d = 1$, it is known that $\{X_x\}_{x \in \mathbf{Z}}$ is K if and only if the right tail

$$\bigcap_{x \geq 1} \sigma(X_x, X_{x+1}, \ldots)$$

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is trivial in that it contains only sets of probability 0 or 1 (and equivalently for the left tail). The proof of this fact is nontrivial (see [**W**]). There is an analogous result in higher dimensions (see [**C**] or [**S**]), although the analogue of a right tail is somewhat cumbersome. Restricting to $d = 2$ for simplicity, the result becomes the following, which can be proved in a manner similar to that for $d = 1$.

THEOREM 1.14. *A $\mathbf{Z}^2$-invariant $\mu \in P(F^{\mathbf{Z}^2})$ is K if and only if the σ-algebra T' is trivial, where T' is defined by*

$$T' = \sigma\left(\bigcup_{m\geq 1} T'_m\right)$$
$$T'_m = \bigcap_{n\geq 1} T'_{m,n},$$

with $T'_{m,n}$ being the σ-algebra generated by the coordinates $\{(x_1, x_2) : x_2 \leq -n$ or $(x_1 \leq -n$ and $x_2 \leq m)\}$.

The last property to be recalled is the following.

Definition 1.15. A stationary process $\{X_x\}_{x\in\mathbf{Z}^d}$ is said to have a *trivial full tail* (TFT) if $T = \bigcap_{m\geq 1} T_m$ is trivial, where $T_m = \sigma(X_x, x \in B_m^c)$.

From Theorem 1.14 and Definition 1.15 it is obvious that if a stationary process has a TFT, then it is K. On the other hand, the construction in [**OW2**] again shows that the converse is false.

There is no ordering between VWB and TFT, i.e. neither implies the other (see [**OS**] and [**OW2**]).

2. *The main theorems*

The goal of this paper is to show that the converses of 'FI implies VWB' and 'TFT implies K', though not true in general as we said above, are true for Markov random fields.

Definition 2.1. A stationary process $\{X_x\}_{x\in\mathbf{Z}^d}$ is called a *Markov random field* (MRF) if for all n the conditional distribution of X_{B_n} given $X_{B_n^c}$ is the same as the conditional distribution of X_{B_n} given $X_{\partial(B_n)}$ (with the abbreviation $X_U = \{X_x\}_{x\in U}$ for $U \subseteq \mathbf{Z}^d$).

We now state two of our main theorems.

THEOREM 2.2. *If a MRF is VWB, then it is FI.*

THEOREM 2.3. *If a MRF is K, then it has a TFT.*

Recalling Definition 1.6, Theorems 1.8 and 1.14 (together with Kolmogorov's 0–1 law for independent random variables and the fact that K is an isomorphism invariant), we see that VWB implies K. The reverse is not true in general (see [**OS**]). It was conjectured by J. P. Thouvenot that the reverse holds for MRF's, but this conjecture is still open (except in $d = 1$ of course). If this conjecture were true, then Theorems 2.2 and 2.3 would yield that K, TFT, VWB and FI are all equivalent for MRF's.

It does not follow from anything above that FI implies TFT in general. We now state this fact where the Markovian assumption is *not* required.

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THEOREM 2.4. *If a $\mathbf{Z}^d$-invariant $\mu \in P(F^{\mathbf{Z}^d})$ is FI, then it has a TFT.*

We point out that if Thouvenot's conjecture were true, then together with Theorems 2.2 and 2.4 it would imply Theorem 2.3. However, we prove Theorem 2.3 directly.

The rest of this paper is devoted to the proofs of these theorems.

3. *Proofs*

3.1. *Proof of Theorem 2.2.* If a process is VWB, then it is Bernoulli by Theorem 1.8. However, Bernoulli is in turn equivalent to the following condition, called *extremality* (see [**K**] or [**OW1**]; this condition is actually called weak extremality in [**K**]).

Definition 3.1. A $\mathbf{Z}^d$-invariant $\mu \in P(F^{\mathbf{Z}^d})$ is called *extremal* if for all $\epsilon > 0$ there exists a $\delta > 0$ such that: for all large n and for all decompositions of μ_{B_n} of the form

$$\mu_{B_n} = \sum_{i=1}^{M} p_i \mu_i,$$

with $(p_1, \ldots, p_M)$ a probability vector and $M \leq 2^{\delta |B_n|}$, most of the μ_i's are $\bar{d}$-close to μ_{B_n} in the sense that

$$\sum_{i:\bar{d}(\mu_{B_n},\mu_i)<\epsilon} p_i > 1 - \epsilon.$$

In other words, extremality means that any 'not too large' decomposition of the measure on large blocks must have almost all components close to the original measure. This may be compared with the (much weaker) notion of ergodicity, which says that there is no nontrivial decomposition of the measure into $\mathbf{Z}^d$-invariant measures on the whole space.

So we must show that a translation invariant MRF μ on $F^{\mathbf{Z}^d}$ that is extremal is also FI. This proceeds as follows. We first prove a technical lemma.

LEMMA 3.2. *If $S \subseteq B_n^c$ is finite and $\bar{d}(\mu|_{B_n}, \mu|_{B_n}/A) < \epsilon$ for all $A \in \mathrm{Atom}(S)$ except for an ϵ-portion as measured by μ, then the same is true for any $S' \subseteq S$ with ϵ replaced by $2\sqrt{\epsilon}$.*

Proof. Let (A_i) be the collection of atoms in $\mathrm{Atom}(S)$. Call A_i 'bad' if $\bar{d}(\mu|_{B_n}, \mu|_{B_n}/A_i) \geq \epsilon$, and let $a = \{1 \leq i \leq |F|^{|S|} : A_i \text{ is bad}\}$, $A = \bigcup_{i \in a} A_i$. Pick any $S' \subseteq S$ and let (B_j) be the collection of atoms in $\mathrm{Atom}(S')$. Call B_j 'bad' if $\mu(B_j \cap A)/\mu(B_j) \geq \sqrt{\epsilon}$, and let $b = \{1 \leq j \leq |F|^{|S'|} : B_j \text{ is bad}\}$, $B = \bigcup_{j \in b} B_j$. We shall show that:

(1) $\mu(B) \leq \sqrt{\epsilon}$;

(2) for $j \notin b$: $\bar{d}(\mu|_{B_n}, \mu|_{B_n}/B_j) < \epsilon + \sqrt{\epsilon}$.

This will prove the claim.

Property (1) follows from $\epsilon \geq \mu(A) \geq \sum_{j \in b} \mu(B_j \cap A) \geq \sqrt{\epsilon} \sum_{j \in b} \mu(B_j) = \sqrt{\epsilon}\mu(B)$. Property (2) goes as follows. Each atom B_j in the coarser partition is a union $\bigcup_{i \in I(j)} A_i$ of atoms in the finer partition. By coupling within ϵ on the A_i's with $i \in I(j) \cap a^c$ and coupling arbitrarily on the A_i's with $i \in I(j) \cap a$, we obtain

$$\bar{d}(\mu|_{B_n}, \mu|_{B_n}/B_j) \leq \frac{1}{\mu(B_j)} \left[\epsilon \mu \left(\bigcup_{i \in I(j) \cap a^c} A_i \right) + 1 \mu \left(\bigcup_{i \in I(j) \cap a} A_i \right) \right].$$

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However, if $j \notin b$, then $\mu(\bigcup_{i\in I(j)\cap a} A_i) = \mu(B_j \cap A) < \sqrt{\epsilon}\mu(B_j)$ by the definition of b. □

Thus, by Lemma 3.2, to verify the FI-property it suffices to consider sets $S \subseteq B_n^c$ that contain $\partial(B_n)$. However, by the Markov property, if $S \subseteq B_n^c$ contains $\partial(B_n)$, then conditioning on S is the same as conditioning $\partial(B_n)$. Therefore, we need only verify the property when $S = \partial(B_n)$.

Now let $\epsilon > 0$ and choose δ as in the definition of extremality. For large n, the property in the definition of extremality comes into play and we have $|F|^{|\partial(B_n)|} \leq 2^{\delta|B_n|}$. For such n, we therefore write

$$\mu_{B_n} = \sum_{\eta\in F^{\partial(B_n)}} \mu(\eta)\mu^\eta,$$

where μ^η is the conditional distribution of μ on B_n given that the configuration on $\partial(B_n)$ is η. This is a special 'not too large' decomposition, and so by the extremality we have $\bar{d}(\mu|_{B_n}, \mu^\eta) < \epsilon$ for all η in $F^{\partial(B_n)}$, except for an ϵ-portion. This proves Theorem 2.2. □

3.2. *Proof of Theorem 2.3.* We shall prove the result only for $d = 2$, the extension to higher dimensions being straightforward. By Theorem 1.14 and Definition 1.15, we must show that $T = T'$ a.s. for MRF's, where T and T' are the σ-algebras defined in the respective statements. In order to do so, we appeal to Lemma 2.10 in [**BH**] (which is stated there only for $d = 1$ but whose proof for higher dimensions is identical). According to this lemma, since $T' \subseteq T$ it suffices to show that

$$H(X_{B_n}|T') = H(X_{B_n}|T) \quad \text{for all } n \geq 0.$$

Fix $n \geq 0$. Since $T'_n \subseteq T' \subseteq T$, it suffices to show that

$$H(X_{B_n}|T'_n) \leq H(X_{B_n}|T).$$

Let $k > 2n$ and $l \geq 0$ be integers. Let

$$C_{k,l} = \bigcup_{x,y:\,|x|\leq l,|y|\leq l} \{B_n + (kx, ky)\}.$$

The $(2l+1)^2$ translates of B_n comprising $C_{k,l}$ are disjoint and have distance $k - 2n$ between them. Let $r = k(l+1) - n$ and define

$$D_r = \{(i, j) : (i = \pm r, |j| < r) \text{ or } (|i| \leq r, j = r)\}.$$

(D_r consists of the left, right and top segments of the box B_r centered at the origin. The boundary of this box encloses $C_{k,l}$ and is a distance $k - 2n$ away from it.) Also define

$$E_r = \{(i, j) : j \leq -r\}.$$

We next order the $(2l+1)^2$ translates of B_n in $C_{k,l}$ in the usual ergodic theoretic way. Namely, we say that the box $B_n + (x, y)$ precedes $B_n + (x', y')$ if $(x, y) \prec^2 (x', y')$ in the sense of the linear ordering defined in §1.2. In this way, we get an ordering of the

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translates of B_n, which we enumerate as $B^1, B^2, \dots, B^{(2l+1)^2}$. The idea of the proof is to compute the conditional entropy

$$(*) = H(X_{C_{k,l}} \vee X_{D_r} | X_{E_r})$$

in *two different ways*, to derive an upper, respectively, lower bound for the two resulting expressions, and in this way obtain an inequality between these bounds. This inequality will then be exploited to complete the proof.

The first step is

$$(*) = H(X_{C_{k,l}} | X_{E_r}) + H(X_{D_r} | X_{E_r} \vee X_{C_{k,l}}).$$

Drop the second term. Develop the first term as

$$\begin{aligned} H(X_{C_{k,l}} | X_{E_r}) &= H\Big(\bigvee_{i=1}^{(2l+1)^2} X_{B^i} \Big| X_{E_r} \Big) \\ &= \sum_{i=1}^{(2l+1)^2} H(X_{B^i} | X_{E_r} \vee X_{B^1 \cup \cdots \cup B^{i-1}}). \end{aligned}$$

Clearly, each of the terms in the sum is bounded below by $H(X_{B_n} | T'_{n,k-n})$ (recall $T'_{n,n-k}$ from Theorem 1.14) and so

$$(*) \geq (2l+1)^2 H(X_{B_n} | T'_{n,k-n}).$$

The second step is

$$(*) = H(X_{D_r} | X_{E_r}) + H(X_{C_{k,l}} | X_{E_r} \vee X_{D_r}).$$

The first term is at most $|D_r| H(X_0) = (6r-1) H(X_0)$. The second term is, by the Markov property, equal to $H(X_{C_{k,l}} | T_r)$ (recall T_r from Definition 1.15). Develop this term as

$$\begin{aligned} H(X_{C_{k,l}} | T_r) &= H\Big(\bigvee_{i=1}^{(2l+1)^2} X_{B^i} \Big| T_r \Big) \\ &\leq \sum_{i=1}^{(2l+1)^2} H(X_{B^i} | T_r) \\ &\leq (2l+1)^2 H(X_{B_n} | T_{2r-(k-n)}). \end{aligned}$$

Here we use that the largest distance between $\partial(B_r)$ and the center of a translate B^i is $2r-(k-n)$. Thus we have

$$(*) \leq (6r-1) H(X_0) + (2l+1)^2 H(X_{B_n} | T_{2r-(k-n)}).$$

Altogether, the above two bounds for $(*)$ give us (recall the definition of r)

$$H(X_{B_n} | T'_{n,k-n}) \leq H(X_{B_n} | T_{2kl+k-n}) + \frac{(6k(l+1) - 6n - 1)}{(2l+1)^2} H(X_0).$$

Letting $l \to \infty$ while keeping k fixed, we get (via the backwards martingale convergence theorem) that

$$H(X_{B_n} | T'_{n,k-n}) \leq H(X_{B_n} | T).$$

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Next letting $k \to \infty$, we arrive at

$$H(X_{B_n}|T'_n) \leq H(X_{B_n}|T),$$

which was to be proved. □

3.3. *Proof of Theorem 2.4.* We shall again prove the result only for $d = 2$, the extension to higher dimensions being straightforward. To show that μ has a TFT, it suffices to show that for any k the following holds. For all $\epsilon > 0$ there exists an $n > k$ such that for any $S \subseteq B_n^c$ finite

$$\bar{d}(\mu|_{B_k}, \mu|_{B_k}/A) < \epsilon$$

for all $A \in \mathrm{Atom}(S)$, except for an ϵ-portion as measured by μ. (The equivalence of this property to TFT follows from standard arguments about tail σ-algebras; see e.g. Proposition 7.9 in [**G**].) Note that the above statement differs from the definition of FI in that the two measures are required to be $\bar{d}$-close on the box $B_k \subseteq B_n$ rather than on the box B_n. Since $\bar{d}$ measures the *fraction* of errors in the box, being $\bar{d}$-close on B_n does *not immediately* imply being $\bar{d}$-close on B_k. Below we shall show that being $\bar{d}$-close on B_k is nevertheless true, but this requires some argument.

Assume that we have an integer k and an $\epsilon > 0$ given to us. Choose an m satisfying $2m + 1 = l(2k + 1)$ for some positive integer l in such a way that the condition in the definition of FI holds on B_m but with ϵ replaced by $(\epsilon/|B_k|)^2$. Moreover, put $n = 3m+1$. (With such a choice of m and n, note that B_m can be perfectly tiled by l^2 translates of B_k, and B_n can be perfectly tiled by 9 translates of B_m.) Below we shall show that this n satisfies the following.

LEMMA 3.3. *If $S \subseteq B_n^c$ is finite, then*

$$\bar{d}(\mu|_{B_k}, \mu|_{B_k}/A) < 2(2\epsilon)^{1/4}$$

for all $A \in \mathrm{Atom}(S)$, except for a $2(2\epsilon)^{1/4}$-portion as measured by μ.

Clearly, this claim completes the proof of Theorem 2.4. □

3.4. *Proof of Lemma 3.3.* Let $S \subseteq B_n^c$. Then $S + B_m \subseteq B_m^c$ (because $n = 3m + 1$). Hence, by the choice of m and the definition of FI, we have

$$\bar{d}(\mu|_{B_m}, \mu|_{B_m}/A) < \left(\frac{\epsilon}{|B_k|}\right)^2$$

for all $A \in \mathrm{Atom}(S + B_m)$ except for an $(\epsilon/|B_k|)^2$-portion as measured by μ. Now fix an atom $A \in \mathrm{Atom}(S + B_m)$ satisfying $\bar{d}(\mu|_{B_m}, \mu|_{B_m}/A) < (\epsilon/|B_k|)^2$, and let ν^A denote a coupling of $\mu|_{B_m}$ and $\mu|_{B_m}/A$ satisfying this inequality, i.e.

$$\frac{1}{|B_m|} \sum_{x \in B_m} \nu^A(\eta_x \neq \xi_x) < \left(\frac{\epsilon}{|B_k|}\right)^2$$

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(recall Definition 1.1). Call a point $x \in B_m$ 'A-good' if $\nu^A(\eta_x \neq \xi_x) < \epsilon/|B_k|$. Then, by the above inequality, all but (at most) a fraction $\epsilon/|B_k|$ of the points $x \in B_m$ are A-good.

Next, let $B^1, \ldots, B^{l^2}$ denote the translates of B_k that tile B_m. Since $|B_m|(\epsilon/|B_k|) = \epsilon l^2$, it follows from the last observation that all but (at most) a fraction ϵ of the B^j's are such that all $x \in B^j$ are A-good. For such a B^j, it is clear that

$$\bar{d}(\mu|_{B^j}, \mu|_{B^j}/A) < \frac{\epsilon}{|B^j|} \leq \epsilon.$$

Thus we have proved that for all $A \in \text{Atom}(S + B_m)$ except for an ϵ-portion as measured by μ, all but a fraction ϵ of the B^j's have the property that $\bar{d}(\mu|_{B^j}, \mu|_{B^j}/A) < \epsilon$. The following lemma turns this statement around.

LEMMA 3.4. *For all but a fraction $\sqrt{2\epsilon}$ of the B^j's: $\bar{d}(\mu|_{B^j}, \mu|_{B^j}/A) \leq \epsilon$ for all $A \in \text{Atom}(S + B_m)$ except for a $\sqrt{2\epsilon}$-portion as measured by μ.*

Proof. (Fubini type argument.) Let (A_i) be the collection of atoms in $\text{Atom}(S + B_m)$. Define

$$c = \{(i, j) : 1 \leq i \leq |F|^{|S+B_m|}, 1 \leq j \leq l^2, \bar{d}(\mu|_{B^j}, \mu|_{B^j}/A_i) \geq \epsilon\}.$$

Then we are given that

$$\sum_i 1_{\{|\{j:(i,j)\in c\}| > \epsilon l^2\}} \mu(A_i) \leq \epsilon.$$

Hence (since $\sum_i \mu(A_i) = 1$),

$$\frac{1}{l^2} \sum_{i,j} 1_{\{(i,j)\in c\}} \mu(A_i) \leq 2\epsilon.$$

The last inequality gives us

$$\frac{1}{l^2} \left| \left\{ j : \sum_i 1_{\{(i,j)\in c\}} \mu(A_i) \geq \sqrt{2\epsilon} \right\} \right| \leq \sqrt{2\epsilon}.$$

This says that for all but a fraction $\sqrt{2\epsilon}$ of the B^j's: $\bar{d}(\mu|_{B^j}, \mu|_{B^j}/A_i) \geq \epsilon$ for all A_i except for a $\sqrt{2\epsilon}$-portion as measured by μ. □

Now fix a box B^j with the property as stated in Lemma 3.4. Let $x \in B_m$ be such that $B^j = B_k + x$. It then follows from Lemma 3.2 that $\bar{d}(\mu|_{B^j}, \mu|_{B^j}/A) < 2(2\epsilon)^{1/4}$ for all $A \in \text{Atom}(S + x)$ except for a $2(2\epsilon)^{1/4}$-portion as measured by μ. By stationarity this in turn gives that $\bar{d}(\mu|_{B_k}, \mu|_{B_k}/A) < 2(2\epsilon)^{1/4}$ for all $A \in \text{Atom}(S)$ except for a $2(2\epsilon)^{1/4}$-portion. This completes the proof of Lemma 3.3. □

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Note added in proof. Chris Hoffman has recently disproved Thouvenot's conjecture by constructing a MRF in $d \geq 2$ that is K but not VWB.

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REFERENCES

[A] S. Adams. Følner independence and the amenable Ising model. *Ergod. Th. & Dynam. Sys.* **12** (1992), 633–657.

[BH] H. C. P. Berbee and W. Th. F. den Hollander. Tail triviality for sums of stationary random variables. *Ann Probab.* **17** (1989), 1635–1645.

[C] J. P. Conze. Entropie d'un groupe abelien de transformations. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **25** (1972), 11–30.

[G] H.-O. Georgii. *Gibbs Measures and Phase Transitions*. De Gruyter, New York, 1988.

[K] J. Kammeyer. A complete classification of the two-point extensions of a multi-dimensional Bernoulli shift. *J. d'Analyse Math.* **54** (1990), 113–163.

[KW] Y. Katznelson and B. Weiss. Commuting measure-preserving transformations. *Israel J. Math.* **12** (1972), 161–173.

[O] D. S. Ornstein. *Ergodic Theory, Randomness and Dynamical Systems*. Yale University Press, New Haven, 1974.

[OS] D. S. Ornstein and P. Shields. An uncountable family of K-automorphisms. *Adv. Math.* **10** (1973), 63–88.

[OW1] D. S. Ornstein and B. Weiss. Entropy and isomorphism theorems for actions of amenable groups. *J. d'Analyse Math.* **48** (1987), 1–141.

[OW2] D. S. Ornstein and B. Weiss. Every transformation is bilaterally deterministic. *Israel J. Math.* **24** (1975), 154–158.

[OW3] D. S. Ornstein and B. Weiss. Finitely determined implies very weak Bernoulli. *Israel J. Math.* **17** (1974), 94–104.

[OW4] D. S. Ornstein and B. Weiss. $\mathbf{Z}^d$-actions and the Ising model. Unpublished manuscript, 1973.

[S] K. Schmidt. *Dynamical Systems of Algebraic Origin*. Birkhäuser, 1995.

[T] J. P. Thouvenot. Convergence en moyenne de l'information pour l'action de $\mathbf{Z}^2$. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **24** (1972), 135–137.

[W] P. Walters. *An Introduction to Ergodic Theory*. Springer, New York, 1975.

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