Mathematika 65 (2019) 557–587 © 2019 University College London
doi:10.1112/S0025579319000019

POINT DISTRIBUTIONS IN TWO-POINT HOMOGENEOUS SPACES

M. M. SKRIGANOV

Abstract. We consider point distributions in compact connected two-point homogeneous spaces (Riemannian symmetric spaces of rank one). All such spaces are known: they are the spheres in the Euclidean spaces, the real, complex and quaternionic projective spaces and the octonionic projective plane. For all such spaces the best possible bounds for the quadratic discrepancies and sums of pairwise distances are obtained in the paper (Theorems 2.1 and 2.2). Distributions of points of t-designs on such spaces are also considered (Theorem 2.3). In particular, it is shown that the optimal t-designs meet the best possible bounds for quadratic discrepancies and sums of pairwise distances (Corollary 2.1). Our approach is based on the Fourier analysis on two-point homogeneous spaces and explicit spherical function expansions for discrepancies and sums of distances (Theorems 4.1 and 4.2).

Contents

1 Introduction and preliminaries **557**
1.1 Introduction 557
1.2 Discrepancies and metrics. 559
2 Main results **563**
2.1 Two-point homogeneous spaces 563
2.2 Bounds for discrepancies and sums of distances 565
2.3 Applications to t-designs 568
3 Proof of Theorem 2.1 **571**
4 Proof of Theorem 2.2 and 2.3 **574**
4.1 Fourier analysis on two-point homogeneous spaces 574
4.2 Spherical function expansions for discrepancies and metrics . . 579
4.3 Bounds for Fourier–Jacobi coefficients 583
4.4 Proof of Theorems 2.2 and 2.3 585
Acknowledgement **586**
References **586**

§1. *Introduction and preliminaries.*

1.1. *Introduction.* Let $S^d = \{x \in \mathbb{R}^{d+1} : \|x\| = 1\}$ be the standard d-dimensional unit sphere in $\mathbb{R}^{d+1}$ with the geodesic (great circle) metric θ and

Received 29 August 2018, published online 26 March 2019.
MSC (2010): 11K38 (primary), 22F30, 52C99 (secondary).

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

the Lebesgue measure μ normalized by $\mu(S^d) = 1$. We write $\mathcal{C}(y,t) = \{x \in S^d : (x,y) > t\}$ for the spherical cap of height $t \in [-1,1]$ centered at $y \in S^d$; here $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^{d+1}$.

For an N-point subset $\mathcal{D}_N \subset S^d$, the spherical cap quadratic discrepancy is defined by

$$\lambda^{\mathrm{cap}}[\mathcal{D}_N] = \int_{-1}^{1} \int_{S^d} (\#\{\mathcal{C}(y,t) \cap \mathcal{D}_N\} - N\mu(\mathcal{C}(y,t)))^2 \, d\mu(y)\, dt. \tag{1.1}$$

We introduce the following sum of pairwise Euclidean distances $\|\cdot\|$ between points of $\mathcal{D}_N$:

$$\tau[S^d, \mathcal{D}_N] = \frac{1}{2} \sum_{x_1,x_2 \in \mathcal{D}_N} \|x_1 - x_2\| = \sum_{x_1,x_2 \in \mathcal{D}_N} \sin\frac{1}{2}\theta(x_1,x_2). \tag{1.2}$$

We write $\langle\tau\rangle$ for the average value distance on S^d:

$$\langle\tau\rangle_{S^d} = \frac{1}{2} \iint_{S^d \times S^d} \|y_1 - y_2\| \, d\mu(y_1)\, d\mu(y_2). \tag{1.3}$$

The study of the quantities (1.1) and (1.2) falls within the subjects of the discrepancy theory and geometry of distances; see [**2**, **5**] and references therein. It turns out that the quantities (1.1) and (1.2) are not independent and are intimately related by the remarkable identity

$$\gamma(S^d)\lambda^{\mathrm{cap}}[\mathcal{D}_N] + \tau[S^d, \mathcal{D}_N] = \langle\tau\rangle_{S^d} N^2 \tag{1.4}$$

for an arbitrary N-point subset $\mathcal{D}_N \subset S^d$. Here $\gamma(S^d)$ is a positive constant independent of $\mathcal{D}_N$,

$$\gamma(S^d) = \frac{d\sqrt{\pi}\,\Gamma(d/2)}{2\Gamma((d+1)/2)}. \tag{1.5}$$

The identity (1.4) was established by Stolarsky in [**30**], and is known in the literature as Stolarsky's invariance principle; see also [**8**] and [**12**].

For large N the exact order of the discrepancies (1.1) and sums of distances (1.2) is known. Alexander [**1**] has proved the existence of N-point subsets $\mathcal{D}_N \subset S^d$ such that

$$\tau[S^d, \mathcal{D}_N] > \langle\tau\rangle_{S^d} N^2 - CN^{1-1/d}. \tag{1.6}$$

By Stolarsky's invariance principle (1.4), this implies the bound

$$\lambda^{\mathrm{cap}}[\mathcal{D}_N] < CN^{1-1/d}. \tag{1.7}$$

On the other hand, Beck [**4**] (see also [**5**]) has established that

$$\lambda^{\mathrm{cap}}[\mathcal{D}_N] > cN^{1-1/d} \tag{1.8}$$

for any N-point subsets $\mathcal{D}_N \subset S^d$. In view of (1.4), this implies the bound

$$\tau[S^d, \mathcal{D}_N] < \langle\tau\rangle_{S^d} N^2 - cN^{1-1/d} \tag{1.9}$$

for any N-point subsets $\mathcal{D}_N \subset S^d$. The constants in the bounds (1.6)–(1.9) are independent of N. We refer to [**2**, **5**, **8**, **13**] for more details.

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

Spheres as homogeneous spaces $S^d = \mathrm{SO}(d+1)/\mathrm{SO}(d)$ are the simplest examples of compact connected two-point homogeneous spaces (compact Riemannian symmetric spaces of rank one). All such spaces are known [**6**, **21**, **22**, **33**]. Besides the spheres there are the real, complex, and quaternionic projective spaces and the octonionic projective plane.

The aim of the present paper is to extend the above sharp bounds (1.6)–(1.9) to all compact connected two-point homogeneous spaces. Stolarsky's invariance principle (1.4) was recently extended to projective spaces in [**29**].

The paper is organized as follows. The necessary facts on point distributions in compact metric measure spaces are collected in §1.2.

Our main results are stated and discussed in §2. First of all, in §2.1 we recall the basic definitions and facts on compact two-point homogeneous spaces. The extension of the bounds (1.6) and (1.7) to such spaces is stated in Theorem 2.1. The proof of Theorem 2.1 is given in §3. In fact, Theorem 2.1 is a direct corollary of a general result from [**27**, Theorem 4.2] on the existence of uniformly distributed point subsets in compact d-rectifiable spaces. Notice that the existence of very well-distributed point subsets is now established for many classes of metric measure spaces; see [**11**, **27**, **28**].

The extension of the bounds (1.8) and (1.9) is contained in Theorem 2.2. The proof of Theorem 2.2 is given in §4 based on Fourier analysis and the theory of spherical functions on two-point homogeneous spaces. Notice that the original proof of the bound (1.8) in [**4**, **5**] is based on Fourier analysis on $\mathbb{R}^{d+1}$. Spherical functions on S^d were used in [**7**, **8**] to prove Beck's bound (1.8).

Many specific point configurations on spheres and other two-point homogeneous spaces are described in the literature; see [**3**, **10**, **11**, **13**–**16**, **24**, **26**]. The question arises of whether the points of such specific configurations are distributed uniformly. In the present paper we consider this question for t-designs. In Theorem 2.3 the quadratic discrepancies and the sums of pairwise distances are estimated precisely for t-designs on spheres and projective spaces. In particular, we show in Corollary 2.1 that optimal t-designs on such spaces meet the best possible bounds for the quadratic discrepancies and the sums of pairwise distances. The proof of Theorem 2.3 is given in §4 also based on spherical functions.

In §4.1 the main formulas on spherical functions on two-point homogeneous spaces are given and discussed in considerable detail. These formulas are used in §4.2 to derive explicit spherical function expansions for the quadratic discrepancies and the sums of pairwise distances; see Theorems 4.1 and 4.2. In fact, these expansions are Fourier–Jacobi series. The exact bounds for the coefficients in such series are given in §4.3 in Lemma 4.2. Relying on these bounds, we complete the proofs of Theorems 2.2 and 2.3 in §4.4.

1.2. *Discrepancies and metrics.* Let $\mathcal{M}$ be a compact metric measure space with a fixed metric θ and a finite Borel measure μ, normalized by

$$\mathrm{diam}(\mathcal{M}, \theta) = \pi, \qquad \mu(\mathcal{M}) = 1, \tag{1.10}$$

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

where $\operatorname{diam}(\mathcal{E}, \rho) = \sup\{\rho(x_1, x_2) : x_1, x_2 \in \mathcal{E}\}$ denotes the diameter of a subset $\mathcal{E} \subseteq \mathcal{M}$ with respect to a metric ρ.

We write $B(r, y) = \{x : \theta(x, y) < r\}$ for the ball of radius $r \in \mathcal{I}$ centered at $y \in \mathcal{M}$ and of volume $v(r, y) = \mu(B(r, y))$; here $\mathcal{I} = \{r = \theta(x_1, x_2) : x_1, x_2 \in \mathcal{M}\}$ is the set of all possible radii. If $\mathcal{M}$ is connected, then $\mathcal{I} = [0, \pi]$. We can conveniently write $B(y, r) = \emptyset$ and $v(y, r) = 0$ if $r \leqslant 0$ and $B(y, r) = \mathcal{M}$ and $v(y, r) = 1$ if $r \geqslant \pi$.

Let $\mathcal{D}_N \subset \mathcal{M}$ be a subset (distribution) of N points, not necessarily different. The *local discrepancy* of $\mathcal{D}_N$ in the ball $B(r, y)$ is defined by

$$\Lambda[B(r, y), \mathcal{D}_N] = \#\{B(r, y) \cap \mathcal{D}_N\} - N v(r, y) = \sum_{x \in \mathcal{D}_N} \Lambda(B(r, y), x), \quad (1.11)$$

where

$$\Lambda(B(r, y), x) = \chi(B(r, y), x) - v(r, y)$$

and $\chi(\mathcal{E}, x)$ denotes the characteristic function of a subset $\mathcal{E} \subset \mathcal{M}$. The *quadratic discrepancy* is defined by

$$\lambda[\xi, \mathcal{D}_N] = \lambda[\mathcal{M}, \xi, \mathcal{D}_N] = \iint_{\mathcal{M} \times \mathcal{I}} \Lambda[B(r, y), \mathcal{D}_N]^2 \, d\mu(y) \, d\xi(r)$$
$$= \sum_{x_1, x_2 \in \mathcal{D}_N} \lambda(\xi, x_1, x_2), \quad (1.12)$$

where ξ is a measure on the set of radii $\mathcal{I}$ and

$$\lambda(\xi, y_1, y_2) = \int_{\mathcal{I}} \lambda_r(y_1, y_2) \, d\xi(r), \quad (1.13)$$

with

$$\lambda_r(y_1, y_2) = \int_{\mathcal{M}} \Lambda(B(r, y), y_1) \Lambda(B(r, y), y_2) \, d\mu(y). \quad (1.14)$$

The quantity $\lambda[\xi, \mathcal{D}_N]^{1/2}$ is known as the L_2-discrepancy. In the present paper it is more convenient to deal with the quadratic discrepancies (1.12).

We introduce the extremal characteristic

$$\lambda_N(\xi) = \lambda_N(\mathcal{M}, \xi) = \inf_{\mathcal{D}_N} \lambda[\xi, \mathcal{D}_N], \quad (1.15)$$

where the infimum is taken over all N-point subsets $\mathcal{D}_N \subset \mathcal{M}$.

It is worth noting that for the sphere S^d the quadratic discrepancies (1.1) and (1.12) are related by

$$\lambda^{\mathrm{cap}}[\mathcal{D}_N] = \lambda[S^d, \xi^\natural, \mathcal{D}_N] \quad (1.16)$$

with the specific measure $d\xi^\natural(r) = \sin r \, dr$.

For an arbitrary metric ρ on $\mathcal{M}$ we define the *sum of pairwise distances*

$$\rho[\mathcal{D}_N] = \sum_{x_1, x_2 \in \mathcal{D}_N} \rho(x_1, x_2), \quad (1.17)$$

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

and introduce the extremal characteristic

$$\rho_N = \rho_N(\mathcal{M}) = \sup_{\mathcal{D}_N} \rho[\mathcal{D}_N], \tag{1.18}$$

where the supremum is taken over all N-point subsets $\mathcal{D}_N \subset \mathcal{M}$. We write $\langle\rho\rangle$ for the average value of a metric ρ,

$$\langle\rho\rangle = \iint_{\mathcal{M}\times\mathcal{M}} \rho(y_1, y_2)\, d\mu(y_1)\, d\mu(y_2). \tag{1.19}$$

The *symmetric difference metric* $\theta^\Delta(\xi)$ on the space $\mathcal{M}$ is defined by

$$\theta^\Delta(\xi, y_1, y_2) = \int_{\mathcal{I}} \theta_r^\Delta(y_1, y_2)\, d\xi(r), \tag{1.20}$$

where

$$\theta_r^\Delta(y_1, y_2) = \frac{1}{2}\int_{\mathcal{M}} \chi(B(r, y_1)\Delta B(r, y_2), y)\, d\mu(y) = \frac{1}{2}\mu(B(r, y_1)\Delta B(r, y_2)), \tag{1.21}$$

and

$$B(r, y_1)\Delta B(r, y_2) = B(r, y_1) \cup B(r, y_2) \setminus B(r, y_1) \cap B(r, y_2)$$

is the symmetric difference of the balls $B(r, y_1)$ and $B(r, y_2)$. We have

$$\chi(B(r, y_1)\Delta B(r, y_2), y) = |\chi(B(r, y_1), y) - \chi(B(r, y_2), y)|. \tag{1.22}$$

Therefore,

$$\theta^\Delta(\xi, y_1, y_2) = \frac{1}{2}\iint_{\mathcal{M}\times\mathcal{I}} |\chi(B(r, y_1), y) - \chi(B(r, y_2), y)|\, d\mu(y)\, d\xi(r) \tag{1.23}$$

is the L_1-distance between the characteristic functions of the balls $B(r, y_1)$ and $B(r, y_2)$. On the other hand, we have

$$\begin{aligned}\chi(B(r, y_1)\Delta B(r, y_2), y) = {}& \chi(B(r, y_1), y) + \chi(B(r, y_2), y) \\ & - 2\chi(B(r, y_1), y)\chi(B(r, y_2), y).\end{aligned} \tag{1.24}$$

Substituting (1.24) into (1.21), we obtain

$$\begin{aligned}\theta^\Delta(\xi, y_1, y_2) = \frac{1}{2}\iint_{\mathcal{M}\times\mathcal{I}} {}& \big(\chi(B(r, y_1), y) + \chi(B(r, y_2), y) \\ & - 2\chi(B(r, y_1), y)\chi(B(r, y_2), y)\big)\, d\mu(y)\, d\xi(r).\end{aligned} \tag{1.25}$$

The symmetry of the metric θ implies the relation

$$\chi(B(r, y), x) = \chi(B(r, x), y) = \chi(r - \theta(x, y)) = \chi_r(\theta(x, y)), \tag{1.26}$$

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

where $\chi(z)$, $z \in \mathbb{R}$, is the characteristic function of the half-axis $(0, \infty)$, and $\chi_r(\cdot)$ is the characteristic function of the interval $[0, r)$, $0 \leqslant r \leqslant \pi$. Using (1.26) in (1.25), we obtain the following formula for the average values (1.19) of the metric (1.21):

$$\langle \theta^{\Delta}(\xi) \rangle = \langle \theta^{\Delta}(\mathcal{M}, \xi) \rangle = \int_{\mathcal{I}} \langle \theta_r^{\Delta} \rangle \, d\xi(r), \tag{1.27}$$

where

$$\langle \theta_r^{\Delta} \rangle = \langle \theta_r^{\Delta}(\mathcal{M}) \rangle = \int_{\mathcal{M}} (v(r, y) - v(r, y)^2) \, d\mu(y) = \int_{\mathcal{M}} v(r, y) v'(r, y) \, d\mu(y); \tag{1.28}$$

here $v'(r, y) = 1 - v(r, y) = \mu(B'(r, y))$, and $B'(r, y) = \mathcal{M} \setminus B(r, y)$ denotes the complement of $B(r, y)$.

A metric measure space $\mathcal{M}$ is called *distance-invariant* if the volume of any ball $v(r) = v(r, y)$ is independent of $y \in \mathcal{M}$; see [**24**, p. 504]. For such spaces the above formulas for the discrepancies and the symmetric difference metrics can be essentially simplified. A simple calculation leads to

$$\left.\begin{aligned} &\lambda_r(y_1, y_2) = \mu(B(r, y_1) \cap B(r, y_2)) - v(r)^2, \\ &\theta_r^{\Delta}(y_1, y_2) = v(r) - \mu(B(r, y_1) \cap B(r, y_2)), \\ &\langle \theta_r^{\Delta} \rangle = v(r) - v(r)^2 = v(r) v'(r). \end{aligned}\right\} \tag{1.29}$$

Integrating these relations with a measure ξ, one can obtain the corresponding formulas for the quantities (1.13), (1.20), and (1.27). Comparing the formulas (1.29), we obtain the identities

$$\left.\begin{aligned} &\lambda_r(y_1, y_2) + \theta_r^{\Delta}(y_1, y_2) = \langle \theta_r^{\Delta} \rangle, \\ &\lambda(\xi, y_1, y_2) + \theta^{\Delta}(\xi, y_1, y_2) = \langle \theta^{\Delta}(\xi) \rangle. \end{aligned}\right\} \tag{1.30}$$

In line with the definitions (1.17) and (1.18), we put

$$\theta^{\Delta}[\xi, \mathcal{D}_N] = \sum_{x_1, x_2 \in \mathcal{D}_N} \theta^{\Delta}(\xi, x_1, x_2), \tag{1.31}$$

$$\theta_N^{\Delta}(\xi) = \theta_N^{\Delta}(\xi, \mathcal{M}) = \sup_{\mathcal{D}_N} \theta^{\Delta}[\xi, \mathcal{D}_N], \tag{1.32}$$

where the supremum is taken over all N-point subsets $\mathcal{D}_N \subset \mathcal{M}$.

Comparing the identities (1.30) with the definitions (1.12) and (1.31), we obtain the L_1-invariance principle

$$\lambda[\xi, \mathcal{D}_N] + \theta^{\Delta}[\xi, \mathcal{D}_N] = \langle \theta^{\Delta}(\xi) \rangle N^2, \tag{1.33}$$

for an arbitrary N-point subset $\mathcal{D}_N \in \mathcal{M}$. Our terminology for L_q-invariance principles will be explained below in the comments on the relation (2.9). From (1.15), (1.32) and (1.33), we also obtain the identity

$$\lambda_N(\mathcal{M}, \xi) + \theta_N^{\Delta}(\mathcal{M}, \xi) = \langle \theta^{\Delta}(\mathcal{M}, \xi) \rangle N^2. \tag{1.34}$$

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

The typical examples of distance-invariant spaces are homogeneous spaces $\mathcal{M} = G/K$, where G is a compact group, $K \subset G$ is a closed subgroup, while θ and μ are the G-invariant metric and measure on $\mathcal{M}$. In this case, the quantities (1.13), (1.14) and (1.20), (1.21) are also G-invariant.

§2. *Main results.*

2.1. *Two-point homogeneous spaces.* Recall the definition and necessary facts on two-point homogeneous spaces; for more details, we refer to [6, 21–23, 33, 34]. Let $G = G(\mathcal{M})$ be the group of isometries of a metric space $\mathcal{M}$ with a metric θ, that is, $\theta(gx_1, gx_2) = \theta(x_1, x_2)$ for all $x_1, x_2 \in \mathcal{M}$ and $g \in G$. The space $\mathcal{M}$ is called *two-point homogeneous* if for any two pairs of points x_1, x_2 and y_1, y_2 with $\theta(x_1, x_2) = \theta(y_1, y_2)$ there exists an isometry $g \in G$, such that $y_1 = gx_1$, $y_2 = gx_2$. In this case, the group G is obviously transitive on $\mathcal{M}$ and $\mathcal{M} = G/K$ is a homogeneous space, where the subgroup $K \subset G$ is the stabilizer of a point $x_0 \in \mathcal{M}$. Furthermore, the homogeneous space $\mathcal{M}$ is symmetric, that is, for any two points $y_1, y_2 \in \mathcal{M}$ there exists an isometry $g \in G$, such that $gy_1 = y_2$, $gy_2 = y_1$.

We consider compact connected two-point homogeneous spaces $\mathcal{M} = G/K$. It is known that for such spaces G and $K \subset G$ are Lie groups and $\mathcal{M} = G/K$ are Riemannian symmetric spaces of rank one. This means that all flat totally geodesic submanifolds in $\mathcal{M}$ are one-dimensional and coincide with geodesics. This also means that all G-invariant differential operators on $\mathcal{M}$ are polynomials of the Laplace–Beltrami operator on $\mathcal{M}$. All such spaces are classified completely; see, for example, [33, §8.12]. They are as follows:

(i) the d-dimensional Euclidean spheres $S^d = \mathrm{SO}(d+1)/\mathrm{SO}(d) \times \{1\}$, $d \geqslant 2$, and $S^1 = O(2)/O(1) \times \{1\}$;
(ii) the real projective spaces $\mathbb{R}P^n = O(n+1)/O(n) \times O(1)$;
(iii) the complex projective spaces $\mathbb{C}P^n = U(n+1)/U(n) \times U(1)$;
(iv) the quaternionic projective spaces $\mathbb{H}P^n = \mathrm{Sp}(n+1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1)$;
(v) the octonionic projective plane $\mathbb{O}P^2 = F_4/\mathrm{Spin}(9)$.

Here we use the standard notation from the theory of Lie groups; in particular, F_4 is one of the exceptional Lie groups in Cartan's classification.

As a compact Riemannian manifold the projective space $\mathbb{F}P^n$ has dimension

$$d = \dim_{\mathbb{R}} \mathbb{F}P^n = nd_0, \qquad d_0 = \dim_{\mathbb{R}} \mathbb{F}, \tag{2.1}$$

where $d_0 = 1, 2, 4, 8$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, correspondingly.

For the spheres S^d we put $d_0 = d$ by definition. Projective spaces of dimension d_0 ($n = 1$) are isomorphic to the spheres S^{d_0}: $\mathbb{R}P^1 \approx S^1$, $\mathbb{C}P^1 \approx S^2$, $\mathbb{H}P^1 \approx S^4$, $\mathbb{O}P^1 \approx S^8$. We can conveniently agree that $d > d_0$ ($n \geqslant 2$) for projective spaces, while the equality $d = d_0$ holds only for spheres. Under this convention, the dimensions $d = nd_0$ and d_0 define uniquely (up to isomorphism) the corresponding two-point homogeneous space which we denote by $Q = Q(d, d_0)$.

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

We consider $Q(d, d_0)$ as a metric measure space with a metric θ and a measure μ proportional to the invariant Riemannian distance and measure on $Q(d, d_0)$. The coefficients of proportionality are defined to satisfy (1.10). In what follows we always assume that $n = 2$ if $\mathbb{F} = \mathbb{O}$, since projective spaces $\mathbb{O}P^n$ do not exist for $n > 2$. Any space $Q(d, d_0)$ is distance-invariant and the volume of balls is given by

$$v(r) = \kappa(d, d_0) \int_0^r \left(\sin\tfrac{1}{2}u\right)^{d-1} \left(\cos\tfrac{1}{2}u\right)^{d_0-1} du, \quad r \in [0, \pi], \tag{2.2}$$

where $\kappa(d, d_0) = B(d/2, d_0/2)^{-1}$; $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ and $\Gamma(a)$ are the beta and gamma functions (see [19, pp. 177–178], [22, pp. 165–168], [24, pp. 508–510]).

The following extension of Stolarsky's invariance principle (1.4) to all two-point homogeneous spaces $Q(d, d_0)$ is given in [29]. The *chordal metric* on the spaces $Q(d, d_0)$ can be defined by

$$\tau(x_1, x_2) = \sin\tfrac{1}{2}\theta(x_1, x_2), \quad x_1, x_2 \in Q(d, d_0). \tag{2.3}$$

Notice that expression (2.3) defines a metric because the function $\varphi(\theta) = \sin\theta/2$, $0 \leqslant \theta \leqslant \pi$, is concave, increasing and $\varphi(0) = 0$, which implies the triangle inequality; see, for example, [17, Lemma 9.0.2]. For the sphere S^d, we have

$$\left.\begin{aligned} &\cos\theta(x_1, x_2) = (x_1, x_2), x_1, x_2 \in S^d, \\ &\tau(x_1, x_2) = \sin\tfrac{1}{2}\theta(x_1, x_2) = \tfrac{1}{2}\|x_1 - x_2\|, \end{aligned}\right\} \tag{2.4}$$

where $(\cdot, \cdot)$ is the inner product and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^{d+1}$.

Each projective space $\mathbb{F}P^n$ can be canonically embedded into the unit sphere

$$\Pi : Q(d, d_0) \ni x \to \Pi(x) \in S^{m-1} \subset \mathbb{R}^m, \quad m = \tfrac{1}{2}(n+1)(d+2), \tag{2.5}$$

such that

$$\tau(x_1, x_2) = \frac{1}{\sqrt{2}}\|\Pi(x_1) - \Pi(x_2)\|, \quad x_1, x_2 \in \mathbb{F}P^n, \tag{2.6}$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^m$. Hence, the metric $\tau(x_1, x_2)$ coincides with the Euclidean length of a segment joining the points $\Pi(x_1)$ and $\Pi(x_2)$ on the unit sphere and normalized by $\operatorname{diam}(Q(d, d_0), \tau) = 1$. The space $Q(d, d_0)$ canonically embedded by (2.5) is a smooth connected compact affine algebraic submanifold in $\mathbb{R}^m$. An explicit description of the embedding (2.5) in the terms of the present paper can be found in [29].

The chordal metric τ on the complex projective space $\mathbb{C}P^n$ is known as the Fubini–Study metric. In connection with special point configurations in two-point homogeneous spaces the chordal metric on projective spaces has been discussed in the papers [14, 15]; see also the paper [16], where the chordal metric has been defined for Grassmannian manifolds.

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

A careful analysis of the embedding (2.5) shows that for any space $Q = Q(d, d_0)$ the chordal metric (2.3) and the symmetric difference metric (1.20) are related by

$$\tau(x_1, x_2) = \gamma(Q)\theta^{\Delta}(\xi^{\natural}, x_1, x_2), \quad x_1, x_2 \in Q, \tag{2.7}$$

where $d\xi^{\natural}(r) = \sin r\, dr$, $r \in \mathcal{I}$, and

$$\gamma(Q(d, d_0)) = \frac{\sqrt{\pi}}{4}(d + d_0)\frac{\Gamma(d_0/2)}{\Gamma((d_0+1)/2)} = \frac{d + d_0}{2d_0}\gamma(S^{d_0}), \tag{2.8}$$

with $\gamma(S^{d_0})$ defined by (1.5). Comparing (2.7) with (1.33) and (1.34), we obtain the L_2-invariance principle

$$\gamma(Q)\lambda[\xi^{\natural}, \mathcal{D}_N] + \tau[\mathcal{D}_N] = \langle\tau\rangle N^2, \tag{2.9}$$

for an arbitrary N-point subset $\mathcal{D}_N \subset Q$, and

$$\gamma(Q)\lambda_N(\xi^{\natural}) + \tau_N(Q) = \langle\tau\rangle N^2. \tag{2.10}$$

For the spheres S^d relation (2.9) coincides with (1.4), while for the spaces $Q(d, d_0)$ with $d > d_0$ this relation can be thought of as an extension of Stolarsky's invariance principle to projective spaces.

Recall that a metric space $\mathcal{M}$ with a metric ρ is called isometrically L_q-embeddable ($q = 1, 2$), if there exists a map $\varphi : \mathcal{M} \ni x \to \varphi(x) \in L_q$, such that $\rho(x_1, x_2) = \|\varphi(x_1) - \varphi(x_2)\|_{L_q}$ for all $x_1, x_2 \in \mathcal{M}$. Notice that L_2-embeddability is stronger and implies L_1-embeddability; see [**17**, Theorem 6.3.1]. A compact metric space $\mathcal{M}$ is isometrically L_1-embeddable with respect to a symmetric difference metric $\theta^{\Delta}(\xi)$; see (1.23). At the same time, a two-point homogeneous space $Q(d, d_0)$ is isometrically L_2-embeddable with respect to the chordal metric τ; see (2.4) and (2.6). This explains our terminology of L_1- and L_2-invariance principles. It should be mentioned that the above discussion raises the question whether there are other measures $\xi \neq \xi^{\natural}$ for which the spaces $Q(d, d_0)$ with the metrics $\theta^{\Delta}(\xi)$ are also L_2-embeddable.

2.2. *Bounds for discrepancies and sums of distances.* We introduce the following classes of measures on the set of radii. The class $W(a, b)$, $a \geqslant b \geqslant 1$, consists of (non-negative) measures on $\mathcal{I} = [0, \pi]$ such that

$$w(a, b, \xi) = \int_0^{\pi} \left(\sin\tfrac{1}{2}r\right)^{a-1}\left(\cos\tfrac{1}{2}r\right)^{b-1} d\xi(r) < \infty. \tag{2.11}$$

The subclass $AW(a, b) \subset W(a, b)$ consists of absolutely continuous measures. In other words, $d\xi(r) = \eta(r)\, dr$, where η is a locally integrable function on the interval $(0, 1)$ and

$$w(a, b, \xi) = \int_0^{\pi} \left(\sin\tfrac{1}{2}r\right)^{a-1}\left(\cos\tfrac{1}{2}r\right)^{b-1}\eta(r)\, dr < \infty. \tag{2.12}$$

Notice that the functions η can have rather large singularities at $r = 0$ and $r = \pi$.

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

THEOREM 2.1. *For a space $Q = Q(d, d_0)$ and a measure $\xi \in W(d, d_0)$, we have the bounds*

$$\lambda_N(Q, \xi) < CN^{1-1/d} \tag{2.13}$$

and

$$\theta_N^{\Delta}(Q, \xi) > \langle \theta^{\Delta}(\xi) \rangle N^2 - CN^{1-1/d}. \tag{2.14}$$

The proof of Theorem 2.1 is given in §3.

The following theorem shows that the bounds (2.13) and (2.14) have the best possible order if the measure ξ is absolutely continuous.

THEOREM 2.2. *For a space $Q = Q(d, d_0)$ and a measure $\xi \in AW(d, d_0)$ with $w(d, d_0, \xi) > 0$, we have the two-sided bounds*

$$cN^{1-1/d} < \lambda_N(Q, \xi) < CN^{1-1/d} \tag{2.15}$$

and

$$\langle \theta^{\Delta}(\xi) \rangle N^2 - cN^{1-1/d} > \theta_N^{\Delta}(Q, \xi) > \langle \theta^{\Delta}(\xi) \rangle N^2 - CN^{1-1/d}. \tag{2.16}$$

In particular, for the chordal metric on Q, we have

$$\langle \tau \rangle N^2 - cN^{1-1/d} > \tau_N(Q) > \langle \tau \rangle N^2 - CN^{1-1/d}. \tag{2.17}$$

The bounds (2.17) can be thought of as an extension of the bounds (1.6) and (1.9) to projective spaces. At the same time, the bounds (2.15) and (2.16) with an arbitrary measure $\xi \in AW(d, d_0)$ are new even in the case of spheres.

The proof of Theorem 2.2 is given in §4. It is clear that the right bounds in (2.15) and (2.16) follow immediately from Theorem 2.1. In §4 we prove the left bound in (2.15). By the invariance principle (1.34), this immediately implies the left bound in (2.16). Finally, the bounds (2.17) follow from (2.16) and the invariance principle (2.10).

The following example shows that Theorem 2.2 fails if $\xi \in W(d, d_0)$ is not absolutely continuous. It is known that for spheres the geodesic distance is a symmetric difference metric

$$\theta(y_1, y_2) = \pi\{1 - 2\mu(B(\pi/2, y_1) \cap B(\pi/2, y_2))\}, \tag{2.18}$$

where the balls $B(\pi/2, y) \subset S^d$ are hemispheres; see [**17**, §6.4]. In this case, the invariance principle (1.34) takes the form

$$\theta_N(S^d) + 2\pi\lambda_N(S^d, \delta_{\pi/2}) = \langle \theta \rangle N^2, \tag{2.19}$$

where $\langle \theta \rangle = \pi/2$ and $\delta_{\pi/2}$ is the Dirac mass at the point $r = \pi/2$.

On the other hand, one can easily check that $\lambda_N(S^d, \delta_{\pi/2}) = 0$ for even N and $\lambda_N(S^d, \delta_{\pi/2}) \leqslant \pi/2$ for odd N. Hence, the bounds (2.15) and (2.16) fail for the singular measure $\delta_{\pi/2}$. This example is considered in more detail in [**8**, **28**].

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

It can be proved that the statements of Theorem 2.2 always hold for any measure $\xi \in W(d, d_0)$ such that the Fourier integrals

$$\int_{c_0}^{\pi - c_0} \exp(\mathrm{i}ur)\, d\xi(r) \to 0 \quad \text{as } u \to \infty, \tag{2.20}$$

for each arbitrarily small $c_0 > 0$; see the proof of Lemma 4.2 below. Theorem 2.2 is stated above for measures $\xi \in AW(d.d_0)$, since in this case the condition (2.20) obviously holds by the Riemann–Lebesgue lemma.

One further comment should be made to mention relations with the theory of random fields on manifolds. Recall that the Lévi–Schoenberg kernel for a metric ρ on a space $\mathcal{M}$ is defined by

$$k(\rho, y_1, y_2) = \tfrac{1}{2}(\rho(y_1, y_0) + \rho(y_2, y_0) - \rho(y_1, y_2)), \tag{2.21}$$

where $y_0 \in \mathcal{M}$ is a fixed point; see [**19**]. The metric ρ can be recovered from the kernel k by $\rho(y_1, y_2) = k(\rho, y_1, y_1) + k(\rho, y_2, y_2) - 2k(\rho, y_1, y_2)$.

If the kernel (2.21) is positive definite (i.e. $\sum_{1 \leqslant i, j \leqslant N} \bar{c}_i c_j k(\rho, y_i, y_j) \geqslant 0$ for any points $y_1, \ldots, y_N \in \mathcal{M}$ and complex numbers $c_1, \ldots, c_N$), then it can be thought of as the covariance of a Gaussian process (a random field) on $\mathcal{M}$. The standard methods of probability theory enable one to construct such a random field as a mapping $W : \mathcal{M} \ni y \to W(y) = W(y, \omega) \in L_2(\Omega, d\omega)$, with $W(y_0) = 0$, $\mathbb{E}W(y_1) = 0$, $\mathbb{E}W(y_1)W(y_2) = k(\rho, y_1, y_2)$, and $\mathbb{E}(W(y_1) - W(y_2))^2 = \rho(y_1, y_2)$. Here $L_2(\Omega, d\omega)$ is the Hilbert space of real-valued square-integrable random variables on a probability space Ω with a probability measure $d\omega$, and $\mathbb{E}$ denotes the expectation on $L_2(\Omega, d\omega)$. Furthermore, if $\mathcal{M}$ is a Riemannian manifold and ρ is Hölder continuous with respect to the geodesic distance θ (i.e. $\rho(y_1, y_2) \lesssim \theta(y_1, y_2)^\beta$, $\beta > 0$), then trajectories of the random field $W(y, \omega)$ are continuous functions of $y \in \mathcal{M}$ for almost all $\omega \in \Omega$. For more details; see, for example, [**19**, **25**].

For the symmetric difference metrics (1.20) and (1.21) the definition (2.21) takes the form

$$k(\theta^\Delta(\xi), y_1, y_2) = \int_{\mathcal{I}} k(\theta_r^\Delta, y_1, y_2)\, d\xi(r), \tag{2.22}$$

where

$$k(\theta_r^\Delta, y_1, y_2) = \frac{1}{2} \int_{\mathcal{M}} F_r(y_1, y) F_r(y_2, y)\, d\mu(y), \tag{2.23}$$

and $F_r(x, y) = \chi(B(r, x), y) - \chi(B(r, y_0), y)$. Therefore, the Lévi–Schoenberg kernels are positive definite for all symmetric difference metrics. In particular, in view of (2.18), for the geodesic metric on S^d, the formula (2.23) can be written as

$$k(\theta, y_1, y_2) = \pi \int_{S^d} F_{\pi/2}(y_1, y) F_{\pi/2}(y_2, y)\, d\mu(y). \tag{2.24}$$

Therefore, the kernel $\theta(y_1, y_0) + \theta(y_2, y_0) - \theta(y_1, y_2)$ is positive definite. This is the well-known theorem of Lévy; see [**19**]. Originally, its proof was obtained

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

in terms of "white noise" integrals for random fields on S^d. A direct proof was given in [19, §4] in terms of an expansion of the metric θ in Gegenbauer polynomials. The above proof is likely to be the simplest.

Notice that in contrast to the spheres S^d, the geodesic metrics θ on the projective spaces $\mathbb{R}P^d$, $\mathbb{C}P^n$, $\mathbb{H}P^n$, and $\mathbb{O}P^2$ are not symmetric difference metrics, and analogs of Lévy's theorem are not true for projective spaces. This follows from [19, §4, pp. 225–226]. At the same time, the Lévi–Schoenberg kernel $k(\tau, y_1, y_2)$ for the chordal metric τ is positive definite for all spaces $Q(d, d_0)$; see (2.7).

Hence the kernels (2.22) and (2.23) are covariances of random fields on the spaces $Q(d, d_0)$. Notice that trajectories of such fields are continuous almost surely. This follows from Lemma 3.1(ii) given below. A comprehensive analysis of the random fields with the covariances (2.22) and (2.23) is beyond the scope of the present paper.

2.3. *Applications to t-designs.* Consider a smooth compact connected manifold $\mathcal{M}$ in $\mathbb{R}^m$ of dimension d equipped with a smooth Riemannian structure, and the geodesic distance θ and the Riemanniam measure μ normalized by (1.10). An N-point subset $\mathcal{D}_N \subset \mathcal{M}$ is called a *t-design* if the exact quadrature formula

$$\sum_{x \in \mathcal{D}_N} F(x) = N \int_{\mathcal{M}} F(y)\, d\mu(y) \tag{2.25}$$

holds for all polynomials $F(x)$, $x \in \mathbb{R}^m$, of total degree not exceeding t.

It is known (see, for example, [18]) that any N-point t-design $\mathcal{D}_N \subset \mathcal{M}$ satisfies the bound $N \gtrsim t^d$. An N-point t-design $\mathcal{D}_N \subset \mathcal{M}$ is called *optimal* if

$$c_+ t^d \geqslant N \geqslant c_- t^d \tag{2.26}$$

with some positive constants c_+ and c_- independent of N and t. Actually, in this definition we deal with sequences of N-point t-designs $\mathcal{D}_N$ as $N \to \infty$.

The above definition of t-design is applicable to any space $Q(d, d_0)$, since it can be realized as a Riemannian submanifold in $\mathbb{R}^m$ by the embedding (2.5). However, for the spaces $Q(d, d_0)$, one can give an equivalent definition in invariant terms; see [3, 24]. An N-point subset $\mathcal{D}_N \subset Q(d, d_0)$ is a t-design if the exact quadrature formula

$$\sum_{x_1, x_2 \in \mathcal{D}_N} f(\cos\theta(x_1, x_2)) = N^2 \iint_{Q \times Q} f(\cos\theta(y_1, y_2))\, d\mu(y_1)\, d\mu(y_2) \tag{2.27}$$

holds for all polynomials $f(z)$, $z \in \mathbb{C}$, of degree not exceeding t. The formula (2.27) is also equivalent to the quadrature formula

$$\sum_{x \in \mathcal{D}_N} f(\cos\theta(x, y)) = N \int_Q f(\cos\theta(x, y))\, d\mu(x), \tag{2.28}$$

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

for all $y \in Q$. Another equivalent definition of t-designs can be given in terms of spherical functions on the spaces $Q(d, d_0)$; see (4.32) below.

For any N-point subset $\mathcal{D}_N \subset \mathcal{M}$ we put

$$\left.\begin{aligned} \nu_1[\mathcal{D}_N, r] &= \int_Q (\#\{B(r, y) \cap \mathcal{D}_N\})^2 \, d\mu(y), \\ \nu_\infty[\mathcal{D}_N, r] &= \Big(\max_{y \in Q} \#\{B(r, y) \cap \mathcal{D}_N\}\Big)^2, \end{aligned}\right\} \tag{2.29}$$

and $\nu_1[\mathcal{D}_N, r] = \nu_\infty[\mathcal{D}_N, r] = N^2$ if $r \geqslant \pi$. Obviously, $\nu_1[\mathcal{D}_N, r] \leqslant \nu_\infty[\mathcal{D}_N, r]$.

Our result on t-designs can be stated as follows.

THEOREM 2.3. *Let $\xi \in AW(d, d_0)$. Then the following statements hold.*

(i) *There exists a constant $L \geqslant 1$ depending only on d and d_0, such that for any N-point t-design $\mathcal{D}_N \subset Q(d, d_0)$, we have*

$$\lambda[\xi, \mathcal{D}_N] < Ct^{d-1} \nu_1[\mathcal{D}_N, Lt^{-1}]. \tag{2.30}$$

(ii) *For an optimal N-point t-design $\mathcal{D}_N \subset Q(d, d_0)$ the bound (2.30) takes the form*

$$\lambda[\xi, \mathcal{D}_N] < CN^{1-1/d} \nu_1[\mathcal{D}_N, c_+^{1/d} LN^{-1/d}], \tag{2.31}$$

where c_+ is the constant in the definition (2.26).

The constants C in the bounds (2.30) and (2.31) depend only on d, d_0 and ξ.

Inequality (2.31) follows immediately from (2.30) and definition (2.26). The proof of the bound (2.30) is given in §4. The proof relies on the theory of spherical functions on $Q(d, d_0)$.

If the factor ν_1 in (2.31) is bounded from above by a constant independent of N, then the order of the bound (2.31) is best possible. Two simple sufficient conditions for this are given below in Lemma 2.1. We introduce some definitions. For an arbitrary N-point subset $\mathcal{D}_N \subset Q = Q(d, d_0)$, we put

$$\delta[\mathcal{D}_N] = \tfrac{1}{2} \min\{\theta(x_1, x_2) : x_1, x_2 \in \mathcal{D}_N, x_1 \neq x_2\}. \tag{2.32}$$

The balls $B(\delta, x)$, $\delta = \delta[\mathcal{D}_N]$, $x \in \mathcal{D}_N$, do not overlap. Therefore, $\delta \lesssim N^{-1/d}$, since the volume $v(r) \simeq r^d$. An N-point subset $\mathcal{D}_N \subset Q$ is called *well separated* if

$$C_1 N^{-1/d} \geqslant \delta[\mathcal{D}_N] \geqslant c_1 N^{-1/d} \tag{2.33}$$

with constants $C_1 \geqslant c_1 > 0$ independent of N.

Consider an equal-measure partition $\mathcal{P}_N = \{P_i\}_1^N$ of Q,

$$Q = \bigcup_{i=1}^N P_i, \qquad \mu(P_i \cap P_j) = 0, \qquad i \neq j, \ \mu(P_i) = 1/N,$$

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

and put $\mathrm{Diam}(\mathcal{P}_N, \theta) = \max_{1 \leqslant i \leqslant N} \mathrm{diam}(P_i, \theta)$. We say that an equal-measure partition $\mathcal{P}_N$ is of *small diameter* if

$$\mathrm{Diam}(\mathcal{P}_N, \theta) \leqslant C_2 N^{-1/d} \tag{2.34}$$

with a constant $C_2 > 0$ independent of N. Constructions of equal-measure partitions of small diameter are known for a large class of compact metric measure spaces, including all compact Riemannian manifolds; see [**20**].

We say that an N-point subset $\mathcal{D}_N = \{x_i\}_1^N \in Q$ is *subordinated* to a partition $\mathcal{P}_N = \{P_i\}_1^N$ of Q if $x_i \in P_i$, $i = 1, \dots, N$.

LEMMA 2.1. *Suppose that an N-point subset $\mathcal{D}_N \subset Q(d, d_0)$ satisfies one of the following conditions:*

(i) *$\mathcal{D}_N$ is well separated;*
(ii) *$\mathcal{D}_N$ is subordinated to an equal-measure partition of small diameter.*

Then, for any constant $c > 0$, there exists a constant $C = C(c)$ independent of N such that

$$\nu_\infty[\mathcal{D}_N, cN^{-1/d}] \leqslant C. \tag{2.35}$$

Proof. For brevity, we write $a = cN^{-1/d}$. Consider the ball $B(a, y)$ centered at an arbitrary point $y \in Q$ and put $\mathcal{E} = B(a, y) \cap \mathcal{D}_N$, $K = \#\{\mathcal{E}\}$. Enumerate points of $\mathcal{D}_N = \{x_i\}_1^N$ such that $\mathcal{E} = \{x_i\}_1^K$.

(i) By the definition of a well-separated subset $\mathcal{D}_N$, the balls $B(\delta, x)$, $\delta = \delta[\mathcal{D}_N]$, $x \in \mathcal{E}$, do not overlap and all these balls are contained in the ball $B(a + \delta, y)$. Therefore, $K\nu(\delta) \leqslant \nu(a + \delta)$. Since $\nu(r) \simeq r^d$, we have $K \lesssim \nu(a + \delta)/\nu(\delta) \simeq ((c + C_1)/c_1)^d$, and (2.35) follows.

(ii) By the definition of a subset $\mathcal{D}_N$ subordinated to an equal-measure partition $\mathcal{P}_N = \{P_i\}_1^N$ of small diameter $b = C_2 N^{-1/d}$, each subset P_i, $1 \leqslant i \leqslant K$ is contained in the ball $B(a + b, y)$. Therefore, $N^{-1}K \leqslant \nu(a + b)$ and $K \leqslant N\nu(a + b) \simeq (c + C_2)^d$, which proves (2.35). □

Comparing Theorem 2.3 with Lemma 2.1, and taking into account the left bounds of Theorem 2.2, we arrive at the following result.

COROLLARY 2.1. *Let $\xi \in AW(d, d_0)$ with $w(d, d_0, \xi) > 0$ and an N-point subset $\mathcal{D}_N \subset Q(d, d_0)$ be an optimal t-design. Suppose that $\mathcal{D}_N$ satisfies one of the conditions (i) or (ii) of Lemma 2.1. Then, for all sufficiently large N, we have the two-sided bounds*

$$\langle \theta^\Delta(\xi) \rangle N^2 - cN^{1-1/d} > \theta^\Delta[\xi, \mathcal{D}_N] > \langle \theta^\Delta(\xi) \rangle N^2 - CN^{1-1/d}, \tag{2.36}$$

$$cN^{1-1/d} < \lambda[\xi, \mathcal{D}_N] < CN^{1-1/d}. \tag{2.37}$$

In particular, for the chordal metric τ on $Q(d, d_0)$, we have

$$\langle r \rangle N^2 - cN^{1-1/d} > \tau[\mathcal{D}_N] > \langle r \rangle N^2 - CN^{1-1/d}. \tag{2.38}$$

The positive constants in (2.36)–(2.38) are independent of N.

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

The existence of optimal t-designs was a long-standing open problem known as the Korevaar–Meyers conjecture. In the papers [9, 10] by Bondarenko, Radchenko and Viazovska, the problem was solved for spherical t-designs. In [9] the existence of optimal t-designs $\mathcal{D}_N \subset S^d$ was proved for all sufficiently large N. Such optimal t-designs were constructed in [9], as subsets subordinated to an equal-measure partition of small diameter. In the subsequent paper [10], the authors proved that optimal spherical t-designs can be constructed as well-separated subsets. Therefore, Corollary 2.1 is applicable for the spheres S^d.

Using optimal spherical t-designs $\mathcal{D}_N \subset S^d$, one can easily construct optimal $[t/2]$-designs $\mathcal{D}_N^\circ$ on the real projective space $\mathbb{R}P^d = Q(d, 1)$, merely as a set of one-dimensional subspaces in $\mathbb{R}^d$ containing points $x \in \mathcal{D}_N$. One can easily check that if $\mathcal{D}_N$ is well separated on S^d, then $\mathcal{D}_N^\circ$ satisfies the bound (2.35) on $\mathbb{R}P^d$. Therefore, the optimal $[t/2]$-designs $\mathcal{D}_N^\circ \subset \mathbb{R}P^d$ satisfy the bounds (2.36)–(2.38).

The corresponding generalization to the projective spaces $\mathbb{C}P^n$, $\mathbb{H}P^n$, and $\mathbb{O}P^2$ is not so straightforward. Using the mapping (2.5), one can realize any space $Q(d, d_0)$ as a smooth connected compact affine algebraic submanifold in $\mathbb{R}^m$. For such manifolds the existence of optimal t-designs as subsets subordinated to an equal-measure partition of small diameter has been established in [18]. Therefore, Corollary 2.1 is also applicable to the projective spaces $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{O}P^2$.

§3. *Proof of Theorem 2.1.* The existence of uniformly distributed subsets has been established in [27, Theorem 4.2] for general compact d-rectifiable metric measure spaces. Here we state this result only for the special case of compact Riemannian manifolds.

PROPOSITION 3.1. *Let $\mathcal{M}$ be a compact d-dimensional Riemannian manifold with a metric θ and a measure μ proportional to the invariant Riemannian distance and measure on $\mathcal{M}$ and normalized by (1.10). Then there exists a positive constant $C_{\mathcal{M}}$ depending only on $\mathcal{M}$ such that the following statements hold.*

(i) *If a metric ρ on $\mathcal{M}$ satisfies the inequality*

$$\rho(x_1, x_2) \leqslant c_0 \theta(x_1, x_2) \tag{3.1}$$

with a constant $c_0 > 0$, then

$$\rho_N \geqslant \langle \rho \rangle N^2 - c_0 C_{\mathcal{M}} N^{1-1/d}. \tag{3.2}$$

(ii) *If the metric $\theta^\Delta(\xi)$ satisfies the inequality*

$$\theta^\Delta(\xi, x_1, x_2) \leqslant c_0 \theta(x_1, x_2) \tag{3.3}$$

with a constant $c_0 > 0$, then

$$\theta_N^\Delta(\xi) \geqslant \langle \theta^\Delta(\xi) \rangle N^2 - c_0 C_{\mathcal{M}} N^{1-1/d} \tag{3.4}$$

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

and

$$\lambda_N(\xi) \leqslant c_0 C_{\mathcal{M}} N^{1-1/d}. \tag{3.5}$$

Notice that condition (3.1) is essential only for closely spaced points x_1 and x_2. For example, for the metric $\tau = \sin\theta$, condition (3.1) holds, while for the metrics θ^a, $0 < a < 1$, it fails. Notice also that (3.2) implies the right bound in (2.17).

The following lemma contains precise estimates for the kernels (1.13), (1.14) and the symmetric difference metrics (1.20), (1.21).

LEMMA 3.1. *For any space $Q(d, d_0)$ the following statements hold.*

(i) *The kernel $\lambda_r(y_1, y_2)$ and the metric $\theta_r^{\Delta}(y_1, y_2)$ satisfy the bounds*

$$\left.\begin{aligned} |\lambda_r(y_1, y_2)| &\leqslant C\left(\sin\tfrac{1}{2}r\right)^d\left(\cos\tfrac{1}{2}r\right)^{d_0}, \\ \theta_r^{\Delta}(y_1, y_2) &\leqslant C\left(\sin\tfrac{1}{2}r\right)^d\left(\cos\tfrac{1}{2}r\right)^{d_0}. \end{aligned}\right\} \tag{3.6}$$

If $\xi \in W(d+1, d_0+1)$, then the kernel $\lambda(\xi, y_1, y_2)$ and the metric $\theta^{\Delta}(\xi, y_1, y_2)$ satisfy the bounds

$$\left.\begin{aligned} |\lambda(\xi, y_1, y_2)| &\leqslant C w(d+1, d_0+1, \xi), \\ \theta^{\Delta}(\xi, y_1, y_2) &\leqslant C w(d+1, d_0+1, \xi). \end{aligned}\right\} \tag{3.7}$$

(ii) *The metric $\theta_r^{\Delta}(y_1, y_2)$ satisfies the bound*

$$\theta_r^{\Delta}(y_1, y_2) \leqslant C\left(\sin\tfrac{1}{2}r\right)^{d-1}\left(\cos\tfrac{1}{2}r\right)^{d_0-1}\theta(y_1, y_2). \tag{3.8}$$

If $\xi \in W(d, d_0)$, then the metric $\theta^{\Delta}(\xi, y_1, y_2)$ satisfies the bound

$$\theta^{\Delta}(\xi, y_1, y_2) \leqslant C w(d, d_0, \xi)\theta(y_1, y_2). \tag{3.9}$$

Constants in the bounds (3.6)–(3.9) depend only on d and d_0.

It follows from Lemma 3.1(i) that the kernel (1.13) and the symmetric difference metric (1.20) are well defined for any measure $\xi \in W(d+1, d_0+1)$.

It follows from Lemma 3.1(ii) that inequality (3.3) is satisfied, and Theorem 2.1 follows immediately from Proposition 3.1(ii).

Proof of Lemma 3.1. (i) Putting $y_1 = y_2 = y$ in the first equality in (1.29), we obtain

$$\lambda_r(y, y) = v(r) - v(r)^2 = v(r)v'(r). \tag{3.10}$$

Applying the Cauchy–Schwarz inequality to (1.13), we obtain

$$|\lambda_r(y_1, y_2)| \leqslant (\lambda_r(y_1, y_2)\lambda_r(y_2, y_2))^{1/2} = v(r)v'(r). \tag{3.11}$$

Using the invariance principle (1.30), the third equality in (1.29) and the bound (3.11), we obtain

$$\theta_r^{\Delta}(y_1, y_2) \leqslant 2v(r)v'(r). \tag{3.12}$$

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

From the formula (2.2) we obtain the following two-sided bounds

$$v(r) \simeq r^d, \qquad v'(r) = 1 - v(r) \simeq (\pi - r)^{d_0}, \quad r \in [0, \pi]. \tag{3.13}$$

We also have

$$\sin \tfrac{1}{2}r \simeq r, \qquad \cos \tfrac{1}{2}r \simeq \pi - r, \quad r \in [0, \pi]. \tag{3.14}$$

We substitute the bounds (3.13) into (3.11) and (3.12). Next we use (3.14) to obtain (3.6). Integrating (3.6) with $\xi \in W(d+1, d_0+1)$, we obtain (3.7).

(ii) We can assume that $0 < r < \pi$, since $\theta_r^{\Delta}(y_1, y_2) = 0$ identically for $r = 0$ and $r = \pi$. For brevity, we write $\delta = \theta(y_1, y_2)/2$. The parameters r and δ vary in the region $0 < r < \pi$, $0 \leqslant \delta \leqslant \pi/2$. This rectangular region can be represented as a disjoint union of three triangular regions:

(a) $0 < r < \delta$, $0 \leqslant \delta \leqslant \pi/2$;
(b) $\pi > r \geqslant \pi - \delta$, $0 \leqslant \delta \leqslant \pi/2$;
(c) $r > \delta$, $0 < r < \pi - \delta$, $0 \leqslant \delta < \pi/2$.

In each of these regions we shall prove the bound (3.8). Notice that for $r \in [0, \pi]$, the function $\sin r/2$ increases while $\cos r/2$ decreases.

(a) Using (3.12), (2.2) and (3.14), we obtain

$$\begin{aligned} \theta_r^{\Delta}(y_1, y_2) \leqslant 2v(r) &\simeq \int_0^r \left(\sin \tfrac{1}{2}u\right)^{d-1} \left(\cos \tfrac{1}{2}u\right)^{d_0-1} du \\ &\lesssim \int_0^r \left(\sin \tfrac{1}{2}u\right)^{d-1} du \simeq \left(\sin \tfrac{1}{2}r\right)^{d-1} r \\ &\lesssim \left(\sin \tfrac{1}{2}r\right)^{d-1} \left(\cos \tfrac{1}{2}r\right)^{d_0-1} \delta. \end{aligned} \tag{3.15}$$

(b) Similarly, using (3.12), (2.2) and (3.14), we obtain

$$\begin{aligned} \theta_r^{\Delta}(y_1, y_2) \leqslant 2v'(r) &\simeq \int_r^{\pi} \left(\sin \tfrac{1}{2}u\right)^{d-1} \left(\cos \tfrac{1}{2}u\right)^{d_0-1} du \\ &\lesssim \int_r^{\pi} \left(\cos \tfrac{1}{2}u\right)^{d_0-1} du \simeq \left(\cos \tfrac{1}{2}r\right)^{d_0-1} (\pi - r) \\ &\lesssim \left(\sin \tfrac{1}{2}r\right)^{d-1} \left(\cos \tfrac{1}{2}r\right)^{d_0-1} \delta. \end{aligned} \tag{3.16}$$

(c) It is known (see [**21**, Ch. VII, §10]) that for the spaces $Q(d, d_0)$ all geodesics are simply closed, and if $0 < \theta(y_1, y_2) < \pi$, then there exists the unique geodesic $\gamma \subset Q(d, d_0)$ of length $\theta(y_1, y_2)$ joining points y_1, y_2. Let y_0 denote its midpoint (i.e. $y_0 \in \gamma$, $\theta(y_1, y_0) = \theta(y_2, y_0) = \delta$). The triangle inequality for the metric θ implies that the ball $B(r - \delta, y_0)$ is contained in the intersection $B(r, y_1) \cap B(r, y_2)$. Hence

$$\mu(B(r, y_1) \cap B(r, y_2)) \geqslant v(r - \delta). \tag{3.17}$$

Using (3.17) together with (3.12), (2.2) and (3.14), we obtain

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

$$\begin{aligned}\theta_r^{\Delta}(y_1, y_2) \leqslant v(r) - v(r-\delta) &\simeq \int_{r-\delta}^{r} \bigl(\sin\tfrac{1}{2}u\bigr)^{d-1}\bigl(\cos\tfrac{1}{2}u\bigr)^{d_0-1}\,du \\ &\lesssim \bigl(\sin\tfrac{1}{2}r\bigr)^{d-1}\bigl(\cos\tfrac{1}{2}(r-\delta)\bigr)^{d_0-1}\delta \\ &\simeq \bigl(\sin\tfrac{1}{2}r\bigr)^{d-1}(\pi - r + \delta)^{d_0-1}\delta \\ &\simeq \bigl(\sin\tfrac{1}{2}r\bigr)^{d-1}(\pi - r)^{d_0-1}\left(1 + \frac{\delta}{\pi - r}\right)^{d_0-1}\delta \\ &\lesssim \bigl(\sin\tfrac{1}{2}r\bigr)^{d-1}(\pi - r)^{d_0-1}\delta \\ &\simeq \bigl(\sin\tfrac{1}{2}r\bigr)^{d-1}\bigl(\cos\tfrac{1}{2}r\bigr)^{d_0-1}\delta. \end{aligned} \tag{3.18}$$

Now, the bound (3.8) follows from the bounds (3.15), (3.16) and (3.18). Integrating (3.8) with $\xi \in W(d, d_0)$, we obtain the bound (3.9). □

§4. *Proof of Theorem 2.2 and 2.3.*

4.1. *Fourier analysis on two-point homogeneous spaces.* The spaces $Q(d, d_0)$ belong to a specific and very important class of commutative spaces. The general theory of commutative spaces can be found in [**34**]; see also [**22**, **23**] and [**32**, vol. III, Ch. 17]. In the case of compact groups the theory becomes significantly simpler. In this section we outline the necessary facts and formulas in the explicit form convenient for subsequent calculations.

Let G be a compact group and $K \subset G$ a closed subgroup. Denote by μ_G and μ_K Haar measures on the groups G and K; correspondingly, $\mu_G(G) = \mu_K(K) = 1$. As before, μ denotes the invariant measure on the homogeneous space $Q = G/K$, and $\mu_G = \mu_K \times \mu$. $L_q(G)$, $q = 1, 2$, denote spaces of functions on G integrable with the power q with respect to the Haar measure. $L_q(G/K)$ and $L_q(K \setminus G/K)$ denote subspaces in $L_q(G)$ consisting of functions such that $f(gk) = f(g)$, $k \in K$, and, correspondingly, $f(k_1 g k_2) = f(g)$, $k_1, k_2 \in K$. Obviously, functions in these subspaces can be thought of as functions on $Q = G/K$. The spaces $L_1(K \setminus G/K) \subset L_1(G/K) \subset L_1(G)$ are associative Banach algebras with the convolution product

$$f_1 * f_2(g) = \int_G f_1(gh^{-1}) f_2(h)\, d\mu_G(h). \tag{4.1}$$

If the algebra $L_1(K \setminus G/K)$ is commutative, the pair of groups $K \subset G$ is called a *Gelfand pair* and the corresponding homogeneous space $Q = G/K$ is called a *commutative space*; see [**34**]. Riemannian symmetric spaces and two-point homogeneous spaces are two large classes of commutative spaces; see [**22**, **34**]. The spaces $Q(d, d_0)$ belong to both of these classes.

Consider the unitary representation of a group G in the space $L_2(G/K)$,

$$T(g)f(h) = f(g^{-1}h), \quad f \in L_2(G/K), \quad g, h \in G, \tag{4.2}$$

and its decomposition into the orthogonal sum, $T = \widehat{\bigoplus_{l\geqslant 0}} T_l$, $L_2(G/K) = \widehat{\bigoplus_{l\geqslant 0}} V_l$, of unitary irreducible representations T_l in finite-dimensional spaces V_l. Let $m_l = \dim V_l$, and $(\cdot, \cdot)$ denote the inner product in V_l.

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

If $Q = G/K$ is a commutative space, then the irreducible representations T_l are pairwise non-equivalent and each subspace V_l contains a single K-invariant unit vector $\mathbf{e}^{(l)}$, that is, $T_l(k)\mathbf{e}^{(l)} = \mathbf{e}^{(l)}$ for all $k \in K$. Fix an orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_{m_l}$ in the space V_l, such that $\mathbf{e}_1 = \mathbf{e}^{(l)}$ and define the matrix elements $t_{ij}^{(l)}(g) = (T_l(g)\mathbf{e}_i, \mathbf{e}_j)$. Then we have

$$t_{ij}^{(l)}(g_1 g_2) = \sum_{p=1}^{m_l} t_{ip}^{(l)}(g_1) t_{pj}^{(l)}(g_2) \quad \text{and} \quad t_{ij}^{(l)}(g^{-1}) = \overline{t_{ji}^{(l)}(g)}. \tag{4.3}$$

We also have the orthogonality relations

$$\int_G t_{ij}^{(l)}(g) \overline{t_{ij}^{(l')}(g)}\, d\mu_G(g) = m_l^{-1} \delta_{ll'} \delta_{ii'} \delta_{jj'}, \tag{4.4}$$

where δ_{ab} is Kronecker's symbol. The sets of functions $\{m_l^{1/2} t_{1j}^{(l)}(g),\ j = 1, \dots, m_l,\ l \geqslant 0\}$ and $\{m_l^{1/2} t_{11}^{(l)}(g),\ l \geqslant 0\}$ are orthonormal bases in the spaces $L_2(G/K)$ and $L_2(K \setminus G/K)$, respectively; see [**32**, vol. I, §2.3] (notice that in [**32**] the subgroup K in a Gelfand pair $K \subset G$ is called *massive*).

The matrix elements $\varphi_l(g) = t_{11}^{(l)}(g)$ are called *zonal spherical functions* or simply *spherical functions* (the matrix elements $t_{1j}^{(l)}(g)$, $j = 2, \dots, m_l$ are called associated spherical functions). The definition of $\varphi_l(g)$ and formula (4.3) immediately imply that the spherical functions are continuous, $\varphi_l(e) = 1$, where e is the unit element in G, $|\varphi_l(g)| \leqslant 1$ for all $g \in G$, and

$$\varphi_l(g_1 g_2^{-1}) = \sum_{j=1}^{m_l} t_{1j}^{(l)}(g_1) \overline{t_{1j}^{(l)}(g_2)} \quad \text{and} \quad \varphi_l(g) = \overline{\varphi_l(g^{-1})}. \tag{4.5}$$

It follows from (4.5) that all φ_l are positive definite:

$$\sum_{1 \leqslant i,j \leqslant N} \overline{c_i} c_j \varphi_l(g_i^{-1} g_j) \geqslant 0 \tag{4.6}$$

for any $g_1, \dots, g_N \in G$ and any complex numbers $c_1, \dots, c_N$.

From (4.1), (4.4), and (4.5), we obtain the following relation for the convolution of two spherical functions:

$$(\varphi_l * \varphi_{l'})(g) = \delta_{ll'} m_l^{-1} \varphi_l(g). \tag{4.7}$$

Putting $g = e$ in (4.7), we obtain the following formula for the dimensions m_l:

$$m_l = \left(\int_G |\varphi_l(g)|^2\, d\mu_G(g) \right)^{-1}. \tag{4.8}$$

Functions $f \in L_2(K \setminus G/K)$ have the expansions

$$f(g) \sim \sum_{l \geqslant 0} m_l c_l(f) \varphi_l(g), \tag{4.9}$$

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

where $\sim$ denotes L_2-convergence, Fourier coefficients are given by

$$c_l(f) = \int_G f(g)\overline{\varphi_l(g)}\, d\mu_G(g), \tag{4.10}$$

and Parseval's equality has the form $\int_G |f(g)|^2\, d\mu_G(g) = \sum_{l\geqslant 0} m_l\, |c_l(f)|^2$. Actually, this is the Peter–Weyl theorem written for the space $L_2(K \setminus G/K)$; see [**32**, vol. I, Ch. 2].

Substituting expansion (4.9) for two functions $f_1, f_2 \in L_2(K \setminus G/K)$ into (4.1) and using the relation (4.7), we obtain

$$f_1 * f_2(g) = \sum_{l\geqslant 0} m_l c_l(f_1) c_l(f_2) \varphi_l(g). \tag{4.11}$$

Applying the Cauchy–Schwarz inequality to (4.11), we observe that the series (4.11) converges absolutely. Since the spherical functions φ_l are continuous and $|\varphi_l(g)| \leqslant 1$, we conclude that the convolution $f_1 * f_2$ is a continuous function.

The facts listed above are true for all compact commutative spaces. Now we wish to specify these facts for two-point homogeneous spaces. Let $K \subset G$ be compact groups and $Q = G/K$ a two-point homogeneous space with a G-invariant metric θ. Suppose that K is the stabilizer of a fixed point $y_0 \in Q$. It follows from the definition that the subgroup K is transitive on each sphere $\Sigma(r, y_0) = \{y : \theta(y, y_0) = r\} \subset Q, r \in \mathcal{I}$. Thus, any function $f \in L_q(K\setminus G/K)$, as a function on Q, is constant on each sphere $\Sigma(r, y_0)$, and we can write

$$f(g) = F(\theta(gy_0, y_0)) \tag{4.12}$$

with a function $F(r)$, $r \in \mathcal{I}$. In other words, the set of double cosets $K \setminus G/K$ is in one-to-one correspondence with the set of radii $\mathcal{I}$.

Using (4.12), the convolution (4.1) can be written in the form

$$\begin{aligned}(f_1 * f_2)(g_1^{-1}g_2) &= \int_G F_1(\theta(g_1y_0, gy_0))F_2(\theta(gy_0, g_2y_0))\, d\mu(g)\\ &= \int_Q F_1(\theta(y_1, y))F_2(\theta(y, y_2))\, d\mu(y),\end{aligned} \tag{4.13}$$

where $y_1 = g_1y_0$, $y_2 = g_2y_0$. For a function of the form (4.12) we have

$$\int_G |f(g)|^2\, d\mu_G(g) = \int_Q |F(\theta(y, y_0))|^2\, d\mu(y) = \int_{\mathcal{I}} |F(r)|^2\, dv(r), \tag{4.14}$$

where the last integral is thought of as a Stieltjes integral with the non-decreasing function $v(r) = \mu(B(r, y_0))$, $r \in \mathcal{I}$. It follows from (4.12) and (4.14) that the mapping $f \to F$ is an isometry of the space $L_2(K \setminus G/K)$ onto the space $L_2(\mathcal{I}, v)$ of functions $F(r)$, $r \in \mathcal{I}$, with the norm $\|F\| = (\int_{\mathcal{I}} |F(r)|^2\, dv(r))^{1/2}$.

Since the spherical functions $\varphi_l \in L_2(K \setminus G/K)$, they can be written in the form (4.12),

$$\varphi_l(g) = \Phi_l(\theta(gy_0, y_0)), \tag{4.15}$$

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

where $\Phi_l \in L_2(\mathcal{I}, v)$, and putting $y_1 = g_1 y_0$, $y_2 = g_2 y_2$, $g_1, g_2 \in G$, we can write

$$\varphi_l(g_1^{-1} g_2) = \Phi_l(\theta(g_1 y_0, g_2 y_0)) = \Phi_l(\theta(y_1, y_2)). \tag{4.16}$$

It follows from the properties of φ_l that Φ_l are continuous and real-valued, $\Phi_l(0) = 1$, $|\Phi_l(r)| \leqslant 1$, $r \in \mathcal{I}$. The set of functions $\{m_l^{1/2} \Phi_l, l \geqslant 0\}$ is an orthonormal basis in the space $L_2(\mathcal{I}, v)$, and the expansion (4.9) for $F \in L_2(\mathcal{I}, v)$ takes the form

$$F(r) \sim \sum_{l \geqslant 0} m_l c_l(F) \Phi_l(r) \tag{4.17}$$

with the Fourier coefficients

$$c_l(F) = \int_{\mathcal{I}} F(r) \Phi_l(r)\, dv(r) \tag{4.18}$$

and Parseval's equality $\int_{\mathcal{I}} |F(r)|^2\, dv_r = \sum_{l \geqslant 0} m_l\, |c_l(F)|^2$.

Comparing relations (4.11) and (4.13), we arrive at the formula

$$\int_Q F_1(\theta(y_1, y)) F_2(\theta(y, y_2))\, d\mu(y) = \sum_{l \geqslant 0} m_l c_l(F_1) c_l(F_2) \Phi_l(\theta(y_1, y_2)). \tag{4.19}$$

For the spaces $Q = Q(d, d_0)$ the matrix elements $t_{1j}^{(l)}(g)$ are eigenfunctions of the Laplace–Beltrami operator on Q, and the spherical functions $\varphi_l(g) = t_{11}^{(l)}(g)$ are eigenfunctions of the radial part of this operator and can be found explicitly; see [**19**, p. 178], [**22**, Ch. V, Theorem 4.5], [**23**], [**24**, pp. 514–512, 543–544], [**34**, Theorem 11.4.21]. For the functions Φ_l in (4.15), we have

$$\Phi_l(r) = \Phi_l^{(\alpha,\beta)}(r) = \frac{P_l^{(\alpha,\beta)}(\cos r)}{P_l^{(\alpha,\beta)}(1)}, \quad r \in \mathcal{I} = [0, \pi], \tag{4.20}$$

where $P_l^{(\alpha,\beta)}(z)$ are the standard Jacobi polynomials of degree l normalized by

$$P_l^{(\alpha,\beta)}(1) = \binom{\alpha + l}{l} = \frac{(\alpha + 1) \cdots (\alpha + l)}{l!} \simeq l^{\alpha}. \tag{4.21}$$

Jacobi polynomials can be defined by Rodrigues' formula (see [**31**, equation (4.3.1)]),

$$P_l^{(\alpha,\beta)}(z) = \frac{(-1)^l}{2^l l!} (1 - z)^{-\alpha} (1 + z)^{-\beta} \frac{d^l}{dz^l} \{(1 - z)^{l+\alpha} (1 + z)^{l+\beta}\}. \tag{4.22}$$

The parameters α, β in (4.21) and dimensions d, d_0 in $Q(d, d_0)$ are related by

$$\alpha = \tfrac{1}{2} d - 1, \qquad \beta = \tfrac{1}{2} d_0 - 1. \tag{4.23}$$

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

In what follows, we assume that (4.23) always holds. Hence, $\alpha \geqslant \beta \geqslant -1/2$, and $|P_l^{(\alpha,\beta)}(z)| \leqslant P_l^{(\alpha,\beta)}(1)$, $z \in [-1, 1]$; see [**31**, Theorem 7.32.1].

We have the following orthogonality relations for Jacobi polynomials (see [**31**, equation (4.3.3)]):

$$\begin{aligned}\int_0^\pi P_l^{(\alpha,\beta)}(\cos u)\, P_{l'}^{(\alpha,\beta)}(\cos u)\bigl(\sin\tfrac{1}{2}u\bigr)^{d-1}\bigl(\cos\tfrac{1}{2}u\bigr)^{d_0-1}\,du \\ = \bigl(\tfrac{1}{2}\bigr)^{\alpha+\beta+1}\int_{-1}^1 P_l^{(\alpha,\beta)}(z)P_{l'}^{(\alpha,\beta)}(z)(1-z)^\alpha(1+z)^\beta\,dz \\ = M_l^{-1}\delta_{ll'},\end{aligned} \tag{4.24}$$

where $M_0 = \kappa(d, d_0)$ and

$$M_l = (2l+\alpha+\beta+1)\frac{\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}{\Gamma(l+\alpha+1)\Gamma(l+\beta+1)} \simeq l, \quad l \geqslant 1. \tag{4.25}$$

Substituting (4.15) and (4.20) into (4.8) and using (4.24), we obtain the following explicit formula for the dimensions m_l:

$$m_l = M_l\, B(d/2, d_0/2)\binom{\alpha+l}{l}^2 \simeq l^{d-1}. \tag{4.26}$$

For functions $F \in L_2([0,\pi], v_r)$ the expansion (4.17) takes the form

$$F(r) \sim \sum_{l\geqslant 0} M_l C_l(F) P_l^{(\alpha,\beta)}(\cos r), \tag{4.27}$$

with the Fourier–Jacobi coefficients

$$C_l(F) = \int_0^\pi F(u) P_l^{(\alpha,\beta)}(\cos u)\bigl(\sin\tfrac{1}{2}u\bigr)^{d-1}\bigl(\cos\tfrac{1}{2}u\bigr)^{d_0-1}\,du, \tag{4.28}$$

and Parseval's equality $\int_{\mathcal{I}} |F(r)|^2\, dv(r) = \kappa(d, d_0)\sum_{l\geqslant 0} M_l\, |C_l(F)|^2$. The Fourier–Jacobi coefficients (4.28) and Fourier coefficients (4.18) are related by

$$c_l(F) = C_l(F)\frac{\kappa(d, d_0)}{P_l^{(\alpha,\beta)}(1)}, \quad l \geqslant 0. \tag{4.29}$$

Using the relations (4.20) and (4.29), we can write formula (4.19) in the form

$$\begin{aligned}\int_Q F_1(\theta(y_1, y))F_2(\theta(y, y_2))\,d\mu(y) \\ = \kappa(d, d_0)\sum_{l\geqslant 0} M_l C_l(F_1)C_l(F_2)\frac{P_l^{(\alpha,\beta)}(\cos\theta(y_1, y_2))}{P_l^{(\alpha,\beta)}(1)}.\end{aligned} \tag{4.30}$$

This formula will be used in the next section to obtain spherical function expansions for discrepancies and metrics.

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

The condition of positive definiteness (4.6) for the spherical functions (4.15) and (4.21) will be used in §4 in the following special form:

$$\varphi_l[\mathcal{D}_N] = \sum_{x_1, x_2 \in \mathcal{D}_N} \frac{P_l^{(\alpha,\beta)}(\cos\theta(x_1, x_2))}{P_l^{(\alpha,\beta)}(1)} \geqslant 0, \tag{4.31}$$

where $\mathcal{D}_N \subset Q(d, d_0)$ is an arbitrary N-point subset.

One can easily check (see also [**3**, **24**]) that conditions (2.27) and (2.28) in the definition of t-designs $\mathcal{D}_N \subset Q(d, d_0)$ are equivalent to the equalities

$$\varphi_l[\mathcal{D}_N] = 0, \quad l = 1, \dots, t, \tag{4.32}$$

4.2. *Spherical function expansions for discrepancies and metrics.* In this section we obtain explicit spherical function expansions for the kernels (1.13) and (1.14) and the symmetric difference metrics (1.20) and (1.21) on the spaces $Q(d, d_0)$.

First of all, we recall the main facts on Jacobi polynomials $P_l^{(\alpha,\beta)}(z)$, $z \in [-1, 1]$, $\alpha \geqslant -1/2$, $\beta \geqslant -1/2$, as $l \to \infty$. It is well known (see [**31**]) that Jacobi polynomials behave extremely irregularly on the interval $z \in [-1, 1]$: inside the interval they oscillate and are of order $l^{-1/2}$, while in neighborhoods of the end points $z = 1$ and $z = -1$ they grow rapidly and are of order l^α and l^β, respectively. It is convenient to describe such behavior in terms of the functions

$$J_l^{(\alpha,\beta)}(r) = \left(\sin\tfrac{1}{2}r\right)^{\alpha+1/2}\left(\cos\tfrac{1}{2}r\right)^{\beta+1/2} P_l^{(\alpha,\beta)}(\cos r), \quad r \in [0, \pi]. \tag{4.33}$$

(a) In the interval $r \in [c_0 l^{-1}, \pi - c_0 l^{-1}]$, where $c_0 > 0$ is an arbitrary fixed constant, we have the asymptotic formula

$$J_l^{(\alpha,\beta)}(r) = (\pi l)^{-1/2}\{\cos[(l + l_0)r + r_0] + O((l \sin r)^{-1})\}, \tag{4.34}$$

where $l_0 = (\alpha + \beta + 1)/2$, $r_0 = -\pi(2\alpha + 1)/4$; see [**31**, Theorem 8.21.3].

(b) In the intervals $r \in [0, c_0 l^{-1}]$ and $r \in [\pi - c_0 l^{-1}, \pi]$, we have the bound $J_l^{(\alpha,\beta)}(r) = O(l^{-1/2})$; see [**31**, Theorem 7.32.2]. This bound, together with (4.34), implies the bound

$$|J_l^{(\alpha,\beta)}(r)| < c(l + 1)^{-1/2}, \quad l \geqslant 0, \tag{4.35}$$

which holds uniformly for all $r \in [0, \pi]$, with the constant c depending only on α, β.

Consider the measure of the intersection of two balls $B(r, y_1)$ and $B(r, y_2)$ in the space $Q = Q(d, d_0)$,

$$\mu_r(y_1, y_2) = \mu(B(r, y_1) \cap B(r, y_2)) = \int_Q \chi_r(\theta(y_1, y))\chi_r(\theta(y, y_2))\, d\mu(y), \tag{4.36}$$

where $\chi_r(\cdot)$ is the characteristic function of the interval $[0, r]$, $0 \leqslant r \leqslant \pi$; see (1.26).

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

LEMMA 4.1. *The kernel (4.36) has the spherical function expansion*

$$\mu_r(y_1, y_2) = v(r)^2 + \kappa(d, d_0) \sum_{l \geqslant 1} l^{-2} M_l a_l(r) \frac{P_l^{(\alpha,\beta)}(\cos\theta(y_1, y_2))}{P_l^{(\alpha,\beta)}(1)}, \quad (4.37)$$

where

$$\begin{aligned} a_l(r) &= \left(\sin\tfrac{1}{2}r\right)^{2d} \left(\cos\tfrac{1}{2}r\right)^{2d_0} \{P_{l-1}^{(\alpha+1,\beta+1)}(\cos r)\}^2 \\ &= \left(\sin\tfrac{1}{2}r\right)^{d-1} \left(\cos\tfrac{1}{2}r\right)^{d_0-1} \{J_{l-1}^{(\alpha+1,\beta+1)}(r)\}^2. \end{aligned} \quad (4.38)$$

The coefficients in (4.37) satisfy

$$M_l a_l(r) \leqslant c\left(\sin\tfrac{1}{2}r\right)^{d-1} \left(\cos\tfrac{1}{2}r\right)^{d_0-1}, \quad (4.39)$$

with a constant depending only on d and d_0. Furthermore, we have the equality

$$\kappa(d, d_0) \sum_{l \geqslant 1} l^{-2} M_l a_l(r) = v(r) - v(r)^2 = v(r)v'(r). \quad (4.40)$$

Proof. Applying the expansion (4.30) to the integral (4.36), we obtain

$$\mu_r(y_1, y_2) = \kappa(d, d_0) \sum_{l \geqslant 0} M_l \{C_l(\chi_r)\}^2 \frac{P_l^{(\alpha,\beta)}(\cos\theta(y_1, y_2))}{P_l^{(\alpha,\beta)}(1)}, \quad (4.41)$$

where $C_l(\chi_r)$ are Fourier–Jacobi coefficients (4.28) of the characteristic function χ_r,

$$\begin{aligned} C_l(\chi_r) &= \int_0^r P_l^{(\alpha,\beta)}(\cos u) \left(\sin\tfrac{1}{2}u\right)^{d-1} \left(\cos\tfrac{1}{2}u\right)^{d_0-1} du \\ &= \left(\tfrac{1}{2}\right)^{(d-1)/2+(d_0-1)/2} \int_{\cos r}^1 (1-z)^\alpha (a+z)^\beta P_l^{(\alpha,\beta)}(z)\, dz. \end{aligned} \quad (4.42)$$

In view of (2.2), we have $C_0(\chi_r) = \kappa(d, d_0)^{-1} v(r)$. Substituting Rodrigues' formula (4.22) into (4.42), we obtain

$$\begin{aligned} &\int_{\cos r}^1 (1-z)^\alpha (1+z)^\beta P_l^{(\alpha,\beta)}(z)\, dz \\ &\quad = (2l)^{-1} (1-\cos r)^{\alpha+1} (1+\cos r)^{\beta+1} P_{l-1}^{(\alpha+1,\beta+1)}(\cos r) \\ &\quad = 2^{\alpha+\beta+1} l^{-1} \left(\sin\tfrac{1}{2}r\right)^{2\alpha+2} \left(\cos\tfrac{1}{2}r\right)^{2\beta+2} P_{l-1}^{(\alpha+1,\beta+1)}(\cos r). \end{aligned}$$

In view of (4.33) and (4.23), we have

$$\begin{aligned} C_l(\chi_r) &= l^{-1} \left(\sin\tfrac{1}{2}r\right)^d \left(\cos\tfrac{1}{2}r\right)^{d_0} P_{l-1}^{(\alpha+1,\beta+1)}(\cos r) \\ &= l^{-1} \left(\sin\tfrac{1}{2}r\right)^{(d-1)/2} \left(\cos\tfrac{1}{2}r\right)^{(d_0-1)/2} J_{l-1}^{(\alpha+1,\beta+1)}(r). \end{aligned} \quad (4.43)$$

Substituting (4.43) into (4.41), we obtain the formulas (4.37) and (4.38).

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

The bound (4.39) follows from (4.38), since $M_l \simeq l$ (see (4.25)), and $J_{l-1}^{(\alpha+1,\beta+1)}(r) \lesssim l^{-1/2}$ (see (4.35)).

From (4.36), we obtain $\mu_r(y, y) = v(r)$. Putting $y_1 = y_2 = y$ in (4.37), we obtain (4.40). In fact, formula (4.40) is Parseval's equality for the expansion (4.27) of the characteristic function χ_r. □

The following theorem is an immediate corollary of Lemma 2.1.

THEOREM 4.1. *For any space $Q(d, d_0)$ the following spherical function expansions hold.*

(i) *For the kernels $\lambda_r(y_1, y_2)$ (see (1.14)) and the metrics $\theta_r^{\Delta}(y_1, y_2)$ (see (1.21)), we have*

$$\lambda_r(y_1, y_2) = \kappa(d, d_0) \sum_{l \geqslant 1} l^{-2} M_l a_l(r) \frac{P_l^{(\alpha,\beta)}(\cos\theta(y_1, y_2))}{P_l^{(\alpha,\beta)}(1)}, \tag{4.44}$$

$$\begin{aligned} \theta_r^{\Delta}(y_1, y_2) &= \langle \theta_r^{\Delta} \rangle - \kappa(d, d_0) \sum_{l \geqslant 1} l^{-2} M_l a_l(r) \frac{P_l^{(\alpha,\beta)}(\cos\theta(y_1, y_2))}{P_l^{(\alpha,\beta)}(1)}, \\ &= \kappa(d, d_0) \sum_{l \geqslant 1} l^{-2} M_l a_l(r) \left[1 - \frac{P_l^{(\alpha,\beta)}(\cos\theta(y_1, y_2))}{P_l^{(\alpha,\beta)}(1)} \right], \end{aligned} \tag{4.45}$$

where $\langle \theta_r^{\Delta} \rangle = v(r)\, v'(r)$ is the average value of metric θ_r^{Δ}; see (1.29).

(ii) *If $\xi \in W(d, d_0)$, then for the kernels $\lambda(\xi, y_1, y_2)$ (see (1.13)) and the metrics $\theta^{\Delta}(\xi, y_1, y_2)$ (see (1.20)), we have*

$$\lambda(\xi, y_1, y_2) = \kappa(d, d_0) \sum_{l \geqslant 1} l^{-2} M_l A_l(\xi) \frac{P_l^{(\alpha,\beta)}(\cos\theta(y_1, y_2))}{P_l^{(\alpha,\beta)}(1)}, \tag{4.46}$$

$$\begin{aligned} \theta^{\Delta}(\xi, y_1, y_2) &= \langle \theta^{\Delta}(\eta) \rangle - \kappa(d, d_0) \sum_{l \geqslant 1} l^{-2} M_l A_l(\xi) \frac{P_l^{(\alpha,\beta)}(\cos\theta(y_1, y_2))}{P_l^{(\alpha,\beta)}(1)}, \\ &= \kappa(d, d_0) \sum_{l \geqslant 1} l^{-2} M_l A_l(\xi) \left[1 - \frac{P_l^{(\alpha,\beta)}(\cos\theta(y_1, y_2))}{P_l^{(\alpha,\beta)}(1)} \right], \end{aligned} \tag{4.47}$$

where $\langle \theta^{\Delta}(\xi) \rangle$ is the average value of the metric $\theta^{\Delta}(\xi)$ and the coefficients $A_l(\xi)$ are defined by

$$A_l(\xi) = \int_0^{\pi} a_l(u)\, d\xi(u). \tag{4.48}$$

The series (4.44)–(4.47) converge absolutely and uniformly.

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

Proof. (i) Substituting the expansion (4.37) into (1.29), we obtain the expansions (4.44) and (4.45). Notice that in the second formula in (4.45) equality (4.40) has been taken into account.

(ii) In view of (4.39), the series (4.44) and (4.45) can be integrated term by term with $\xi \in W(d, d_0)$. This gives the expansions (4.46) and (4.47). □

It is worth noting the following. The chordal metric τ is a symmetric difference metric (1.20) with the specific measure $\xi^\natural$; see (2.7). Therefore, it has the expansion (4.47). At the same time, the chordal metric can be written as

$$\tau(y_1, y_2) = c(\alpha, \beta)\left[1 - \frac{P_1^{(\alpha,\beta)}(\cos\theta(y_1, y_2))}{P_1^{(\alpha,\beta)}(1)}\right]^{1/2}, \tag{4.49}$$

with $c(\alpha, \beta) = (\alpha + 1/\alpha + \beta + 2)^{1/2} = (d/d + d_0)^{1/2}$. Indeed, by Rodrigues' formula (4.22), $P_1^{(\alpha,\beta)}(z) = \frac{1}{2}(\alpha + \beta + 2)z + \frac{1}{2}(\alpha - \beta)$, and

$$\frac{1}{2}(1 - z) = \frac{\alpha + 1}{\alpha + \beta + 2}\left[1 - \frac{P_1^{(\alpha,\beta)}(z)}{P_1^{(\alpha,\beta)}(1)}\right]. \tag{4.50}$$

On the other hand, by definition (2.3),

$$\tau(y_1, y_2) = \sin\tfrac{1}{2}\theta(y_1, y_2) = \left[\tfrac{1}{2}(1 - \cos\theta(y_1, y_2))\right]^{1/2}. \tag{4.51}$$

Then formula (4.49) follows from (4.50) and (4.51).

Comparing Theorem 4.1(ii) with definitions (1.12) and (1.17), we arrive at the following theorem.

THEOREM 4.2. *If $\xi \in W(d, d_0)$, then for an arbitrary N-point subset $\mathcal{D}_N \in Q(d, d_0)$, we have the expansions*

$$\lambda[\xi, \mathcal{D}_N] = \kappa(d, d_0) \sum_{l \geqslant 1} l^{-2} M_l A_l(\xi) \varphi_l[\mathcal{D}_N] \tag{4.52}$$

and

$$\begin{aligned} \theta^\Delta[\xi, \mathcal{D}_N] &= \langle \theta^\Delta(\eta) \rangle - \kappa(d, d_0) \sum_{l \geqslant 1} l^{-2} M_l A_l(\xi) \varphi_l[\mathcal{D}_N] \\ &= \kappa(d, d_0) \sum_{l \geqslant 1} l^{-2} M_l A_l(\xi)[1 - \varphi_l[\mathcal{D}_N]], \end{aligned} \tag{4.53}$$

where the coefficients $\varphi_l[\mathcal{D}_N] \geqslant 0$ are defined by (4.31). The series (4.52) and (4.53) converge absolutely.

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

4.3. *Bounds for Fourier–Jacobi coefficients.* In this section we estimate the coefficients

$$\left.\begin{aligned} a_l(r) &= \left(\sin\tfrac{1}{2}r\right)^{d-1}\left(\cos\tfrac{1}{2}r\right)^{d_0-1}\{J_{l-1}^{(\alpha+1,\beta+1)}(r)\}^2, \\ A_l(\chi_r) &= \int_0^\pi \chi_r(u)a_l(u)\,du = \int_0^r a_l(u)\,du, \\ A_l\,(\xi) &= \int_0^\pi a_l(u)\eta(u)\,du, \end{aligned}\right\} \tag{4.54}$$

where $\xi \in AW(d, d_0)$, $d\xi(r) = \eta(r)\,d(r)$. In fact, we shall prove special weighted bounds for Jacobi polynomials.

LEMMA 4.2. *Let $\xi \in AW(d, d_0)$, $w(\xi, d, d_0) > 0$, then the following bounds hold.*

(i) *For $0 < r \leqslant \pi$ and $l \geqslant 1$, we have*

$$A_l(\xi) > cr^{-d+1}a_l(r). \tag{4.55}$$

(ii) *There exists a constant $L \geqslant 1$, depending only on α and β, such that for $0 < r \leqslant \pi/2$ and $lr > L$, we have*

$$A_l(\xi) < Cr^{-d}A_l(\chi_r). \tag{4.56}$$

The positive constants c and C in (4.55) and (4.56) depend only on α, β, and ξ.

Proof. The asymptotic formula (4.34) implies that

$$\{J_{l-1}^{(\alpha+1,\beta+1)}(r)\}^2 = (\pi l)^{-1}\left\{\tfrac{1}{2} - \tfrac{1}{2}\cos 2[(l + l_0)r + r_0] + R_l(r)\right\}, \tag{4.57}$$

where the error term $R_l(r)$ satisfies

$$R_l(r) = \begin{cases} O(l^{-1}), & \text{for } 0 < c_0 \leqslant r \leqslant \pi - c_0, \\ O((lr)^{-1}), & \text{for } l^{-1} \leqslant r \leqslant \pi/2, \end{cases} \tag{4.58}$$

with $0 < c_0 < \pi/2$ arbitrarily fixed.

(i) Since $d\xi(r) = \eta(r)dr$ and $w(\xi, d, d_0) > 0$ (see (2.12)), a sufficiently small constant $0 < c_0 < \pi/2$ can be chosen to satisfy

$$\int_{c_0}^{\pi-c_0}\left(\sin\tfrac{1}{2}u\right)^{d-1}\left(\cos\tfrac{1}{2}u\right)^{d_0-1}\eta(u)\,du \geqslant \frac{1}{2}w(\xi, d, d_0) > 0. \tag{4.59}$$

Using (4.59) and (4.57) and the first bound in (4.58), we obtain

$$\begin{aligned} A_l(\xi) &\geqslant \int_{c_0}^{\pi-c_0}\left(\sin\tfrac{1}{2}u\right)^{d-1}\left(\cos\tfrac{1}{2}u\right)^{d_0-1}\{J_{l-1}^{(\alpha+1,\beta+1)}(u)\}^2\eta(u)\,du \\ &= (\pi l)^{-1}\left\{\tfrac{1}{4}w(\xi, d, d_0) - I_l + O(l^{-1})\right\}, \end{aligned} \tag{4.60}$$

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

where

$$I_l = \frac{1}{2}\int_{c_0}^{\pi - c_0} \left(\sin\tfrac{1}{2}u\right)^{d-1}\left(\cos\tfrac{1}{2}u\right)^{d_0-1}\cos 2[(l+l_0)u + r_0]\eta(u)\,du.$$

By the Riemann–Lebesgue lemma, $I_l \to 0$ as $l \to \infty$, and from (4.60), we obtain the bound

$$A_l(\xi) > (8\pi l)^{-1} w(\xi, d, d_0) \tag{4.61}$$

for all sufficiently large $l > l_1$, where l_1 depends only on α, β and ξ. Since $A_l(\xi) > 0$ for all $l \geqslant 1$, we have the bound

$$A_l(\xi) > c_1 l^{-1} \tag{4.62}$$

for all $l \geqslant 1$ with a constant $c_1 > 0$ depending only on α, β and ξ.

On the other hand, the bound (4.35) implies

$$r^{-d+1}a_l(r) = r^{-d+1}\left(\sin\tfrac{1}{2}r\right)^{d-1}\left(\cos\tfrac{1}{2}r\right)^{d_0-1}\{J_{l-1}^{(\alpha+1,\beta+1)}(r)\}^2 \leqslant c_2 l^{-1}. \tag{4.63}$$

Comparing the bounds (4.62) and (4.63), we obtain the bound (4.55) with $c = c_1 c_2^{-1}$.

(ii) Let $0 < r \leqslant \pi/2$ and $lr \geqslant L$, where $L \geqslant 1$ is a constant which will be fixed later. From definitions (4.54), we obtain

$$\begin{aligned} r^{-d}A_l(\chi_r) &\geqslant r^{-d}\int_{r/2}^{r} a_l(u)\,du \\ &\geqslant r^{-d}\left(\sin\frac{1}{4}r\right)^{d-1}\left(\cos\frac{1}{2}r\right)^{d_0-1}\int_{r/2}^{r}\{J_{l-1}^{(\alpha+1,\beta+1)}(u)\}^2\,du \\ &> c_1 r^{-1}\int_{r/2}^{r}\{J_{l-1}^{(\alpha+1,\beta+1)}\}^2\,du, \end{aligned} \tag{4.64}$$

where one can put $c_1 = (1/8)^{d-1}(1/2)^{d_0-1}$. Using the asymptotic formula (4.57) and the second bound in (4.58), we obtain

$$\begin{aligned} &r^{-1}\int_{r/2}^{r}\{J_{l-1}^{(\alpha+1,\beta+1)}(u)\}^2\,du \\ &\quad = (\pi l)^{-1}\left\{\frac{1}{4} - \frac{1}{2}r^{-1}\int_{r/2}^{r}\cos 2[(l+l_0)u + r_0]\,du + O(L^{-1})\right\}. \end{aligned} \tag{4.65}$$

The integral on the right-hand side in (4.65) is of order $O((rl)^{-1}) \lesssim O(L^{-1})$. Substituting (4.65) into (4.64), we obtain

$$r^{-d}A_l(\chi_r) > c_1(4\pi l)^{-1}\{1 + O(L^{-1})\}. \tag{4.66}$$

Now, in view of (4.66), we can fix a sufficiently large constant L, depending only on α and β, to satisfy

$$r^{-d}A_l(\chi_r) > c_1(8\pi l)^{-1} = c_2 l^{-1}. \tag{4.67}$$

On the other hand, the bound (4.35) implies

$$A_l(\xi) \leqslant C_2 w(\xi, d, d_0) l^{-1} = C_3 l^{-1}. \tag{4.68}$$

Comparing (4.67) and (4.68), we obtain the bound (4.56) with $C = C_3 c_2^{-1}$. □

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

4.4. *Proof of Theorems 2.2 and 2.3.* By Theorem 4.1, we have

$$\left.\begin{aligned}\lambda_r[\mathcal{D}_N] &= \kappa(d,d_0)\sum_{l\geqslant 1} l^{-2}M_l a_l(r)\varphi_l[\mathcal{D}_N],\\ \lambda[\chi_r,\mathcal{D}_N] &= \kappa(d,d_0)\sum_{l\geqslant 1} l^{-2}M_l A_l(\chi_r)\varphi_l[\mathcal{D}_N],\\ \lambda[\xi,\mathcal{D}_N] &= \kappa(d,d_0)\sum_{l\geqslant 1} l^{-2}M_l A_l(\xi)\varphi_l[\mathcal{D}_N],\end{aligned}\right\} \tag{4.69}$$

where $\mathcal{D}_N \subset Q(d,d_0)$ is an arbitrary N-point subset. The series (4.69) converge and all their terms are non-negative.

LEMMA 4.3. *Let $\xi \in AW(d,d_0)$, $w(\xi,d,d_0) > 0$. Then the following bounds hold.*

(i) *For any N-point subset $\mathcal{D}_N \subset Q(d,d_0)$ and an arbitrary r, $0 < r \leqslant \pi$, we have*

$$\lambda[\xi,\mathcal{D}_N] > cr^{-d+1}\lambda_r[\mathcal{D}_N]. \tag{4.70}$$

(ii) *There exists a constant $L \geqslant 1$, depending only d and d_0, such that for any N-point t-design $\mathcal{D}_N \subset Q(d,d_0)$, we have*

$$\lambda[\xi,\mathcal{D}_N] < Cr^{-d}\lambda[\chi_r,\mathcal{D}_N], \quad r = Lt^{-1}. \tag{4.71}$$

The positive constants c and C in (4.70) and (4.71) depend only on d, d_0 and ξ.

Proof. (i) Applying the bound (4.55) to the series (4.69), we obtain the bound (4.70).

(ii) If $\mathcal{D}_N \subset Q(d,d_0)$ is a t-design, then $\varphi[\mathcal{D}_N] = 0$ for $l = 1,\dots,t$; see (4.32). Hence, the summation in all series (4.69) is taken over $l > t$.

Let L be chosen as the constant indicated in Lemma 4.2(ii). If $r = Lt^{-1}$, then we have $0 < r \leqslant \pi/2$ for $t \geqslant 2L/\pi$ and $lr > L$ for $l > t$. Applying the bound (4.56) to the series (4.69), we obtain the bound (4.71). □

Now we are in position to prove Theorems 2.2 and 2.3.

Proof of Theorem 2.2. It suffices to prove only the left bound in (2.15) (as was explained in the comments to Theorem 2.2). From the definitions of discrepancies (1.11) we conclude that $\lambda_r[\mathcal{D}_N] \geqslant \langle Nv_r\rangle^2$, where $\langle z\rangle = \min\{|z-n|, n\in\mathbb{Z}\}$ denotes the distance of $z\in\mathbb{R}$ from the nearest integer. Define r by $Nv_r = 1/2$. Then $\lambda_r[\mathcal{D}_N] \geqslant 1/2$. In view of (3.13), $r \simeq N^{-1/d}$ and the bound (4.70) implies the left bound in (2.15). □

Proof of Theorem 2.3. First of all, we notice that

$$\int_Q (\#\{B(r,y)\cap\mathcal{D}_N\})^2\,d\mu(y) = \int_Q \left(\sum_{y_1\in\mathcal{D}_N}\chi(B(r,y),y_1)\right)^2 d\mu(y)$$

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

$$= \int_Q \Big(\sum_{y_1 \in \mathcal{D}_N} \chi(B(r, y_1), y) \Big)^2 d\mu(y) = \sum_{y_1, y_2 \in \mathcal{D}_N} \mu(B(r, y_1) \cap B(r, y_2)),$$

where formula (1.26) has been used. Comparing these equalities with (1.29), we obtain

$$\lambda_r[\mathcal{D}_N] < \int_Q (\#\{B(r, y) \cap \mathcal{D}_N\})^2 \, d\mu(y) = \nu_1[\mathcal{D}_N, r].$$

Therefore,

$$\lambda[\chi_r, \mathcal{D}_N] = \int_0^r \lambda_u[\mathcal{D}_N] \, du < r\nu_1[\mathcal{D}_N, r], \tag{4.72}$$

since $\nu_1[\mathcal{D}_N, r]$ is a non-decreasing function of r. Substituting (4.72) into (4.71), we obtain

$$\lambda[\xi, \mathcal{D}_N] < Cr^{-d+1}\nu_1[\mathcal{D}_N, r]. \tag{4.73}$$

If $r = Lt^{-1}$, then the bound (4.73) coincides with the bound (2.30). □

Acknowledgement. The author is very grateful to the referee for many useful comments and remarks.

References

1. J. R. Alexander, On the sum of distances between n points on a sphere. *Acta Math. Hungar.* **23**(3–4) (1972), 443–448.
2. J. R. Alexander, J. Beck and W. W. L. Chen, Geometric discrepancy theory and uniform distributions. In *Handbook of Discrete and Computational Geometry*, 3rd edn. (eds C. D. Toth, J. E. Goodman and J. O'Rourke), Taylor and Francis (Boca Raton, FL, 2017), 279–304.
3. E. Banai, On extremal finite sets in the sphere and other metric spaces. In *Algebraic, Extremal and Metric Combinatorics* (eds M. M. Deza, P. Frankl and I. G. Rosenberg), Cambridge University Press (Cambridge, 1988).
4. J. Beck, Sums of distances between points on a sphere: an application of the theory of irregularities of distributions to distance geometry. *Mathematika* **31** (1984), 33–41.
5. J. Beck and W. W. L. Chen, *Irregularities of Distribution* (*Cambridge Tracts in Mathematics* **89**), Cambridge University Press (Cambridge, 1987).
6. A. L. Besse, *Manifolds All of Whose Geodesics are Closed* (*A Series of Modern Surveys in Mathematics* **93**), Springer (Berlin, 1978).
7. D. Bilyk and F. Dai, Geodesic distance Riesz energy on the sphere. *Trans. Amer. Math. Soc.* (2018), published on line, https://dx.doi.org/10.1090/tran/7711.
8. D. Bilyk, F. Dai and R. Matzke, Stolarsky principle and energy optimization on the sphere. *Constr. Approx.* **48**(1) (2018), 31–61.
9. A. Bondarenko, D. Radchenko and M. Viazovska, Optimal asymptotic bounds for spherical designs. *Ann. of Math.* **178**(2) (2013), 443–452.
10. A. Bondarenko, D. Radchenko and M. Viazovska, Well-separated spherical designs. *Constr. Approx.* **41**(1) (2015), 93–112.
11. L. Brandolini, W. W. L. Chen, L. Colzani, G. Gigante and G. Travaglini, Discrepancy and numerical integration on metric measure spaces. *J. Geom. Anal.* **29**(1) (2019), 328–369.
12. J. S. Brauchart and J. Dick, A simple proof of Stolarsky's invariance principle. *Proc. Amer. Math. Soc.* **141** (2013), 2085–2096.
13. J. S. Brauchart and P. J. Grabner, Distributing many points on spheres. *J. Complexity* **31**(3) (2015), 293–326.
14. H. Cohn and A. Kumar, Universally optimal distribution of points on spheres. *J. Amer. Math. Soc.* **20**(1) (2006), 99–147.
15. H. Cohn, A. Kumar and G. Minton, Optimal simplices and codes in projective spaces. *Geom. Topol.* **20** (2016), 1289–1357.

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press

16. J. Conway, R. Hardin and N. J. A. Sloane, Packing lines, planes, etc.: packing in Grassmannian spaces. *Exp. Math.* **5** (1996), 139–159.
17. M. M. Deza and M. Laurent, *Geometry of Cuts and Metrics*, Springer (Berlin, 1997).
18. U. Etayo, J. Marzo and J. Ortega-Cerdà, Asymptotically optimal designs on compact algebraic manifolds. *Monatsh. Math.* **186**(2) (2018), 235–248.
19. R. Gangolli, Positive definite kernels on homogeneous spaces and certain stochastic processes related to Lévy's Brownian motion of several parameters. *Ann. Inst. H. Poincaré* **III**(2) (1967), 121–325.
20. W. Gigante and P. Leopardi, Diameter bounded equal measure partitions of Alfors regular metric measure spaces. *Discrete Comput. Geom.* **57** (2017), 419–430.
21. S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press (New York, 1978).
22. S. Helgason, *Groups and Geometric Analysis. Integral Geometry, Invariant Differential Operators, and Spherical Functions*, Academic Press (London, 1984).
23. T. H. Koornwinder, Jacobi functions and analysis on noncompact semisimple Lie groups. In *Special Functions: Group Theoretical Aspects and Application* (eds R. A. Askey, T. H. Koornwinder and W. Schempp), D. Reidel (Dordrecht, 1984), 1–85.
24. V. I. Levenshtein, Universal bounds for codes and designs. In *Handbook of Coding Theory* (eds V. S. Pless and W. C. Huffman), Elsevier (Amsterdam, 1998), 499–648.
25. A. Malyarenko, *Invariant Random Fields on Spaces with a Group Action*, Springer (Berlin, 2013).
26. E. B. Saff and A. B. J. Kwijlaars, Distributing many points on a sphere. *Math. Intelligencer* **19**(1) (1997), 5–11.
27. M. M. Skriganov, Point distributions in compact metric spaces. *Mathematika* **63**(3) (2017), 1152–1171.
28. M. M. Skriganov, Bounds for L_p-discrepancies of point distributions in compact metric measure spaces. *Preprint*, 2018, arXiv:1802.01577.
29. M. M. Skriganov, Stolarsky's invariance principle for projective spaces. *Preprint*, 2018, arXiv:1805.03541.
30. K. B. Stolarsky, Sums of distances between points on a sphere, II. *Proc. Amer. Math. Soc.* **41** (1973), 575–582.
31. G. Szegö, *Orthogonal Polynomials*, American Mathematical Society (Providence, RI, 1950).
32. N. J. Vilenkin and A. U. Klimyk, *Representation of Lie Groups and Special Functions,* vols. 1–3, Kluwer Academic (Dordrecht, 1991–1992).
33. J. A. Wolf, *Spaces of Constant Curvature*, University of California (Berkley, 1972).
34. J. A. Wolf, *Harmonic Analysis on Commutative Spaces* (*Mathematical Surveys and Monographs* **142**), American Mathematical Society (Providence, RI, 2007).

M. M. Skriganov,
St. Petersburg Department,
Steklov Mathematical Institute,
Russian Academy of Sciences,
St. Petersburg 191023,
Russia
E-mail: mmskrig@gmail.com

https://doi.org/10.1112/S0025579319000019 Published online by Cambridge University Press