Recycling Stirling's series acceleration technique

PAUL LEVRIE and AMRIK SINGH NIMBRAN

Introduction

Infinite series are an important topic in mathematical analysis and convergence is the most crucial concept in the theory of infinite series. The speed at which the sequence of the partial sums of a series approaches its limiting sum has been a subject of investigation for many a mathematician. Euler [1], Kummer [2] and Markoff [3] all developed techniques for accelerating the convergence of slowly converging series. Euler's method only works for alternating series, Kummer's and Markoff's are suitable for general series with a special form.

Gosper [4] illustrates how the rate of convergence of infinite series can be accelerated by a suitable splitting of each term into two parts and then combining the second part of the n th term with the first part of the $n + 1$ th term and leaving the first part of the first term. Repeated application of this process yields a new series which approaches 0 and the series of the left out first parts ('orphans') that converges faster than the original series.

Something similar had already been done by Stirling, in his *Methodus Differentialis* dating from 1730 [5, 6] (see also [7, 8]). Stirling applied his technique to the series $\sum_{n=1}^{\infty} 1/n^2$. Using it he was able to calculate this sum to 8 decimal places. At that time the exact value was not known. The problem of calculating the sum of this series, now known as the Basel problem, was first posed by Pietro Mengoli in 1644. Jakob, Johann, and Daniel Bernoulli, Leibniz, Stirling and de Moivre all attempted to solve the problem. They all failed. It was only in 1735 that Euler obtained its true value, $\pi^2/6$. One of the problems in finding the exact value was the fact that the series converges very slowly.

In 1890 Markoff [3] gave a very general scheme for accelerating the convergence of series, now known as Markoff's transformation. His scheme encompasses Stirling's method.

In this paper we look at some special cases of Markoff's transformation that are very close to Stirling's own method. Stirling used inverse factorial series (see [9]); we will use finite inverse factorial sums instead.

In the first part of this paper we explain the method used by Stirling to accelerate the convergence of the series $\sum_{n=1}^{\infty} 1/n^2$. Schur and Knopp [10] adapted Stirling's method and found the following accelerated series:

$$\zeta(2) = 3 \sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}}.$$

Knopp [12] later used this adaptation as an example of Markoff's transformation. We will show that a lot of other series for $\zeta(2)$ can be obtained in a similar way.

https://doi.org/10.1017/mag.2017.7 Published online by Cambridge University Press

In the second part we apply the same method to accelerate the convergence of the series for $\zeta(k)$, with $k \geqslant 3$. This leads to accelerated series containing the (unsigned) Stirling numbers of the first kind.

1. *Stirling's convergence acceleration method for* $\zeta(2)$ *extended*

The method described here is an adaptation of the following scheme devised by Stirling [5, 6].

It is easy to verify that

$$\begin{aligned}
\frac{1}{n} &= \frac{1}{n+1} + \frac{1}{n(n+1)}, \\
\frac{1}{n(n+1)} &= \frac{1}{(n+1)(n+2)} + \frac{2}{n(n+1)(n+2)}, \\
\frac{1}{n(n+1)(n+2)} &= \frac{1}{(n+1)(n+2)(n+3)} + \frac{3}{n(n+1)(n+2)(n+3)} \\
&\vdots
\end{aligned} \tag{1}$$

Proceeding in this manner and combining the different equations, we get a series for $\frac{1}{n}$:

$$\frac{1}{n} = \sum_{k=1}^{\infty} \frac{(k-1)!}{(n+1)(n+2)\ldots(n+k)}.$$

Then dividing both sides by n, we get:

$$\frac{1}{n^2} = \sum_{k=1}^{\infty} \frac{(k-1)!}{n(n+1)(n+2)\ldots(n+k)}. \tag{2}$$

Stirling used this to accelerate the series for $\zeta(2)$ in the following way [5, pp. 28-29].

EXEMPLUM VI.

Summanda ſit Series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+$ &c. ubi Denominatores ſunt Quadrata numerorum 1, 2, 3, 4, &c. & Terminus in genere eſt $\frac{1}{zz}$: hic reductus in formam ſummabilem, evadit

$$\frac{1}{z.z+1}+\frac{1}{z.z+1.z+2}+\frac{1.2}{z.z+1.z+2.z+3}+\frac{1.2.3}{z.z+1.z+2.z+3.z+4}+\text{ &c.}$$

Ergo Summa =

$$\frac{1}{z}+\frac{1}{2z.z+1}+\frac{1.2}{3z.z+1.z+2}+\frac{1.2.3}{4z.z+1.z+2.z+3}+\text{ &c.}$$

FIGURE 1: Stirling's acceleration of $\zeta(2)$

https://doi.org/10.1017/mag.2017.7 Published online by Cambridge University Press

Stirling keeps the first terms of the series and replaces the terms from $1/z^2$ on by another series:

$$\begin{aligned}
\frac{1}{1^2} &= \frac{1}{1^2}\\
\frac{1}{2^2} &= \frac{1}{2^2}\\
&\vdots\\
\frac{1}{(z-1)^2} &= \frac{1}{(z-1)^2}\\
\frac{1}{z^2} &= \frac{1}{z(z+1)} + \frac{1}{z(z+1)(z+2)} + \frac{2}{z(z+1)(z+2)(z+3)} + \cdots\\
\frac{1}{(z+1)^2} &= \frac{1}{(z+1)(z+2)} + \frac{1}{(z+1)(z+2)(z+3)}\\
&\quad + \frac{2}{(z+1)(z+2)(z+3)(z+4)} + \cdots\\
&\vdots
\end{aligned}$$

He then sums everything at the right hand side, column by column. The first $z-1$ terms in the first column are kept apart. The other terms in the first column telescope, giving

$$\frac{1}{z(z+1)} + \frac{1}{(z+1)(z+2)} + \frac{1}{(z+2)(z+3)} + \cdots = \frac{1}{z}.$$

This is a consequence of:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

More generally, we have

$$\frac{1}{n(n+1)\ldots(n+m)} = \frac{1}{m}\left(\frac{1}{n\ldots(n+m-1)} - \frac{1}{(n+1)\ldots(n+m)}\right),$$

which we use for the other columns. This is the final result:

$$\zeta(2) = \frac{1}{1^2} + \cdots + \frac{1}{(z-1)^2} + \frac{1}{z} + \frac{1}{2}\frac{1}{z(z+1)} + \frac{1}{3}\frac{1\cdot 2}{z(z+1)(z+2)} + \cdots + \frac{1}{n}\frac{(n-1)!}{z(z+1)\ldots(z+n-1)} + \cdots.$$

Stirling used this series with $z = 13$ and computed $\zeta(2)$ correct to nine decimal places of which eight were correct, by using 25 terms.

https://doi.org/10.1017/mag.2017.7 Published online by Cambridge University Press

Now we adapt Stirling's method so that we get other series. We look at two cases.

Case 1: *All the squares*

The first case is based on the defining series for $\zeta(2)$:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Instead of using the full series (2), we now use only the first m equations of (1). We then get the following expression for $\frac{1}{n}$:

$$\frac{1}{n} = \sum_{k=1}^{m} \frac{(k-1)!}{(n+1)(n+2)\dots(n+k)} + \frac{m!}{n(n+1)(n+2)\dots(n+m)}$$

and, after dividing by n,

$$\frac{1}{n^2} = \sum_{k=1}^{m} \frac{(k-1)!}{n(n+1)(n+2)\dots(n+k)} + \frac{m!}{n^2(n+1)(n+2)\dots(n+m)}. \qquad (3)$$

As can be seen above, in his example Stirling used this expression for $n = 1, 2, \dots, 12$ with $m = 0$, and for $n > 12$ with $m = \infty$. However, other combinations are possible. For instance, if we use $m = 0$ for the first term of the series for $\zeta(2)$, $m = 1$ for the second term, and so on, we get this

$$\begin{aligned}
\frac{1}{1^2} &= \frac{0!}{1^2} \\
\frac{1}{2^2} &= \frac{0!}{2\cdot 3} + \frac{1!}{2^2\cdot 3} \\
\frac{1}{3^2} &= \frac{0!}{3\cdot 4} + \frac{1!}{3\cdot 4\cdot 5} + \frac{2!}{3^2\cdot 4\cdot 5} \\
\frac{1}{4^2} &= \frac{0!}{4\cdot 5} + \frac{1!}{4\cdot 5\cdot 6} + \frac{2!}{4\cdot 5\cdot 6\cdot 7} + \frac{3!}{4^2\cdot 5\cdot 6\cdot 7} \\
&\vdots
\end{aligned}$$

Adding the two sides vertically and using telescoping if possible, we are left with

$$\zeta(2) = \left(1 + \frac{1}{2}\right) + \left(\frac{1!}{2^2\cdot 3} + \frac{1!}{2\cdot 3\cdot 4}\right) + \left(\frac{2!}{3^2\cdot 4\cdot 5} + \frac{2!}{3\cdot 4\cdot 5\cdot 6}\right) + \dots.$$

Here the general term is

$$\frac{m!}{(m+1)^2(m+2)\dots(2m+1)} + \frac{1}{m+1}\frac{m!}{(m+2)\dots(2m+2)}$$

which is

$$3\,\frac{m!}{(m+1)(m+2)\dots(2m+2)}.$$

https://doi.org/10.1017/mag.2017.7 Published online by Cambridge University Press

Hence, we find that

$$\zeta(2) = 3\,\frac{0!}{2} + 3\,\frac{1!}{2\cdot 3\cdot 4} + 3\,\frac{2!}{3\cdot 4\cdot 5\cdot 6} + \ \dots\ + 3\,\frac{m!}{(m+1)\dots(2m+2)} + \ \dots$$

or, after changing the summation index,

$$\zeta(2) \ = \ 3\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}}.$$

This series converges geometrically with rate 1/4 and gives an average of $0.6\,(\approx \log_{10} 4)$ new decimal digits per term. (The convergence rate of a series $\sum a_n$ is defined as $\lim_{n\to\infty} a_{n+1}/a_n$ if this limit exists.) The series can be found in a paper by Catalan [11] dating from 1867. Catalan used Kummer's transformation. It reappears in a paper by Schur and Knopp [10] in 1918 and in Knopp's book [12]. Schur and Knopp use the method described above, as an example of Markoff's transformation of series [3]. The series was probably already known to Euler [13].

If we use (3) with $m = 0$ for the first two terms of the series for $\zeta(2)$, with $m = 1$ for the next two, and so on, we get this:

$$\begin{aligned}
\frac{1}{1^2} &= \frac{1}{1^2}\\
\frac{1}{2^2} &= \frac{1}{2^2}\\
\frac{1}{3^2} &= \frac{1}{3\cdot 4} + \frac{1}{3^2\cdot 4}\\
\frac{1}{4^2} &= \frac{1}{4\cdot 5} + \frac{1}{4^2\cdot 5}\\
\frac{1}{5^2} &= \frac{1}{5\cdot 6} + \frac{1}{5\cdot 6\cdot 7} + \frac{2}{5^2\cdot 6\cdot 7}\\
\frac{1}{6^2} &= \frac{1}{6\cdot 7} + \frac{1}{6\cdot 7\cdot 8} + \frac{2}{6^2\cdot 7\cdot 8}\\
\frac{1}{7^2} &= \frac{1}{7\cdot 8} + \frac{1}{7\cdot 8\cdot 9} + \frac{2}{7\cdot 8\cdot 9\cdot 10} + \frac{2\cdot 3}{7^2\cdot 8\cdot 9\cdot 10}\\
\frac{1}{8^2} &= \frac{1}{8\cdot 9} + \frac{1}{8\cdot 9\cdot 10} + \frac{2}{8\cdot 9\cdot 10\cdot 11} + \frac{2\cdot 3}{8^2\cdot 9\cdot 10\cdot 11}\\
&\ \ \vdots
\end{aligned}$$

https://doi.org/10.1017/mag.2017.7 Published online by Cambridge University Press

Every couple of rows is given by:

$$\frac{1}{(2n+1)^2} = \sum_{k=1}^{n} \frac{(k-1)!}{\prod_{j=1}^{k+1}(2n+j)} + \frac{n!}{(2n+1)\prod_{k=1}^{n+1}(2n+k)}$$

$$\frac{1}{(2n+2)^2} = \sum_{k=1}^{n} \frac{(k-1)!}{\prod_{j=1}^{k+1}(2n+j+1)} + \frac{n!}{(2n+2)\prod_{k=1}^{n+1}(2n+k+1)}.$$

If we add the terms column by column, and use telescoping, three terms per column remain:

$$\frac{m!}{(2m+1)^2(2m+2)\dots(3m+1)} + \frac{m!}{(2m+2)^2(2m+3)\dots(3m+2)} + \frac{m!}{(m+1)(2m+3)\dots(3m+3)}.$$

Bringing them on the same denominator, summing and putting $m = n-1$, we obtain this series:

$$\zeta(2) = \frac{1}{4}\sum_{n=1}^{\infty} \frac{46n^2 - 34n + 7}{n^2(2n-1)^2\binom{3n}{n}}. \tag{4}$$

Its convergence rate is $2^2/3^3$.

If we use (3) with $m = 1$ for the first term of the series for $\zeta(2)$, with $m = 3$ for the second, $m = 5$ for the third, thus

$$\frac{1}{1^2} = \frac{1}{1\cdot 2} + \frac{1}{1^2\cdot 2}$$

$$\frac{1}{2^2} = \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{2}{2\cdot 3\cdot 4\cdot 5} + \frac{2\cdot 3}{2^2\cdot 3\cdot 4\cdot 5}$$

$$\frac{1}{3^2} = \frac{1}{3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \frac{2}{3\cdot 4\cdot 5\cdot 6} + \frac{2\cdot 3}{3\cdot 4\cdot 5\cdot 6\cdot 7} + \frac{2\cdot 3\cdot 4}{3\cdot\dots\cdot 8} + \frac{2\cdot 3\cdot 4\cdot 5}{3^2\cdot 4\cdot\dots\cdot 8}$$

$$\vdots$$

we get the same series (4).

Other choices are possible of course. If we use $m = 0$ for the first two, $m = 1$ for the next one, $m = 2$ for the next two, and so on, alternating 2 and 1, then we have to add all the terms of two consecutive columns, columns I+II, columns III+IV, and so on. The general term in the resulting series consists of 5 terms in this case. We obtain this series:

$$\zeta(2) = \frac{1}{36}\sum_{n=1}^{\infty} \frac{P_6(n)}{n^2(2n-1)^2(3n-2)^2(3n-1)^2\binom{5n}{2n}}$$

with

$$P_6(n) = 54306n^6 - 123798n^5 + 112233n^4 - 51426n^3 + 12493n^2 - 1528n + 76.$$

https://doi.org/10.1017/mag.2017.7 Published online by Cambridge University Press

Note that the binomial factor in the denominator determines the speed of convergence of the series. It is not difficult to see that the multiple of n at the bottom of the binomial coefficient equals the number of columns we have to take together, and the multiple of n at the top is given by the number of terms that remain after the telescoping. As a consequence the fastest convergent series are those where the number of columns is half the number of terms that form the general term. Here are some examples:

Convergence rate $\frac{1}{16}$

The next series was derived by using $m = 0$ for the first two terms of the series for $\zeta(2)$, $m = 2$ for the next two, $m = 4$ for the next two, $m = 6$ for the next two, and so on, and then adding pairs of columns (I+II, III+IV, etc.). The general term is then a sum of four terms. This is the resulting series:

$$\zeta(2) = \frac{3}{4}\sum_{n=1}^{\infty}\frac{20n^2 - 8n + 1}{n^2(2n-1)^2\binom{4n}{2n}}.$$

This series can be found in Mohammed [14, p. 16]. Mohammed used the Markoff-WZ Method [15].

Convergence rate $\frac{1}{64}$

Choosing $m = 0$ for the first three, $m = 3$ for the next three, $m = 6$ for the next three, and so on, and then adding three successive columns (I+II+III, IV+V+VI, etc.), leads to this series:

$$\zeta(2) = \frac{1}{3}\sum_{n=1}^{\infty}\frac{1701n^4 - 1944n^3 + 729n^2 - 96n + 4}{n^2(3n-1)^2(3n-2)^2\binom{6n}{3n}}.$$

We can go on like this. In general the series with convergence rate $\left(\frac{1}{4}\right)^{\ell}$ is given by

$$\zeta(2) = \sum_{n=0}^{\infty}\frac{P_{\ell}(n)(\ell n)!^2}{(2\ell(n+1))!}$$

with $P_{\ell}(n) =$

$$\sum_{k=1}^{\ell}\frac{\Gamma(\ell n+k)\Gamma(2\ell n+2\ell+1)\left(\Gamma(2\ell n+k+1)\Gamma(\ell n+\ell+1)+\Gamma(2\ell n+k+\ell+1)\Gamma(\ell n+1)\right)}{\Gamma(2\ell n+k+1)(\ell n+k)(\Gamma(\ell n+1))^2\Gamma(2\ell n+k+\ell+1)}.$$

If we use 10 terms of this series, with $\ell = 100$, the approximation of $\zeta(2)$ we get is correct to more than 600 digits.

https://doi.org/10.1017/mag.2017.7 Published online by Cambridge University Press

Case 2: *Only odd squares*

For the second case we start from

$$\frac{3}{4}\zeta(2) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Again, it is easy to see that

$$\frac{1}{n} = \frac{1}{n+2} + \frac{2}{n(n+2)},$$

$$\frac{1}{n(n+2)} = \frac{1}{(n+2)(n+4)} + \frac{4}{n(n+2)(n+4)},$$

$$\frac{1}{n(n+2)(n+4)} = \frac{1}{(n+2)(n+4)(n+6)} + \frac{6}{n(n+2)(n+4)(n+6)}$$

and so on.

Using these relations, we can write $1/n$ as the sum of any number of terms. For instance, writing $1/(2m+1)^2$ as the sum of $m+1$ terms, we have:

$$\begin{aligned}
\frac{1}{1^2} &= \frac{2^0\cdot 0!}{1^2} \\
\frac{1}{3^2} &= \frac{2^0\cdot 0!}{3\cdot 5} + \frac{2^1\cdot 1!}{3^2\cdot 5} \\
\frac{1}{5^2} &= \frac{2^0\cdot 0!}{5\cdot 7} + \frac{2^1\cdot 1!}{5\cdot 7\cdot 9} + \frac{2^2\cdot 2!}{5^2\cdot 7\cdot 9} \\
\frac{1}{7^2} &= \frac{2^0\cdot 0!}{7\cdot 9} + \frac{2^1\cdot 1!}{7\cdot 9\cdot 11} + \frac{2^2\cdot 2!}{7\cdot 9\cdot 11\cdot 13} + \frac{2^2\cdot 3!}{7^2\cdot 9\cdot 11\cdot 13} \\
&\vdots
\end{aligned}$$

Then, summing the columns, we get

$$\begin{aligned}
\frac{\pi^2}{8} &= 2^0 0!\,\frac{1^2+2\cdot 3}{1^2\cdot 2\cdot 3} + 2^1 1!\,\frac{3^2+4\cdot 7}{3^2\cdot 4\cdot 5\cdot 7} + 2^2 2!\,\frac{5^2+6\cdot 11}{5^2\cdot 6\cdot 7\cdot 9\cdot 11} + \ldots \\
&= \sum_{n=1}^{\infty} \frac{2^{n-1}(n-1)!\left[(2n-1)^2+2n(4n-1)\right]}{2n(2n-1)^2(2n+1)(2n+3)\ldots(4n-1)} \\
&= \sum_{n=1}^{\infty} \frac{2^n n!\,(12n^2-6n+1)(2n-1)!!}{4n^2(2n-1)^2(4n-1)!!}
\end{aligned}$$

or, after some manipulations

$$\zeta(2) = \frac{1}{3}\sum_{n=1}^{\infty} \frac{4^n(12n^2-6n+1)}{n^2(2n-1)^2\binom{4n}{2n}}.$$

As before, there are a lot of other possibilities. If we look at the same special case as before, we get series with convergence rate $1/4^\ell$ of which the previous one is an example. In general, we find

$$\zeta(2) = \frac{4}{3}\sum_{n=1}^{\infty} \frac{4^{\ell n} Q_\ell(n)\,(2\ell n)!^2}{(4\ell^\ell(n+1))!},$$

https://doi.org/10.1017/mag.2017.7 Published online by Cambridge University Press

with $Q_\ell(n) = \dfrac{\Gamma(4m\ell + 4\ell + 1)}{(\Gamma(2m\ell + 1))^2 4^{m\ell+1}} \times$

$$\sum_{i=0}^{\ell}\left(\frac{2\Gamma(m\ell+1)\Gamma(m\ell+i+1/2)}{(2m\ell+2i+1)\Gamma(2m\ell+3/2+i)} + \frac{\Gamma(m\ell+i+1)\Gamma(m\ell+\ell+1/2)}{(m\ell+i+1)\Gamma(2m\ell+i+3/2+\ell)}\right).$$

For instance, if we take $\ell = 2$ we obtain the following series:

$$\zeta(2) = \frac{1}{3}\sum_{n=1}^{\infty}\frac{16^n p_6(n)}{n^2(4n-3)^2(4n-2)^2(4n-1)^2\binom{8n}{4n}}$$

with $p_6(n) = 15360n^6 - 33024n^5 + 27584n^4 - 11248n^3 + 2316n^2 - 225n + 9$.

2. *Stirling's convergence acceleration method for* $\zeta(k)$

The same method can be used to accelerate the convergence of series involving the reciprocals of higher powers of natural numbers. Again the inspiration comes from Stirling himself [5, p. 11]:

Tabula posterior.

1									
1	1								
2	3	1							
6	11	6	1						
24	50	35	10	1					
120	274	225	85	15	1				
720	1764	1624	735	175	21	1			
5040	13068	13132	6769	1960	322	28	1		
40320	109584	105056	67284	22449	4536	546	36	1	
&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.

Deinde fumendo Coefficientes ex columnis deſcendentibus, obtinebis valores Dignitatum,

$$\frac{1}{z^2} = \frac{1}{z.z+1} + \frac{1}{z.z+1.z+2} + \frac{2}{z.z+1.z+2.z+3} + \frac{6}{z.z+1.z+2.z+3.z+4} + \&c.$$

$$\frac{1}{z^3} = \frac{1}{z.z+1.z+2} + \frac{3}{z.z+1.z+2.z+3} + \frac{11}{z.z+1.z+2.z+3.z+4} + \&c.$$

$$\frac{1}{z^4} = \frac{1}{z.z+1.z+2.z+3} \text{ in } 1 + \frac{6}{z+4} + \frac{35}{z+4.z+5} + \frac{225}{z+4.z+5.z+6} + \&c.$$

FIGURE 2: Unsigned Stirling numbers of the first kind and $\zeta(k)$

https://doi.org/10.1017/mag.2017.7 Published online by Cambridge University Press

In this triangular array Stirling defines what is now known as the (unsigned) Stirling numbers of the first kind, and uses these to derive the inverse factorial series for $1/z^k$.

We will treat the series

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

Note that the following identities are easy to prove (see Theorem 1):

$$\begin{aligned}
\frac{1}{n^3} &= \frac{1}{(n)_3} + \frac{3n+2}{n^2.(n)_3} \\
\frac{1}{n^3} &= \frac{1}{(n)_3} + \frac{3}{(n)_4} + \frac{11n+6}{n^2.(n)_4} \\
\frac{1}{n^3} &= \frac{1}{(n)_3} + \frac{3}{(n)_4} + \frac{11}{(n)_5} + \frac{50n+24}{n^2.(n)_5} \\
&\vdots
\end{aligned}$$

Here we have used the Pochhammer symbol to denote the rising factorial: $(z)_k = z(z+1)\ldots(z+k-1)$. In general we have

$$\frac{1}{n^3} = \frac{1!H_1}{(n)_3} + \frac{2!H_2}{(n)_4} + \ldots + \frac{(k-1)!H_{k-1}}{(n)_{k+1}} + \frac{k!H_k \cdot n + k!}{n^2 \cdot (n)_{k+1}} \qquad (5)$$

where $H_m = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m}$.

Hence we can write:

$$\begin{aligned}
\frac{1}{1^3} &= \frac{1}{1^3} \\
\frac{1}{2^3} &= \frac{1}{2\cdot3\cdot4} + \frac{3\cdot2+2}{2^3\cdot3\cdot4} \\
\frac{1}{3^3} &= \frac{1}{3\cdot4\cdot5} + \frac{3}{3\cdot4\cdot5\cdot6} + \frac{11\cdot3+6}{3^3\cdot4\cdot5\cdot6} \\
\frac{1}{4^3} &= \frac{1}{4\cdot5\cdot6} + \frac{3}{4\cdot5\cdot6\cdot7} + \frac{11}{4\cdot5\cdot6\cdot7\cdot8} + \frac{50\cdot4+24}{4^3\cdot5\cdot6\cdot7\cdot8} \\
&\vdots
\end{aligned}$$

Summing the terms of the columns, letting the telescoping do its work, and taking the terms with and without H_n together, we find this series:

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3\binom{2n}{n}} + \sum_{n=1}^{\infty} \frac{3n^2+3n+1}{n^2(n+1)(2n+1)\binom{2n}{n}} H_n.$$

Note that if we only use the odd cubes, as in 2.2, we get this series:

$$\frac{7}{8}\zeta(3) = \sum_{n=1}^{\infty} \frac{4^n}{(2n-1)^3\binom{4n}{2n}} + \frac{3}{4}\sum_{n=1}^{\infty} \frac{(4n^2+2n+1)4^n}{(2n-1)^2(n+1)(4n+1)\binom{4n}{2n}} H_n.$$

https://doi.org/10.1017/mag.2017.7 Published online by Cambridge University Press

Other schemes are possible. For instance, if we use this one:

$$\begin{aligned}
\frac{1}{1^3} &= \frac{1}{1^3}\\
\frac{1}{2^3} &= \frac{1}{2^3}\\
\frac{1}{3^3} &= \frac{1}{3\cdot4\cdot5} + \frac{3}{3\cdot4\cdot5\cdot6} + \frac{11\cdot3+6}{3^3\cdot4\cdot5\cdot6}\\
\frac{1}{4^3} &= \frac{1}{4\cdot5\cdot6} + \frac{3}{4\cdot5\cdot6\cdot7} + \frac{11\cdot4+6}{4^3\cdot5\cdot6\cdot7}\\
\frac{1}{5^3} &= \frac{1}{5\cdot6\cdot7} + \frac{3}{5\cdot6\cdot7\cdot8} + \frac{11}{5\cdot6\cdot7\cdot8\cdot9} + \frac{50}{5\cdot\ldots\cdot10} + \frac{274\cdot5+120}{5^3\cdot6\cdot\ldots\cdot10}\\
\frac{1}{6^3} &= \frac{1}{6\cdot7\cdot8} + \frac{3}{6\cdot7\cdot8\cdot9} + \frac{11}{6\cdot7\cdot8\cdot9\cdot10} + \frac{50}{6\cdot\ldots\cdot11} + \frac{274\cdot6+120}{6^3\cdot7\cdot\ldots\cdot11}\\
&\vdots
\end{aligned}$$

and combine the columns like this: I+II, III+IV, …, we obtain this series:

$$\zeta(3) = \frac{1}{4}\sum_{n=1}^{\infty}\frac{208n^5-8n^4-12n^3+10n^2-1}{n^3(2n-1)^3(2n+1)(4n+1)\binom{4n}{2n}} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{240n^4+24n^3-40n^2+2n+3}{n^2(2n-1)^2(2n+1)(4n+1)\binom{4n}{2n}}H_{2n-1}.$$

Other values of the zeta function can be treated in the same way.

The expression (5) can be generalised for $k = 2, 3, 4, \ldots$.

Theorem 1: For $\ell = k-2, k-1, k, \ldots$

$$\frac{1}{n^k} = \sum_{m=k}^{\ell+1}\frac{c(m-1,k-1)}{(n)_m} + \frac{1}{n^{k-1}(n)_{\ell+1}}\sum_{m=1}^{k-1}c(\ell+1,m)\,n^{m-1}$$

where the $c(i,j)$ denote the unsigned Stirling numbers of the first kind.

Proof: We will use induction on ℓ. For $\ell = k-2$ the right-hand side reduces to the left-hand side since the first sum is empty and the numerator in the second sum is equal to $\frac{1}{n}(n)_{k-1}$. This is a consequence of the definition of the unsigned Stirling numbers of the first kind. They satisfy [16, p. 263]

$$\sum_{m=1}^{k-1}c(k-1,m)\,n^m = (n)_{k-1}.$$

https://doi.org/10.1017/mag.2017.7 Published online by Cambridge University Press

Let us now assume that we have that

$$\frac{1}{n^k} - \sum_{m=k}^{\ell} \frac{c(m-1,k-1)}{(n)_m} = \frac{1}{n^{k-1}(n)_\ell} \sum_{m=1}^{k-1} c(\ell, m) n^{m-1}.$$

Then

$$\begin{aligned}
&\frac{1}{n^k} - \sum_{m=k}^{\ell+1} \frac{c(m-1,k-1)}{(n)_m} \\
&= \frac{1}{n^k} - \sum_{m=k}^{\ell} \frac{c(m-1,k-1)}{(n)_m} - \frac{c(\ell,k-1)}{(n)_{\ell+1}} \\
&= \frac{1}{n^{k-1}(n)_\ell} \sum_{m=1}^{k-1} c(\ell,m) n^{m-1} - \frac{c(\ell,k-1)}{(n)_{\ell+1}} \\
&= \frac{1}{n^{k-1}(n)_{\ell+1}} \left(\sum_{m=1}^{k-1} (n+\ell) c(\ell,m) n^{m-1} - n^{k-1} c(\ell,k-1) \right).
\end{aligned}$$

For the expression between brackets we use the recurrence relation for the Stirling numbers [16, p. 259]:

$$c(\ell+1, m) = \ell c(\ell, m) + c(\ell, m-1), \qquad c(\ell, 0) = 0.$$

Using this, it reduces to

$$\begin{aligned}
&\sum_{m=1}^{k-1} (n+\ell) c(\ell, m) n^{m-1} - n^{k-1} c(\ell, k-1) \\
&= \sum_{m=1}^{k-1} \left(c(\ell,m) n^m + \left(c(\ell+1,m) - c(\ell,m-1) \right) n^{m-1} \right) - n^{k-1} c(\ell,k-1) \\
&= \sum_{m=1}^{k-1} c(\ell+1,m) n^{m-1}
\end{aligned}$$

after some rearranging. This proves (6).

Note that

$$c(i, 1) = (i-1)!, \qquad c(i, 2) = (i-1)! H_{i-1}$$

proving (5).

If we proceed in the same way as before, and use (6) for $n = 1$ with $\ell = k-2$, for $n = 2$ with $\ell = k-1$, and so on, then we obtain the following formula by summing the columns as before:

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{n!}{(2n+k-3)!} \left(\sum_{m=1}^{k-1} \frac{c(n+k-2,m)}{n^{k-m+1}} + \frac{c(n+k-2,k-1)}{(n+k-2)(2n+k-2)} \right).$$

https://doi.org/10.1017/mag.2017.7 Published online by Cambridge University Press

For the first few values of k we get:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{n!}{(2n-1)!}\left(\frac{(n-1)!}{n^2} + \frac{(n-1)!}{2n^2}\right) = 3\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}}$$

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{n!}{(2n)!}\left(\frac{n!}{n^3} + \frac{n!\,H_n}{n^2} + \frac{n!\,H_n}{(n+1)(2n+1)}\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^3\binom{2n}{n}} + \sum_{n=1}^{\infty} \frac{3n^2+3n+1}{n^2(n+1)(2n+1)\binom{2n}{n}} H_n$$

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}\left(\frac{(n+1)!}{n^4} + \frac{(n+1)!H_{n+1}}{n^3} + \frac{c(n+2,\,3)}{n^2} + \frac{c(n+2,\,3)}{(n+2)(2n+2)}\right).$$

Note that the series for $k = 2$ and $k = 3$ are the same as the ones obtained before.

Acknowledgement: We would like to thank the anonymous referee for useful suggestions.

References

1. L. Euler, *Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum* (1755) Part II, chapter I, available at eulerarchive.maa.org/page/E212.html
2. E. Kummer, Eine neue Methode, die numerischen Summen langsam convergirender Reihen zu berechnen, *J. Reine Angew. Math.* **16** (1837) pp. 206-214.
3. A. Markoff, Mémoire sur la transformation des séries peu convergentes en séries très convergentes, *Mémoires de L'Académie Impériale des Sciences de St.-Pétersbourg*, VII^{E} Série. Tome XXXVII, No. 9 (1890).
4. R. Gosper, *Acceleration of series*, Massachusetts Institute of Technology (1974) available at http://dspace.mit.edu/bitstream/handle/1721.1/6088/AIM-304.pdf
5. J. Stirling, *Methodus differentialis: sive tractatus de summatione et interpolatione serierum infinitarum*, Londini: Typis Gul. Bowyer; impensis G. Strahan (1730).
6. I. Tweddle, *James Stirling's Methodus Differentialis: an annotated translation of Stirling's text*, Springer-Verlag (2003).
7. D. Romik, Some comments on Euler's series for $\frac{\pi^2}{6}$, *Math. Gaz.* **86** (July 2002) pp. 281-285.

https://doi.org/10.1017/mag.2017.7 Published online by Cambridge University Press

8, R. Schumacher, Rapidly convergent summation formulas involving Stirling series, *Cornell University Library* (2016), available at http://arxiv.org/abs/1602.00336.

9. N. Nielsen, *Die Gammafunktion*, Chelsea, New York (1965).

10. I. Schur and K. Knopp, Über die Herleitung der Gleichung $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, *Arch. Math. Phys.* **3**, 27 (1918) pp. 174-176.

11. E. Catalan, Mémoire sur la transformation des séries, et sur quelques intégrales définies, *Mémoires Couronnés et Mémoires des Savants Étrangers* **33** (1867) pp. 1-50.

12. K. Knopp, *Theory and application of infinite series*, (2nd edn.), (4th reprint, 1954) pp. 266-267.

13. L. Euler, Démonstration de la somme de cette suite $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} +$ etc., *Journ. lit. d'Allemagne, de Suisse et du Nord*, **2**, 1 (1743), pp. 115-127, available at: eulerarchive.maa.org/page/E212.html

14. M. Mohammed, Infinite families of accelerated series for some classical constants by the Markov-WZ Method, *Discrete Math. Theor. Comput. Sci.* **7** (2005) pp. 11-24.

15. M. Kondratieva and S. Sadov, Markov's transformation of series and the WZ-method, *Adv. in Applied Math.* **34** (2005), pp. 393-407, available at: arXiv.org/abs/math/0405592

16. R. Graham, D. Knuth and O. Patashnik, *Concrete mathematics*, (2nd edn.), Addison-Wesley, Reading, MA (1994).

10.1017/mag.2017.7

PAUL LEVRIE[1,2] and
AMRIK SINGH NIMBRAN[3]

[1]*Faculty of Applied Engineering, University of Antwerp, B-2020 Antwerpen, Belgium*

[2]*Department of Computer Science, KU Leuven, P.O. Box 2402, B-3001 Heverlee, Belgium*

e-mail: *paul.levrie@cs.kuleuven.be*

[3]*B3-304, Palm Grove Heights, Ardee City, Gurgaon, Haryana, India*

e-mail: *amrikn622@gmail.com*

https://doi.org/10.1017/mag.2017.7 Published online by Cambridge University Press