WHEN IS $A + xA = \mathbb{R}$

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Abstract. For a subgroup A of $(\mathbb{R}, +)$ and a real x, define $A + xA = \{a + xb : a, b \in A\}$ and $X_A = \{x \in \mathbb{R} : A + xA = \mathbb{R}\}$. We show that there is an F_σ subgroup A of $(\mathbb{R}, +)$ such that $\dim_{\mathrm{H}}(A) \leq \frac{1}{2}$ and $X_A \neq \emptyset$. However, if $A \subseteq \mathbb{R}$ is a subring of $\mathbb{R}$ and $X_A \neq \emptyset$, then $A = \mathbb{R}$. Moreover, assuming the continuum hypothesis, there is a subgroup A of $(\mathbb{R}, +)$ with $\dim_{\mathrm{H}}(A) = 0$ such that $X_A = \mathbb{R}\backslash\mathbb{Q}$. The proof of this theorem combines several techniques in recursion theory and algorithmic dimension. Several other theorems on analytic subgroups and subfields of the reals are presented. We also discuss some of these results in the p-adics.

§1. Introduction. In this article we prove several results concerning the "sizes" of subgroups and subrings of the reals. Here by "size," we typically refer to the Hausdorff measure and dimension of such objects. The story of "sizes" has a rich history, which begins with a classical result in real analysis.

THEOREM 1.1 [Steinhaus [18]]. *Suppose $A \subseteq \mathbb{R}^n$ is Lebesgue measurable and has positive measure. Then the difference set $A - A := \{x - y : x, y \in A\}$ contains a ball with positive radius whose center is at the origin.*

The next corollary follows immediately.

COROLLARY 1.2. *If* A *is a measurable proper subgroup of $(\mathbb{R}, +)$, then* A *is null.*[1]

Subsequently, Volkmann and Erdős initiated the study of the dimension of subgroups/rings of the reals in the 1960s. In [20] they showed that for each $\alpha \in [0, 1]$, there is a Borel subgroup of $(\mathbb{R}, +)$ with Hausdorff dimension α. Edgar and Miller [3], and independently Bourgain [1], showed that an *analytic* (see the definition before Fact 2.5) subring of $\mathbb{R}$ either has Hausdorff dimension 0 or is all of $\mathbb{R}$. Mauldin [14] showed that assuming the continuum hypothesis (CH), for each α, $0 \leqslant \alpha \leqslant 1$, there is a subfield of $\mathbb{R}$ with Hausdorff dimension α. Following the strategy of "discretization" used by Bourgain in [1], de Saxcé [16] considered the problem in the setting of a connected simple real Lie group G endowed with a Riemannian metric and showed that there is no Borel measurable dense subgroup of G with Hausdorff dimension strictly between 0 and $\dim_{\mathrm{H}}(G)$. For a more detailed discussion on the early history, see [3, p. 1122]. The subgroups and subrings of the reals are also natural objects appearing in geometric measure theory (see, e.g., [4, Section 12.4]).

Received July 30, 2025 UTC.
2020 *Mathematics Subject Classification.* Primary 28A05, Secondary 03D80, 12L99.
Key words and phrases. subgroups of the reals, Hausdorff dimension, the continuum hypothesis.
[1] We call a set $A \subseteq \mathbb{R}$ *null* if it has Lebesgue measure zero.

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0022-4812/00/0000-0000
DOI:10.1017/jsl.2026.10232

https://doi.org/10.1017/jsl.2026.10232 Published online by Cambridge University Press

In this article we consider the following related question.

DEFINITION 1.3. Given a subgroup A of $(\mathbb{R}, +)$ and a real x, define

$$A + xA = \{a + xb : a, b \in A\}$$

and

$$X_A = \{x \in \mathbb{R} : A + xA = \mathbb{R}\}.$$

QUESTION 1. *Suppose $A \subseteq (\mathbb{R}, +, \cdot)$ is a subobject in some algebraic sense, e.g., subgroup, subring, or subfield. Under what conditions on* A *is X_A nonempty? And if $X_A \neq \emptyset$, what do we know about the size of* A*? What do we know about X_A?*

We prove the following main results.

THEOREM 1.4. *There is an F_σ subgroup* A *of $(\mathbb{R}, +)$ such that $\dim_{\mathrm{H}}(A) \leq \frac{1}{2}$ and $X_A \neq \emptyset$.*

THEOREM 1.5. *Assume* CH*, holds: $2^{\aleph_0} = \aleph_1$. There is a group* A *with $\dim_{\mathrm{H}}(A) = 0$ such that $X_A = \mathbb{R}\backslash\mathbb{Q}$.*

Theorem 1.4 is proved via a concrete construction of A and $x \in X_A$. For Theorem 1.5, we construct A using the concept of genericity and algorithmic dimension. We believe that this method may be used to construct other exotic subsets of reals.

We also consider what would happen if we restrict Question 1 to the analytic subgroups. By the Marstrand projection theorem, if A is an analytic subgroup of the reals and $\dim_{\mathrm{H}}(A) > \frac{1}{2}$, then X_A is co-null[2] (Proposition 3.4). Our main results improve this proposition in terms of dimension. The restriction on analytic sets also enables us to use descriptive set theory. For example, if A is an analytic subgroup of the reals and $x \in X_A$, then there is an F_σ subgroup $B \subseteq A$ such that $x \in X_B$ (Theorem 3.8).

For subrings, the condition is different. If A is a subring of $\mathbb{R}$ and $X_A \neq \emptyset$, then $A = \mathbb{R}$ (Proposition 2.4).

Besides Question 1, we also study the maximal subfields of $\mathbb{R}$ without a given point because the proof techniques suggest so. Given a field K and $a \in K$, a *maximal subfield of K without a* is a maximal (with respect to inclusion) element of $\{L \subseteq K : L \text{ is a subfield of } K \text{ and } a \notin L\}$. Such subfields have been well investigated in the literature. For example, Quigley [15] described the structure of K/M in detail where K is an algebraically closed field and M is a maximal subfield of K without a given point a, and used Galois theory to give existence proofs of M *which he characterized as* more precise than that trivially given by Zorn's lemma. We restrict our attention to the case that $K = \mathbb{R}$. We show that the usage of the axiom of choice (AC) is necessary for the existence of a maximal subfield of $\mathbb{R}$ without a given point (Corollary 2.7). Hence, the claim that Quigley's proof is "more precise" is arguable. On the other hand, by assuming CH, we construct a maximal subfield of $\mathbb{R}$ with Hausdorff dimension 0 such that some given point is not in its algebraic closure relative to $\mathbb{R}$ (Proposition 4.9).

The structure of the article is as follows: Section 2 deals with subrings and subfields of $\mathbb{R}$. In Section 3 we discuss some simple results concerning the basic

[2]We call a set $A \subseteq \mathbb{R}$ *co-null* if its complement is null.

https://doi.org/10.1017/jsl.2026.10232 Published online by Cambridge University Press

properties of subgroups and prove Theorem 1.4. In Section 4 we assume CH in the constructions and prove Theorem 1.5. We also show that A in Theorem 1.5 cannot be F_σ (Proposition 4.6). Due to the variety of the tools used for the results and the independence of the methods, we postpone the introduction of notation and terminologies, and only recall them when being used.

§2. Subrings and subfields. Recall that an ordered field $(R, +, \cdot, 0, 1, \leq)$ ($\leq$ is a total order on F; if $a \leq b$ then $a + c \leq b + c$; and if $0 \leq a$ and $0 \leq b$ then $0 \leq a \cdot b$) is *real closed* if:

(i) any positive element has a square root in R, and

(ii) any polynomial equation $f(x) = 0$ where $f(x) \in R[x]$ is of odd degree has a root in R.

FACT 2.1 (Artin–Schreier, see [7, p. 674]). *Let* C *be an algebrically closed field and* F *be a proper subfield of* C *such that* C/F *is finite. Then* F *is real closed and* $C = F\left(\sqrt{-1}\right)$.

FACT 2.2 ("Lying-over," see [7, p. 411]). *Let* E *be a commutative ring and* R *a subring such that* E *is integral over* R (*for every element* α *of* E, *there is a monic polynomial* $f(x) \in R[x]$ *such that* $f(\alpha) = 0$). *Then for each prime ideal* p *of* R, *there is a prime ideal* P *of* E *such that* $p = P \cap R$. *In particular, if* E *is a field, so is* R.

Let p be a prime number. Let $a = \frac{b}{c}, b, c \in \mathbb{Z}$ be nonzero. We extract from b and c as high a power of the prime number p as possible, namely choose m an interger such that

$$a = p^m \frac{b'}{c'}, (b'c', p) = 1,$$

and we put the *p-adic absolute value* $|a|_p = \frac{1}{p^m}$. The field of *p-adic numbers* $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the metric $d_p(x, y) = |x - y|_p$.

The following lemma is presumably well-known.

LEMMA 2.3. (i) *If* F *is a subfield of* $\mathbb{R}$ *such that* $\mathbb{R}/F$ *is finite, then* $F = \mathbb{R}$.

(ii) *If* F *is a subfield of* $\mathbb{Q}_p$ *such that* $\mathbb{Q}_p/F$ *is finite, then* $F = \mathbb{Q}_p$.

PROOF. (i) follows from Fact 2.1. For (ii), consider the normal closure K of $\mathbb{Q}_p/F$; K is finite over $\mathbb{Q}_p$ and hence any automorphism of K is continuous. Hence any automorphism of K restricts to the identity on $\mathbb{Q}_p$, which implies that $\mathbb{Q}_p = F = K$. ⊣

Recall that we define $X_A = \{x \in \mathbb{R} : A + xA = \mathbb{R}\}$.

PROPOSITION 2.4. (i) *If* A *is a subring of* $\mathbb{R}$ *and* $X_A \neq \emptyset$, *then* $A = \mathbb{R}$.

(ii) *If* A *is a subring of* $\mathbb{Q}_p$ *and there is* x *such that* $A + xA = \mathbb{Q}_p$, *then* $A = \mathbb{Q}_p$.

PROOF. (i) By our assumption, $\mathbb{R}$ is finite over A and hence by Fact 2.2, A is a field since $\mathbb{R}$ is a field. Then Lemma 2.3(i) implies $A = \mathbb{R}$.

(ii) By Fact 2.2 and our assumption, A is a subfield of $\mathbb{Q}_p$. If $A \neq \mathbb{Q}_p$, then $[\mathbb{Q}_p : A] \neq 1$ is finite. Then Lemma 2.3(ii) implies $A = \mathbb{Q}_p$. ⊣

Both $\mathbb{Q}_p$ and $\mathbb{R}$ are *Polish* (separable completely metrizable) spaces. A subset A of a Polish space X is *analytic* if it is the projection of a Borel set $B \subseteq X \times X$.

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FACT 2.5 (See [8, Theorem 11.18]). *Every analytic subset of* $\mathbb{R}$ *is measurable.*

The proof of this fact is general enough to work for other measures in Polish spaces. For example, in the following Proposition 2.6, we will use the fact that every analytic subset of $\mathbb{Q}_p$ is μ-measurable where μ is the Haar measure on $\mathbb{Q}_p$.

Given a field K and $a \in K$, a *maximal subfield of* K *without* a is a maximal (with respect to inclusion) element of $\{L \subseteq K : L \text{ is a subfield of } K \text{ and } a \notin L\}$.

PROPOSITION 2.6. *Let* $K = \mathbb{Q}_p$ *or* $\mathbb{R}$. *Let* F *be an analytic subfield of* K *with some* $x \in K \setminus F$. *Then there is* $y \in K \setminus F$ *such that* $x \notin F(y)$. *Hence* F *is not a maximal subfield of* K *without* x.

PROOF. Assume otherwise, then F is a maximal subfield of K avoiding x. This implies that K/F is algebraic. Indeed, if x is not algebraic over F, then $x \notin F(x^2)$. Now if K/F is not algebraic, then K contains a copy of $F(X)$ for X transcendental over F, which does not contain x.

So each $y \in K \setminus F$ satisfies a polynomial over F. Let D_n denote the set of elements in K whose minimal polynomial over F is of degree at most n, this is an analytic set and $\bigcup_{n\in\mathbb{N}} D_n = K$. In particular, D_n is measurable by Fact 2.5. It is clear that D_n is closed under multiplication by -1. Note further that the difference of 2 elements of degree at most n over F has degree at most n^2. Moreover, there must be n such that $\mu(D_n) > 0$ where μ denote the Haar measure on K. So $D_{n^2} = K$ by the Steinhaus theorem for Haar measure on locally compact groups (see [19]). This means that any element in K/F is of degree at most n for some $n \in \mathbb{N}$. By the primitive element theorem, we have that $[K : F]$ is at most n. Then by Lemma 2.3, $K = F$, a contradiction. ⊣

The assumption that F is analytic is used to guarantee that D_n is measurable, so that we can use the Steinhaus theorem. If D_n's are measurable a priori, then the proof will go through as is. It follows from AC that there is a non-measurable set. Under some weaker set theoretic axioms, the situation will be different. We use DC to denote *the axiom of dependent choice*, that is: suppose R is a relation on a nonempty set X; if for every $a \in X$, there is a $b \in X$ such that aRb, then there is a sequence $\{x_n\}_{n\in\omega}$ in X such that $x_n R x_{n+1}$ for all $n \in \omega$. DC follows from AC trivially. Solovay [17] proved that ZF + DC + Every subset of$\mathbb{R}$is measurable is consistent.

By the above discussion and the proof of Proposition 2.6, we have the following conclusion: without AC, one cannot construct a maximal subfield of $\mathbb{R}$ without a given point.

COROLLARY 2.7. *Assume* ZF + DC + *Every subset of* $\mathbb{R}$ *is measurable. For any subfield* F *of* $\mathbb{R}$ *and* $x \in \mathbb{R} \setminus F$, F *is not a maximal subfield of* K *without* x.

§3. Subgroups. We recall some terminology and notation from [4]. Let $A \subseteq X$, where (X, d_X) is a metric space. Let the diameter of A, written $|A|$, be the supremum of the distances between any two points in A, i.e., $|A| = \sup_{x,y\in A} d_X(x, y)$. Suppose $s \geq 0$. For each $\delta > 0$, we define the *s-dimensional Hausdorff measure* of A by

$$\mathcal{H}^s(A) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(A) = \lim_{\delta\to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a cover of } A, 0 < |U_i| \leqslant \delta, \forall i \right\}.$$

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$\mathcal{H}^s(A)$ can be (and usually is) 0 or ∞. There is a critical value of s at which $\mathcal{H}^s(A)$ "jumps" from ∞ to 0. This critical value is called the *Hausdorff dimension* of A, written $\dim_{\mathrm{H}}(A)$. In other words,

$$\dim_{\mathrm{H}}(A) = \inf\{s : \mathcal{H}^s(A) = 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\}.$$

We gather some of the basic properties of subgroups of the reals. Recall that we define $X_A = \{x \in \mathbb{R} : A + xA = \mathbb{R}\}$.

PROPOSITION 3.1. *Suppose that* A *is a subgroup of* $\mathbb{R}$.

(i) *For all* $x \in \mathbb{R}$, $A + xA$ *is a subgroup of* $\mathbb{R}$.
(ii) *If* $p \in \mathbb{Q}$ *and* $p \in X_A$, *then* $A = \mathbb{R}$.
(iii) *If* $k \in \mathbb{N}$, $p_0, \ldots, p_k \in \mathbb{Q}$ *and* $A + \sum_{i=0}^{k} p_i A = \mathbb{R}$, *then* $A = \mathbb{R}$.
(iv) *Define the tensor product*

$$\mathbb{Q} \otimes_{\mathbb{Z}} A = \left\{ \sum_{i=0}^{k} p_i a_i : k \in \mathbb{N}, p_i \in \mathbb{Q}, a_i \in A \text{ for } i = 0, \ldots, k \right\}.$$

Then $\dim_{\mathrm{H}}(\mathbb{Q} \otimes_{\mathbb{Z}} A) = \dim_{\mathrm{H}}(A)$.

(v) *If* $x \in X_A$, *then for all* $m \in \mathbb{Z} - \{0\}$, $mx \in X_A$.
(vi) *If* $x \in X_A$, *then for all* $m, n \in \mathbb{Z} - \{0\}$, $x + \frac{n}{m} \in X_A$.

PROOF. (i) By the definition of a subgroup.

(ii) Suppose that $p = \frac{m}{n}$ for some integers m and $n \neq 0$. Then for any $c \in \mathbb{R}$, there are $a, b \in A$ such that $a + \frac{mb}{n} = c$. So $nc = na + mb \in A$. Thus $\mathbb{R} = \frac{1}{n}A$. Since $\mathrm{Char}(\mathbb{R}) = 0$, we have $\mathbb{R} = n\mathbb{R} = A$.

(iii) Suppose that $p_i = \frac{m_i}{n_i}$ for some integers m_i and $n_i \neq 0$. Let n be the least common multiple of those n_i. Then $\mathbb{R} = A + \sum_{i=0}^{k} p_i A \subseteq \frac{1}{n}A$, so $A = \mathbb{R}$.

(iv) Note that $\mathbb{Q} \otimes_{\mathbb{Z}} A \subseteq \bigcup_{n \geq 1} \frac{1}{n}A$. But it is clear that $\dim_{\mathrm{H}}(A) = \dim_{\mathrm{H}}(\frac{1}{n}A)$ for any $n \geq 1$ since $f : \mathbb{R} \to \mathbb{R}, x \mapsto nx$ is bi-Lipschitz. So

$$\dim_{\mathrm{H}}(A) \leqslant \dim_{\mathrm{H}}(\mathbb{Q} \otimes_{\mathbb{Z}} A) \leqslant \dim_{\mathrm{H}}\left(\bigcup_{n \geq 1} \frac{1}{n}A\right) = \sup_i \dim_{\mathrm{H}}(\frac{1}{n}A) = \dim_{\mathrm{H}}(A).$$

(v) Note that

$$A + mxA = m(\frac{A}{m} + xA) \supseteq m(A + xA) = m\mathbb{R}.$$

Since $\mathrm{Char}(\mathbb{R}) = 0$, we have $A + mxA = \mathbb{R}$.

(vi) Note that

$$A + (x + \frac{n}{m})A = A + (mx + n)\frac{A}{m} \supseteq A + (mx + n)A = A + mxA.$$

By (v), we have $A + (x + \frac{n}{m})A = \mathbb{R}$. ⊣

REMARK 3.2. Let X be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ with $\dim_{\mathrm{H}}(X) = 0$ (such a basis exists by Lutz et al. [11]), and let A be the group generated by X. Then by Proposition 3.1(iv),

$$\dim_{\mathrm{H}}(A) = \dim_{\mathrm{H}}(\mathbb{R}) = 1 > \dim_{\mathrm{H}}(X).$$

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Hence the additive group generated by a basis X may have Hausdorff dimension greater than $\dim_{\mathrm{H}}(X)$.

We can easily deduce from the Marstrand projection theorem: if A is "regular" and "large" enough, then there is x such that $A + xA = \mathbb{R}$.

FACT 3.3 (Marstrand [13]). *Let $F \subseteq \mathbb{R}^2$ be an analytic set with $\dim_{\mathrm{H}}(F) > 1$. Then for almost all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\mathrm{proj}_\theta(F)$ (the projection of* F *on the line $l_\theta : y = \tan\theta \cdot x$) has positive length.*

PROPOSITION 3.4. *If* A *is an analytic subgroup of $(\mathbb{R}, +)$ with $\dim_{\mathrm{H}}(A) > \frac{1}{2}$, then X_A is co-null.*

This is a minor generalization of [4, Exercise 6.8], which implies that for almost all x, $A + xA$ has Hausdorff dimension 1.

PROOF. By the product formula of Hausdorff dimension (see [4, Chapter 7]),

$$\dim_{\mathrm{H}}(A \times A) \geq \dim_{\mathrm{H}}(A) + \dim_{\mathrm{H}}(A) > 1.$$

Hence for almost all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\mathrm{proj}_\theta(A \times A)$ has positive length. Let $f(t) = \arctan t$ for all $t \in \mathbb{R}$. Define $\Theta = \{\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) : \mathrm{proj}_\theta(A \times A) \text{ has positive length}\}$. Since Θ is co-null (and hence measurable), $f^{-1}(\Theta)$ is measurable. We claim that $f^{-1}(\Theta)$ is co-null. Suppose not, then $\mathbb{R}\backslash f^{-1}(\Theta)$ has positive measure. Then there is $N \in \mathbb{R}$ such that $\mathbb{R}\backslash f^{-1}(\Theta) \cap [-N, N]$ has positive measure. But $\inf_{t\in[-N,N]} f'(t) > 0$, then $f(f^{-1}(\Theta) \cap [-N, N])$ has positive measure, a contradiction.

Suppose $(a, b), (c, d) \in A \times A$, $a \neq c$ and $b \neq d$. Then the line passing through these two points is

$$l_{(a,b),(c,d)} : \frac{y-d}{x-c} = \frac{y-b}{x-a}.$$

The slope of $l_{(a,b),(c,d)}$ is $\frac{b-d}{a-c}$. And if $t \neq 0$ and $a + tb = c + td$, then $t = -\frac{a-c}{b-d}$. So for $(a, b), (c, d)$, and t as above, $a + tb = c + td$ if and only if $l_{(a,b),(c,d)}$ is perpendicular to $l_{\arctan t} : y = t \cdot x$. In particular, $l_{\arctan t}$ and $A + tA$ are homeomorphic via:

$$(a, b) \mapsto a + bt.$$

And

$$|a + bt| = (1 + t^2)|a| = \sqrt{1 + t^2}\sqrt{a^2 + b^2} = \sqrt{1 + t^2}|\overrightarrow{0, (a, b)}|.$$

Hence the measure of $A + tA$ is equal to the measure of $\mathrm{proj}_{\arctan t}(A \times A)$ times $\sqrt{1 + t^2}$ provided $t \neq 0$. By Proposition 3.1(i), $A + tA$ is a group. Then by Corollary 1.2, $A + tA = \mathbb{R}$ if $\mathrm{proj}_{\arctan t}(A \times A)$ has positive length. Since $f^{-1}(\Theta)$ is co-null, $\mathrm{proj}_{\arctan t}(A \times A)$ has positive length for almost all t. ⊣

Next, we prove Theorem 1.4, which refines Proposition 3.4.

PROOF OF THEOREM 1.4. As a convention, we consider a real a and its binary expansion $m + 0.a_1a_2 \ldots a_n \ldots$ with infinitely many 0s to be equivalent, where $m \in \mathbb{Z}$ is the integer part of a and the numbers $a_i \in \{0, 1\}$ express its binary decimal part.

We first prove a weaker result: there is an F_σ subgroup A of $(\mathbb{R}, +)$ and $x \in \mathbb{R}$ such that $A + xA = \mathbb{R}$ and A is null.

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Define $Q_n = \{m \in \mathbb{N} : 3^{n-1} \leqslant m < 3^n\}$ for $n \geq 1$. Then each Q_n contains $2 \cdot 3^{n-1}$ numbers. Divide $\{1, 2, ...\}$ into two parts P_1, P_2 such that for all $n \geq 1$, the first 3^{n-1} numbers of Q_n are in P_1, and the other numbers of Q_n are in P_2. Formally,

$$P_1 = \{k : \exists n \exists i \in [0, 3^n)(k = 3^n + i)\},$$
$$P_2 = \{k : \exists n \exists i \in [3^n, 2 \cdot 3^n)(k = 3^n + i)\}.$$

Define

$$A_0 = \{m + 0.a_1 a_2 \cdots a_n \cdots : m \in \mathbb{Z} \wedge \forall n \in P_2(a_n = 0)\}.$$

A_0 consists of reals with segments of 0s of length $3^k (k \in \mathbb{N})$, and ahead of each segment of 0s there is an arbitrary segment of the same length. Define for each n,

$$A_n = \overbrace{A_0 \pm A_0 \pm \cdots \pm A_0}^{n} = \{\alpha_1 \pm \alpha_2 \pm \cdots \pm \alpha_n : \alpha_i \in A_0, 1 \leqslant i \leqslant n\}.$$

Fix $n \geq 1$ and $\varepsilon > 0$. Choose $k \geq 1$ such that

(i) $2^{-\frac{1}{2} \cdot 3^k + 1} < \varepsilon$;

(ii) for all $a = m + 0.a_1 a_2 \cdots \in A_n$,

$$\forall j \in \left[2 \cdot 3^{k-1}, \frac{5}{2} \cdot 3^{k-1}\right](a_j = 0) \vee \forall j \in \left[2 \cdot 3^{k-1}, \frac{5}{2} \cdot 3^{k-1}\right](a_j = 1).$$

Then the measure of A_n is no more than $2 \cdot 2^{-\frac{1}{2} \cdot 3^k} < \varepsilon$. Since ε is arbitrary, A_n is null. Let A be the subgroup of $(\mathbb{R}, +)$ generated from A_0. Then $A = \bigcup_{n \geq 1} A_n$ is null. Since A_0 is compact, A is F_σ. Define $x = 0.x_1 x_2 ...$ such that $x_n = 1$ if and only if $\exists k (n = 3^k)$. Given any real $y = m + 0.y_1 y_2 ...$, we shall define a real $b = 0.b_1 b_2 \cdots \in A_0$ such that $b \cdot x$ and y are equal on the nth binary place for each $n \in P_2$. Then we can pick $c \in A_0$ such that $c + bx = y$, so $A + xA = \mathbb{R}$.

Intuitively, in the calculation $b \cdot x$, $x_{3^k} = 1$ causes the binary places of b to shift to the right 3^k places, i.e.,

$$0.\overbrace{0 ... 0}^{3^k - 1} 1 \times 0.b_1 b_2 \cdots = 0.\overbrace{0 ... 0}^{3^k} b_1 b_2 ... .$$

Denote $bx = 0.c_1 c_2 ...$. By recursion on n, one can choose $\{b_m : m \in Q_n \cap P_1\}$ such that $c_m = y_m$ for all $m \in Q_n \cap P_2$ as follows: suppose $\{b_m : m \in Q_k \cap P_1, k < n\}$ have been chosen so that $c_m = y_m$ for all $k < n$ and $m \in Q_k \cap P_2$. Whatever $\{b_m : m \in Q_k \cap P_1, k \geq n + 1\}$ are chosen, we have

$$0.\overbrace{0 ... 0}^{3^n - 1} b_{3^n} b_{3^n + 1} b_{3^n + 2} \cdots \times x < 0.\overbrace{0 ... 0}^{3^n - 1} 1.$$

Hence $\{c_m : m \in Q_n \cap P_2\}$ is determined only by $\{b_m : m \in Q_k \cap P_1, k \leqslant n\}$. Now it is clear that we can choose $\{b_m : m \in Q_n \cap P_1\}$ to ensure $c_m = y_m$ for all $m \in Q_n \cap P_2$. This ends the proof of the weaker result.

Achieving $\dim_{\mathrm{H}}(A) = \frac{1}{2}$ requires a bit more effort. Let $\{p_n\}_{n \geq 1}$ be an increasing sequence of even numbers such that $\sum_{k<n} p_k = o(p_n)$ and $3^n = o(p_n)$. Define

$$Q_n = \left\{ m \in \mathbb{N} : 1 + \sum_{k<n} p_k \leq m < 1 + \sum_{k \leq n} p_k \right\}.$$

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Divide $\{1,2,...\}$ into two parts P_1, P_2 such that for all $n \geq 1$, the first half of the numbers of Q_n are in P_1, and the other numbers of Q_n are in P_2. Define

$$A_0 = \{m + 0.a_1a_2\cdots a_n\cdots : m \in \mathbb{Z} \wedge \forall n \in P_2(a_n = 0)\}.$$

Define $A_n = \overbrace{A_0 \pm A_0 \pm \cdots \pm A_0}^{n}$ and $A = \bigcup_{n\geq 1} A_n$. As before, A is an F_σ subgroup of $(\mathbb{R}, +)$ and there is x such that $A + xA = \mathbb{R}$. It remains to show that $\dim_{\mathrm{H}}(A_n) = \frac{1}{2}$ for all n. Fix n. For all $k > n$ and $a = m + 0.a_1a_2\cdots \in A_n$,

$$\forall j \in \left[\sum_{j<k} p_j + \frac{p_k}{2}, \sum_{j\leq k} p_j - n\right](a_j = 0) \vee \forall j \in \left[\sum_{j<k} p_j + \frac{p_k}{2}, \sum_{j\leq k} p_j - n\right](a_j = 1).$$

Let $\delta = 2^{-\sum_{j\leq k} p_j}$. If $s > \frac{1}{2}$, then

$$\mathcal{H}^s_\delta(A_n) \leq 2^{1+\sum_{j<k} p_j + \frac{p_k}{2}} 2^{n+3} (2^{-\sum_{j\leq k} p_j})^s \leq 2^{n+4+\sum_{j<k} p_j - (s-\frac{1}{2})p_k}.$$

Since $\sum_{j<k} p_j = o(p_k)$, $\mathcal{H}^s(A_n) = 0$. So $\dim_{\mathrm{H}}(A_n) \leq \frac{1}{2}$. ⊣

REMARK 3.5. Note that the analogue of the above holds in the p-adics as well. Namely, there is an F_σ subgroup $A \subseteq (\mathbb{Z}_p, +)$ and $x \in \mathbb{Z}_p$ such that $\mu(A) = 0$ and $A + xA = \mathbb{Z}_p$, where μ denotes the Haar measure on $\mathbb{Z}_p$. Moreover, one can also achieve that the Hausdorff dimension is $1/2$. The proof follows from the same strategy as the previous proof. We include the details for the sake of completeness.

PROOF. We only include the proof for the existence of such A, the calculation of the Hausdorff dimension of A is left to the reader as it follows from the same argument as in $\mathbb{R}$.

Note that every element $a \in \mathbb{Z}_p$ has a unique expansion

$$a = \sum_{n=0}^{\infty} a_n p^n, \qquad a_n \in \{0, \ldots, p-1\}.$$

Define

$$Q_n = \{m \in \mathbb{N} : p^{n-1} \leq m < p^n\}.$$

Put

$$q_n = p^{n-1}, \qquad N_n = |Q_n| = (p-1)p^{n-1},$$

and let

$$L_n = \left\lfloor \frac{N_n}{2} \right\rfloor, \qquad s_n = N_n - L_n = \left\lceil \frac{N_n}{2} \right\rceil.$$

Thus the last part of Q_n has length L_n, while the first part has length s_n, and $s_n + L_n = N_n$. Partition the coordinates by putting the first s_n indices of Q_n into P_1 and the remaining L_n indices into P_2. Formally,

$$P_1 \cap Q_n = \{q_n, q_n + 1, \ldots, q_n + s_n - 1\}, \qquad P_2 = \mathbb{N} \setminus P_1.$$

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Define

$$A_0 = \left\{ \sum_{n=0}^{\infty} a_n p^n : \forall n \in P_2 \ (a_n = 0) \right\}.$$

Thus elements of A_0 have arbitrary digits on P_1 and vanish on P_2. For each $m \geq 1$, define

$$A_m = \underbrace{A_0 \pm A_0 \pm \cdots \pm A_0}_{m}.$$

Let

$$A = \bigcup_{m \geq 1} A_m.$$

Since A_0 is compact, every A_m is compact, hence A is an F_σ subgroup of $(\mathbb{Z}_p, +)$.

We now make explicit the carry estimate used in the measure calculation. Fix m. In any signed sum of at most m elements of A_0, the carry sequence is bounded in absolute value by a constant B_m depending only on m and p. Indeed, at each digit the non-carry part is between $-m(p-1)$ and $m(p-1)$, so the usual base-p carrying recursion keeps all carries in a fixed finite interval. Now suppose that the summand digits are all zero on a consecutive interval $J \subseteq P_2$. After at most

$$d_m = \lceil \log_p(B_m + 1) \rceil + 1$$

positions, the carry has become either 0 or -1. Consequently all later output digits in J are, respectively, all 0 or all $p-1$.

Apply this to the interval $P_2 \cap Q_k$, whose length is L_k. For all sufficiently large k, every element of A_m has, on a terminal subinterval $I_k \subseteq P_2 \cap Q_k$ of length at least $L_k - d_m$, either only digits 0 or only digits $p-1$. Therefore A_m can be covered by at most

$$2p^{p^k - (L_k - d_m)}$$

balls of radius p^{-p^k}. Since such a ball has Haar measure p^{-p^k}, we get

$$\mu(A_m) \leq 2p^{d_m} p^{-L_k} \to 0.$$

Hence every A_m is Haar null, and therefore A is Haar null.

Define

$$x = \sum_{k=1}^{\infty} p^{s_k}.$$

Notice that the shift used in each k-th block is s_k, the length of the first part of Q_k; this is exactly what sends the first L_k free coordinates of $P_1 \cap Q_k$ onto $P_2 \cap Q_k$.

Let

$$y = \sum_{n=0}^{\infty} y_n p^n \in \mathbb{Z}_p.$$

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We recursively construct

$$b = \sum b_n p^n \in A_0$$

such that bx agrees with y on all coordinates in P_2. Suppose that the coordinates of b in $P_1 \cap Q_i$ have already been chosen for all $i < k$ so that bx agrees with y on $P_2 \cap Q_i$. Later choices have indices at least q_k and all shifts in x are non-negative, so they cannot change any digit below q_k; hence the previously treated blocks remain fixed. It remains to choose the free coordinates in $P_1 \cap Q_k$. The term p^{s_k} in x gives the direct shift

$$p^{s_k} \sum_{i=0}^{L_k-1} b_{q_k+i} p^{q_k+i} = \sum_{i=0}^{L_k-1} b_{q_k+i} p^{q_k+s_k+i},$$

whose exponents are precisely the coordinates in $P_2 \cap Q_k$. Since the digits of bx in $P_2 \cap Q_k$ depend only on the choice of $\{b_i\}_{i \le q_k+L_k-1}$, we may choose

$$\{b_n : n \in P_1 \cap Q_k\}$$

so that $(bx)_n = y_n$ for all $n \in P_2 \cap Q_k$. Proceeding block by block gives $b \in A_0$ with bx agreeing with y on P_2.

Define $c = y - bx$. Since bx already agrees with y on P_2, the element c has vanishing digits on P_2, hence $c \in A_0$. Therefore $y = c + bx \in A + xA$. Since $y \in \mathbb{Z}_p$ was arbitrary, $A + xA = \mathbb{Z}_p$. ⊣

In the proof of Theorem 1.4, both A and x depend on the intervals Q_n. What is the relation between A and elements of X_A in general?

Proposition 3.6. *If* A *is an analytic subgroup of* $(\mathbb{R}, +)$ *and* $x \in X_A$, *then there are* $c, d \in A$ *such that* $x = \frac{c}{d}$. *In particular,* $cA + dA = \mathbb{R}$.

This result is an immediate corollary of the following theorem of Le Gac.

Fact 3.7 (Le Gac [5]). *If* G *and* H *are analytic subgroups of* $(\mathbb{R}, +)$ *such that* $G + H = \mathbb{R}$ *and* $G \cap H = \{0\}$, *then either* $G = \mathbb{R}$ *or* $G = \{0\}$.

Recall that for each $\alpha \in [0, 1]$, there is a Borel subgroup A_α of $(\mathbb{R}, +)$ with $\dim_{\mathrm{H}}(A_\alpha) = \alpha$ [20]. For $\alpha > \frac{1}{2}$, X_{A_α} is co-null by Proposition 3.4. The following theorem claims that for such groups A_α, there is an F_σ subgroup $B \subseteq A_\alpha$ such that X_B is co-null. It also reduces the problem of constructing an F_σ proper subgroup B of the reals with $X_B \neq \emptyset$ to the problem of constructing an analytic proper subgroup A of the reals with $X_A \neq \emptyset$.

Theorem 3.8. *Suppose that* A *is an analytic subgroup of* $(\mathbb{R}, +)$.

(i) *If* $x \in X_A$, *then there is an* F_σ *subgroup* $B \subseteq A$ *such that* $x \in X_B$.
(ii) *If* X_A *is co-null, then there is an* F_σ *subgroup* $B \subseteq A$ *such that* X_B *is co-null.*

By Proposition 3.1(vi), $X_A = \bigcup_{q \in \mathbb{Q}} q + X_A$. Thus if X_A has positive measure, it is automatically co-null. We need the following uniformization theorem in the proof.

Fact 3.9 (Jankov–von Neumann, see [9, Theorem 18.1]). *Let* X, Y *be standard Borel spaces and* $P \subseteq X \times Y$ *be analytic. Then there is a* Σ*-measurable function* $f : X \to Y$ *such that* $(x, f(x)) \in P$ *for all* $x \in proj_X(P)$ *(the projection from* P *to* X*), where* Σ *is the* σ*-algebra generated by the analytic sets.*

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In particular, if X and Y are Borel subsets of Euclidean spaces, then the function f is Lebesgue measurable by Fact 2.5.

PROOF OF THEOREM 3.8. For (i), suppose that A is an analytic subgroup of $(\mathbb{R}, +)$ and $x \in X_A$. Then the graph of the function $\varphi : A \times A \to \mathbb{R}$ so that $(a, b) \mapsto a + xb$ is an analytic subset of $\mathbb{R}^3$. By Fact 3.9, there is a Lebesgue measurable function $f : \mathbb{R} \to A \times A$ such that for any $z \in \mathbb{R}$, $\varphi(f(z)) = z$. By the Lusin theorem, there is a compact set $P \subseteq \mathbb{R}$ with positive measure and a continuous function $\psi : P \to A \times A$ so that for any $z \in P$, $\varphi(\psi(z)) = z$. Let

$$C_0 = \{a \in \mathbb{R} : \exists z \in P \exists b \in \mathbb{R}(\psi(z) = (a, b))\},$$

and

$$C_1 = \{b \in \mathbb{R} : \exists z \in P \exists a \in \mathbb{R}(\psi(z) = (a, b))\}.$$

Both C_0 and C_1 are compact and so is $C_0 \cup C_1$. Let B be the group generated by $C_0 \cup C_1$. Then B is F_σ and $B \subseteq A$. Moreover, the image $\varphi(B \times B) \supseteq P$ has positive measure. Since $\varphi(B \times B)$ is a subgroup of $\mathbb{R}$, by Corollary 1.2, $\varphi(B \times B) = \mathbb{R}$. Hence $x \in X_B$.

For (ii), suppose that A is an analytic subgroup of $(\mathbb{R}, +)$ and X_A is co-null. By the regularity of Lebesgue measure, there is a co-null F_σ subset $F \subseteq X_A$. It suffices to show that there is an F_σ subgroup $B \subseteq A$ such that for almost all $x \in F$, we have $x \in X_B$.

The graph of the function

$$\varphi : A \times A \times F \to \mathbb{R} \times F, (a, b, x) \mapsto (a + xb, x)$$

is an analytic subset of $\mathbb{R}^5$. By Fact 3.9, there is a Lebesgue measurable function

$$f : \mathbb{R} \times F \to A \times A \times F$$

such that for any $z \in \mathbb{R} \times F$, $\varphi(f(z)) = z$. By the Lusin theorem, for all n, there is a compact subset K_n of $\mathbb{R} \times F$ such that $\psi_n := f \restriction K_n$ is continuous and

$$\mu((\mathbb{R} \times F) \cap B_n(0) \backslash K_n) < \frac{1}{n},$$

where μ is the Lebesgue measure on $\mathbb{R}^2$ and $B_n(0) \subseteq \mathbb{R}^2$ is the closed ball with radius n and center 0. Let

$$C_0 = \{a \in \mathbb{R} : \exists n \exists z \in K_n \exists b, x \in \mathbb{R}(\psi_n(z) = (a, b, x))\},$$

and

$$C_1 = \{b \in \mathbb{R} : \exists n \exists z \in K_n \exists a, x \in \mathbb{R}(\psi(z) = (a, b, x))\}.$$

Both C_0 and C_1 are F_σ and so is $C_0 \cup C_1$. Let B be the additive group generated by $C_0 \cup C_1$. Then B is an F_σ set and $B \subseteq A$. Moreover, the projection of the image $\varphi(B \times B \times \{x\})$ to the first coordinate has positive measure for almost all $x \in F$ by the Fubini theorem since $(\mathbb{R} \times F) \backslash \cup_n K_n$ is null. So X_B is conull. ⊣

§4. Constructions assuming CH. The main goal of this section is to prove Theorem 1.5. It requires notions from recursion theory and algorithmic dimension.

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A standard reference is [2]. We consider the elements of the Cantor space 2^ω as reals. We denote the set of binary strings of finite length by $2^{<\omega}$. Given $\sigma, \tau \in 2^{<\omega}$, we write $\sigma \prec \tau$ if σ is a proper initial segment of τ. The same notation is applied when τ is replaced by a real $x \in 2^\omega$. We write $\sigma\tau$ to denote the string obtained by concatenating σ and τ. The Cantor space is equipped with a topology generated by the basic clopen sets $I_\sigma = \{\sigma\alpha : \alpha \in 2^\omega\}$ for $\sigma \in 2^{<\omega}$. It is also a measure space: the Lebesgue measure $\mu(I_\sigma) = 2^{-|\sigma|}$, where $|\sigma|$ is the length of the string σ. For a co-infinite set $z \subseteq \omega$, define

$$F(z) = \sum_{i \in z} 2^{-i-1} \in [0, 1) \subseteq \mathbb{R}.$$

F is an "isometry" between the co-null subset of 2^ω consisting of the co-infinite sets and the interval $[0, 1)$. Note that under F, the measure μ on 2^ω turns into the Lebesgue measure on $\mathbb{R}$. Similarly the Hausdorff dimension is also preserved between 2^ω and $\mathbb{R}$. For a rigorous proof of this fact, see [12, Section 4]. There the authors proved that F preserves the property of having positive Hausdorff measure. Since the set of all the co-finite elements in 2^ω is countable, it is μ-null. Hence we assume that every real $x \in 2^\omega$ we deal with is co-infinite and consider the arithmetic operations on 2^ω to be the same as arithmetic operations on $\mathbb{R}/\mathbb{Z}$. Given reals x, y. We write $x \leq_T y$ if x is Turing reducible to y. Given a real y, if $W \subseteq 2^{<\omega}$ is r.e. (recursively enumerable) in y, then the set $U \subseteq 2^\omega$ of reals with an initial segment in W is called a $\Sigma^0_1(y)$ *set*. Given $\sigma \in 2^{<\omega}$ and $x \in 2^\omega$, let $K(\sigma)$ be the prefix-free Kolmogorov complexity of σ and $K^x(\sigma)$ be the prefix-free Kolmogorov complexity of σ relativized to x (see [2]). Given reals $x, y \in 2^\omega$, define the real $x \oplus y$ such that for each n, $x \oplus y(2n) = x(n)$ and $x \oplus y(2n+1) = y(n)$. Note that operation $\oplus$ is not associative. However, it is invariant under Turing degree. For example, we have $(x_1 \oplus x_2) \oplus x_3 \equiv_T x_1 \oplus (x_2 \oplus x_3)$. Then we define by recursion that

$$x_1 \oplus \cdots \oplus x_n = x_1 \oplus (x_2 \oplus x_3 \oplus \cdots \oplus x_n).$$

Definition 4.1. (i) A set $S \subseteq 2^{<\omega}$ is *dense* if for every $\sigma \in 2^{<\omega}$, there is a string $\tau \in S$ such that $\tau \succ \sigma$.

(ii) Given reals x and y. We say that x is *y-generic*[3] if for every $\Sigma^0_1(y)$ dense set $S \subseteq 2^{<\omega}$, there is a string $\sigma \prec x$ such that $\sigma \in S$. We say that x is *generic* if x is x_0-generic where $x_0(n) = 0$ for all $n \in \omega$. In particular, if x is y-generic for some y, then x is generic.

Fact 4.2 (Hölzl et al. [6]). *There is a constant* c *such that for every generic real* x *and* $i \in \{0, 1\}$, *there are infinitely many* n *such that*

$$\forall m \in \left[n, 2^{2^{2^n}}\right] (K(G \upharpoonright m) \leqslant K(m) + c \wedge G(m) = i).$$

Fact 4.3 (The point-to-set principle, Lutz and Lutz [10]). *For every set* $E \subseteq \mathbb{R}$,

$$\dim_{\mathrm{H}}(E) = \min_{A \subseteq \omega} \sup_{x \in E} \liminf_{n \to \infty} \frac{K^A(x \upharpoonright n)}{n}.$$

[3]Our definition of y-generic reals should be named as the *weakly 1-y-generic reals* according to the tradition. Since there is no need for the more general concept of n-genericity in this article, we state the definition as above for convenience.

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LEMMA 4.4 (Folklore). *Let* $G = \{x \in 2^\omega : x \text{ generic}\}$, *then* $\dim_{\mathrm{H}}(G) = 0$.

PROOF. Note that $K(n) \leqslant \log n + 2\log\log n + O(1)$ for every $n \in \omega$ (see [2]). Then by Fact 4.2, we have $\liminf_{n\to\infty} \frac{K(x\restriction n)}{n} = 0$ for every generic real x. Hence by Fact 4.3, $\dim_{\mathrm{H}}(G) = 0$. ⊣

The following lemma reflects the intuition that genericity is preserved under arithmetic operations. Recall that we consider the arithmetic operations on 2^ω to be the same as arithmetic operations on $\mathbb{R}/\mathbb{Z}$.

LEMMA 4.5. *Given* $x, a, b, g \in 2^\omega\backslash\{0\}$, *if* g *is* $a \oplus b \oplus x$*-generic, then* $g + b$, $a \cdot g$, $g^n (n \in \mathbb{Z}\backslash\{0\})$ *and* $a \cdot g + b$ *are* $a \oplus b \oplus x$*-generic.*

PROOF. We proof that $g + b$ is $a \oplus b \oplus x$-generic first. Fix a $\Sigma^0_1(a \oplus b \oplus x)$ dense set S and $\{S_s\}_{s\in\omega}$ an $a \oplus b \oplus x$-recursive enumeration of S ($S_n \subseteq S_{n+1}$ for each n and there is exactly one element in $S_{n+1}\backslash S_n$). We inductively define S'_s at stage $s + 1$ as follows:

Suppose $\sigma \in S_{s+1}\backslash S_s$. Since there are infinitely many 0s in the sequence b, there is τ such that $\forall y \succ \tau(y + b \succ \sigma)$. Choose τ of the least length and enumerate it into S'.

It is clear that S' is $\Sigma^0_1(a \oplus b \oplus x)$. Fix $\gamma \in 2^{<\omega}$. Choose γ_1 such that $\{y + b : y \succ \gamma\} \supseteq I_{\gamma_1}$. Since S is dense, there is $\rho \in S$ such that $\rho \succ \gamma_1$. By the definition of S', there is $\tau \in S'$ such that $\forall y \succ \tau(y + b \succ \rho)$. Then $\tau \succeq \gamma$. Hence S' is dense. Since g is $a \oplus b \oplus x$-generic, there is $\tau \prec g$ such that $\tau \in S'$. By the definition of S', there is $\sigma \in S$ such that $\sigma \prec g + b$. So $g + b$ is $a \oplus b \oplus x$-generic.

Similarly one can prove that $a \cdot g$, g^{-1} and g^2 are $a \oplus b \oplus x$-generic, which implies g^n and $a \cdot g + b$ also are. ⊣

PROOF OF THEOREM 1.5. Fix $\{(x_\alpha, y_\alpha)\}_{\alpha<\aleph_1}$ an enumeration of $(\mathrm{dom}(F)\backslash F^{-1}(\mathbb{Q})) \times \mathrm{dom}(F)$ (F is defined at the beginning of this section). We construct a sequence of pairs of reals $\{(g_\alpha, h_\alpha)\}_{\alpha<\aleph_1}$ by induction on $\alpha < \aleph_1$.

Stage α. Define

$$G_\alpha = \{x_\beta \oplus y_\beta \oplus g_\beta \oplus h_\beta \oplus x_\alpha \oplus y_\alpha : \beta < \alpha\}.$$

Let

$$I_\alpha = \{x : (\exists n)(\exists a_0, \dots, a_n \in G_\alpha)[x \leq_T a_0 \oplus a_1 \oplus \cdots \oplus a_n]\},$$

the Turing ideal generated by G_α. Since I_α is countable, there is a real g_α that is z-generic for all $z \in I_\alpha$. Let $h_\alpha = \frac{y_\alpha - g_\alpha}{x_\alpha}$.

Let A_0 be the group generated by $\{g_\alpha : \alpha < \aleph_1\} \cup \{h_\alpha : \alpha < \aleph_1\}$. We claim that A_0 contains only generic reals. For any $g \in A_0$, there are finite sequences $\{g_{\alpha_i}, h_{\alpha_i}, s_i, t_i\}_{0\leqslant i\leqslant n}$, where s_i, $t_i \in \mathbb{Z}$, and ordinals $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ such that $s_n^2 + t_n^2 \neq 0$ and

$$g = \sum_{i=0}^{n} (s_i g_{\alpha_i} + t_i h_{\alpha_i}).$$

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Define $c = (\oplus_{0\leqslant i\leqslant n-1}(g_{\alpha_i} \oplus h_{\alpha_i})) \oplus x_{\alpha_n} \oplus y_{\alpha_n}$. By the construction, g_{α_n} is c-generic. Also

$$s_n g_{\alpha_n} + t_n h_{\alpha_n} = t_n \frac{y_{\alpha_n}}{x_{\alpha_n}} + \left(s_n - \frac{t_n}{x_{\alpha_n}}\right) g_{\alpha_n},$$

so

$$g = \left(s_n - \frac{t_n}{x_{\alpha_n}}\right) g_{\alpha_n} + \left(t_n \frac{y_{\alpha_n}}{x_{\alpha_n}} + \sum_{i=0}^{n-1} (s_i g_{\alpha_i} + t_i h_{\alpha_i})\right).$$

Since x_{α_n} is irrational, we have that $s_n - \frac{t_n}{x_{\alpha_n}} \neq 0$. By Lemma 4.5, g is c-generic and hence generic. Thus $\dim_{\mathrm{H}}(A_0) = 0$ by Lemma 4.4. Let A be the subgroup of $\mathbb{R}$ generated by $F(A_0)$. Then $\dim_{\mathrm{H}}(A) = 0$. For any pair $(x, y) \in [0,1] \times [0,1]$ with $x \notin \mathbb{Q}$, there are members $g, h \in A$ so that $h = \frac{y-g}{x}$ and so $g + x \cdot h = y$. So $\mathbb{R}\backslash\mathbb{Q} \subseteq X_A$. By Proposition 3.1(ii), $\mathbb{Q} \cap X_A = \emptyset$. ⊣

QUESTION 2. (i) *Can* CH *in Theorem 1.5 be removed*?
(ii) *Can* A *in Theorem 1.5 be Borel*?

The complexity of A in Theorem 1.5 is not evident from the construction. The following proposition shows that at least it is impossible for A to be F_σ.

PROPOSITION 4.6. *Suppose $A \subseteq \mathbb{R}$ is F_σ and null,*[4] *then there is an irrational* x *such that $A + xA$ is null. Moreover, the set $\{x : A + xA \text{ is null}\}$ is comeager.*[5]

Let μ be the Lebesgue measure on $\mathbb{R}$. The following fact is clear.

FACT 4.7. *Let $\{I_i\}_{i\leq n}$ be a finite set of open intervals and* J *an open interval, then*

$$\mu((\bigcup_{i\leq n} I_i) + J) \leq n\mu(J) + \sum_{k\leq n} \mu(I_i).$$

PROOF OF PROPOSITION 4.6. Suppose $A = \bigcup_{n\geq 1} A_n$ is an F_σ null subset of $\mathbb{R}$ with each A_n compact. We claim that for each $m, n \geq 1$, the set

$$D_{m,n} = \left\{x \in \mathbb{R} : \mu(A_n + xA_n) < \frac{1}{m}\right\}$$

contains a dense open subset of $\mathbb{R}$.

Using this claim, the set $\{x : \mu(A + xA) = 0\}$ is comeager by the Baire category theorem. It reclaims to prove the claim.

Given $m, n \geq 1$, we shall show that $D_{m,n}$ contains a dense open set in $(0,1)$. It is not hard to generalize the proof to show that $D_{m,n}$ contains a dense open set in $\mathbb{R}$.

Let σ be a binary string and $q \in \mathbb{N}$. Fix a real $x = 0.\sigma 0^q x^* \in (0,1)$, where $x^* \in 2^\omega$ is the tail of the binary expansion of x. Define $y = 0.x^* \in (0,1)$. Then

$$A_n + xA_n \subseteq A_n + \sum_{0\leqslant i\leqslant |\sigma|-1} \sigma(i) 2^{-i-1} A_n + 2^{-|\sigma|-q} y A_n.$$

[4] A is not necessarily a group.
[5] A *comeager* set is the complement of a set of the first category.

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The set $A_{n,\sigma} := A_n + \sum_{0\leqslant i\leqslant|\sigma|-1}\sigma(i)2^{-i-1}A_n$ is compact and

$$A_{n,\sigma} \subseteq A + \sum_{0\leqslant i\leqslant|\sigma|-1}\sigma(i)2^{-i-1}A$$

is null by Proposition 3.1(i) and Corollary 1.2. So there is a finite open interval cover $\{I_j\}_{1\leqslant j\leqslant k}$ of $A_{n,\sigma}$ such that $\sum_{1\leqslant j\leqslant k}|I_j| < \frac{1}{2m}$. Suppose that $A_n \subseteq [-N, N]$ and so $yA_n \subseteq [-N, N]$. Then if $2^{-q+1}Nk < \frac{1}{2m}$, by Fact 4.7, we have

$$\mu(A_n + xA_n) \leq \mu(A_{n,\sigma} + 2^{-|\sigma|-q}yA_n) \leq \sum_{0\leq j\leq k}|I_j| + k2^{-q-|\sigma|}2N < \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m}.$$

Since σ is arbitrary, $D_{m,n}$ contains a dense open set in $(0, 1)$. ⊣

Assuming CH, we can also construct a maximal subfield of $\mathbb{R}$ with Hausdorff dimension 0 such that some given real is not in its algebraic closure. Given a set $A \subseteq \mathbb{R}$, let $F(A)$ be the field generated by A. Denote the relative algebraic closure of $F(A)$ in $\mathbb{R}$ by $\mathrm{acl}_{\mathbb{R}}(A)$.

Fact 4.8 (Exchange of algebraic closure). *For a set $A \subseteq \mathbb{R}$ and reals* a *and* b, *if $b \in \mathrm{acl}_{\mathbb{R}}(A \cup \{a\})$ and $b \notin \mathrm{acl}_{\mathbb{R}}(A)$, then $a \in \mathrm{acl}_{\mathbb{R}}(A \cup \{b\})$.*

Proposition 4.9. *Assume* CH. *Given* x *a transcendental number, there is a subfield* A *of $\mathbb{R}$ such that*

(i) $\dim_{\mathrm{H}}(A) = 0$;
(ii) $x \notin \mathrm{acl}_{\mathbb{R}}(A)$;
(iii) *for any $y \notin \mathrm{acl}_{\mathbb{R}}(A)$, $x \in \mathrm{acl}_{\mathbb{R}}(A \cup \{y\})$.*

Proof. Fix $\{y_\alpha\}_{\alpha<\aleph_1}$ an enumeration of $\mathbb{R}$. Fix x a transcendental number. Define $A = \bigcup_{\alpha<\aleph_1} A_\alpha$ by induction on stages $\alpha < \aleph_1$:

Stage 0. Define $A_0 = \mathbb{Q}$. $x \notin \mathrm{acl}_{\mathbb{R}}(A_0)$ since it is transcendental.

Stage $\alpha > 0$. Recall that G is the set of generic reals (see Lemma 4.4). Suppose by induction that we have a countable field

$$B_\alpha = \bigcup_{\beta<\alpha} A_\beta \subseteq G \cup \mathbb{Q}$$

such that $x \notin \mathrm{acl}_{\mathbb{R}}(B_\alpha)$. Choose the least $\gamma < \aleph_1$ such that $y_\gamma \notin \mathrm{acl}_{\mathbb{R}}(B_\alpha)$ and $x \notin \mathrm{acl}_{\mathbb{R}}(B_\alpha \cup \{y_\gamma\})$ (if such a γ does not exist, end the whole construction). And if such a γ exists, we say that γ *acts* at stage α. Let I_α be the Turing ideal generated by $B_\alpha \cup \{y_\gamma, x\}$. Choose a real g_α such that

(i) g_α is y-generic for all $y \in I_\alpha$;
(ii) $g_\alpha \notin \mathrm{acl}_{\mathbb{R}}(B_\alpha \cup \{y_\gamma, x\})$.

Let A_α be the subfield of $\mathbb{R}$ generated by $B_\alpha \cup \{g_\alpha, \frac{g_\alpha - x}{y_\gamma}\}$.

We first show that $x \notin \mathrm{acl}_{\mathbb{R}}(A_\alpha)$ if there is an ordinal γ which acts at stage α. By induction, we assume that $x \notin \mathrm{acl}_{\mathbb{R}}(B_\alpha)$. Let $h_\alpha = \frac{g_\alpha - x}{y_\gamma}$. Then $x \in \mathrm{acl}_{\mathbb{R}}(B_\alpha \cup \{g_\alpha, h_\alpha\})$ but $x \notin \mathrm{acl}_{\mathbb{R}}(B_\alpha \cup \{g_\alpha\})$ ($x \notin \mathrm{acl}_{\mathbb{R}}(B_\alpha)$ by the induction hypothesis, and

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$g_\alpha \notin \mathrm{acl}_{\mathbb{R}}(B_\alpha \cup \{x\})$ by its choice, then by Fact 4.8, $x \notin \mathrm{acl}_{\mathbb{R}}(B_\alpha \cup \{g_\alpha\}))$. So $h_\alpha \in \mathrm{acl}_{\mathbb{R}}(B_\alpha \cup \{g_\alpha, x\})$ by Fact 4.8. Then $y_\gamma \in \mathrm{acl}_{\mathbb{R}}(B_\alpha \cup \{g_\alpha, x\})$. And by Fact 4.8 again, since $y_\gamma \notin \mathrm{acl}_{\mathbb{R}}(B_\alpha)$, we have $y_\gamma \notin \mathrm{acl}_{\mathbb{R}}(B_\alpha \cup \{x\})$. So by Fact 4.8 again, $g_\alpha \in \mathrm{acl}_{\mathbb{R}}(B_\alpha \cup \{y_\gamma, x\})$, a contradiction.

By Lemma 4.5, $A_\alpha \subseteq G \cup \mathbb{Q}$ for all α. So the induction hypothesis holds at any stage before the construction ends.

Now we define $A = \bigcup_{\alpha<\aleph_1} A_\alpha$ and verify that A satisfies the conditions in the statement. We may assume the construction does not end at any stage $\alpha < \aleph_1$ (otherwise A is countable and it is clear that the conditions hold). Since $A = \bigcup_{\alpha<\aleph_1} A_\alpha \subseteq G \cup \mathbb{Q}$, $\dim_{\mathrm{H}}(A) = 0$. If $y \notin \mathrm{acl}_{\mathbb{R}}(A)$, choose $\gamma < \aleph_1$ such that $y = y_\gamma$. Then $x \in \mathrm{acl}_{\mathbb{R}}(A \cup \{y\})$. Otherwise, γ acts at some stage $\alpha < \aleph_1$. So g_α, $\frac{g_\alpha - x}{y}$ and y are in $\mathrm{acl}_{\mathbb{R}}(A \cup \{y\})$. Then x is also in $\mathrm{acl}_{\mathbb{R}}(A \cup \{y\})$, a contradiction. ⊣

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