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Invariant measures of full dimension for some expanding maps

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Abstract. It is an open problem to determine for which maps f, any compact invariant set K carries an ergodic invariant measure of the same Hausdorff dimension as K. If f is conformal and expanding, then it is a known consequence of the thermodynamic formalism that such measures do exist. (We give a proof here under minimal smoothness assumptions.) If f has the form $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$, where f_1 and f_2 are conformal and expanding maps satisfying $\inf|Df_1| \geq \sup|Df_2|$, then for a large class of invariant sets K, we show that ergodic invariant measures of dimension arbitrarily close to the dimension of K do exist. The proof is based on approximating K by self-affine sets.

1. *Introduction*

Let M be a connected compact Riemannian manifold and let $f : M \to M$ be an expanding C^2-diffeomorphism. Then there exists a unique probability measure μ on M which is f-invariant, ergodic and equivalent to Lebesgue measure ([**KS**]). While measures of maximal entropy are more natural from a purely dynamical perspective, this absolutely-continuous measure is geometrically the most significant among the f-invariant measures on M, since it describes the asymptotic distribution of 'typical' orbits (where the starting point is chosen according to Lebesgue measure).

Now suppose that K is a compact set of Lebesgue measure zero in a manifold, and that $f : K \to K$ is an expanding map. Then the closest analogue to the above measure is an f-invariant probability measure μ of the same Hausdorff dimension as K.

In general, it is not known when such measures exist. The only maps f for which it is known that *every* compact f-invariant set supports an ergodic measure of the same

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Hausdorff dimension are either conformal or linear. Our main result, Theorem 2 below, concerns a class of non-conformal non-linear maps: for these maps the dimension of *any* compact invariant set, satisfying specification in its symbolic dynamics, can be approximated by the dimensions of the invariant ergodic measures it supports.

If f is conformal, then *any* compact invariant set K on which f is expanding carries an f-invariant probability measure of the same Hausdorff dimension as K. This is known for many pairs of f and K (see the remarks after Theorem 1). A proof of the general statement is given in Theorem 1 under minimal smoothness assumptions ($f \in C^1$).

1.1. *Definitions.* By a probability measure on a metric space X we will always mean a *Borel* probability measure. The Hausdorff dimension of a probability measure μ on X is defined by

$$\dim(\mu) = \inf\{\dim B : B \text{ Borel}, \mu(B) = 1\}.$$

Throughout the paper dim will denote Hausdorff dimension. Call a measure μ with $\dim(\mu) = \dim X$ a measure of *full* dimension for X. Note that if X is compact, then by regularity

$$\dim(\mu) = \inf\{\dim F : F \text{ is } \sigma\text{-compact}, \mu(F) = 1\}. \tag{1.1}$$

We recall the definition of Minkowski dimension (also referred to as box dimension) of a compact set F in a metric space X. It is defined to be the limit, when it exists, of the quantity $\log n(\epsilon)/\log(\epsilon^{-1})$, where $n(\epsilon)$ is the minimum number of ϵ-balls needed to cover F. When the limit does not exist one replaces lim by lim sup and lim inf and speaks of upper and lower Minkowski dimensions respectively.

Specification. This property of topological dynamical systems may be viewed as a quantitative version of topological mixing; for subshifts, it takes a particularly simple form.

Let $\mathcal{L}$ be a finite alphabet. A compact shift-invariant set $Y \subset \mathcal{L}^{\mathbb{N}_0}$ is called a *subshift*. The subshift Y *satisfies specification* iff there exists an $m \in \mathbb{N}_0$ such that for any two blocks ξ_1 and ξ_2 in $\cup_{n\geq 0}\mathcal{L}^n$ with each ξ_i extendable to a sequence in Y, there exists a block $\eta \in \mathcal{L}^m$, such that $\xi_1\eta\xi_2$ extends to a sequence in Y.

It follows by induction that for any $k \geq 2$ and for any k blocks $\xi_1, \ldots, \xi_k$ with each ξ_i extendable to a sequence in Y, there exist blocks $\eta_1, \ldots, \eta_{k-1} \in \mathcal{L}^m$ such that $\xi_1\eta_1\xi_2 \ldots \eta_{k-1}\xi_k$ extends to a sequence in Y. Following [**HR**] we have dropped a periodicity requirement that was part of Bowen's original definition.

It is easily seen that topologically mixing subshifts of finite type satisfy specification, and this extends to their symbolic factors, the mixing sofic shifts (see [**DGS**]).

We now give an example of a subshift that has the specification property but is not sofic (cf. [**We** p. 472]). Let X be an arbitrary subshift of $\{1, 2\}^{\mathbb{N}_0}$. Let Y be the subshift of $\{1, 2, 3\}^{\mathbb{N}_0}$ consisting of all sequences that are concatenations of blocks that are legal in X and blocks of 3's. In particular, Y contains all sequences in X. Then Y has the specification property, but Y is sofic only if X is sofic.

1.2. *Statement of results.* Throughout the paper the term manifold will refer to a connected C^1 manifold.

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Let M be a Riemannian manifold, let U be an open subset of M and let $f : U \to M$ be a C^1 map. Call a compact set Λ an *invariant repeller* for the map f if the following conditions hold:

- there exist constants $C > 0$ and $\lambda > 1$ such that

$$\|Df^n(x)(u)\| \geq C\lambda^n\|u\|$$

for all $x \in \Lambda$, u in the tangent space T_xM and $n \in \mathbb{N}$;

- $f(\Lambda) = \Lambda$;
- there is an open neighborhood V of Λ such that $f^{-1}(\Lambda) \cap V \subset \Lambda$.

Note. In the definition of an invariant repeller we require that $f(\Lambda) = \Lambda$. In general, however, K is f-invariant will mean $f(K) \subset K$.

Recall that the C^1 map $f : U \to M$ is *conformal* if for each $x \in U$ the derivative map $Df(x)$ is a scalar multiple of an isometry. We denote the magnitude of this scalar by $|Df(x)|$. Moreover, if f has an invariant repeller Λ, then $f|_\Lambda$ has Markov partitions of arbitrarily small size (see for example [**Ru3**]). In particular $f|_\Lambda : \Lambda \to \Lambda$ is a factor of a subshift of finite type. We say that an invariant subset K of Λ *satisfies specification in its symbolic dynamics* if there exists a Markov partition for Λ such that the preimage of K by the factor map satisfies specification.

THEOREM 1. *Let U be an open set in a Riemannian manifold M and let $f : U \to M$ be a conformal C^1 map, with invariant repeller Λ. Then for any compact f-invariant set $K \subset \Lambda$:*

(i) *there exists an f-invariant ergodic probability measure μ on K such that* $\dim(\mu) = \dim K$*;*

(ii) *the Hausdorff dimension $\theta = \dim K$ and the Minkowski dimension of K coincide, and are given by the Bowen–Ruelle formula $P(-\theta(\log|Df|)|_K) = 0$, where P is topological pressure;*

(iii) *if the modulus of continuity*

$$w(\epsilon) = \sup_{d(x,y)<\epsilon} ||Df(x)| - |Df(y)||$$

of $|Df|$ satisfies

$$\int_0^1 \frac{w(\epsilon)}{\epsilon}\, d\epsilon < \infty, \tag{1.2}$$

then K has positive Hausdorff measure in its dimension θ;

(iv) *if (1.2) holds and K satisfies specification in its symbolic dynamics, then K has finite Hausdorff measure in its dimension and the measure μ in Theorem 1(i)* is unique among ergodic measures. *Furthermore, μ is equivalent to θ-dimensional Hausdorff measure $\mathcal{H}_\theta$, with Radon–Nikodym derivative bounded away from* 0 *and* ∞.

Remarks. The pressure formula $P(-\theta\log|Df|) = 0$ was first obtained by Bowen [**Bo2**] in a different setting; Ruelle [**Ru3**] showed that Bowen's proof works in the context of Theorem 1. In particular, Ruelle proved the pressure formula for the case $f \in C^{1+\alpha}$ (i.e. $|Df|$ is Hölder continuous) and K a mixing repeller for f (so $K = \Lambda$ and Λ is

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topologically mixing, in the notation of Theorem 1), by showing that the unique Gibbs state associated with the function $-\theta \log |Df|$ is equivalent to θ-dimensional Hausdorff measure. This also implies Theorem 1(i) in the case considered by Ruelle. Note that if $f \in C^{1+\alpha}$, then clearly the modulus of continuity $w(\cdot)$ of $|Df|$ satisfies condition (1.2). Also, for K a mixing repeller, K satisfies specification in its symbolic dynamics as it is a factor of a topologically mixing subshift of finite type. For other related work see the papers of Furstenberg [**Fu**], Hofbauer [**Ho**] and Raith [**Ra**].

We now turn to our main result, Theorem 2 below, dealing with a class of non-conformal (not necessarily linear) maps.

Let f_1 and f_2 be two conformal C^1 maps with invariant repellers Λ_1 and Λ_2 respectively. Set $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$. Since each $f_i|_{\Lambda_i}$ is a factor of a subshift of finite type, so is $f|_{\Lambda_1 \times \Lambda_2}$; denote the factor map by π.

THEOREM 2. *For $i = 1, 2$, let U_i be an open subset of a Riemannian manifold M_i. Let $f_i : U_i \to M_i$ be C^1 and conformal with invariant repeller Λ_i. Set $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$. If*

$$\min_{x_1 \in \Lambda_1} |Df_1(x_1)| \geq \max_{x_2 \in \Lambda_2} |Df_2(x_2)|,$$

then for any compact f-invariant set $K \subset \Lambda_1 \times \Lambda_2$, for which $\pi^{-1}(K)$ satisfies specification,

$$\dim K = \sup\{\dim(\mu) : \mu(K) = 1,\ \mu \text{ is } f\text{-invariant and ergodic}\}.$$

Remarks. (1) Even when the maps f_1, f_2 are *linear*, the Hausdorff dimension of K may be strictly smaller than the Minkowski dimension (cf. [**McM**] and [**Be**]). Moreover, the Hausdorff measure of K in its dimension may be infinite even when K corresponds to a full-shift in its symbolic dynamics; see [**Pe**].

(2) In general, an f-invariant set K *need not* be a Cartesian product of f_i-invariant sets. Indeed, f-invariant sets which are Cartesian products are easy to analyze and cannot exhibit the interesting behaviour described in the previous remark.

(3) For *piecewise linear maps* f_i, and certain invariant sets K (which correspond to full-shifts in their symbolic dynamics), Gatzouras and Lalley [**GL**] proved that K actually carries an ergodic measure of full dimension. Kenyon and Peres [**KP**] proved the same result for *linear* f_i and *arbitrary* compact invariant sets K. In both cases, invariant sets corresponding to full-shifts are called self-affine. Here we use the result of [**GL**] and the idea of approximating arbitrary invariant sets by self-affine sets from [**KP**], to prove Theorem 2. (Nonlinearity requires a further approximation.) Both papers mentioned above are rooted in the works of McMullen [**McM**] and Bedford [**Be**].

(4) Bedford and Urbanski [**BU**] consider certain non-linear maps f of the plane for which there exists a Markov partition $\{R_i\}$ of the corresponding repeller, and a different $C^{1+\alpha}$ contraction of the form $\varphi_i(x, y) = (\psi_i(x), r_i(y))$ for each i, such that $f = \varphi_i^{-1}$ on R_i. It is also assumed that $\sup_x |D\psi_i(x)| < \inf_y |Dr_i(y)|$ for all i. The repeller here corresponds to a full shift in its symbolic representation, and must be the graph of a function $[0, 1] \to \mathbb{R}$. For such maps, Bedford and Urbanski consider the Gibbs state associated to a certain potential and prove that, if the projection of this measure onto the y-axis has dimension 1, then the Gibbs state itself is of full Hausdorff dimension.

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(5) Finally, we emphasize that we are only concerned with expanding maps. There exist Axiom-A diffeomorphisms f, with corresponding basic sets Λ, such that the strict inequality

$$\dim \Lambda > \sup\{\dim(\mu) : \mu(\Lambda) = 1,\ \mu \text{ ergodic}\}$$

holds (cf. [**MM**]). Other relevant examples of non-expanding maps are in Aaronson *et al* [**ADU**].

Most of the work in this paper is done on sequence spaces. In §§2 and 4 we study suitable sequence-space analogues of the conformal and non-conformal cases respectively. These are transferred to our original settings in §5, where Theorems 1 and 2 are proved.

2. *The conformal case*

Throughout this section we fix a finite alphabet $\{1, \ldots, L\}$, and endow the sequence space $\Omega = \{1, \ldots, L\}^{\mathbb{N}_0}$ with the usual product topology. Denote the left shift on Ω by T. For $\omega \in \Omega$, let $\mathcal{C}_n(\omega)$ be the cylinder set

$$\mathcal{C}_n(\omega) = \{\omega' \in \Omega : \omega'(i) = \omega(i) \text{ for all } 0 \le i < n\} \quad (n \in \mathbb{N}).$$

Let $\varphi : \Omega \to \mathbb{R}$ be a continuous function. Define $S_0\varphi = S_0^*\varphi = 0$,

$$S_n\varphi = \sum_{i=0}^{n-1} \varphi \circ T^i \quad (n \in \mathbb{N})$$

and

$$S_n^*\varphi(\omega) = \min_{\omega' \in \mathcal{C}_n(\omega)} S_n\varphi(\omega') \quad (\omega \in \Omega,\ n \in \mathbb{N}).$$

Thus $S_n^*\varphi$ only depends on the first n coordinates of its argument. Also define

$$\operatorname{Var}_n \varphi = \sup\{|\varphi(\omega) - \varphi(\omega')| : \omega(i) = \omega'(i) \text{ for all } 0 \le i < n\} \quad (n \in \mathbb{N})$$

and notice that by uniform continuity, $\lim_{n\to\infty} \operatorname{Var}_n \varphi = 0$. As a consequence

$$0 \le \frac{1}{n}(S_n\varphi(\omega) - S_n^*\varphi(\omega)) \le \frac{1}{n}\sum_{i=1}^{n} \operatorname{Var}_i \varphi \to 0 \quad (\omega \in \Omega). \tag{2.1}$$

Given a strictly positive continuous function $\varphi : \Omega \to \mathbb{R}$, associate with it a metric $[\varphi]$ on Ω defined by

$$[\varphi](\omega, \omega') = \exp(-S_n^*\varphi(\omega)),$$

where $n = \min\{i \in \mathbb{N}_0 : \omega(i) \ne \omega'(i)\}$. Observe that $[\varphi]$ induces the product topology on Ω, since the strict positivity of φ implies that $S_n^*\varphi(\omega) \to +\infty$ for all $\omega \in \Omega$.

Since we shall be using many different metrics on Ω, we denote the Hausdorff dimension of a set $A \subset \Omega$ with respect to a metric ρ by $\dim_\rho A$, and similarly for measures. Finally, denote by $|\ |_\rho$ the diameter of a set with respect to the metric ρ. With these definitions

$$|\mathcal{C}_n(\omega)|_{[\varphi]} = \exp(-S_n^*\varphi(\omega)) \quad (\omega \in \Omega,\ n \in \mathbb{N}).$$

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THEOREM 2.1. *Let K be a subshift, and let $\varphi : \Omega \to \mathbb{R}$ be a strictly positive continuous function. Endow Ω with the metric $[\varphi]$. Then:*

(i) *there exists a shift-invariant ergodic probability measure ν on K, such that $\dim_{[\varphi]}(\nu) = \dim_{[\varphi]} K$;*

(ii) *the Hausdorff dimension $\theta = \dim K$ and the Minkowski dimension of K coincide, and θ satisfies $P(-\theta\varphi|_K) = 0$, where P is topological pressure;*

(iii) *a sufficient condition for K to have positive Hausdorff measure in its dimension is that*

$$\sum_{n=1}^{\infty} \mathrm{Var}_n \varphi < \infty; \tag{2.2}$$

(iv) *if K satisfies specification and (2.2) holds, then the Hausdorff measure of K in its dimension is finite and the measure ν in (i)* is unique among ergodic measures. *Furthermore, ν has the Gibbs property, i.e. for some constant $0 < A < \infty$,*

$$A^{-1} e^{-\theta S_n \varphi(\omega)} \le \nu(\mathcal{C}_n(\omega)) \le A e^{-\theta S_n \varphi(\omega)} \quad (n \in \mathbb{N},\ \omega \in K).$$

In the proof of this theorem we shall use the following standard lemma, which is a variant of a well known lemma for Euclidean space (see [**Bi**, §14], or [**Yo**]).

LEMMA 2.2. *Let (X, ρ) be a compact metric space in which any two balls are either disjoint or one is contained in the other. Let ν be a probability measure on X, and suppose that there exists a constant $D = D(\nu)$ such that for ν-a.a. x,*

$$\lim_{\varepsilon \to 0} \frac{\log \nu(B_\varepsilon(x))}{\log \varepsilon} = D,$$

where $B_\varepsilon(x)$ is the ball of radius ε and center x. Then $\dim_\rho(\nu) = D$.

Remark. The limit in the above lemma may be replaced by a lim inf; we shall only use the lemma when the limit exists, and then the dimension of ν is a more complete summary of its scaling properties.

COROLLARY 2.3. *Let ν be a shift invariant ergodic probability measure on Ω and let $\varphi : \Omega \to \mathbb{R}$ be strictly positive and continuous. Then*

$$\dim_{[\varphi]}(\nu) = \frac{h_\nu(T)}{\int \varphi \, d\nu}.$$

(Here $h_\nu(T)$ is the measure-theoretic entropy of T.)

Proof. By the Shannon–McMillan–Breiman theorem

$$\lim_{n \to \infty} -\frac{1}{n} \log \nu(\mathcal{C}_n(\omega)) = h_\nu(T) \quad (\nu\text{-a.a. } \omega) \tag{2.3}$$

and by the ergodic theorem

$$\lim_{n \to \infty} \frac{1}{n} S_n \varphi(\omega) = \int \varphi d\nu \quad (\nu\text{-a.a. } \omega). \tag{2.4}$$

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Consequently, by (2.1), (2.3) and (2.4)

$$\lim_{n\to\infty} \frac{\log \nu(\mathcal{C}_n(\omega))}{\log |\mathcal{C}_n(\omega)|_{[\varphi]}} = \lim_{n\to\infty} \frac{\log \nu(\mathcal{C}_n(\omega))}{-S_n^*\varphi(\omega)} = \frac{h_\nu(T)}{\int \varphi \, d\nu} \quad (\nu\text{-a.a. } \omega).$$

Since the balls of the metric space $(\Omega, [\varphi])$ are precisely the cylinder sets $\mathcal{C}_n(\omega)$, applying Lemma 2.2 concludes the proof. □

We need the following (standard) lemma in the proof of equality of Hausdorff and Minkowski dimensions.

LEMMA 2.4. *Let X and Y be metric spaces and let $\pi : X \to Y$ be a continuous mapping satisfying a Hölder condition with exponent $a > 0$. Then for any $E \subset X$ and any $s \geq 0$, we have $\mathcal{H}_s(E) < \infty \Rightarrow \mathcal{H}_{s/a}(\pi(E)) < \infty$. Hence,* $\dim \pi(E) \leq a^{-1} \dim E$. *Analogous inequalities hold for the upper Minkowski dimension of a compact set F and its image $\pi(F)$ and, if X is compact, for the dimension of a probability measure μ and its image $\mu \circ \pi^{-1}$.*

Proof. The proofs of the inequalities regarding Hausdorff measure and dimensions of sets are routine. For the inequality about dimensions of measures use (1.1): for any σ-compact $F \subset X$ with $\mu(F) = 1$, we have that $\pi(F)$ is σ-compact and $\mu \circ \pi^{-1}(\pi(F)) = 1$. Hence, from the inequality about dimensions of sets $\dim(\mu \circ \pi^{-1}) \leq \dim \pi(F) \leq a^{-1} \dim F$. □

The key ingredients of the proof of Theorem 2.1 are the notion of topological pressure of a continuous function $\psi : K \to \mathbb{R}$, introduced in [**Ru1**] and the variational principle for pressure. We refer to the paper of Walters [**Wa1**] and his book [**Wa2**, Ch 9] for the necessary background.

Proof of Theorem 2.1. (i) Define ξ to be the collection of cylinders $\mathcal{C}_1(\omega) \cap K$, for $\omega \in K$. Then ξ is an open cover of K. Let $\xi_n := \vee_{i=0}^{n-1} (T|_K)^{-i} \xi$; ξ_n is the open cover of K consisting of the cylinders $\mathcal{C}_n(\omega) \cap K$, $\omega \in K$. Now use the facts that ξ is a one-sided generator for $T|_K$ and that $T|_K$ is positively expansive to apply Theorem 1.9 of [**Wa1**]: the pressure $P(\psi)$ of any continuous function $\psi : K \to \mathbb{R}$ is given by

$$P(\psi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathcal{C}\in\xi_n} \sup_{\omega\in\mathcal{C}} e^{S_n\psi(\omega)} \tag{2.5}$$

(The right-hand side of (2.5) is $P(T|_K, \psi, \xi)$, in the notation of Walters [**Wa1**, Lemma 1.5].)

Next consider the function $t \mapsto P(-t\varphi|_K)$ for $t \in [0, +\infty)$. By Theorem 2.1 of [**Wa1**], it is continuous and convex and

$$h(T|_K) - t \sup \varphi \leq P(-t\varphi|_K) \leq h(T|_K) - t \inf \varphi \quad (t \geq 0).$$

($h(T|_K)$ denotes topological entropy.) Since $0 \leq h(T|_K) < \infty$, there exists a unique $\theta \geq 0$ such that $P(-\theta\varphi|_K) = 0$ and we fix this θ for the rest of the proof.

The variational principle for pressure now states that $P(-\theta\varphi|_K) = \sup\{h_\nu(T|_K) + \int_K -\theta\varphi \, d\nu\}$ over T-invariant measures ν supported by K. Since T is expansive, the

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supremum is actually attained (see [**Wa2**, Theorem 9.13(iv)]). Thus, there exists a T-invariant ergodic measure ν supported by K such that

$$0 = P(-\theta\varphi|_K) = h_\nu(T|_K) + \int_K (-\theta\varphi)\, d\nu = h_\nu(T) - \theta \int \varphi\, d\nu.$$

Then by Corollary 2.3, $\dim_{[\varphi]}(\nu) = \theta$. We will next show that $\dim_{[\varphi]} K \le \theta$ and this will complete the proof.

Choose $\varepsilon > 0$. By (2.5) applied to $\psi = -\theta\varphi|_K$, and the definition of θ, there exists an $N \in \mathbb{N}$ such that

$$\sum_{\mathcal{C}\in\xi_n} \sup_{\omega\in\mathcal{C}} e^{-\theta S_n\varphi(\omega)} < e^{\varepsilon n} \quad \forall n \ge N.$$

But each element $\mathcal{C}$ of $\xi_n = \vee_{i=0}^{n-1}(T|_K)^{-i}\xi$ is a cylinder set of the form $\mathcal{C}_n(\omega) \cap K$ and so

$$|\mathcal{C}|_{[\varphi]} \le e^{-S_n^*\varphi(\omega)} = \sup_{\omega'\in\mathcal{C}_n(\omega)} e^{-S_n\varphi(\omega')} \le \sup_{\omega'\in\mathcal{C}} e^{-S_n\varphi(\omega')} \exp\left(\sum_{i=1}^n \operatorname{Var}_i \varphi\right). \tag{2.6}$$

Set $c = 3/\inf\varphi$ and choose m large so as to have $n^{-1}\sum_{i=1}^n \operatorname{Var}_i\varphi < \varepsilon/(\theta + c\varepsilon)$ for $n \ge m$ (use (2.1)). Then

$$\sum_{\mathcal{C}\in\xi_n} |\mathcal{C}|_{[\varphi]}^{\theta+c\varepsilon} \le \exp\left(-nc\varepsilon \inf\varphi + (\theta + c\varepsilon)\sum_{i=1}^n \operatorname{Var}_i\varphi\right) \sum_{\mathcal{C}\in\xi_n} \sup_{\omega\in\mathcal{C}} e^{-\theta S_n\varphi(\omega)} \le e^{-n\varepsilon}$$

for all large enough n. Thus, $\dim_{[\varphi]} K \le \theta + c\varepsilon$. As $\varepsilon > 0$ was arbitrary, we are done.

(ii) The pressure formula for the Hausdorff dimension was explained above. The equality between the Hausdorff and Minkowski dimensions of K was established by Furstenberg [**Fu**] in the case that φ is constant, and the argument extends to the general case. A similar argument was used by Falconer [**Fa1**] to prove more general statements, and the present assertion follows from Falconer's Theorem 3.

First assume $\sum_{i=1}^\infty \operatorname{Var}_i\varphi < \infty$ and set $a = \exp(-\sum_{i=1}^\infty \operatorname{Var}_i\varphi)$. For an arbitrary subset F of K, with $0 < |F|_{[\varphi]} < 1$, use the mapping T^n to map F into K, where n is determined by the requirement that F be contained in some $\mathcal{C}_n(\omega)$ but F is not contained in any of the cylinder sets $\mathcal{C}_{n+1}(\omega')$ that partition $\mathcal{C}_n(\omega)$. Then

$$a[\varphi](\omega', \omega'') \le |\mathcal{C}_n(\omega)|_{[\varphi]}[\varphi](T^n\omega', T^n\omega'') = |F|_{[\varphi]}[\varphi](T^n\omega', T^n\omega'')$$

holds for all ω', ω'' in F. Theorem 3 of [**Fa1**] may now be applied to the metric space (K, ρ), where ρ is the restriction of the metric $[\varphi]$ to K.

If $\sum_{i=1}^\infty \operatorname{Var}_i\varphi = \infty$, approximate φ by functions φ_n with $\sum_{i=1}^\infty \operatorname{Var}_i\varphi_n < \infty$. For example, take $\varphi_n(\omega) = \inf_{\omega'\in\mathcal{C}_n(\omega)} \varphi(\omega')$. Then, for each n, the Hausdorff and Minkowski dimensions of K in the metric $[\varphi_n]$ agree. On the other hand, a simple calculation (see §3 for similar arguments) shows that the identity mapping $\mathrm{id} : (\Omega, [\varphi]) \to (\Omega, [\varphi_n])$ is bi-Hölder with respect to the metrics $[\varphi]$ and $[\varphi_n]$, with exponent tending to 1 as $n \to \infty$. In particular, for $a_n = \operatorname{Var}_n\varphi/\inf\varphi$ we have

$$[\varphi](\omega, \omega') \le [\varphi_n](\omega, \omega') \le ([\varphi](\omega, \omega'))^{1-a_n}$$

for all ω, ω'. This, together with Lemma 2.4, imply that the Hausdorff and Minkowski dimensions of K also coincide in the metric $[\varphi]$.

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(iii) This follows in the same way as the first case of (ii) from McLaughlin [**McL**]; see also Theorems 1 and 2 of Falconer [**Fa1**].

(iv) This follows from a general result of Bowen [**Bo1**] showing that specification and (2.2) imply that equilibrium measures have the Gibbs property, and that the equilibrium measure is unique.

Call a subset E of K n-separated if for any two $\omega, \omega' \in E$ there exists $0 \le i < n$ such that $\omega(i) \ne \omega'(i)$. For a continuous function $\psi : K \to \mathbb{R}$ set

$$Z_n(\psi) = \sup\left\{ \sum_{\omega \in E} e^{S_n \psi(\omega)} : E \text{ an } n\text{-separated subset of } K \right\}.$$

Then $Z_n(\psi) = \sum_{\mathcal{C} \in \xi_n} \sup_{\omega \in \mathcal{C}} e^{S_n \psi(\omega)}$. In particular, $n^{-1} \log Z_n(\psi) \to P(\psi)$, by (2.5).

Now assume K satisfies specification and that $\sum_{i=1}^{\infty} \mathrm{Var}_i\, \psi < \infty$. Then following the steps of Lemmas 2 and 3 of [**Bo1**], one verifies that $c^{-1} e^{nP(\psi)} \le Z_n(\psi) \le c e^{nP(\psi)}$ for some constant $0 < c < \infty$. Applying this to the function $\psi = -\theta\varphi|_K$ and using $P(-\theta\varphi|_K) = 0$ and (2.6), we get

$$\begin{aligned} \sum_{\mathcal{C} \in \xi_n} |\mathcal{C}|^{\theta}_{[\varphi]} &\le \sum_{\mathcal{C} \in \xi_n} \sup_{\omega \in \mathcal{C}} e^{-\theta S_n \varphi(\omega)} \exp\left(\theta \sum_{i=1}^{n} \mathrm{Var}_i\, \varphi \right) \\ &= Z_n(-\theta\varphi|_K) \exp\left(\theta \sum_{i=1}^{n} \mathrm{Var}_i\, \varphi \right) \le c \exp\left(\theta \sum_{i=1}^{\infty} \mathrm{Var}_i\, \varphi \right), \end{aligned}$$

where ξ_n is as in (i). This shows that K has finite Hausdorff measure in its dimension.

For the uniqueness of the measure ν in (i) observe that if ν' is any ergodic measure with $\dim_{[\varphi]}(\nu') = \dim_{[\varphi]} K$ then, by Corollary 2.3, ν' must satisfy

$$\int_K (-\theta\varphi)\, d\nu' + h_{\nu'}(T|_K) = 0 = P(-\theta\varphi|_K).$$

Hence one only needs to show that under (2.2) and specification there is only one equilibrium measure for the function $-\theta\varphi|_K$. This is essentially Lemma 8 of [**Bo1**]; however, having dropped the periodicity requirement in the definition of specification, we must modify Bowen's proof. Instead of considering measures concentrated on the set of periodic points, we fix, for each n and each $\mathcal{C} \in \xi_n$, a sequence $\omega_{\mathcal{C}} \in \mathcal{C}$ with $S_n(-\theta\varphi)(\omega_{\mathcal{C}}) = \sup_{\mathcal{C}}(-\theta S_n \varphi)$. Let $E_n = \{\omega_{\mathcal{C}} : \mathcal{C} \in \xi_n\}$. Then

$$\nu_n = \frac{1}{Z_n(-\theta\varphi|_K)} \sum_{\omega \in E_n} e^{-\theta S_n \varphi(\omega)} \delta_\omega, \quad (\delta_\omega = \text{point mass at } \omega)$$

defines a probability measure on K. Now fix a subsequence $\{\nu_{n_k}\}$ such that the sequence $n_k^{-1} \sum_{i=0}^{n_k - 1} \nu_{n_k} \circ T^{-i}$ converges weakly to some probability measure ν. Then ν is T-invariant and one can show as in Lemma 5 of [**Bo1**] that ν has the Gibbs property, i.e. for some constant A positive and finite,

$$A^{-1} e^{-\theta S_n \varphi(\omega)} \le \nu(\mathcal{C}) \le A e^{-\theta S_n \varphi(\omega)} \tag{2.7}$$

for all $n \in \mathbb{N}$, $\mathcal{C} \in \xi_n$ and $\omega \in \mathcal{C}$. Since $c^{-1} < Z_n(-\theta\varphi|_K) < c$, (2.7) implies that ν is an equilibrium state for $-\theta\varphi|_K$. As in Lemma 6 of [**Bo1**], ν is also ergodic. Hence, if ν' is any other ergodic measure, then ν' is singular with respect to ν. But then, as in Lemma 8 of [**Bo1**], $h_{\nu'}(T|_K) + \int_K (-\theta\varphi)\, d\nu' < P(-\theta\varphi|_K) = 0$ and hence $\dim_{[\varphi]}(\nu') < \theta$. □

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Note that specification in part (iv) of Theorem 2.1 is really needed. Let K be a subshift such that the number of distinct n-blocks that may appear as parts of sequences in K is $n+1$. (For an example of such a subshift see [**Qu**, p. 105] and [**CH**]). Take $\varphi \equiv 1$. Then $\dim_{[\varphi]} K = 0$ and K has infinite Hausdorff measure in its dimension. For an example of a subshift with positive Hausdorff dimension and infinite measure take a direct product with a full shift.

3. *Two renormalization lemmas*

In this section we prove two elementary lemmas concerning the effect of renormalization (or 'blocking'): viewing blocks of a fixed length as new symbols. They will be used in §4. Throughout this section we retain the notation fixed in §2.

Let $J_n : \Omega \to (\{1, \dots, L\}^n)^{\mathbb{N}_0}$ be the natural 'blocking' map which partitions any infinite sequence in Ω into blocks of length n. If φ is a positive continuous function on Ω, it also induces a metric, namely $[\varphi \circ J_n^{-1}]$, on $(\{1, \dots, L\}^n)^{\mathbb{N}_0}$:

$$[\varphi \circ J_n^{-1}](J_n\omega, J_n\omega') = \exp\left(- \inf_{\omega'' \in \mathcal{C}_{nk}(\omega)} \sum_{i=0}^{k-1} \varphi \circ T^{in}(\omega'') \right),$$

where k is the least positive integer for which

$$((T^{nk}\omega)(0), \dots, (T^{nk}\omega)(n-1)) \neq ((T^{nk}\omega')(0), \dots, (T^{nk}\omega')(n-1))$$

and $[\varphi \circ J_n^{-1}](J_n\omega, J_n\omega') = 1$ if $k = 0$.

LEMMA 3.1. *Let $\varphi : \Omega \to \mathbb{R}$ be a strictly positive continuous function, let $\varepsilon \geq 0$ and let φ_n be the function that is induced on $(\{1, \dots, L\}^n)^{\mathbb{N}_0}$ by the function $S_n^*\varphi + \varepsilon$ on Ω, i.e.*

$$\varphi_n((i_1, \dots, i_n), (j_1, \dots, j_n), \dots) = S_n^*\varphi(i_1, \dots, i_n, j_1, \dots, j_n, \dots) + \varepsilon.$$

Then the blocking map $J_n : \Omega \to (\{1, \dots, L\})^{\mathbb{N}_0}$ is bi-Hölder with respect to the metrics $[\varphi]$ and $[\varphi_n]$, with exponents tending to 1. More precisely, for some constants $0 < c_n < \infty$ and $\alpha_n > 0$, with $\alpha_n \to 0$,

$$c_n^{-1}([\varphi](\omega, \omega'))^{1+\alpha_n} \leq [\varphi_n](J_n\omega, J_n\omega') \leq c_n([\varphi](\omega, \omega'))^{1-\alpha_n} \quad (n \in \mathbb{N};\ \omega, \omega' \in \Omega).$$

Proof. Fix $n \in \mathbb{N}$ and let $\omega, \omega' \in \Omega$. Let $m = \min\{i \in \mathbb{N}_0 : \omega(i) \neq \omega'(i)\}$ and write $m = kn + v$ for some $k \in \mathbb{N}_0$ and $0 \leq v < n$. Assume $m \geq n$ so that $k \geq 1$. Then by definition of the metric $[\varphi_n]$,

$$[\varphi_n](J_n\omega, J_n\omega') = \exp\left(- \sum_{i=0}^{k-1} S_n^*\varphi(T^{in}\omega) - \varepsilon k \right),$$

since $S_n^*\varphi$ depends only on the first n coordinates of its argument. Hence,

$$\left| \log \frac{[\varphi_n](J_n\omega, J_n\omega')}{[\varphi](\omega, \omega')} \right| = \left| S_m^*\varphi(\omega) - \sum_{i=0}^{k-1} S_n^*\varphi(T^{in}\omega) - \varepsilon k \right|.$$

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By (2.1) we then have

$$\begin{aligned}\left|\log\frac{[\varphi_n](J_n\omega,\,J_n\omega')}{[\varphi](\omega,\,\omega')}\right| &\le \left|S_m\varphi(\omega)-\sum_{i=0}^{k-1}S_n\varphi(T^{in}\omega)\right|+\sum_{i=1}^{m}\mathrm{Var}_i\,\varphi+k\sum_{i=1}^{n}\mathrm{Var}_i\,\varphi+\varepsilon k\\ &= \sum_{i=kn}^{m-1}\varphi\circ T^i(\omega)+\sum_{i=1}^{m}\mathrm{Var}_i\,\varphi+k\sum_{i=1}^{n}\mathrm{Var}_i\,\varphi+\varepsilon k\\ &\le n\sup\varphi+\sum_{i=1}^{m}\mathrm{Var}_i\,\varphi+k\sum_{i=1}^{n}\mathrm{Var}_i\,\varphi+\varepsilon k. \qquad (3.1)\end{aligned}$$

Now let

$$\alpha_n=\frac{2}{\inf\varphi}\left(\frac{1}{n}\sum_{i=1}^{n}\mathrm{Var}_i\,\varphi+\frac{\varepsilon}{n}\right)\quad(n\in\mathbb{N}). \qquad (3.2)$$

For each $n\in\mathbb{N}$ choose m_n large enough so that for $m\ge m_n$,

$$\frac{1}{m}\sum_{i=1}^{m}\mathrm{Var}_i\,\varphi\le\frac{\alpha_n\inf\varphi}{2}. \qquad (3.3)$$

Finally, set

$$c_n'=n\sup\varphi+\sum_{i=1}^{m_n}\mathrm{Var}_i\,\varphi. \qquad (3.4)$$

Since $S_m^*\varphi(\omega)\ge m\inf\varphi$ and $k\le mn^{-1}$, we then have by (3.1), (3.2), (3.3) and (3.4),

$$\begin{aligned}\left|\log\frac{[\varphi_n](J_n\omega,\,J_n\omega')}{[\varphi](\omega,\,\omega')}\right| &\le c_n'+\alpha_nS_m^*\varphi(\omega)\\ &= c_n'-\alpha_n\log[\varphi](\omega,\,\omega').\end{aligned}$$

The result now follows. □

LEMMA 3.2. *Let $\varphi:\Omega\to\mathbb{R}$ be a strictly positive continuous function and let $m\in\mathbb{N}$ be a fixed integer. For each $n\in\mathbb{N}$ define a map $h_n:\Omega\to\Omega$ as follows. For an arbitrary $\omega\in\Omega$ write $\omega=\xi_1\eta_1\xi_2\eta_2\ldots$, where each ξ_i is a finite word of length n and each η_i is a finite word of length m, and define $h_n(\omega)=\xi_1\xi_2\ldots$. Then the functions h_n are Hölder continuous with exponents tending to 1, i.e. for some $0<C_n<\infty$ and $\beta_n>0$ with $\beta_n\to 0$,*

$$[\varphi](h_n(\omega),h_n(\omega'))\le C_n([\varphi](\omega,\omega'))^{1-\beta_n}\quad(\omega,\omega'\in\Omega;\ n\in\mathbb{N}).$$

Proof. Let $\omega,\omega'\in\Omega$ and let k be the smallest non-negative integer for which $\omega(k)\ne\omega'(k)$. Write $k=j(n+m)+v$, with $0\le v<n+m$. Then $\omega=\xi_1\eta_1\xi_2\ldots\eta_{j-1}\xi_j\eta$ and $\omega'=\xi_1\eta_1\xi_2\ldots\eta_{j-1}\xi_j\eta'$, where each word ξ_i has length n, each word η_i has length m, and η,η' are infinite sequences. Clearly, $h_n(\omega)$ and $h_n(\omega')$ agree in their first jn coordinates. By (2.1),

$$\log[\varphi](h_n(\omega),h_n(\omega'))\le-\sum_{i=0}^{j-1}S_n\varphi(T^{i(n+m)}\omega)+j\sum_{i=1}^{n}\mathrm{Var}_i\,\varphi.$$

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Since

$$0 \le \sum_{i=0}^{k-1} \varphi \circ T^i(\omega) - \sum_{i=0}^{j-1} S_n\varphi(T^{i(n+m)}\omega) \le (jm + v) \sup \varphi,$$

it follows that

$$[\varphi](h_n(\omega), h_n(\omega')) \le [\varphi](\omega, \omega') \exp\left((jm + v) \sup \varphi + j \sum_{i=1}^{n} \mathrm{Var}_i\, \varphi\right).$$

Recalling that m is fixed and taking n large easily concludes the proof. □

4. *The non-conformal case*

Let L_1 and L_2 be positive integers. Define $\Omega_j = \{1, \dots, L_j\}^{\mathbb{N}_0}$, for $j = 1, 2$ and $\Omega = (\{1, \dots, L_1\} \times \{1, \dots, L_2\})^{\mathbb{N}_0}$. Let $\mathcal{P}_j : \Omega \to \Omega_j$ denote the projection

$$\omega = \{(\omega_1(i), \omega_2(i))\}_{i\in\mathbb{N}_0} \xmapsto{\mathcal{P}_j} \omega_j = \{\omega_j(i)\}_{i\in\mathbb{N}_0} \tag{4.1}$$

and $T : \Omega \to \Omega$, the left shift on Ω. Observe that (Ω, T) is a full shift on L_1L_2 symbols.

Given a function $\phi : \Omega \to \mathbb{R}^2$ of the form

$$\phi(\omega) = (\varphi_1 \circ \mathcal{P}_1(\omega), \varphi_2 \circ \mathcal{P}_2(\omega)) \quad (\omega \in \Omega), \tag{4.2}$$

with $\varphi_j : \Omega_j \to \mathbb{R}$ strictly positive and continuous for $j = 1, 2$, we associate with it the metric $[\phi] = [\varphi_1, \varphi_2]$ on Ω defined by

$$[\phi](\omega, \omega') = \max_{j=1,2} [\varphi_j](\mathcal{P}_j(\omega), \mathcal{P}_j(\omega')).$$

In this section we prove the following theorem.

THEOREM 4.1. *Let L_1 and L_2 be positive integers, let $\Omega = (\{1, \dots, L_1\} \times \{1, \dots, L_2\})^{\mathbb{N}_0}$ and let T be the shift on Ω. Let $K \subset \Omega$ be a compact shift-invariant set ($TK \subset K$) and let $\varphi_j : \Omega_j \to \mathbb{R}$ be strictly positive and continuous for $j = 1, 2$. If the subshift $(K, T|_K)$ satisfies the specification property and if*

$$\sup_{\omega\in\Omega} \varphi_2 \circ \mathcal{P}_2(\omega) \le \inf_{\omega\in\Omega} \varphi_1 \circ \mathcal{P}_1(\omega) \tag{4.3}$$

then

$$\dim_{[\phi]} K = \sup\{\dim_{[\phi]}(\mu) : \mu(K) = 1, \mu \circ T^{-1} = \mu\}, \tag{4.4}$$

where $\phi = (\varphi_1 \circ \mathcal{P}_1, \varphi_2 \circ \mathcal{P}_2)$.

Our proof of Theorem 4.1 is based on the following known case.

THEOREM 4.2. (Gatzouras and Lalley [**GL**]) *Let $\Omega = (\{1, \dots, L_1\} \times \{1, \dots, L_2\})^{\mathbb{N}_0}$ and let T be the shift on Ω. Let D be any subset of $\{1, \dots, L_1\} \times \{1, \dots, L_2\}$ and let $\varphi_j : \Omega_j \to \mathbb{R}$ be a strictly positive function depending only on the first coordinate of its argument for each $j = 1, 2$. If*

$$\sup_{\omega\in D^{\mathbb{N}_0}} \varphi_2 \circ \mathcal{P}_2(\omega) < \inf_{\omega\in D^{\mathbb{N}_0}} \varphi_1 \circ \mathcal{P}_1(\omega),$$

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then $D^{\mathbb{N}_0}$ supports a T-invariant ergodic probability measure μ such that

$$\dim_{[\phi]}(\mu) = \dim_{[\phi]}(D^{\mathbb{N}_0}),$$

where $\phi = (\varphi_1 \circ \mathcal{P}_1, \varphi_2 \circ \mathcal{P}_2)$.

The proof of this theorem is in Gatzouras and Lalley [**GL**, Theorem 5.3]. The initial setting there is $\mathbb{R}^2$, but the proofs are in sequence spaces.

In the proof of Theorem 4.1, we also need the following (standard) fact.

LEMMA 4.3. *Let X and Y be compact metric spaces and let $\tau : X \to X$ and $\sigma : Y \to Y$ be continuous functions. Let $h : X \to Y$ be a continuous mapping of X onto Y and let ν be a Borel probability measure on Y. Then:*

(i) *there exists a Borel probability measure μ on X such that $\nu = \mu \circ h^{-1}$;*

(ii) *if ν is σ-invariant and $h \circ \tau = \sigma \circ h$, then we can choose μ in (i) to be τ-invariant.*

Proof. (i) This is a well-known consequence of the Hahn–Banach theorem.

(ii) By compactness, some subsequence of $\mu_n = n^{-1} \sum_{i=0}^{n-1} \mu \circ \tau^{-i}$ has a weak*-limit μ^*, and μ^* is τ-invariant. On the other hand, $\mu_n \circ h^{-1} = \nu$ for all n since $h \circ \tau = \sigma \circ h$, and ν is σ-invariant. Hence, $\mu^* \circ h^{-1} = \nu$. □

Now let K be a compact shift-invariant subset of Ω. For each $n \in \mathbb{N}$ define the nth full-shift approximation to K to be the set

$$A_n(K) = \bigcap_{j=0}^{\infty} \{\omega \in \Omega : (T^{jn}\omega)|_n \in K|_n\}. \tag{4.5}$$

Here, $\omega|_n$ is the restriction of $\omega \in \Omega$ to the first n coordinates, i.e. $\omega|_n = (\omega(0), \ldots, \omega(n-1))$. Thus, $\omega \in A_n(K)$ iff for each $j \in \mathbb{N}_0$ the finite word consisting of the first n coordinates of $T^{jn}\omega$ can be extended (on the right) to a sequence in K. Observe that $A_n(K)$ is T^n-invariant, and that the T-invariance of K implies that $K \subset A_n(K)$ for every $n \in \mathbb{N}$.

Proof of Theorem 4.1 Observe that Ω_j and $(\{1, \ldots, L_j\}^n)^{\mathbb{N}_0}$ are naturally isomorphic and let $J_n^{(j)}$ denote the isomorphism $(\{1, \ldots, L_j\}^n)^{\mathbb{N}_0} \to \Omega_j$ (inverse of the blocking map). Recall that $S_n^*\varphi_j$ depends only on the first n coordinates of its argument $\omega_j = \mathcal{P}_j(\omega)$ and may therefore be viewed as a function on $\{1, \ldots, L_j\}^n$; this function induces, in an obvious way, a function on $(\{1, \ldots, L_j\}^n)^{\mathbb{N}_0}$, depending only on the first coordinate (n-block) of its argument. Fix $\varepsilon > 0$, and for each n define $\psi_n^{(1)} = S_n^*\varphi_1 \circ J_n^{(1)} + \varepsilon$ and $\psi_n^{(2)} = S_n^*\varphi_2 \circ J_n^{(2)}$, i.e.

$$\psi_n^{(1)}((i_1, \ldots, i_n), (k_1, \ldots, k_n), \ldots) = S_n^*\varphi_1(i_1, \ldots, i_n, k_1, \ldots, k_n, \ldots) + \varepsilon,$$
$$\psi_n^{(2)}((i_1, \ldots, i_n), (k_1, \ldots, k_n), \ldots) = S_n^*\varphi_2(i_1, \ldots, i_n, k_1, \ldots, k_n, \ldots).$$

By (4.3) we have that $\sup \psi_n^{(2)} < \inf \psi_n^{(1)}$. Now apply Theorem 4.2 with $\{1, \ldots, L_1\}^n \times \{1, \ldots, L_2\}^n$ in place of $\{1, \ldots, L_1\} \times \{1, \ldots, L_2\}$, for the set $D = K|_n$ and $\psi_n^{(1)}$, $\psi_n^{(2)}$ in place of φ_1, φ_2. We obtain an ergodic probability measure ν_n on $(K|_n)^{\mathbb{N}_0}$, which is invariant under the shift T_n on $((\{1, \ldots L_1\} \times \{1, \ldots L_2\})^n)^{\mathbb{N}_0}$ (i.e. T_n shifts by an n-block), such that

$$\dim_{[\psi_n]}(\nu_n) = \dim_{[\psi_n]}(K|_n)^{\mathbb{N}_0}, \tag{4.6}$$

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where $\psi_n = (\psi_n^{(1)}, \psi_n^{(2)})$. We have identified the product $\{1, \dots, L_1\}^n \times \{1, \dots, L_2\}^n$ with $(\{1, \dots, L_1\} \times \{1, \dots, L_2\})^n$ here; the latter contains $K|_n$. Now ν_n induces a T^n-invariant measure μ_n' on Ω: if J_n is the 'inverse blocking map' $(\{1, \dots, L_1\}^n \times \{1, \dots, L_2\}^n)^{\mathbb{N}_0} \to \Omega$, we define $\mu_n' = \nu_n \circ J_n^{-1}$. Since J_n maps $(K|_n)^{\mathbb{N}_0}$ onto $A_n(K)$, the measure μ_n' is supported by $A_n(K)$. Recall that $K \subset A_n(K)$.

Define for each n, a mapping $h_n : \Omega \to \Omega$ as follows. Let m be as in the definition of specification for K. For each $\omega \in \Omega$ let $h_n(\omega)$ be the infinite sequence which is obtained by keeping the first n coordinates of ω, throwing away the next m coordinates, keeping again the following n, etc. In other words, $h_n(\omega)$ is defined by the requirement that

$$T^{jn}(h_n(\omega))|_n = T^{j(n+m)}(\omega)|_n \quad j = 0, 1, \dots.$$

By the choice of m and a routine compactness argument the map h_n maps K onto $A_n(K)$. Since $h_n \circ T^{n+m} = T^n \circ h_n$, we can use Lemma 4.3 to find a T^{n+m}-invariant probability measure μ_n'', supported on K, such that $\mu_n'' \circ h_n^{-1} = \mu_n'$. Finally, we pass to an average: $\mu_n = (n+m)^{-1} \sum_{i=0}^{n+m-1} \mu_n'' \circ T^{-i}$ to obtain a T-invariant ergodic measure on K.

LEMMA 4.4. *The mapping $h_n : \Omega \to \Omega$ is Hölder with exponent tending to* 1, *i.e. for some constants $C_n < \infty$ and $\beta_n > 0$ with $\beta_n \to 0$,*

$$[\phi](h_n(\omega), h_n(\omega')) \le C_n([\phi](\omega, \omega'))^{1-\beta_n}.$$

LEMMA 4.5. *The mapping $J_n : (\{1, \dots, L_1\}^n \times \{1, \dots, L_2\}^n)^{\mathbb{N}_0} \to \Omega$ is bi-Hölder with respect to the metrics $[\psi_n]$ and $[\phi]$ respectively, with exponent tending to* 1, *i.e. for some constants $0 < c_n < \infty$ and $\alpha_n > 0$ with $\alpha_n \to 0$,*

$$c_n^{-1}([\phi](\omega, \omega'))^{1+\alpha_n} \le [\psi_n](J_n^{-1}(\omega), J_n^{-1}(\omega')) \le c_n([\phi](\omega, \omega'))^{1-\alpha_n}.$$

Before proving the lemmas, let us show how they imply Theorem 4.1.

Clearly, as μ_n is obtained from μ_n'' by averaging, $\mu_n(B) = 1 \Rightarrow \mu_n''(B) = 1$, so

$$\dim_{[\phi]}(\mu_n) \ge \dim_{[\phi]}(\mu_n''). \tag{4.7}$$

By Lemmas 2.4, 4.4 and 4.5

$$\begin{aligned} \dim_{[\phi]}(\mu_n'') &\ge (1-\beta_n)\dim_{[\phi]}(\mu_n'' \circ h_n^{-1}) = (1-\beta_n)\dim_{[\phi]}(\mu_n') \\ &= (1-\beta_n)\dim_{[\phi]}(\nu_n \circ J_n^{-1}) \ge (1-\beta_n)(1-\alpha_n)\dim_{[\psi_n]}(\nu_n). \end{aligned} \tag{4.8}$$

On the other hand, by (4.6) and the lemmas again,

$$\dim_{[\psi_n]}(\nu_n) = \dim_{[\psi_n]}(K|_n)^{\mathbb{N}_0} \ge \frac{1}{1+\alpha_n} \dim_{[\phi]} A_n(K). \tag{4.9}$$

Hence, by (4.7), (4.8) and (4.9), we have $\liminf_{n\to\infty} \dim_{[\phi]}(\mu_n) \ge \dim_{[\phi]} K$. Since every μ_n is supported by K, we actually have equality. □

Proof of Lemmas 4.4 and 4.5. By definition, $[\phi](\omega, \omega') = \max_j [\varphi_j](\mathcal{P}_j(\omega), \mathcal{P}_j(\omega'))$, so it suffices to apply Lemma 3.2 to each coordinate space separately. In particular, h_n induces a mapping $h_n^{(j)} : \Omega_j \to \Omega_j$ and we apply Lemma 3.2 to the mapping $h_n^{(j)}$ and the metric $[\varphi_j]$, for each j. Lemma 4.5 is obtained similarly from Lemma 3.1, since $[\psi_n]$ is also defined as $\max_j [\psi_n^{(j)}]$ and since each 'blocking map' $(J_n^{(j)})^{-1}$ is by Lemma 3.1 bi-Hölder with respect to $[\varphi_j]$ and $[\psi_n^{(j)}]$. □

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5. *Proofs of the main results*

Proof of Theorem 1. We first verify that $(\Lambda, f|_\Lambda)$ has Markov partitions and is therefore a factor of a subshift of finite type. This will then enable us to transfer onto Λ the results of §2 (Theorem 2.1 in particular) via Lemma 2.4 and the following lemma, whose proof is routine.

LEMMA 5.1. *Let X and Y be metric spaces and let $\pi : X \to Y$ be a mapping with the following property. There exist positive b, C and N such that for all sufficiently small $\varepsilon > 0$, the preimage $\pi^{-1}(B)$ of any ε-ball in Y can be covered by at most N sets in X of diameter less than $C\varepsilon^b$. Then for any set $E \subset Y$ we have that $\mathcal{H}_s(\pi^{-1}(E)) > 0 \Rightarrow \mathcal{H}_{sb}(E) > 0$, where $\mathcal{H}_s$ is s-dimensional Hausdorff measure. Hence,* $\dim E \geq b \dim \pi^{-1}(E)$. *Also, if μ is a probability measure on X then* $\dim(\mu \circ \pi^{-1}) \geq b \dim(\mu)$.

By the definition of a compact repeller, $f|_\Lambda : \Lambda \to \Lambda$ is an open mapping. Replacing the map f by a power thereof if necessary, we may assume that for some constants α, β,

$$1 < \alpha \leq |Df(x)| \leq \beta < \infty \quad (x \in \Lambda). \tag{5.1}$$

By the compactness of Λ, there are numbers $r > 0$ and $\gamma > 1$ such that if $x \in \Lambda$ and $y \in M$ satisfy $d(x, y) \leq r$ then

$$d(f(x), f(y)) \geq \gamma d(x, y). \tag{5.2}$$

It now follows from Lemma 1 of [**CR**] that $f|_\Lambda : \Lambda \to \Lambda$ is an 'expanding map' in the sense of [**Ru2**, p. 143] and must therefore have Markov partitions of arbitrarily small size.

Let $\{Q_1, \ldots, Q_L\}$ be such a Markov partition for $f|_\Lambda$, i.e. each Q_i is a closed subset of Λ and $\Lambda = \cup_i Q_i$, the relative interior of Q_i in Λ is dense in Q_i and the (relative) interiors of Q_i and Q_j are disjoint for $i \neq j$ and, finally, each $f(Q_i)$ is a union of sets Q_j.

Set $\Omega = \{1, \ldots, L\}^{\mathbb{N}_0}$ and $\Omega(f) = \{\omega \in \Omega : Q_{\omega(n)} \subset f(Q_{\omega(n-1)})$ for all $n \in \mathbb{N}\}$. Call a finite sequence $(i_0, \ldots, i_{n-1}) \in \{1, \ldots, L\}^n$ *admissible* if $Q_{i_j} \subset f(Q_{i_{j-1}})$ for all $1 \leq j < n$. For an admissible $(i_0, \ldots, i_{n-1})$ set

$$Q(i_0, \ldots, i_{n-1}) = \bigcap_{j=0}^{n-1} f^{-j}(Q_{i_j})$$

and write $Q_n(\omega)$ for $Q(\omega(0), \ldots, \omega(n-1))$ whenever $\omega \in \Omega(f)$. By (5.2), assuming that the Markov partition has been chosen so that $\operatorname{diam}(Q_i) \leq \min\{r, \gamma^{-1}\}$ for each i, we get

$$\operatorname{diam}(Q_n(\omega)) \leq \gamma^{-(n-1)} \operatorname{diam}(Q_{\omega(n-1)}) \leq \gamma^{-n}.$$

Consequently, for each $\omega \in \Omega(f)$ the set $\cap_{n \geq 0} Q_n(\omega)$ is a singleton, which we denote by $\pi(\omega)$. This establishes the existence of a map $\pi : \Omega(f) \to \Lambda$, which is surjective and such that $\pi \circ T = f \circ \pi$ on $\Omega(f)$, where T is the shift on Ω.

We also need a metric on Ω. Define $\varphi : \Omega(f) \to (0, +\infty)$ by $\varphi(\omega) = \log |Df(\pi(\omega))|$, and extend φ arbitrarily to a strictly positive continuous function on all of Ω, denoted by φ again.

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(i) By Theorem 2.1 there exists a shift-invariant ergodic measure ν with $\nu(\pi^{-1}(K)) = 1$ and $\dim_{[\varphi]} \pi^{-1}(K) = \dim_{[\varphi]}(\nu)$. Set $\mu = \nu \circ \pi^{-1}$. Then μ is the desired measure, if f itself satisfies (5.1). Clearly, $\mu(K) = 1$ and μ is f-invariant and ergodic in this case. If f was replaced by f^n for some n so that (5.1) holds, then μ is only f^n-invariant. In this case we pass to a suitable ergodic average of μ to obtain an f-invariant ergodic measure; clearly this averaging will not decrease the dimension of μ. Thus, we only have to show that $\dim(\mu) = \dim K$. This will be done by establishing that the factor map $\pi : (\Omega(f), [\varphi]) \to (\Lambda, d)$, satisfies the hypotheses of Lemmas 2.4 and 5.1 with a and b arbitrarily close to 1. This will imply that $\dim(\mu) = \dim K$.

For $x \in M$ and $\rho > 0$ let $B_\rho(x)$ denote the ball in M with center x and radius ρ. Since f is C^1 and Λ is compact,

$$\epsilon(\rho) := \sup_{x\in\Lambda} \sup_{y,z\in B_\rho(x)} ||Df(y)| - |Df(z)|| \to 0 \quad (\rho \to 0) \tag{5.3}$$

and since $|Df(x)| \geq \alpha > 1$ for $x \in \Lambda$,

$$e^{-\epsilon(\rho)} \leq \frac{|Df(y)|}{|Df(x)|} \leq e^{\epsilon(\rho)} \quad (y \in B_\rho(x),\ x \in \Lambda). \tag{5.4}$$

Using the facts that Λ is compact again, that f is locally a diffeomorphism on Λ and (5.4), we then have, following [**Fa1**], that for all ρ sufficiently small,

$$e^{-\epsilon(\rho)} \leq \frac{d(f(x), f(y))}{|Df(x)|d(x, y)} \leq e^{\epsilon(\rho)}$$

whenever $x \in \Lambda$, $x \neq y$ and $y \in B_\rho(x)$. In particular, we may assume that r in (5.2) has been chosen small enough for this to hold for $\rho = r$. This implies that

$$r \geq d(f^{n-1}(x), f^{n-1}(y)) \geq \gamma^{n-1-i} d(f^i(x), f^i(y)) \tag{5.5}$$

for all $x \in \Lambda$ and $y \in M$ such that $f^i(y) \in B_r(f^i(x))$ and for all $0 \leq i < n$. Thus, if $x \neq y$ then

$$\zeta_n^{-1} \leq \frac{d(f^n(x), f^n(y))}{|Df^n(x)|d(x, y)} \leq \zeta_n, \tag{5.6}$$

where $\zeta_n = \exp(\sum_{i=0}^{n-1} \epsilon(r\gamma^{-i}))$. In particular, by (5.3), $\zeta_n^{1/n} \to 1$.

We now establish that π satisfies the hypotheses of Lemma 2.4. By (5.6)

$$\operatorname{diam}(Q_n(\omega)) \leq \frac{\zeta_{n-1} \operatorname{diam}(Q_{\omega(n-1)})}{|Df^{n-1}(\pi(\omega))|},$$

recalling that $\max_i \operatorname{diam}(Q_i) < \min\{r, \gamma^{-1}\}$. Assuming further that $\operatorname{diam}(Q_i) < \beta^{-1}$ for all i, we have that $\operatorname{diam}(Q_{\omega(n-1)}) < |Df(f^{n-1}(\pi(\omega)))|^{-1}$, and since also $\zeta_{n-1} < \zeta_n$,

$$\operatorname{diam}(Q_n(\omega)) \leq \frac{\zeta_n}{|Df^n(\pi(\omega))|} = \zeta_n e^{-S_n\varphi(\omega)}. \tag{5.7}$$

This clearly implies that π is Hölder with exponent a arbitrarily close to 1.

To show that π satisfies the condition on the ε-balls in Lemma 5.1 we consider, following [**Ru3**], the δ-neighborhood $\tilde{Q}_i$ of Q_i, where δ is small. In particular, we

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assume that δ is small enough to have $\tilde{Q}_i \cap \tilde{Q}_j = \emptyset$ whenever $Q_i \cap Q_j = \emptyset$ and that $\delta \leq r$. For an admissible $(i_0, \ldots, i_{n-1})$ set

$$\tilde{Q}(i_0, \ldots, i_{n-1}) = \bigcap_{j=0}^{n-1} f^{-j}(\tilde{Q}_{i_j}).$$

Let $x \in \Lambda$ and $\varepsilon > 0$. Take any $\omega \in \pi^{-1}(\{x\})$ and choose $n = n(x, \varepsilon)$ so that

$$\frac{\delta}{\zeta_n |Df^n(x)|} < \varepsilon \leq \frac{\delta}{\zeta_{n-1} |Df^{n-1}(x)|}. \tag{5.8}$$

From (5.6) again, the ball (in M) with center x in $Q(i_0, \ldots, i_{n-1})$ and radius $\delta/(\zeta_{n-1}|Df^{n-1}(x)|)$ is contained in $\tilde{Q}(i_0, \ldots, i_{n-1})$. Consequently, $B_\varepsilon(x) \subset \tilde{Q}_n(\omega)$ also. Since $\pi(\mathcal{C}_j(\tilde{\omega})) = Q_j(\tilde{\omega})$ for all j and $\tilde{\omega}$, the set $\pi^{-1}(B_\varepsilon(x))$ is covered by the cylinders $\mathcal{C}_n(\omega^{(1)}), \ldots, \mathcal{C}_n(\omega^{(k)})$, where $\omega^{(1)}, \ldots, \omega^{(k)} \in \Omega(f)$ are such that $Q_n(\omega^{(j)}) \cap Q_n(\omega) \neq \emptyset$. Furthermore, we may assume that the sequences $\omega^{(j)}$ labeling the cylinders $\mathcal{C}_n(\omega^{(j)})$ have been chosen so that $\pi(\omega^{(j)}) \in Q_n(\omega)$. Observe that $k = k_n(x)$ depends on x and n. Hence, for each j,

$$\begin{aligned} |\mathcal{C}_n(\omega^{(j)})|_{[\varphi]} &\leq \exp\left(- S_n\varphi(\omega^{(j)}) + \sum_{i=1}^{n} \mathrm{Var}_i\, \varphi \right) \\ &= \frac{\exp\left(\sum_{i=1}^{n} \mathrm{Var}_i\, \varphi\right)}{|Df^n(\pi(\omega^{(j)}))|} \leq \frac{\zeta_n \exp\left(\sum_{i=1}^{n} \mathrm{Var}_i\, \varphi\right)}{|Df^n(x)|}, \end{aligned} \tag{5.9}$$

the last inequality being a consequence of (5.4), (5.5) and the fact that $x, \pi(\omega^{(j)}) \in Q_n(\omega)$.

Let $0 < b < 1$. Recall that both $\zeta_n^{1/n} \to 1$ and $n^{-1}\sum_{i=1}^{n} \mathrm{Var}_i\, \varphi \to 0$, and $1 < \alpha \leq \min_\Lambda |Df|$. Hence, for all sufficiently large n one has

$$|\mathcal{C}_n(\omega^{(j)})|_{[\varphi]} \leq \frac{\delta^b \alpha^{n(1-b)}}{\zeta_n^b |Df^n(x)|} \leq \frac{\delta^b}{\zeta_n^b |Df^n(x)|^b} < \varepsilon^b,$$

by (5.8). Thus, for any $0 < b < 1$, the preimage $\pi^{-1}(B_\varepsilon(x))$ can be covered by $k = k_n(x)$ balls of $\Omega(f)$, of diameter $< \varepsilon^b$, provided ε is small enough.

It remains to show that the number $k_n(x)$ of cylinders required to cover $\pi^{-1}(B_\varepsilon(x))$ is bounded. This is a consequence of the following fact.

If the size of the Markov partition is sufficiently small, then

$$\tilde{Q}(i_0, \ldots, i_n) \cap Q(j_0, \ldots, j_n) \neq \emptyset \quad \textit{iff} \quad Q(i_0, \ldots, i_n) \cap Q(j_0, \ldots, j_n) \neq \emptyset$$

and for any given $Q(i_0, \ldots, i_n)$ the number of distinct sets $Q(j_0, \ldots, j_n)$ that intersect it is bounded by $N = \max_i \#\{j : Q_i \cap Q_j \neq \emptyset\}$. In particular, the bound N is independent of n.

This is from [**Ru3**] and is implied by $f|_\Lambda : \Lambda \to \Lambda$ being an 'expanding map' in the sense of [**Ru2**], i.e. in addition to (5.2) there also exists a $\varrho > 0$ such that for any $x, y \in \Lambda$ with $d(f(x), y) < \varrho$, the set $f^{-1}(\{y\})$ has precisely one element in $B_\varrho(x) \cap \Lambda$.

The proof of (i) is now complete.

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(ii) By Theorem 2.1(ii), the Hausdorff and Minkowski dimensions of $\pi^{-1}(K)$ agree for the metric $[\varphi]$. Let $\overline{\dim}_{\mathrm{M}}$ denote upper Minkowski dimension. Then by Lemmas 2.4 and 5.1

$$\overline{\dim}_{\mathrm{M}} K \le a^{-1}\overline{\dim}_{[\varphi]}\pi^{-1}(K) = a^{-1}\dim_{[\varphi]}\pi^{-1}(K) \le \frac{1}{ab}\dim K.$$

Since a, b may be chosen arbitrarily close to 1, the Minkowski dimension of K exists as a limit and equals the Hausdorff dimension of K.

For a continuous function $\psi : K \to \mathbb{R}$ denote by $P(f|_K, \psi)$ the pressure of ψ as a function in the system $(K, f|_K)$. Since π is continuous and $\pi \circ T = f \circ \pi$, we have by Theorem 2.2(vi) of [**Wa1**] that $P(f|_K, \psi) \le P(T|_{\pi^{-1}(K)}, \psi \circ \pi)$. By the variational principle, and since π is finite-to-one, we also have the reverse inequality so that $P(f|_K, \psi) = P(T|_{\pi^{-1}(K)}, \psi \circ \pi)$ for any continuous ψ. This, in conjunction with Theorem 2.1(ii), gives the pressure formula for the dimension θ, provided (5.1) holds. If we have to replace f by f^n for some n in order to have (5.1), then the above argument gives the equation $P(f^n|_K, -\theta \log |Df^n|) = 0$ for θ. However, by Theorem 2.2(i) of [**Wa1**] the left-hand side of this equation equals $nP(f|_K, -\theta \log |Df|)$ and this again gives the pressure formula for the dimension.

(iii) Recall that initially φ was only defined on $\Omega(f)$ and then extended arbitrarily to all of Ω. For $\omega, \omega' \in \Omega(f)$, with $\omega(i) = \omega'(i)$ for all $0 \le i < n$ we have

$$|\varphi(\omega) - \varphi(\omega')| = \left|\log \frac{|Df(\pi(\omega))|}{|Df(\pi(\omega'))|}\right| \le \epsilon(r\gamma^{1-n})$$

by (5.3), (5.4) and (5.5). If the extension of φ is now chosen properly, then we can arrange to have $\mathrm{Var}_n \varphi \le \epsilon(r\gamma^{1-n})$. In particular, this is true if we extend φ by setting

$$\varphi(\omega) = \min_{\omega' \in \mathcal{C}_n(\omega) \cap \Omega(f)} \varphi(\omega'),$$

where $n = n(\omega)$ is maximal so that the intersection $\mathcal{C}_n(\omega) \cap \Omega(f)$ is non-empty.

Clearly (1.2) implies that the sequence ζ_n in (5.6) is bounded and hence, by the previous paragraph, $\sum_{i=1}^{n} \mathrm{Var}_i \varphi$ is also bounded. Hence the map π satisfies Lemma 5.1 with $b = 1$, as both factors in the numerator of the rightmost term in (5.9) are now bounded. By Theorem 2.1(iii), $\sum_{i=1}^{\infty} \mathrm{Var}_i \varphi < \infty$ also implies that $\pi^{-1}(K)$ has positive Hausdorff measure in its dimension θ. But then $\mathcal{H}_\theta(K) > 0$ also, by Lemma 5.1.

(iv) Finite Hausdorff measure follows from Theorem 2.1(iv) and Lemma 2.4, as in (iii) of Theorem 1 above: π is now a Lipshitz map, since ζ_n is bounded in (5.7). The uniqueness of the measure μ in (i) follows from Lemma 4.3 and the uniqueness part of Theorem 2.1(iv).

Finally, the proof of the fact that $\mathcal{H}_\theta$ and μ are equivalent is as in [**Ru3**]. □

Proof of Theorem 2. As in the preceding proof, one has Markov partitions $\{Q_1^{(j)}, \dots, Q_{L_j}^{(j)}\}$ for each $f_j|_{\Lambda_j}$, of arbitrarily small size. Let Ω_j, $\Omega_j(f_j)$, $Q^{(j)}(i_0, \dots, i_{n-1})$ and π_j, be defined as in the proof of Theorem 1 for each f_j, and set $R_{ij} = Q_i^{(1)} \times Q_j^{(2)}$. Define

$$R((i_0 j_0), \dots, (i_{n-1} j_{n-1})) = \bigcap_{k=0}^{n-1} f^{-k}(R_{i_k j_k})$$

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and observe that $R((i_0 j_0), \ldots, (i_{n-1} j_{n-1})) = Q^{(1)}(i_0, \ldots, i_{n-1}) \times Q^{(2)}(j_0, \ldots, j_{n-1})$.

Let

$$\Omega = (\{1, \ldots, L_1\} \times \{1, \ldots, L_2\})^{\mathbb{N}_0},$$
$$\Omega(f) = \{\omega \in \Omega : R_{\omega(n)} \subset f(R_{\omega(n-1)}) \text{ for all } n \in \mathbb{N}\}$$

and write $\pi(\omega)$ for the unique element of $\cap_{n \geq 0} R_n(\omega)$, for $\omega \in \Omega(f)$. Observe that $\pi \circ T = f \circ \pi$ on $\Omega(f)$, where T is the shift on Ω, and that π is surjective again. Finally, define

$$\varphi_j(\omega_j) = \log |Df_j(\pi_j(\omega_j))| \quad (\omega_j \in \Omega_j(f_j);\ j = 1, 2);$$

using compactness, extend each φ_j to a continuous function on all of Ω_j, without altering its maximum and minimum. Set $\phi = (\varphi_1 \circ \mathcal{P}_1, \varphi_2 \circ \mathcal{P}_2)$. By Theorem 4.1, since $\pi^{-1}(K)$ satisfies the specification property, there exists a sequence of shift-invariant measures ν_n with $\nu_n(\pi^{-1}(K)) = 1$ and $\dim_{[\phi]}(\nu_n) \to \dim_{[\phi]} \pi^{-1}(K)$. Set $\mu_n = \nu_n \circ \pi^{-1}$. Then each μ_n is f-invariant and $\mu_n(K) = 1$. We will show that

$$\dim E = \dim_{[\phi]} \pi^{-1}(E) \tag{5.10}$$

for any Borel set $E \subset \Lambda_1 \times \Lambda_2$ and this will then imply that

$$\lim_{n \to \infty} \dim(\mu_n) = \lim_{n \to \infty} \dim_{[\phi]}(\nu_n) = \dim_{[\phi]} \pi^{-1}(K) = \dim K$$

and complete the proof of the theorem. The equality (5.10) will follow from Lemmas 2.4 and 5.1, once we establish that the map π of the metric space $(\Omega(f), [\phi])$ onto the metric space $(\Lambda_1 \times \Lambda_2, d)$ satisfies the assumptions of these lemmas with a and b arbitrarily close to 1.

Observe that $\pi(\omega) = (\pi_1(\omega_1), \pi_2(\omega_2))$, where $\omega_j = \mathcal{P}_j(\omega)$ is the projection (4.1). By the proof of Theorem 1, for any $a < 1$, each $\pi_j : \Omega_j(f_j) \to \Lambda_j$ is Hölder with exponent a. Consequently, so is $\pi : \Omega(f) \to \Lambda_1 \times \Lambda_2$.

We now show that π satisfies the condition in Lemma 5.1. Let $\varepsilon > 0$ and $(x_1, x_2) \in \Lambda_1 \times \Lambda_2$. Choose integers n_1 and n_2 so that

$$\frac{\delta}{\zeta_{n_j} |Df_j^{n_j}(x_j)|} < \varepsilon \leq \frac{\delta}{\zeta_{n_j - 1} |Df_j^{n_j - 1}(x_j)|},$$

where δ and ζ_n are as in the proof of Theorem 1 (they may be chosen to have common values for f_1 and f_2). Note that $n_1 \leq n_2$ since $|Df_1| \geq |Df_2|$. The ball $B_\varepsilon^{(j)}(x_j)$, with center x_j and radius ε, in M_j, is contained in $\tilde{Q}_{n_j}^{(j)}(\omega_j)$, where $\omega_j \in \pi_j^{-1}(x_j)$, and so the ball $B_\varepsilon(x_1, x_2)$ in $M_1 \times M_2$ is contained in $\tilde{Q}_{n_1}^{(1)}(\omega_1) \times \tilde{Q}_{n_2}^{(2)}(\omega_2)$. Thus, $\pi^{-1}(B_\varepsilon(x_1, x_2))$ is covered by at most $N_1 N_2$ sets of the form

$$\mathcal{B}_{n_1, n_2}(\{(i_k, j_k)\}) = \{\omega \in \Omega(f) : \omega_1(k) = i_k \text{ for } 0 \leq k < n_1;\ \omega_2(k) = j_k \text{ for } 0 \leq k < n_2\}.$$

From the proof of Theorem 1, for any number $b < 1$, if ε is chosen sufficiently small, then the diameter of these sets will not exceed ε^b. □

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6. *Concluding remarks*

As in the work of Bedford and Urbanski [**BU**], in the setting of Theorem 2 the Minkowski dimension of K is given by $\dim_{\mathrm{M}} K = r + \theta$, where r and θ solve the equations

$$P(-r \log |Df_2||_{K_2}) = 0 \quad \text{and} \quad P(-r \log |Df_2||_{K_2} - \theta \log |Df_1||_{K_1}) = 0.$$

Here $K_i = \mathrm{proj}_i(K)$, where $\mathrm{proj}_i(x_1, x_2) = x_i$. The formula holds for the setting of Theorem 4.2 (see [**GL**]) and can be proved for the setting in Theorem 1 by approximating $\pi^{-1}(K)$ by the sets $A_n(K)$ of §4 and using the maps h_n. However, the formula has been obtained by Hu [**Hu**] for more general mappings of $\mathbb{R}^2$ (but less general invariant sets) and we omit the details.

The usefulness of the specification property in the conformal and non-conformal settings leads to the following.

Conjecture. Let f be an expanding map. Then any compact invariant set K which satisfies specification, carries a unique ergodic invariant measure μ of full dimension. Moreover, μ is mixing (and perhaps even Bernoulli) for f.

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