Hole dissections for planar figures

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A geometric dissection is a cutting of a geometric figure (or a finite set of figures) into pieces that we can rearrange to form another geometric figure (or finite set of figures). If our figures are required to be polygons, then there is always a dissection that has just a finite number of pieces. This was established by John Lowry [1], William Wallace [2], Farkas Bolyai [3], and Karl Gerwien [4]. The American Sam Loyd [5] and the Englishman Henry Ernest Dudeney [6, 7] emphasised the goal of minimising the number of pieces that resulted from such a standard dissection.

As examples, Figure 1 shows a 7-piece dissection (in [8]) of an 8-pointed star {8/2} to a square, and Figure 2 shows a 5-piece dissection (in [9]) by Henry Perigal of two non-congruent squares to a larger square. Notable surveys of the field include books by Harry Lindgren [10] and Greg Frederickson [8, 11], as well as a comprehensive website by Gavin Theobald [12].

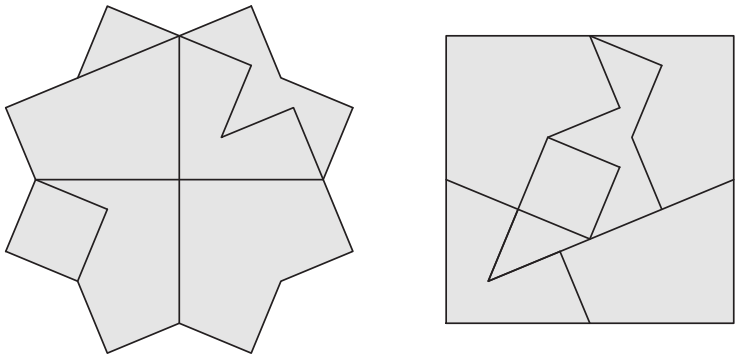


FIGURE 1: My dissection from the 1970's of an {8/2}-star to a square

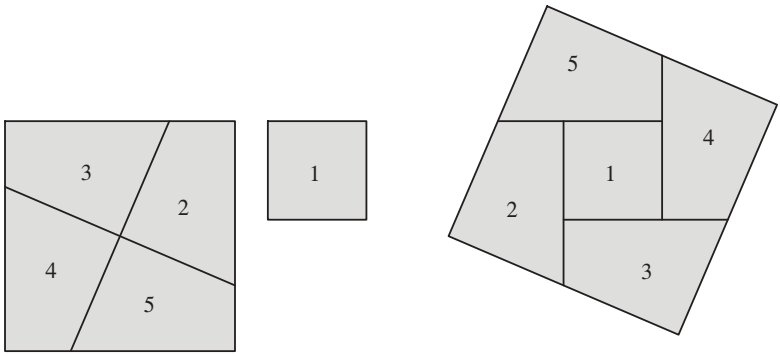


FIGURE 2: Henry Perigal's dissection of two different squares to a larger square

https://doi.org/10.1017/mag.2021.52 Published online by Cambridge University Press

A variation of this notion of a geometric dissection of is what we call a *hole dissection*: Given an (infinite) plane with one or more holes cut out of it, dissect it into pieces that we can rearrange to form another plane with a different set of holes, but such that the total area of the second set of holes equals the total area of the first set.

Since we shall focus on planar objects from which various areas (the holes and their boundaries) have been removed, we shall consider a mathematical model of dissection that is, in some sense, consistent. As discussed near the end of the second chapter of my second book [11], we will understand a piece of the plane to be a connected, open set. Thus its boundary is not part of the piece. When we cut one piece into two pieces, we shall remove a set of points that separate the two pieces, removing what we could term 'sawdust'. When we assemble the open pieces to form a figure, we assume that the curve that had been separating them will be filled in with a 'thin film' of molecules (i.e. a form of glue). The film will be so thin that it will have no area. The sawdust, like 'pixie dust', will also take up no area.

In our diagrams of hole dissections we do not show a complete (i.e. infinite) plane. Instead, we just show a relevant portion of it, by trimming the plane to be a convenient circle.

In the 1970's I found a hole dissection of an {8/2}-star to a square, shown in Figure 3. I thought that I had dreamed up the concept of a hole dissection, only to learn later that Leonardo da Vinci [13] had apparently conceived of such a dissection as a geometric proof of the Pythagorean Theorem. Sam Loyd, in his column 'Mental Gymnastics' in [14] gave a lovely example of a hole dissection of a star and crescent to a 'Crusader's Cross', as shown in Figure 4, which I included on page 169 of [8].

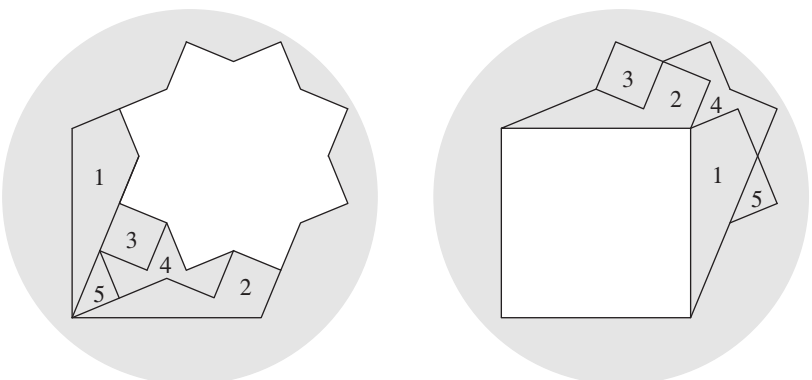


FIGURE 3: My (1970's) hole dissection of an {8/2}-star to a square

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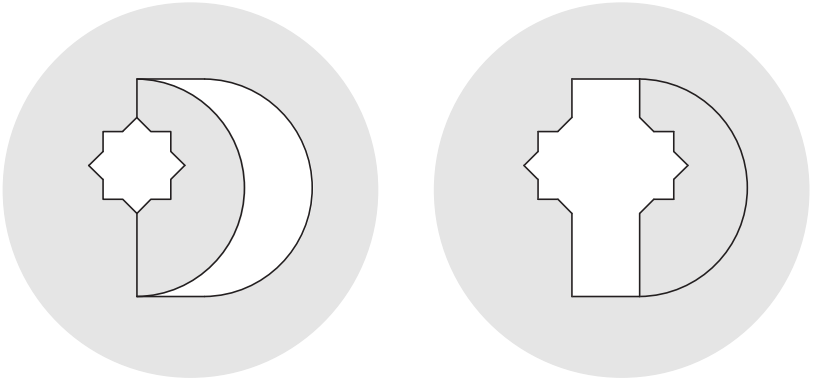


FIGURE 4: Loyd's hole dissection of a star and crescent to a ‘Crusader's Cross’

In 2018, Gavin Theobald came up with a similar idea, which he then employed to produce many interesting examples [12].

Here are several further examples. On the left in Figure 5 is a large square hole cut out of the plane, with non-overlapping pieces 1, 2, 3, and 4 filling in portions of the square hole, but leaving two smaller holes open. On the right in Figure 5 is a rearrangement of the four pieces, leaving a single square hole open. This hole dissection is *translational* since without rotating we can move the pieces in the configuration on the left to get the configuration on the right. Leonardo's hole dissection is not only translational, but also *slidable*, in the sense that the movements can take place within the holes. Just slide piece 2 up into the upper right-hand corner of the hole, slide piece 3 to the lower left-hand corner, and finally slide piece 1 to the lower right-hand corner.

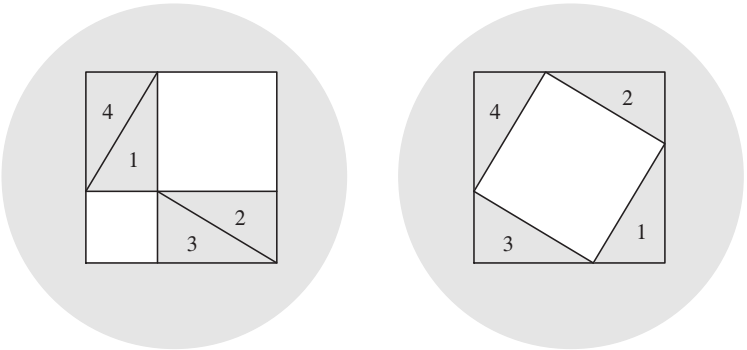


FIGURE 5: Leonardo's hole dissection of two unequal squares to one

We can modify Leonardo's dissection to use only three finite pieces, since piece 4 doesn't actually move in going from one configuration to the other. Thus we can achieve the hole dissection in Figure 6, which is also both translational and slidable.

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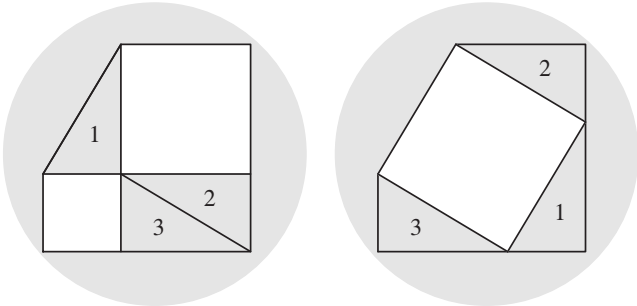


FIGURE 6: A related dissection of two unequal squares to one, with one fewer piece

Finally, if both small square holes are the same size, there is a dissection with two finite pieces that is slidable, though not translational, in Figure 7. To go from the configuration on the left to the configuration on the right, slide piece 1 all the way down and slide piece 2 all the way up. Then slide piece 1 halfway up and to the right, while sliding piece 2 halfway down and to the left, so that the two pieces form a small square in the centre. Rotate this combined square clockwise by 90 degrees, and then slide piece 1 down and to the left as you slide piece 2 up and to the right.

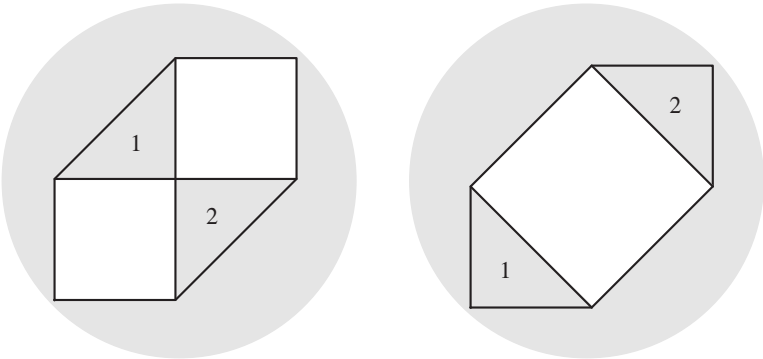


FIGURE 7: A dissection with two small holes of equal size

In March 2019 I learned of some further examples related to the Pythagorean Theorem that readers may find interesting. In particular, Noam Elkies [15] had used the cosine rule $\left(c^2 = a^2 + b^2 - 2ab\cos C\right)$ as the basis for two examples that turn out to be hole dissections. Elkies' observations seem to have been in response to a message from Douglas Rogers, who wrote about a floor-tiling pattern of Leonard Sohncke [16] and also two geometric dissections on pages 38-39 of [8].

Let's examine each of Elkies' dissections in turn. First consider when angle C is an acute angle. Elkies gave what corresponds to a dissection, with finite pieces 1, 2 and 3. The corresponding hole dissection is in Figure 8. On the left-hand side of that figure there is an irregular heptagonal hole, with

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pieces 1, 2 and 3 filling in portions of the hole. This leaves three holes open, with one hole being a square of side length c, and the other two holes being identical parallelograms with side lengths a and b.

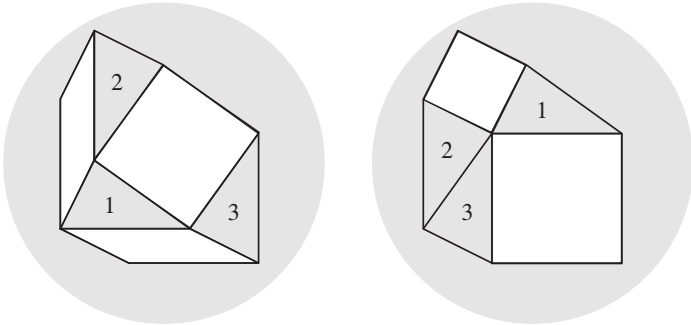


FIGURE 8: Elkies' hole dissection for the acute angle case of the cosine rule

On the right-hand side of Figure 8 we see the same three finite pieces rearranged within the same irregularly-shaped heptagonal hole. Two square holes are left open, with one being of side length a, and the other being of side length b. This hole dissection is translational, since the three pieces that we rearrange can be shifted (without any rotation) from forming the configuration on the left to forming the configuration on the right. Just slide piece 1 up and to the right, bringing its edge on the upper right to be flush with the edge of the hole that is parallel to it. Similarly, slide piece 2 down and slightly to the left, and slide piece 3 horizontally to the left. Thus this hole dissection is also *slidable*.

Next we examine Elkies' second dissection (in Figure 9) for the case in which angle C is an obtuse angle. Elkies' dissection uses two congruent triangular pieces, with sides of lengths a, b and c. The remaining piece is of infinite area and has an irregular hexagonal hole in it. The left-hand side of the figure shows the two finite pieces and a square hole of side length c left open. On the right-hand side of Figure 9, pieces 1 and 2 fill in a portion of the irregular hexagonal hole. We view the remainder of the hole as partitioned into two square holes, one of side length a and the other of side length b, and also two identical parallelograms of side lengths a and b. To go from the configuration on the left to the configuration on the right, slide piece 1 all the way up as you slide it slightly to the right, and then slide piece 2 first to the left and then slightly up.

A reader may note that Figure 9 as drawn is a bit problematic, since it is not clear how the open areas are to be kept separated. If the boundaries between open areas are treated as regions possessing zero area, and we allow them to be shifted similarly to pieces 1 and 2, then it is possible to slide them in a way to effect the transition from one configuration to the other. It is left to the reader to decide whether this is a reasonable assumption.

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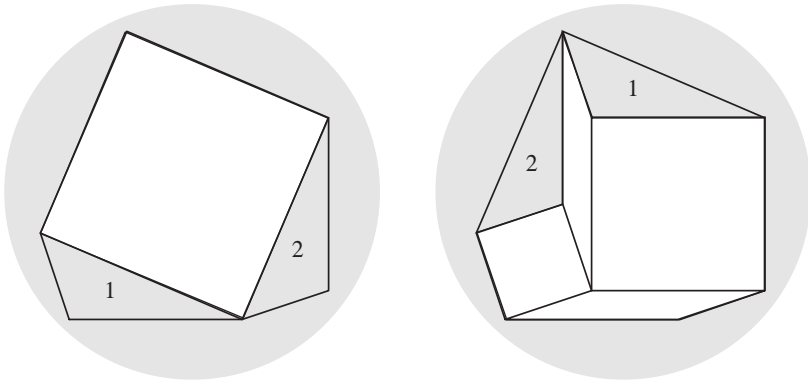


FIGURE 9: Elkies' hole dissection for the obtuse angle case of the cosine rule

Since March 2019 I have drawn inspiration from dissections of identical regular polygons to a larger version of the same polygon. Specifically I have produced hole dissections that combine several congruent regular polygonal holes to a single similar polygonal hole. They handle the cases of three equilateral triangles to one, three regular hexagons to one, and five regular pentagons to one.

On the left-hand side of Figure 10 we see three congruent right-triangular pieces arranged within a rotationally-symmetric 9-sided hole, leaving an equilateral triangular hole. On the right-hand side of this figure, we see the same 9-sided hole filled in with the same right-triangular pieces in different positions. This placement leaves three congruent equilateral triangular holes. Note that the dissection in Figure 10 is not translational. It is also not *slidable*, since the right angles of pieces 1, 2, and 3 are effectively stuck.

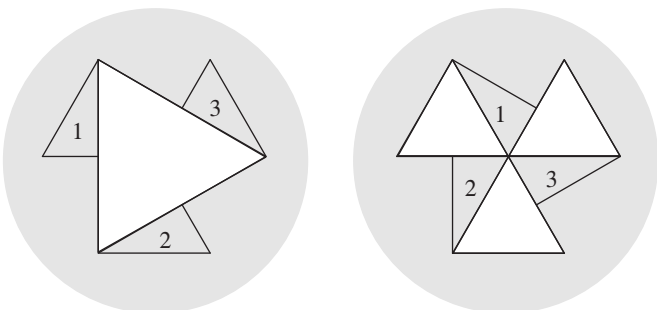


FIGURE 10: A hole dissection for three equilateral triangular holes to one

Next is the case of three congruent regular hexagonal holes to one hexagonal hole (in Figure 11). We start with a rotationally-symmetric 15-sided hole, into which we have arranged three congruent non-regular pentagonal pieces in different positions. This placement will leave a regular hexagonal hole in the left-hand side of the figure. On the right-hand side of

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this figure, we see the same 15-sided hole filled in with the same pentagonal pieces in different positions. This placement leaves three congruent regular hexagonal holes. Note that the dissection is *translational* but not *slidable*, the latter because on the right pieces 1, 2 and 3 are locked.

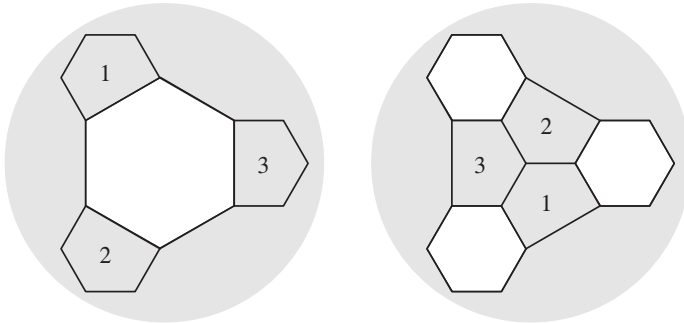


FIGURE 11: A hole dissection for three congruent regular hexagonal holes to one

Finally is the case of five congruent regular pentagonal holes to one (in Figure 12). We start with a rotationally-symmetric 25-sided hole, into which we have arranged five congruent trapezoidal pieces and five congruent isosceles triangles. On the left-hand side is a regular pentagonal hole in the centre.

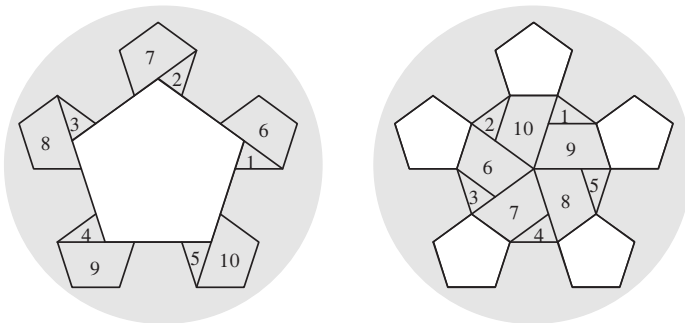


FIGURE 12: A hole dissection for five congruent regular pentagonal holes to one

On the right-hand side we see the same 25-sided hole filled in with the same pieces in different positions. This leaves five smaller congruent regular pentagonal holes. Thus the dissection is neither *translational* nor *slidable* (the latter because pieces 6, 7, 8, 9, and 10 cannot be slid out).

Following in Leonardo's footsteps may have allowed us, surprisingly, to fill in a gaping hole in the history of geometric dissections. It has also let us enjoy what is not there, as well as what is. And a hole dissection may remind us of Gertrude Stein's supposed comment about the city of Oakland in California, that there is no there there!

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10.1017/mag.2021.52

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