Probability in the Engineering and Informational Sciences, **29**, 2015, 623–637.
doi:10.1017/S0269964815000224

COUNTABLE STATE MARKOV PROCESSES: NON-EXPLOSIVENESS AND MOMENT FUNCTION

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The existence of a moment function satisfying a drift function condition is well known to guarantee non-explosiveness of the associated minimal Markov process (cf. [1,9]), under standard technical conditions. Surprisingly, the reverse is true as well for a countable space Markov process. We prove this result by showing that recurrence of an associated jump process, that we call the α-jump process, is equivalent to non-explosiveness. Non-explosiveness corresponds in a natural way to the validity of the Kolmogorov integral relation for the function identically equal to 1. In particular, we show that positive recurrence of the α-jump chain implies that all bounded functions satisfy the Kolmogorov integral relation. We present a drift function criterion characterizing positive recurrence of this α-jump chain.

Suppose that to a drift function V there corresponds another drift function W, which is a moment with respect to V. Via a transformation argument, the above relations hold for the transformed process with respect to V. Transferring the results back to the original process, allows to characterize the V-bounded functions that satisfy the Kolmogorov forward equation.

1. INTRODUCTION

It is a truth universally acknowledged that truths leaping to the eye are obscured by their own proximity. We were surprised to realize that the existence of a moment function for a minimal Markov process satisfying a drift condition is not only sufficient but also necessary for non-explosiveness.

The proof is actually quite simple and relies on a characterization of non-explosiveness of a Markov process by recurrence of so-called α-jump chains. Let us make these statements more precise as well as their motivation.

Consider a continuous time Markov process with values in a countable state space $\boldsymbol{S}$. We will restrict our analysis to Markov processes satisfying the following assumption.

Assumption 1.1: *X is a minimal, standard, stable Markov process with right-continuous sample paths (with respect to the discrete topology), and with a conservative q-matrix Q, in*

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other words $Q = (q_{xy})_{x,y\in S}$, *such that for all* $x \in \boldsymbol{S}$

1. $0 \leq q_x = -q_{x,x} < \infty$;
2. $\sum_y q_{xy} = 0$.

The assumptions imply (cf. Anderson [1] Theorem 2.2.2) that the associated, possibly dishonest, transition functions $P_t = (\, p_{t,xy})_{x,y\in S}$, $t \geq 0$, satisfy the Kolmogorov forward and backward equations, and that they are the minimal non-negative solutions of these equations. If the transition function is honest, then it is the unique solution of these equations.

Following the terminology in Meyn and Tweedie's paper [9] we use the term *non-explosiveness* instead of the more commonly used *regularity* to denote that the process does not explode in finite time.

In [15] I have studied conditions on functions $f : \boldsymbol{S} \to \mathbf{R}$, in order to satisfy an integral form of the Kolmogorov forward equations. Write Qf and $P_t f$ to denote the real-valued functions on $\boldsymbol{S}$ with $Qf(x) = \sum_{y\in S} q_{xy} f(y)$, and $P_t f(x) = \sum_{y\in S} p_{t,xy} f(y)$. For non-negative functions f that satisfy the drift inequality $Qf \leq cf$ (componentwise) for some constant $c \in \mathbf{R}$, it has been proved that the Kolmogorov forward integral equation

$$P_t f = f + \int_0^t P_s(Qf)ds \tag{1.1}$$

holds if and only if an associated Markov process X^f, obtained from X by a transformation induced by f (see Section 2) is non-explosive. This is even true, when X itself is explosive. This is of importance when considering functions of the Markov process, say reward functions associated with a queueing process. It is therefore of interest to have direct tools for characterizing non-explosiveness. To my knowledge the easiest applicable tool is the one described below. We use the notation $\mathbf{R}_+$ for the non-negative reals and $\mathbf{R}_{>0}$ for the positive reals, that is, $\mathbf{R}_+ = [0, \infty)$ and $\mathbf{R}_{>0} = (0, \infty)$.

Definition 1.1:

- $V : \boldsymbol{S} \to \mathbf{R}_+$ *is called a* moment function, *if there exists an increasing sequence* $\{K_n\}_n$ *of finite sets, that is,* $K_n \subset \boldsymbol{S}$, $K_n \subset K_{n+1}$, $n = 1, 2, \ldots$, *such that* $\inf_{x\notin K_n} V(x) \to \infty$, $n \to \infty$.
- $V : \boldsymbol{S} \to \mathbf{R}_{>0}$ *is called a* c-drift function, *if* $QV \leq cV$, *where "*$\leq$*" stand for component-wise ordering and* $c \in \mathbf{R}$.

Note that Meyn and Tweedie [9] use the terminology "norm-like function" to denote a moment function. It has been proved for the first time for general space Markov jump processes by M.F. Chen in 1986 (see, e.g., [4], see also [5] Theorem 2.25) that the existence of a c-drift moment function, for some constant $c \in \mathbf{R}$, implies non-explosiveness of the Markov process. A simple proof for the particular case of countable state Markov processes can be found in the book [1] by Anderson. Meyn and Tweedie [9] treat the general state space case.

As has been mentioned before, the reverse is also true. This will be the main statement of our paper, see Theorem 2.1 (i) $\Longleftrightarrow$ (ii) in Section 2 below. This result has been obtained in M.F. Chen (Remark 3.20 [5]) for the special case of a birth process.

The result follows from a new characterization of non-explosiveness through recurrence of an associated discrete time Markov chain that we will develop shortly.

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To this end, extend $\boldsymbol{S}$ with a coffin state δ and put $\boldsymbol{S}_\delta = \boldsymbol{S} \cup \{\delta\}$. We further use the notation $\mathsf{P}_x\{\cdot\}$, $\mathsf{E}_x\{\cdot\}$ to denote the conditional probability and the conditional expectation respectively given the event $X_0 = x$.

DEFINITION 1.2: *Let p be a probability distribution on $\boldsymbol{S}$, with $p_y > 0$ for all $y \in \boldsymbol{S}_\delta$ and let $\alpha > 0$. The discrete time Markov chain X^α is an α-jump chain (with restart distribution p on $\boldsymbol{S}$) on the extended state space $\boldsymbol{S}_\delta$, if*

1. *The transition matrix, denoted P^α, is given by*

$$p^\alpha_{xy} = \begin{cases} \dfrac{q_{xy}}{q_x + \alpha}, & x, y, \in \boldsymbol{S}, x \neq y, \\ \dfrac{\alpha}{q_x + \alpha}, & x \in \boldsymbol{S}, y = \delta, \\ p_y, & x = \delta, y \in \boldsymbol{S}; \end{cases}$$

2. *The restart distribution p, is assumed such that (a) $\mathsf{E}_x\tau < \infty$ for all $x \in \boldsymbol{S}$ implies $\sum_x p_x \mathsf{E}_x\tau < \infty$ and (b) X^α is aperiodic. The random variable τ stands for the hitting time of state δ.*

Note that X^α is an irreducible, aperiodic Markov chain. We suppress the dependence on the restart distribution p in our notation, since it is irrelevant for the class properties transience, null recurrence and positive recurrence of the Markov chain, given the requirements imposed on p. For $\alpha = 0$ an added coffin state is not necessary and we get the customary jump chain.

Our second main statement is that recurrence of α-jump chain for some $\alpha > 0$ (and then for all) is equivalent to non-explosiveness; cf. Theorem 2.1 (i) $\iff$ (iii).

This result has a resemblance to the well-known assertion (by Tweedie [16] and later stated by Anderson in [1] as Proposition 5.3.2) that recurrence of the jump chain (0-jump chain) is equivalent to recurrence of the minimal Markov process. Thus the latter ensures that recurrence of the jump chain implies the Markov process to be non-explosive. The reverse relation does not hold. As an example consider a birth-death process on the non-negative integers, with birth rates $\lambda_i = 2/3$ for $i = 0, \ldots$ and death rates $\mu_i = 1/3$, $i > 0$. This has a transient jump chain. However, the rates are bounded and hence the Markov process is non-explosive.

Related recent work worth mentioning and pointed out to the author during the refereeing process, has been presented in the paper by Avrachenkov, Piunovskiy and Zhang [2]. In this paper, the authors associate a restarted continuous time process with a general state Markov process, where restarts take place at the epochs of a Poisson process with rate α. They show that the restart process is positive Harris recurrent and exponentially ergodic if the original process is non-explosive. The associated jump chain consequently is recurrent. This associated jump chain is precisely the α-jump chain on $\boldsymbol{S}$ as we defined above. Clearly this indirectly proves one part of our result. Interestingly enough, Example 3.4 shows that the jump chain of the restart process is not necessarily positive Harris recurrent, even though the restart process is exponentially ergodic.

A third characterization of non-explosiveness that we will discuss, is the well-known statement, derived by Reuter [14], that explosiveness is equivalent to the q-matrix having a bounded, non-zero eigenvector to eigenvalue c for some (and hence for all) constant $c > 0$. This is statement (i) $\iff$ (iv) of Theorem 2.1. We provide a simple proof via the α-jump chain.

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The characterization of non-explosiveness via the existence of a c-drift moment function allows to shed some light on various drift criteria for non-explosiveness from the literature. In Section 3 we will compare this result with drift criteria from Piunovskiy and Zhang [11] and Guo and Hernández–Lerma [6]. In the paper [15] we have discussed how non-explosiveness properties are related to the validity of the Kolmogorov forward integral relation. In view of the above theorems the result from [15] can be sharpened. In Section 4, we will discuss this and compare the result to a criterion used by Guo, Hernandez–Lerma and Prieto–Rumeau in [7] and by the two latter authors in [12,13].

2. MAIN THEOREM AND PROOF

Before formulating the main theorem, we will give a precise definition of explosiveness. Let $\tau_0 = 0$, and $\tau_{n+1} = \inf\{t > \tau_n \,|\, X_t \neq X_t^-\}$, if X_{τ_n} is a non-absorbing state. If it is absorbing, put $\tau_k = \infty$ and $X_{\tau_k} = X_{\tau_n}$, $k > n$. Let $J_\infty = \lim_{n\to\infty} \tau_n$. Then X is said to be *explosive*, if there exists $x \in \boldsymbol{S}$ such that $\mathsf{P}_x\{J_\infty < \infty\} > 0$.

Our main theorem is as follows.

THEOREM 2.1: *Let X be a minimal Markov process satisfying Assumption 1.1. The following are equivalent statements.*

(i) *X is non-explosive.*
(ii) *There exists a c-drift moment function for X for some constant $c \in \mathbf{R}$.*
(iii) *There exists $\alpha > 0$, such that the α-jump chain is recurrent.*
(iv) *For no value $\alpha > 0$ does the q-matrix have a bounded non-zero (right) eigenvector.*

Under either condition it holds that for any constant $c > 0$ there exists a c-drift moment function, and for any constant $\alpha > 0$ the α-jump chain is recurrent.

COROLLARY 2.2: *Let X be a minimal Markov process satisfying Assumption 1.1. Then, either Q has a bounded non-zero eigenvector to eigenvalue α for each $\alpha > 0$, or for none.*

Before turning to prove Theorem 2.1, we need introduce the resolvent. Let $\alpha > 0$. The resolvent $R(\alpha)$, $\alpha > 0$ is defined by

$$R_{xy}(\alpha) = \int_0^\infty e^{-\alpha t}\, p_{t,xy} dt,$$

which can be interpreted as the expected time spent in state y before "extinction or explosion", when the initial state is x. Extinction in this case is interpreted to occur at rate α. One can associate a non-conservative q-matrix with the resolvent, such that the corresponding jump chain is the restriction of an α-jump chain to $\boldsymbol{S}$ before reaching the coffin state.

Next we write ${}_\delta P^\alpha$ for the taboo matrix, with taboo state δ, ignoring the jumps towards δ. That is, ${}_\delta p^\alpha_{xy} = 0$, $y = \delta$, and ${}_\delta p^\alpha_{xy} = p^\alpha_{xy}$ for $y \in \boldsymbol{S}$. Put ${}_\delta P^{\alpha,(0)} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix, and generally ${}_\delta P^{\alpha,(t)}$ for the tth iterate. The taboo transition probabilities for states in $\boldsymbol{S}$ are the transition probabilities of the "non-extended" (i.e., without $\{\delta\}$) α-jump chain.

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PROOF OF THEOREM 2.1 (i) $\iff$ (iii): From Proposition 4.1.1 in Anderson [1]

$$R_{xy}(\alpha) = \sum_{t=0}^{\infty} {}_{\delta}p_{xy}^{\alpha,(t)} \frac{1}{\alpha + q_y}. \tag{2.1}$$

By virtue of Theorem 2.2.2 and Propositions 2.2.3, 2.2.4 in Anderson [1] X is explosive if and only if $\sum_y p_{t,xy} < 1$ for some state $x \in \boldsymbol{S}$ and some $t \geq 0$. Since this is a non-increasing function of t, the following fact is simply derived (cf. Proposition 1.3.1 by Anderson [1]): given $\alpha > 0$, the matrix $\alpha R(\alpha)$ is a stochastic matrix if and only if P_t is a stochastic matrix for some $t > 0$. In either case $\alpha R(\alpha)$ is stochastic for all $\alpha > 0$ and P_t is stochastic for all $t \geq 0$.

Multiplying Eq. (2.1) by α, it follows that

$$\alpha \sum_{y \in S} R_{xy}(\alpha) = \sum_{t=0}^{\infty} \sum_{y \in S} {}_{\delta}p_{\alpha,xy}^{(t)} \frac{\alpha}{\alpha + q_y} = \mathsf{P}\{\exists n \geq 1 \text{ s.t. } X_n^{\alpha} = \delta \,|\, X_0^{\alpha} = x\},$$

is the probability of reaching δ in the extended α-jump chain.

This means that the row sums of $\alpha R(\alpha)$ equal 1 if and only if state δ is reached with probability 1 from every state $x \in \boldsymbol{S}$, in the Markov chain X^{α}. By irreducibility, this is true if and only if X^{α} is a recurrent Markov chain. ■

By virtue of the above result, the proof of Theorem 2.1 (i) $\iff$ (ii) now reduces to applying the necessary and sufficient condition for recurrence of irreducible, aperiodic discrete time Markov chains derived by Mertens, Samuel-Cahn and Zamir in [8] Theorem 1, the "if" part of which is the best known and used part. However, the "only if" part of this result does not seem to be as widely known and so we will quote the precise result here. We will also give the proof of the "only if" part here, first because the formulation presented here is slightly different and secondly because it is has some interesting and more detailed consequences.

THEOREM 2.3 Mertens et al. [8] Theorem 1: *Let $X = \{X_n\}_{n=0,\ldots}$ be a discrete time irreducible, aperiodic Markov chain on a countable state space $\boldsymbol{S}$ with transition matrix P. X is recurrent if and only if there exists a moment function $f : \boldsymbol{S} \to \mathbf{R}_+$ and a finite set $K \subset \boldsymbol{S}$, such that*

$$Pf(x) \leq f(x), \quad x \notin K. \tag{2.2}$$

In particular, under either condition the following properties hold:

(i) *for any finite set $K \subset \boldsymbol{S}$ there exists a moment function $f : \boldsymbol{S} : \mathbf{R}_+$ satisfying Eq. (2.2) with $f(x) = 0$ iff $x \in K$;*

(ii) *for any increasing sequence of finite sets $\{K_n\}_n$, $K_n \uparrow \boldsymbol{S}$, there exists a function $f : \boldsymbol{S} \to \mathbf{R}_+$, with $\inf_{x \notin K_n} f(x) \to \infty$, as $n \to \infty$, and $f(x) = 0$ iff $x \in K_1$, such that Eq. (2.2) is satisfied for f and the set K_1.*

PROOF OF THE 'ONLY IF' PART: We assume that X is recurrent. Let $K \subset \boldsymbol{S}$ be an arbitrary finite set. Let $K_n \supset K$, $n = 1, \ldots$, be any increasing sequence of finite sets, with $K_n \uparrow \boldsymbol{S}$ as $n \to \infty$. Define $\tau_n = \inf\{t \geq 0 \,|\, X_t \in K_n^C\}$ and $\tau_K = \inf\{t \geq 0 \,|\, X_t \in K\}$.

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Define

$$\phi^n(x) = \mathsf{P}\{\tau_n \leq \tau_K \,|\, X_0 = x\}.$$

Then $\phi^n(x) = 0$ for all n, and $x \in K$. On the other hand, $\phi^n(x) = 1$ for $x \in K_n^C$. Consequently

$$P\phi^n(x) = \sum_y p_{xy}\phi^n(y) \leq \phi^n(x), \quad x \notin K. \tag{2.3}$$

Hence, ϕ^n satisfies Eq. (2.2) for the set K. However, ϕ^n is bounded. We will construct a moment function by summing an appropriate collection of the functions ϕ^n. To this end, let $x \notin K$ be given. Clearly $n \mapsto \phi^n(x)$ is a non-increasing function of n, since $\{\tau_n \leq \tau_K\} \supset \{\tau_{n+1} \leq \tau_K\}$.

Then $\{\tau_n \leq \tau_K\} \supset \{X_t \notin K,\, t \geq 0\}$, since the Markov chain restricted to states of $K_n \setminus K$ is transient. Consequently, $\cap_n\{\tau_n \leq \tau_K\} \supset \{X_t \notin K,\, t \geq 0\}$. We claim that

$$\cap_n\{\tau_n \leq \tau_K\} = \{X_t \notin K,\, t \geq 0\}.$$

It is sufficient to prove that

$$\cup_n\{\tau_n > \tau_K\} \supset \{\exists t \geq 0 \text{ such that } X_t \in K\}.$$

This is clearly true, because each trajectory leading to set K has finite "length". Now, by monotone convergence and recurrence, we have that

$$\lim_{n\to\infty} \mathsf{P}\{\tau_n \leq \tau_K \,|\, X_0 = x\} = 0, \text{ for each } x \in \boldsymbol{S}.$$

Let $n_0 = 0$. For any $k \geq 1$ choose index $n_k > n_{k-1}$ so that $\phi^{n_k}(x) \leq 2^{-k}$ for $x \in K_k$. Take $n_0 = 0$. Define

$$f(x) = \sum_{k\geq 1} \phi^{n_k}(x).$$

$f(x)$ is finite for each $x \notin K$, and $f(x) = 0$ for $x \in K$. Moreover $f(x) \geq k$ for $x \in K_{n_k}^C$. Hence, f is a moment function. Further, f satisfies Eq. (2.2) since each term in the summation satisfies Eq. (2.2) by virtue of Eq. (2.3).

The assertions (i) and (ii) follow from the above construction. ■

Proof of Theorem 2.1 (i) $\Longleftrightarrow$ (ii): By virtue of the fact that we have already shown that (i) and (iii) are equivalent, it is sufficient to show that the α-jump chain X^α is recurrent if and only if there exists a α-drift moment function for X.

Suppose that such a function exists. Denote it by V. Then $QV \leq \alpha V$ implies that ${}_\delta P^\alpha V(x) \leq V(x)$ for $x \neq \delta$. Setting $V(\delta) = 0$, we get $P^\alpha V(x) \leq V(x)$, $x \neq \delta$. Hence, Eq. (2.2) is satisfied for this function V and set $K = \{\delta\}$, and so X^α is recurrent by virtue of Theorem 2.3.

Vice versa, suppose that X^α is recurrent. By Theorem 2.3(i) it follows that there exists a moment function V with $V(\delta) = 0$ and $V(x) > 0$, $x \in \boldsymbol{S}$, such that ${}_\delta P^\alpha V(x) \leq V(x)$ for $x \neq \delta$. Hence $QV(x) \leq \alpha V(x)$, $x \in \boldsymbol{S}$. ■

Proof of Theorem 2.1 (i) $\Longleftrightarrow$ (iv): Suppose that X is non-explosive and suppose that $g : \boldsymbol{S} \to [-1, 1]$, $g(x) \not\equiv 0$, is an eigenvector of Q to eigenvalue $\alpha > 0$. In other words $\alpha g =$

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Qg. Then

$$g(x) = \sum_{y \in S} p^{\alpha}_{xy} g(y) = \sum_{y} {}_{\delta}p^{\alpha}_{xy} g(y), \quad x \neq \delta,$$

where we take $g(\delta) = 0$. By iteration

$$g(x) = \sum_{y} {}_{\delta}p^{\alpha,(n)}_{xy} g(y), \quad n \geq 1, x \in \boldsymbol{S}.$$

Let $\tau = \min\{n \,|\, X^{\alpha}_{n} = \delta\}$ be the hitting time of δ. For $x \neq \delta$

$$-\mathsf{P}\{\tau > n \,|\, X^{\alpha} = x\} = -\sum_{y} {}_{\delta}p^{\alpha,(n)}_{xy} \leq g(x) \leq \sum_{y} {}_{\delta}p^{\alpha,(n)}_{xy}$$
$$\leq \mathsf{P}\{\tau > n \,|\, X^{\alpha} = x\}, \quad n \geq 1, x \in \boldsymbol{S}. \tag{2.4}$$

If X is non-explosive, the α-jump chain is recurrent, and hence both right-hand and left-hand sides of Eq. (2.4) converge to 0, as $n \to \infty$. Hence, for Q no bounded eigenvector to the positive eigenvalue α can exist. On the other hand, suppose Q has a bounded eigenvector $g \not\equiv 0$ to eigenvalue $\alpha > 0$. We may assume that $\sup_x |g(x)| \leq 1$. Then there exists $x \in \boldsymbol{S}$ such that $g(x) \neq 0$, say $g(x) > 0$. It follows that $\mathsf{P}\{\tau > n \,|\, X^{\alpha} = x\} \geq g(x) > 0$ for all n. Hence the α-jump chain is transient, and so X is explosive. ■

The next section discusses consequences of the main theorems and relations with drift criteria concerning explosiveness and the Dynkin formula from the literature.

3. DRIFT CRITERIA

Recently a new criterion for non-explosiveness has been proposed by Piunovskiy and Zhang [11], for a Markov process with a general state space. The assertion recast in our notation reads as follows.

LEMMA 3.1 Piunovskiy and Zhang [11] Corollary 3.2: *Suppose that Q is conservative and satisfies Assumption 1.1. Let $\alpha > 0$. Assume the existence of a non-negative function V: $\boldsymbol{S} \to \mathbf{R}_{+}$ such that*

$$QV(x) + q_x \leq \alpha V(x), \quad x \in \boldsymbol{S}. \tag{3.1}$$

Then X is non-explosive. In this case, the minimal non-negative solution of Eq. (3.1) is given by

$$V_{\min}(x) = \sum_{t=0}^{\infty} \sum_{y} {}_{\delta}p^{\alpha,(t)}_{xy} \frac{q_y}{\alpha + q_y} = \sum_{y} R_{xy}(\alpha) q_y. \tag{3.2}$$

This lemma does not require that V be a moment function. By virtue of Theorem 2.1 necessarily the condition in the lemma implies the existence of a c-drift moment function. The question of interest is whether Eq. (3.1) is stronger than non-explosiveness or equivalent to it.

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Using τ again to denote the first hitting time of state δ,

$$V_{\min}(x) = \sum_{t=0}^{\infty} \sum_{y} {}_{\delta}p_{xy}^{\alpha,(t)} \frac{q_y}{\alpha + q_y} = \sum_{t=1} \mathsf{P}\{\tau > t \,|\, X^{\alpha}(0) = x\}$$
$$= \mathsf{E}\{\tau \,|\, X^{\alpha}(0) = x\} - 1, \quad x \neq \delta.$$

Hence, if there is a non-negative function satisfying Eq. (3.1) then the α-jump chain is positive recurrent. Vice versa, positive recurrence of the α-jump chain implies that $V(x) = \mathsf{E}\{\tau \,|\, X^{\alpha}(0) = x\}$, $x \in \boldsymbol{S}$, is finite. Then $V(x) - 1$, $x \in \boldsymbol{S}$, is a non-negative solution to Eq. (3.1).

COROLLARY 3.2: *The existence of a non-negative function $V : \boldsymbol{S} \to \mathbf{R}_+$ satisfying Eq. (3.1) is equivalent to positive recurrence of the corresponding α-jump chain.*

Below we will give two examples. One where the α-jump chains are all recurrent, they are null-recurrent for some values of α and positive recurrent for others. In the second example, all α-jump chains are null-recurrent. This illustrates that the condition in Lemma 3.1 is not equivalent to non-explosiveness.

Example 3.3: Let $\boldsymbol{S} = \{1, \ldots\}$. We consider the minimal Markov process X with q-matrix with the only non-zero entries given by $q_{x,x+1} = x = -q_{x,x}$, $x \in \boldsymbol{S}$. X is a non-explosive pure birth process.

The latter follows from the fact that $\sum_x q_x^{-1} = \infty$ (cf. Norris [10], Theorem 2.5.2). Alternatively, choosing moment function $V(x) = x$, yields $QV \leq V$. Consequently, V is a 1-drift moment function.

Hence, the α-jump process is recurrent for any $\alpha > 0$. The expected time to reach the coffin state δ, given the initial state is x, is equal to

$$\mathsf{E}\{\tau \,|\, X^{\alpha}(0) = x\} = \sum_{n=0}^{\infty} \prod_{t=0}^{n} \frac{x+t}{\alpha + x + t}.$$

Put $\alpha = 1$. Then the above reduces to

$$\sum_{n=0}^{\infty} \prod_{t=0}^{n} \frac{x+t}{1+x+t} = \sum_{n=0}^{\infty} \frac{x}{1+x+n},$$

which is divergent. As a consequence, the 1-jump chain is null-recurrent. For any $\alpha \geq 2$ however it is simply deduced that the α-jump chain is positive recurrent.

Note that the Markov process in the above example is a pure birth process, and so the associated jump chain (0-jump chain) is transient. However, the Markov process is non-explosive.

Example 3.4: We refine the previous example slightly, leading to a non-explosive process for which no α-jump chain is positive recurrent. Consider a pure birth process X on $\boldsymbol{S} = \{1, 2, \ldots\}$, with rates $q_{x,x+1} = x \ln x = -q_{xx}$. Again we assume that X is the minimal process.

Similarly to the above example, $\sum_x q_x^{-1} = \infty$, and hence the process is non-explosive. A moment function does not seem easily constructed.

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The expected time to reach the coffin state δ by the α-jump chain, given initial state $x \in \boldsymbol{S}$ is equal to

$$\mathsf{E}\{\tau \mid X^\alpha(0) = x\} = \sum_{n=0}^{\infty} \prod_{t=0}^{n} \frac{(x+t)\ln(x+t)}{\alpha + (x+t)\ln(x+t)}. \tag{3.3}$$

Let $\alpha > 0$ and $x \in \boldsymbol{S}$ with $\ln x \geq \alpha$. Then $(x+t)\ln(x+t) \geq (x+t)\alpha$ and hence

$$\frac{(x+t)\ln(x+t)}{\alpha + (x+t)\ln(x+t)} \geq \frac{x+t}{1+x+t}.$$

It follows that

$$\prod_{t=0}^{n} \frac{(x+t)\ln(x+t)}{\alpha + (x+t)\ln(x+t)} \geq \prod_{t=0}^{n} \frac{x+t}{1+x+t} = \frac{x}{x+n+1}, \quad n = 1, \ldots,$$

so that Eq. (3.3) diverges. In other words, $\mathsf{E}\{\tau \mid X^\alpha(0) = x\} = \infty$, and so the α-jump chain is null-recurrent for any $\alpha > 0$.

In [6] Assumption 2.2, Guo and Hernandez–Lerma formulate the following sufficient criterion for non-explosiveness:

there exist a function $w : \boldsymbol{S} \to [1, \infty)$ and constants $c_0 \neq 0$, $b_0 \geq 0$ and $L_0 > 0$ such that for $x \in \boldsymbol{S}$

$$\left.\begin{aligned} Qw(x) &\leq c_0 w(x) + b_0 \\ q_x &\leq L_0 w(x). \end{aligned}\right\} \tag{3.4}$$

In their paper [11] ("*Discussion about condition 3.1*"), Pionovskiy and Zhang show that Eq. (3.4) implies Eq. (3.1) for the function

$$V(x) = \frac{L_0}{\alpha - c_0}\Big(w(x) + \frac{b_0}{\alpha}\Big), \quad x \in \boldsymbol{S},$$

for any $\alpha > c_0$. The imposed requirement $w(x) \geq 1$ is not used in this derivation, and is therefore redundant. This is easily checked. In particular, it implies positive recurrence of the α-jump chain for any $\alpha > c_0$. In view of Example 3.4, the existence of a function w satisfying Eq. (3.4) is therefore sufficient but not necessary for non-explosiveness.

The following example shows that the condition in Eq. (3.4) is stronger than Eq. (3.1).

Example 3.5: Let X be a minimal Markov process on $\{0, \ldots\}$, with q-matrix given by

$$q_{xy} = \begin{cases} p2^x, & y = x+1, \\ (1-p)2^x & y = x-1, x \neq 0, \\ -2^x, & y = x > 0, \\ -p, & y = x = 0, \end{cases}$$

for some p with $(3 - \sqrt{5})/2 < p < 1/2$.

Let $V : \boldsymbol{S} \to \mathbf{R}_+$ be given by $V(x) = \beta^x$, with $\beta > 1$. We claim that

- there exists β such that Eq. (3.1) is satisfied for the function $V(x) = \beta^x$, $x \geq 0$; and
- there is no function w satisfying Eq. (3.4).

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Clearly, V is a moment function with $QV \le c \cdot V$, with $c = \max\{(p\beta + (1-p)/\beta - 1), p\beta - p\}$. In particular, for $x \ge 1$

$$QV(x) + q_x = (p\beta + (1-p)/\beta - 1)(2\beta)^x + 2^x = (2\beta)^x \left(p\beta + \frac{1-p}{\beta} - 1 + \beta^{-x} \right) \le \alpha \beta^x,$$

for some constant α, provided

$$p\beta + \frac{1-p}{\beta} - 1 + \beta^{-x} \le 0,$$

for x large enough. This means that we should choose $\beta > 1$, such that

$$f(\beta) = p\beta + \frac{1-p}{\beta} - 1 < 0.$$

The zeroes of the polynomial $p\beta^2 - \beta + 1 - p$ are 1 and $(1-p)/p$. This yields that

$$f(\beta) \begin{cases} < 0 & 1 < \beta < (1-p)/p, \\ > 0 & \beta > (1-p)/p. \end{cases}$$

The assumption $(3-\sqrt{5})/2 < p < 1/2$ implies that $(1-p)/p \in (1,2)$. Hence, choosing $\beta \in (1,(1-p)/p)$, gives $\beta < 2$ and so $q_x > V(x)$, for all x. This proves the first assertion.

We next prove the second assertion that there is no function $w : \boldsymbol{S} \to \mathbf{R}_+$ satisfying Eq. (3.4). Suppose that not. In other words, there exists a function $w : \boldsymbol{S} \to \mathbf{R}_+$, with $w(x) \ge 2^x$, and $Qw(x) \le c'w(x) + d'$. Since w is a moment function, this implies the existence of a constant c_0 such that $Qw \le c_0 w$. We distinguish three cases.

(a) There exists an increasing sequence $\{x_n\}_n$, with $x_1 \ge 1$, and $x_n \to \infty$, $n \to \infty$, such that $w(x_n)/w(x_n - 1) \le 2 \le w(x_n + 1)/w(x_n)$. Then

$$\frac{c_0}{2^{x_n}} \ge \frac{Qw(x_n)}{2^{x_n} w(x_n)} = p\frac{w(x_n+1)}{w(x_n)} - 1 + (1-p)\frac{w(x_n-1)}{w(x_n)} \ge 2p - 1 + \frac{1-p}{2} = f(2),$$

for all n. By the assumption $f(2) > 0$, so that the right-hand side of the above equation is strictly positive for all n. However, the left-hand side converges to 0, as $n \to \infty$. A contradiction.

(b) $w(x+1)/w(x) \ge 2$ for $x \ge x_0$, for some $x_0 \in \boldsymbol{S}$.
Put $\theta_x = w(x+1)/(2w(x))$. Then $\theta_x \ge 1$ for $x \ge x_0$. This implies

$$1 + \frac{c_0}{2^x} \ge 2p\theta_x + \frac{1-p}{2\theta_{x-1}}, \quad x \ge x_0. \tag{3.5}$$

For any $\epsilon > 0$ there exists $x(\epsilon) \ge x_0$ such that $c_0/2^x \le \epsilon$ for $x \ge x(\epsilon)$. We will later choose ϵ appropriately.
By Eq. (3.5) $\theta_x \le (1+\epsilon)/(2p)$ for $x > x(\epsilon)$. Hence, $\theta_{x-1} \le (1+\epsilon)/(2p)$ for $x - 1 > x(\epsilon)$. Combining $\theta_x \ge 1$, it follows that

$$(1+\epsilon)^2 \ge 3p - p^2 + 2p\epsilon, \tag{3.6}$$

for $x - 1 > x(\epsilon)$. We have chosen p such that $3p - p^2 > 1$. Consequently, for ϵ sufficiently small Eq. (3.6) is violated.

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(c) $w(x)/w(x-1) \le 2$ for $x \ge x_0$, for some $x_0 \in \boldsymbol{S}$.
Put $\theta_x = 1 - w(x)/(2w(x-1))$ for $x \ge x_0$. Then $w(x)/w(x_0) = 2^{x-x_0} \prod_{y=x_0+1}^{x}(1-\theta_y)$. Since $w(x) \ge 2^x$, it follows that $\prod_{y=x_0+1}^{x}(1-\theta_y) \ge 2^{x_0}/w(x_0)$, $x \ge x_0$. Since $\prod_{y=x_0+1}^{x}(1-\theta_y)$ is a non-increasing, non-negative function of $x > x_0$, $\prod_{y \ge x_0+1}(1-\theta_y)$ must be convergent with limit greater or equal than $2^{x_0}/w(x_0) > 0$. This implies that $\theta_x \to 0$ as $x \to \infty$.
We now get for $x > x_0$

$$\frac{c_0}{2^x} \ge \frac{Qw(x_n)}{2^x w(x_n)} = 2p(1-\theta_{x+1}) - 1 + \frac{1-p}{2(1-\theta_{x-1})}.$$

The left-hand side converges to 0, whereas the right-hand side converges to $2p - 1 + (1-p)/2 = f(2) > 0$, as $x \to \infty$. A contradiction.

4. V-TRANSFORMATIONS AND KOLMOGOROV FORWARD INTEGRAL EQUATION

We will first recall some relevant properties of a Markov process satisfying Assumption 1.1. In the first place, Anderson [1] Proposition 1.2.4 shows that $t \mapsto p_{t,xy}$ is continuously differentiable on $[0,\infty)$, and $x, y \in \boldsymbol{S}$. Moreover, $t \mapsto \sum_y p_{t,xy}$ is non-increasing and by virtue of Proposition 1.2.6 in the same reference it is continuously differentiable on $[0,\infty)$ for each $x \in \boldsymbol{S}$.

Since we are also interested in reward processes, the question arises whether $t \mapsto P_t V(x) = \sum_y p_{t,xy} V(y)$ is continuous, provided of course that $V : \boldsymbol{S} \to \mathbf{R}$ is integrable w.r.t. the kernel P_t, $t \ge 0$. This is true under the assumption that V is a c-drift function, for some constant $c \in \mathbf{R}$. Indeed, by virtue of Anderson [1] Proposition 2.2.13

$$P_t V \le e^{ct} V, \quad t \ge 0, \tag{4.1}$$

since X is the minimal process, if V is a c-drift function. We now invoke a construction from Spieksma [15], see also Anderson [1] Lemma 5.4.2. With X we associate the minimal Markov process X^V with q-matrix Q^V, given by

$$q^V_{xy} = \begin{cases} q_{xy} V(y)/V(x), & x \ne y; x, y \ne \Delta, \\ q_{xx} - c, & x = y; x, y \ne \Delta, \\ c - \sum_{y \in S} q_{xy} V(y)/V(x), & y = \Delta; x \ne \Delta, \\ 0, & y \in \boldsymbol{S}_\Delta; x = \Delta, \end{cases}$$

with Δ an extra state to which all "excessive" rate created by the transformation is directed. This makes Q^V a conservative q-matrix, with Δ an absorbing state. X^V is called a V-transformation of X (it is not unique, since for each larger constant than c we can define such transformation as well).

We will work with the extended state space $\boldsymbol{S}_\Delta$ and denote by $\{P^V_t\}_t$ again the (minimum) transition function on the enlarged state space $\boldsymbol{S} \cup \{\Delta\}$.

As has been pointed out above, $t \mapsto p^V_{t,xy}$, $t \mapsto \sum_{y \in S_\Delta} p^V_{t,xy}$ are continuously differentiable on $[0,\infty)$, for $x, y \in \boldsymbol{S}_\Delta$. Note that (Anderson [1], Lemma 5.4.2)

$$p^V_{t,xy} = e^{-ct}\, p_{t,xy} \frac{V(y)}{V(x)}, \quad x, y \in \boldsymbol{S}, \tag{4.2}$$

which actually sharpens Eq. (4.1). As a consequence of this transformation we can transfer results for a Markov process to its expectation with respect to V. It then follows that

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$t \mapsto P_t V(x)$ is continuously differentiable on $[0,\infty)$, for each $x \in \boldsymbol{S}$. As a side-remark, in [15] I wrote that continuity of $t \mapsto P_t V(x)$ follows from Eq. (4.1) by virtue of the dominated convergence theorem. This, unfortunately, seems to imply only right-continuity.

By the dominated convergence theorem $t \mapsto P_t f$ is continuous for any function $f : \boldsymbol{S} \to \mathbf{R}$ with $\sup_x |f(x)|/V(x) < \infty$. This fact calls for resorting to Banach space terminology here. Introduce the norm $\|\cdot\|_V$ of functions $f : \boldsymbol{S} \to \mathbf{R}$ by

$$\|f\|_V = \sup_{x \in S} \frac{|f(x)|}{V(x)}.$$

Then

$$\ell^\infty(\boldsymbol{S}, V) = \{f : \boldsymbol{S} \to \mathbf{R} \,|\, \|f\|_V < \infty\}$$

is a Banach space. The validity of Eq. (4.1) under Assumption 1.1 for a c-drift function V implies that $P_t : \ell^\infty(\boldsymbol{S}, V) \to \ell^\infty(\boldsymbol{S}, V)$ is a bounded linear operator with respect to this norm.

The above discussion now leads to the following conclusion: *if V is a c-drift function, then $t \mapsto P_t f$ is componentwise continuously differentiable for each $f \in \ell^\infty(\boldsymbol{S}, V)$.*

In [15] I have studied non-explosiveness properties in connection to the problem for what functions $f : \boldsymbol{S} \to \mathbf{R}$ the Kolmogorov forward integral equation holds:

$$P_t f(x) = f(x) + \int_0^t P_s(Qf)(x) ds = f(x) + \mathsf{E}_x \int_0^t Qf(X_s) ds, \quad x \in \boldsymbol{S}. \tag{4.3}$$

In case of f being a c-drift function for some constant c, this equation is holds if and only if the f-transformation X^f is non-explosive (cf. Theorem 3.2 of Spieksma [15]), provided Assumption 1.1 holds.

Using Theorem 2.1 X^f is non-explosive if and only the f-transformed q-matrix Q^f has no bounded, non-zero eigenvector to any positive value $\alpha > c$. In [15] Spieksma shows that $g \in \ell^\infty(\boldsymbol{S}, f)$ is an eigenvector of Q to eigenvalue $\alpha > 0$ if and only if $h : \boldsymbol{S}_\Delta \to \mathbf{R}$ defined by $h(x) = g(x)/f(x)$, $x \in \boldsymbol{S}$, $h(\Delta) = 0$, is a bounded eigenvector of the f-transformed q-matrix Q^f, for $\alpha - c > 0$. Since Δ is an absorbing state for the f-transformation, any eigenvector to Q^f must have value 0 at Δ.

Hence, *X^f non-explosive is equivalent to the non-existence of eigenvectors of Q to any eigenvalue $\alpha > c$, in the space $\ell^\infty(\boldsymbol{S}, f)$.*

Theorem 2.1 provides an alternative characterization of non-explosiveness of X^f, namely the existence of a d-drift moment function W for X^f, for some constant d. We may transform the function W to a function $F : \boldsymbol{S} \to \mathbf{R}$, given by $F(x) = W(x) f(x)$, $x \in \boldsymbol{S}$, which will be called a f-moment function if W is a moment function for X^f.

Definition 4.1: *The function $F : \boldsymbol{S} \to \mathbf{R}_+$ is called an f-moment function, if there exists an increasing sequence $\{K_n\}_n$ of finite sets, $K_n \uparrow \boldsymbol{S}$, such that $\inf_{x \notin K_n} F(x)/f(x) \to \infty$ as $n \to \infty$.*

In view of Theorem 2.1 we may rephrase Theorems 3.2 and 3.4 of Spieksma [15].

Corollary 4.2 to Theorem 2.1 and Spieksma [15] Theorem 3.2: *Suppose that Assumption 1.1 holds true and that V is a c-drift function for some constant c. Then V satisfies Eq. (4.3) if and only if if there exists a d-drift V-moment function W, for some constant d.*

The following theorem specifies a class of functions that satisfy Eq. (4.3).

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THEOREM 4.3 Spieksma [15] Theorem 3.4.: *Let X satisfy Assumption 1.1. Suppose that V is a c-drift function for X and that $W : \boldsymbol{S} \to \mathbf{R}_+$ is a d-drift V-moment function. Then V satisfies Eq. (4.3).*
Additionally, $f \in \ell^\infty(\boldsymbol{S}, V)$ satisfies Eq. (4.3) provided that $\mathsf{E}_x \int_0^t |Qf(X_s)|ds < \infty$, $x \in \boldsymbol{S}$.

Interestingly enough, posing the additional condition that the α-jump chain $X^{V,\alpha}$ is *positive recurrent* for some $\alpha > 0$, all functions $f \in \ell^\infty(\boldsymbol{S}, V)$ satisfy the Kolmogorov integral equation Eq. (4.3). In the formulation of this result we use the drift condition characterization of positive recurrence of α-jump chains from the previous section applied to the V-transformation, but translate it to a drift condition for the original process X.

LEMMA 4.4: *Let X satisfy Assumption 1.1. Let V be a c-drift function for X. Then the following two conditions are equivalent.*

(i) *There exists a function $W : \boldsymbol{S} \to \mathbf{R}_+$ such that*

$$QW(x) + q_x V(x) \leq \alpha W(x), \quad x \in \boldsymbol{S}, \tag{4.4}$$

for some $\alpha > 0$.

(ii) *The α-jump chain $X^{V,\alpha}$ associated with the V-transformation is positive recurrent for some $\alpha > 0$.*

PROOF: First assume (i) the existence of a positive function W satisfying Eq. (4.4). Form the V-transformation and put $W(\Delta) = 0$. Then X^V satisfies Eq. (3.1) for the function $F : \boldsymbol{S}_\Delta \to \mathbf{R}_+$ given by $F(x) = W(x)/V(x)$, $x \in \boldsymbol{S}$, and $F(\Delta) = 0$, and constant $\alpha - c$. The equation for state Δ has 0 on both sides of the equation. Statement (ii) follows from Lemma 3.1.

Next assume (ii) that $X^{V,\alpha}$ is positive recurrent. By the same Lemma there exists a non-negative function F for X^V satisfying Eq. (3.1). Then X satisfies Eq. (3.1) for the function $W(x) = F(x)V(x)$ and constant $\alpha + c$. ■

THEOREM 4.5: *Let the assumptions and conditions of Lemma 4.4 be satisfied. Then $\mathsf{E}_x \int_0^t |Qf(X_s)|ds < \infty$, $x \in \boldsymbol{S}$, for all $f \in \ell^\infty(\boldsymbol{S}, V)$, and so Eq. (4.3) applies for all $f \in \ell^\infty(\boldsymbol{S}, V)$.*

PROOF: We will first consider the V-transformation X^V on the space $\boldsymbol{S}_\Delta$. By the previous Lemma and its proof $X^{V,\alpha}$ is positive recurrent. Let us denote the exit state by δ. By Eq. (3.2)

$$\sum_{y \in S_\Delta} R^V_{xy}(\alpha) q^V_y = \sum_{y \in S} R^V_{xy}(\alpha)(q_y + c) < \infty, \quad x \in \boldsymbol{S}_\Delta.$$

Noting the definition of the resolvent $R^V(\alpha)$ and using that c may be assumed positive, we get

$$\int_0^T \sum_{y \in S} p^V_{t,xy}(q_y + c)\, dt \leq e^{\alpha T} \int_0^T e^{-\alpha t} \sum_{y \in S} p^V_{t,xy}(q_y + c)\, dt \leq \sum_{y \in S} R^V_{xy}(\alpha)(q_y + c) < \infty,$$
$$x \in \boldsymbol{S}. \tag{4.5}$$

Since $q_y + c \geq \sum_{z \in S, z \neq y} q^V_{yz}$, this implies

$$\int_0^T \sum_{y \in S} \sum_{z \in S, z \neq y} p^V_{t,xy} q^V_{yz} dt < \infty, \quad x \in \boldsymbol{S}. \tag{4.6}$$

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Using Eq. (4.2), notice that for $t \leq T$

$$p_{t,xy} V(y) q_y \leq V(x) e^{cT} \, p^V_{t,xy} q_y.$$

Hence, also

$$p_{t,xy} \sum_{z \neq y} q_{yz} V(z) \leq V(x) e^{cT} \, p^V_{t,xy} \sum_{z \neq y, z \in S} q^V_{yz}.$$

Combining with Eq. (4.5) and (4.6) yields for $x \in \boldsymbol{S}$

$$\int_0^T \sum_{y,z} p_{t,xy} \sum_z |q_{yz}| V(y) \, dt = \int_0^T P_t(|Q|V)(x) \, dt < \infty.$$

This implies for any function $f \in \ell^\infty(\boldsymbol{S}, V)$ that

$$\int_0^T P_t |Qf|(x) \leq \int_0^T P_t(|Q|f)(x) \, dt < \infty,$$

for $x \in \boldsymbol{S}$. The results follows. ■

The problem of the validity of the Kolmogorov forward integral relation has been addressed in the context of the optimality equation of average optimality in Markov decision processes, see for example, Guo et al. [7], as well as the recent paper [13] and the book [12] by Hernandez–Lerma and Prieto–Rumeau. These works impose the following conditions for functions in order to satisfy Eq. (4.3), when restricting to the case of one Markov process; cf. Guo et al. [7], Assumptions A, C(3), and Hernandez–Lerma and Prieto–Rumeau [13] Assumption 3.2 and [12] Assumptions 2.1 and 2.2.

Let Assumption 1.1 be satisfied, and let $V : \boldsymbol{S} \to [1, \infty)$ be a c-drift moment function. Further assume that there exists a function $W' : \boldsymbol{S} \to \mathbf{R}_+$, and constants b', c' and M' such that for $x \in \boldsymbol{S}$

$$\left.\begin{aligned} q_x V(x) &\leq M' W'(x) \\ QW'(x) &\leq c' W'(x) + b'. \end{aligned}\right\} \tag{4.7}$$

Then the Kolmogorov forward integral equation Eq. (4.3) holds for any function $f \in \ell^\infty(\boldsymbol{S}, V)$.

Note that we have slightly adapted the formulation of the conditions from these works. By the way, to our knowledge the proof of this statement has not been provided explicitly. The condition of V being a *moment* drift function is not necessary, as we have seen. Moreover, Eq. (4.7) implies that Eq. (3.4) is satisfied by the V-transformation, for some function w, and constants c_0, L_0 and b_0. As we have shown in Example 3.5, this is a stronger requirement than assuming the V-transformation to satisfy Eq. (3.1) for some function w. The latter implies Eq. (4.4) to hold for the function $w \cdot V$. Hence, the above condition implies, and is stronger than, the conditions assumed in Theorem 4.5.

A second comment concerns a general remark on the validity of the Kolmogorov forward integral equation. In Guo and Hernandez–Lerma [6] Appendix C3 the validity is asserted for functions $f : \boldsymbol{S} \to \mathbf{R}$ with $\mathsf{E}_x |f|(X_t) < \infty$ and $\mathsf{E}_x \int_0^t |Qf(X_s)| ds < \infty$ for all $t \geq 0$ and $x \in \boldsymbol{S}$. As we have seen, this is not true for c-drift functions f, for which X^f is explosive. An example has been provided by Spieksma in [15].

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Brémaud makes a similar statement in [3] Theorem 9.2.2. There the Markov process itself is required to be non-explosive, which is not necessary. The theorem in that book states that the condition

$$\mathsf{E}_x \int_0^t q_{X_s} |f|(X(s))ds < \infty$$

is equivalent to $\mathsf{E}_x |f|(X_t) < \infty$. The above mentioned example shows that the second property does not imply the first. If it would imply it, then the Kolmogorov forward integral equation would hold, but we know that it does not.

Clearly the study of countable state Markov processes has some pitfalls.

Acknowledgements

The author would like to thank the referees, Dr. A.B. Piunovskiy and Dr. Y. Zhang for their careful reading of the paper, comment and their pointing out some errors.

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