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Two-Grid Method for Miscible Displacement Problem by Mixed Finite Element Methods and Mixed Finite Element Method of Characteristics

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Abstract. The miscible displacement of one incompressible fluid by another in a porous medium is governed by a system of two equations. One is elliptic form equation for the pressure and the other is parabolic form equation for the concentration of one of the fluids. Since only the velocity and not the pressure appears explicitly in the concentration equation, we use a mixed finite element method for the approximation of the pressure equation and mixed finite element method with characteristics for the concentration equation. To linearize the mixed-method equations, we use a two-grid algorithm based on the Newton iteration method for this full discrete scheme problems. First, we solve the original nonlinear equations on the coarse grid, then, we solve the linearized problem on the fine grid used Newton iteration once. It is shown that the coarse grid can be much coarser than the fine grid and achieve asymptotically optimal approximation as long as the mesh sizes satisfy $h = H^2$ in this paper. Finally, numerical experiment indicates that two-grid algorithm is very effective.

AMS subject classifications: 35M13, 65M12

Key words: Two-grid method, miscible displacement problem, mixed finite element, characteristic finite element method.

1 Introduction

We consider the miscible displacement of one incompressible fluid by another in a reservoir $\Omega \subset \mathbb{R}^2$ of unit thickness. The nonlinear coupled system of equations that describes

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the pressure $p(x,t)$ and the concentration $c(x,t)$ of one of the fluids is given by

$$\phi\frac{\partial c}{\partial t}+\boldsymbol{u}\cdot\nabla c-\nabla\cdot(D\nabla c)=f(c), \tag{1.1}$$

$$\nabla\cdot\boldsymbol{u}=q, \tag{1.2}$$

$$\boldsymbol{u}=-a(c)\nabla p, \tag{1.3}$$

where $x\in\Omega$, $t\in J=[0,T]$ and $a(c)=a(x,c)=\frac{k(x)}{\mu(c)}$. $k(x)$ is the permeability of the porous rock, $\mu(c)$ is the viscosity of the fluid mixture, $\boldsymbol{u}(x,t)$ is the Darcy velocity of the mixture, $q(x,t)$ represents the flow rate at wells and q^+ is the positive part of the q. $f(c)$ may be nonlinear function [17]. $c_0(x)$ is the initial concentration, $\phi(x)$ is the porosity of the rock, and $D(\boldsymbol{u})$ is the coefficient of molecular diffusion and mechanical dispersion of one fluid into the other and it is the 2×2 matrix,

$$D=\phi[d_mI+|\boldsymbol{u}|(d_lE(\boldsymbol{u})+d_tE^{\perp}(\boldsymbol{u}))],$$

where $E(\boldsymbol{u})=\frac{u_iu_j}{|\boldsymbol{u}|^2}$ and $E^{\perp}=I-E$, d_m is the molecular diffusion, d_l and d_t are, respectively, the longitudinal and transverse dispersion coefficients. For convenience, we assume that $D=\phi d_mI$ implies only the molecular diffusion and not the dispersion in this paper.

The system is subjected to boundary conditions:

$$\boldsymbol{u}\cdot\boldsymbol{n}=0,\quad x\in\partial\Omega,\quad t\in J, \tag{1.4}$$

$$D\nabla c\cdot\boldsymbol{n}=0,\quad x\in\partial\Omega,\quad t\in J, \tag{1.5}$$

and an initial condition

$$c(x,0)=c_0(x),\quad x\in\Omega. \tag{1.6}$$

Two-phase flow and transportation of fluids in porous media play an important role in both theoretic and applicative aspects. Petroleum engineers have been interested in efficient oil exploitation and the hydrologists have been concerned about improving the utilization of groundwater resource for a long time. Chen and Ewing [7] have studied the mathematic theory related to these models.

In the past thirty years, numerical approximations have received a great deal of attention for the miscible displacement of one incompressible fluid by another in a reservoir [10–12,14–16,23,24]. Ewing and Wheeler [10] use Galerkin methods for both continuous time and full discrete approximation. Douglas and his coauthors proposed a finite element method based on the use of an elliptic mixed finite element method to approximate the pressure p and the velocity $\boldsymbol{u}$ and a parabolic Galerkin method to approximate the concentration [16]. It is not surprising that a mixed method [18,19], Since only the velocity and not the pressure appears explicitly in Eq. (1.1) for the concentration.

In realistic displacements D is quite small for the concentration equation, so that (1.1) for c is strongly convection-dominated. Standard Galerkin methods produce unacceptable nonphysical and oscillations in the concentration approximations, thus, Douglas

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and his coauthors introduced and analyzed characteristic method for a single convection dominated diffusion equation [20]. This numerical techniques have been introduced to obtain better approximations for (1.1), such as characteristic finite difference method [24], characteristic finite element method [23], the modified method of characteristic finite element method [8,12,14,23]. Mixed finite element method has been proven to be an effective numerical method for solving fluid problems. It has an advantage of approximating the unknown variable and its diffusive flux simultaneously. In this paper, we shall approximate c by mixed finite method of characteristics for the concentration equation.

In this paper, we try to consider an effective algorithm for this essential system. As we known, Xu proposed two-grid algorithm for nonlinear or nonsymmetric and coupled system [13,21,22]. It is a simple but effective algorithm that have been applied to many different kinds of problems, such as Dawson [9], Chen ([3–5]) make a nice work of two-grid method for quasilinear reaction diffusion equations, Zhou and his coauthors presented an efficient two-grid scheme for the Cahn-Hilliard equation in [26]. So, it is a nature idea to use two- grid scheme here for our model problem (1.1)-(1.3). For simplicity, we shall consider the situation where D only related to the molecular diffusion is linear and $f(c)$ is nonlinear function in our paper. We use the mixed finite element method of characteristics for the concentration equation and mixed finite element scheme for the pressure-velocity equations. We first estimate the mixed finite element and the mixed finite element method of characteristics solution in the sense of L^q norm. Then, we present our main algorithm-the two grid methods. There are lots of literatures concern about the miscible displacement problem by different treatments, but, to the best of our knowledge, there are few results about two-grid algorithm is used to cope with such a nonlinear coupled problem. The main idea of our algorithm is to solve a nonlinear coupled system in the very coarse grid and then solve the decoupled linear system on the fine grid. We will prove that under certain assumptions, we can achieve the same accuracy as the finite element method but with much less cost time since we just have to solve a small scale nonlinear problem.

The remainder of the article is organized as follows. In section 2, we introduce a characteristic method for the concentration. In Section 3, we present the weak formulas of our model. Section 4 will make a simple analysis of the finite element solutions of the model. Our main algorithm and its convergence analysis will be advocated in Section 5. In Section 6, the numerical experiment of two gird algorithm is presented in this paper.

2 A characteristic method for the concentration

For the concentration equation (1.1), convection essentially dominates diffusion, and it is natural to seek numerical methods for such problems that reflect their almost hyperbolic nature. We shall consider combining the method of characteristics with finite element to treat Eq. (1.1), then we shall indicate a number of extensions and applications of our concepts.

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Let

$$\psi(x)=[\boldsymbol{u}^2(x)+\phi^2(x)]^{\frac{1}{2}}$$

and let the characteristic direction associated with the operator $\phi\frac{\partial c}{\partial t}+\boldsymbol{u}\nabla c$ be denoted by τ, where

$$\frac{\partial c}{\partial \tau}=\frac{\phi(x)}{\psi(x)}\frac{\partial c}{\partial t}+\frac{\boldsymbol{u}(x)}{\psi(x)}\nabla c.$$

Then (1.1) can be rewritten in the form

$$\psi\frac{\partial c}{\partial \tau}-\nabla\cdot(D\nabla c)=f(c). \tag{2.1}$$

Note that (2.1) has the form of the heat equation, so that its numerical approximations should be better behaved than those of (1.1), if a reasonable treatment of the 'time' derivative $\frac{\partial c}{\partial \tau}$ can be found.

In each of the procedures to be treated below we shall consider a time step $\Delta t>0$ and approximate the solution at times $t^n=n\Delta t$. The characteristic derivative will be approximated basically in the following manner:

Let

$$\bar{x}=x-\frac{\boldsymbol{u}(x)}{\phi(x)}\Delta t, \tag{2.2}$$

and note that

$$\psi\frac{\partial c}{\partial \tau}\approx\phi\frac{c(x,t^n)-c(\bar{x},t^{n-1})}{\Delta t}=\phi\frac{c^n-\bar{c}^{n-1}}{\Delta t}. \tag{2.3}$$

For simplicity, we assume that

$$\partial_\tau c^n=\frac{c^n-\bar{c}^{n-1}}{\Delta t}.$$

Throughout this paper we assume that the coefficients $\phi(x)$ and $\boldsymbol{u}$ are bounded, that ϕ is bounded below by a positive constant, and that

$$\left|\frac{\boldsymbol{u}(x)}{\phi(x)}\right|+\left|\frac{d}{dx}\left(\frac{\boldsymbol{u}(x)}{\phi(x)}\right)\right|\le K. \tag{2.4}$$

In particular, this implies that $|x-\bar{x}|\le K\Delta t$.

3 Weak formulation of the problem and preliminaries

We assume that the function $a(c)$ and $\phi(x)$ are bounded above and below by positive constants

$$a_*\le a(c)\le a^*,$$
$$\phi_*\le\phi(c)\le\phi^*.$$

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The matrix D is uniformly positive definite, that is:

$$\sum_{i,j=1}^{2} D_{i,j}\xi_i\xi_j \geq a_0|\xi|^2, \quad \forall \xi \in \mathbb{R}^2.$$

We also assume there exists a positive constant C such that:

$$\|c_t\|, \|c_{tt}\| \leq C,$$

and $\alpha(c)=a^{-1}(c)$ is twice continuously differentiable with bounded derivative up to the second order.

Assume

$$\|q\|_{L^\infty} \leq C_q, \quad \|\alpha_c\|_{L^\infty} \leq C_\alpha,$$

and

$$\|c\|_{L^\infty(J;H^{l+1})}, \|\mathbf{u}\|_{L^\infty(J;L^\infty)} \leq C.$$

Now, we state some standard notations which will be used in this article. We denote $W^{m,p}(\Omega)$ the Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|\phi\|_{m,p}=\sum_{|\alpha|\leq m}\|D^\alpha\phi\|_{L^p}^p$. We set $H^m(\Omega)=W^{m,2}(\Omega)$, $\|\cdot\|_m=\|\cdot\|_{m,2}$, $\|\cdot\|_\infty=\|\cdot\|_{L^\infty}$ and $\|\cdot\|=\|\cdot\|_{0,2}$. In addition, C denote a general positive constant independent of h.

Let $H(div;\Omega)$ be the set of vector functions $\boldsymbol{v}\in(L^2(\Omega))^2$, such that $\nabla\cdot\boldsymbol{v}\in L^2(\Omega)$ and let

$$\begin{aligned}
&\boldsymbol{V}=H(div;\Omega)\cap\{\boldsymbol{v}\cdot\nu=0 \text{ on } \partial\Omega\},\\
&W=\{w\in L^2(\Omega),(w,1)=0\},\\
&M=\{\varphi\in L^2(\Omega)\},
\end{aligned}$$

equipped with the norm given by:

$$\|\boldsymbol{v}\|_V=\|\boldsymbol{v}\|_{H(div;\Omega)}=(\|\boldsymbol{v}\|^2+\|\nabla\cdot\boldsymbol{v}\|^2)^{\frac{1}{2}},$$

and $\|w\|$. It is clear $\nabla\cdot\boldsymbol{V}=W$ in [25].

Take $\boldsymbol{z}=D\nabla c$. The weak form of (2.1) is equal to the problem of finding a map $(c,\boldsymbol{z})$: $J\to L^2(\Omega)\times\boldsymbol{V}$ such that

$$\left(\psi\frac{\partial c}{\partial\tau},\varphi\right)-(\nabla\cdot\boldsymbol{z},\varphi)=(f(c),\varphi), \quad \forall\varphi\in L^2(\Omega), \tag{3.1}$$

$$(\kappa\boldsymbol{z},\chi)+(c,\nabla\cdot\chi)=0, \quad \forall\chi\in\boldsymbol{V}, \tag{3.2}$$

$$c(x,0)=c_0(x), \quad \boldsymbol{z}(x,0)=D\nabla c(x,0), \quad \forall x\in\Omega, \tag{3.3}$$

where $\kappa=D^{-1}$.

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Then a weak formulation of (3.1)-(3.2) and (1.2)-(1.3) are given by the determination of a map $\{c,\boldsymbol{z},\boldsymbol{u},p\}: J\to L^2(\Omega)\times\boldsymbol{V}\times\boldsymbol{V}\times W$ such that $c(x,0)=c_0$ and for $t\in J$:

$$\left(\psi\frac{\partial c}{\partial\tau},\varphi\right)-(\nabla\cdot\boldsymbol{z},\varphi)=(f(c),\varphi),\quad\forall\varphi\in L^2(\Omega), \tag{3.4}$$

$$(\kappa\boldsymbol{z},\chi)+(c,\nabla\cdot\chi)=0,\quad\forall\chi\in\boldsymbol{V}, \tag{3.5}$$

$$(\nabla\cdot\boldsymbol{u},w)=(q,w),\quad\forall w\in W, \tag{3.6}$$

$$(\alpha(c)\boldsymbol{u},\boldsymbol{v})-(p,\nabla\cdot\boldsymbol{v})=0,\quad\forall\boldsymbol{v}\in\boldsymbol{V}. \tag{3.7}$$

where $\alpha(c)=a(c)^{-1}$.

Let $\mathcal{T}_{h_c}$, $\mathcal{T}_{h_p}$ be a family of regularity finite element partitions of Ω. For a integer $l\geq 1$, let a Rarviart-Thomas mixed finite element space $M_h\times\boldsymbol{V_h}\subset L^2(\Omega)\times\boldsymbol{V}$ of index $l\geq 0$ over the partition $\mathcal{T}_{h_c}$. and that

$$\inf_{\varphi_h\in M_h}\|\varphi-\varphi_h\|_M\leq C\|\varphi\|_{H^{l+1}(\Omega)}h_c^{l+1}, \tag{3.8}$$

$$\inf_{z_h\in V_h}\|\boldsymbol{z}-\boldsymbol{z}_h\|_V\leq C(\|\boldsymbol{z}\|_{H^{l+1}(\Omega)^2}+\|\nabla\cdot\boldsymbol{z}\|_{H^{l+1}(\Omega)})h_c^{l+1}, \tag{3.9}$$

for $\varphi\in H^{l+1}(\Omega)$. This implies approximation in $L^2(\Omega)$ of order $\mathcal{O}(h_c^{l+1})$.

For approximating the pressure and Darcy velocity equations (3.6)-(3.7), we choose a Rarviart-Thomas mixed finite element space $\boldsymbol{V}_h\times W_h\subset\boldsymbol{V}\times W$ of index $k\geq 0$ over the partition $\mathcal{T}_{h_p}$. Thus we preserve the relation $\nabla\cdot\boldsymbol{V}_h=W_h$ and

$$\inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\|\boldsymbol{v}-\boldsymbol{v}_h\|_{(L^2(\Omega))^2}\leq C\|\boldsymbol{v}\|_{H^{k+1}(\Omega)^2}h_p^{k+1}, \tag{3.10}$$

$$\inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\|\boldsymbol{v}-\boldsymbol{v}_h\|_V\leq C(\|\boldsymbol{v}\|_{H^{k+1}(\Omega)^2}+\|\nabla\cdot\boldsymbol{v}\|_{H^{k+1}(\Omega)})h_p^{k+1}, \tag{3.11}$$

$$\inf_{w_h\in W_h}\|w-w_h\|_W\leq C\|w\|_{H^{k+1}(\Omega)^2}h_p^{k+1}, \tag{3.12}$$

whenever the norms on the right hand side are finite.

Let $\Delta t=T/N$ and $t^n=n\Delta t$. The full discrete procedure of (3.4)-(3.7) at $t=t^n$ is given by the finding of $\{c_h^n,\boldsymbol{z}_h^n,\boldsymbol{u}_h^n,p_h^n\}$ such that $c_h(x,0)=c_h^0,(\phi c_h^0,\varphi)=(\phi c^0,\varphi)$.

$$(\phi\partial_\tau c_h^n,\varphi)-(\nabla\cdot\boldsymbol{z}_h^n,\varphi)=(f(c_h^n),\varphi),\quad\forall\varphi\in M_h, \tag{3.13}$$

$$(\kappa\boldsymbol{z}_h^n,\chi)+(c_h^n,\nabla\cdot\chi)=0,\quad\forall\chi\in\boldsymbol{V}_{h_c}, \tag{3.14}$$

$$(\nabla\cdot\boldsymbol{u}_h^n,w)=(q,w),\quad\forall w\in W_h, \tag{3.15}$$

$$(\alpha(c_h^n)\boldsymbol{u}_h^n,\boldsymbol{v})-(p_h^n,\nabla\cdot\boldsymbol{v})=0,\quad\forall\boldsymbol{v}\in\boldsymbol{V}_{h_p}, \tag{3.16}$$

where $\partial_\tau c_h^n=\frac{c_h^n-\bar{c}_h^{n-1}}{\Delta t}$.

In this paper, we assume that function α be twice continuously differentiable with bounded derivatives through the second order and α, κ are square-integrable, symmetric,

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uniformly positive definite variables defined on Ω. Further, for integer $l,k\geq 0$, we assume the solution function $c,\boldsymbol{z},\boldsymbol{u},p$ have the following regularity

$$c\in L^\infty(J,W^{l+1,\infty});\quad \boldsymbol{z}\in(L^\infty(J,W^{k+1,\infty}))^2;\quad \boldsymbol{u}\in(L^\infty(J,W^{k+1,\infty}))^2;\quad p\in L^\infty(J,W^{k+2,\infty}).$$

Let Q_h denote the L^2 projection defined by: for any $\varphi\in L^2(\Omega)$ and $\psi\in(L^2(\Omega))^2$, we have

$$(\varphi,w)=(Q_h\varphi,w),\quad \forall w\in W_h, \tag{3.17}$$
$$(\psi,v)=(Q_h\psi,v),\quad \forall v\in \boldsymbol{V}_h. \tag{3.18}$$

In the paper, we assume that $1<q<\infty$, the L^2 projection operator has the following stable and approximation properties([19]): for $\varphi\in W_{k+1,q}(\Omega)(or\ \psi\in W_{k+1,q}(\Omega)^2)$.

$$\|Q_h\varphi\|_q\leq\|\varphi\|_q,\quad 2\leq q<\infty, \tag{3.19}$$
$$\|\psi-Q_h\psi\|_q\leq\|\psi\|_{r,q}h^r,\quad 0\leq r\leq k+1. \tag{3.20}$$

We will make use of the Fortin projection Π_h onto $\boldsymbol{V}_h$ defined as

$$(\nabla\cdot(\boldsymbol{v}-\Pi_h\boldsymbol{v}),w_h)=0,\quad w_h\in W_h(or\ w_h\in M_h). \tag{3.21}$$

Then we assume the following property holds:

$$\|\boldsymbol{v}-\Pi_h\boldsymbol{v}\|_q\leq Ch_p^{k+1}\|\boldsymbol{v}\|_{k+1,q},\quad 2\leq q\leq\infty. \tag{3.22}$$
$$\|\nabla\cdot(\boldsymbol{v}-\Pi_h\boldsymbol{v})\|_q\leq Ch_p^{k+1}\|\nabla\cdot\boldsymbol{v}\|_{k+1,q}. \tag{3.23}$$

4 L^q error estimate of finite element method

In this section, we try to analyze the convergence of error estimate for the finite element solution both for the concentration and Darcy velocity variables.

First, it is very useful to introduce an elliptic projection of the solution of the differential problem into the finite element space. Consider first the map: $(R_h\boldsymbol{u},R_hp):J\to\boldsymbol{V}_{h_p}\times W_h$ is given to satisfy the following systems:

$$(\nabla\cdot R_h\boldsymbol{u},w)=(q,w),\quad \forall w\in W_h, \tag{4.1}$$
$$(\alpha(c)R_h\boldsymbol{u},\boldsymbol{v})-(R_hp,\nabla\cdot\boldsymbol{v})=0,\quad \forall\boldsymbol{v}\in\boldsymbol{V}_{h_p}. \tag{4.2}$$

We have following properties for the elliptic projection [16]

$$\|\boldsymbol{u}-R_h\boldsymbol{u}\|_V+\|p-R_hp\|_W\leq C\{\inf_{\boldsymbol{v}\in\boldsymbol{V}_h}\|\boldsymbol{u}-\boldsymbol{v}\|_V+\inf_{\phi\in W_h}\|p-\phi\|_W\}, \tag{4.3}$$

where the constant C does not depend on c, since we have assume p is smooth. Then, from the standard interpolation theory, we have:

$$\inf_{\boldsymbol{v}\in\boldsymbol{V}_h}\|\boldsymbol{u}-\boldsymbol{v}\|_V\leq C\|p\|_{k+3}h_p^{k+1},$$
$$\inf_{\phi\in W_h}\|p-\phi\|_W\leq C\|p\|_{k+1}h_p^{k+1},$$

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and it follows that

$$\|\boldsymbol{u}-R_h\boldsymbol{u}\|_{\mathbf{V}}+\|p-R_hp\|_W\leq C\|p\|_{L^\infty(J,H^{k+3})}h_p^{k+1}, \tag{4.4}$$

with C depending only on uniform bounds for $a(c)$ but not on c itself.

Second, we try to get the L^q error estimate of the Darcy velocity. Subtract (3.6)-(3.7) from (4.1)-(4.2) to get the error equation

$$(\nabla\cdot(R_h\mathbf{u}-\mathbf{u}),w)=0,\quad \forall w\in W_h, \tag{4.5}$$

$$(\alpha(c)(R_h\boldsymbol{u}-\boldsymbol{u}),\boldsymbol{v})-(R_hp-p,\nabla\cdot\boldsymbol{v})=0,\quad \forall\boldsymbol{v}\in\mathbf{V}_{h_p}. \tag{4.6}$$

Now, let D_h [6] be the L^2-projection onto the space

$$\bar{\boldsymbol{V}}_h=\{\boldsymbol{v}_h\in\boldsymbol{V}_h:\nabla\cdot\mathbf{v}_h=0\},$$

of the divergence-free vectors. It has the stability property [6]

$$\|D_h\boldsymbol{v}\|_q\leq C\|\boldsymbol{v}\|_q,\quad 2\leq q\leq\infty. \tag{4.7}$$

Lemma 4.1. *$\boldsymbol{u}^n$ is the exact solution of the pressure equation* (1.2)-(1.3) *at $t=t^n$ with $1\leq n\leq N$, $R_h\boldsymbol{u}_h^n$ is the mixed finite element projection solution of the pressure equation* (4.1)-(4.2)*, then, for $2\leq q<\infty$, we have*

$$\|\boldsymbol{u}^n-R_h\boldsymbol{u}^n\|_q\leq C\|\boldsymbol{u}^n\|_{k+1,q}h_p^{k+1}, \tag{4.8}$$

$$\|\nabla\cdot(\boldsymbol{u}^n-R_h\boldsymbol{u}_h^n)\|_q\leq C\|\nabla\cdot\boldsymbol{u}^n\|_{k+1,q}h_p^{k+1}. \tag{4.9}$$

Proof. From(4.6), we have

$$(\boldsymbol{u}^n-R_h\boldsymbol{u}^n,\boldsymbol{v})=0,\quad \forall\boldsymbol{v}\in\bar{\boldsymbol{V}}_h.$$

Then

$$(\boldsymbol{u}^n-Q_h\boldsymbol{u}^n,\boldsymbol{v})-(R_h\boldsymbol{u}^n-Q_h\boldsymbol{u}^n,\boldsymbol{v})=0,\quad \forall\boldsymbol{v}\in\bar{\boldsymbol{V}}_h.$$

Hence

$$R_h\boldsymbol{u}^n-Q_h\boldsymbol{u}^n=D_h(\boldsymbol{u}^n-Q_h\boldsymbol{u}^n),$$

$$\begin{aligned}\|\boldsymbol{u}^n-R_h\boldsymbol{u}^n\|_q&\leq\|\boldsymbol{u}^n-Q_h\boldsymbol{u}^n\|_q+\|Q_h\boldsymbol{u}^n-R_h\boldsymbol{u}^n\|_q\\&\leq C\|\boldsymbol{u}^n-Q_h\boldsymbol{u}^n\|_q\leq C\|\boldsymbol{u}^n\|_{k+1,q}h_p^{k+1}.\end{aligned}$$

From (4.5)

$$(\nabla\cdot(\boldsymbol{u}^n-R_h\boldsymbol{u}_h^n),w_h)=0.$$

From (3.21) and the assumption of (3.23), we have

$$(\nabla\cdot(\Pi_h\boldsymbol{u}^n-R_h\boldsymbol{u}_h^n),w_h)=(\nabla\cdot(\boldsymbol{u}^n-R_h\boldsymbol{u}_h^n),w_h)=0,$$

so that $\nabla\cdot(\Pi_h\boldsymbol{u}^n-R_h\boldsymbol{u}_h^n)=0$. Then we get

$$\|\nabla\cdot(\boldsymbol{u}^n-R_h\boldsymbol{u}_h^n)\|_q\leq\|\nabla\cdot(\Pi_h\boldsymbol{u}^n-\boldsymbol{u}^n)\|_q\leq C\|\boldsymbol{u}^n\|_{k+1,q}h_p^k.$$

This completes the proof. □

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Subtract (3.15)-(3.16) from (4.1)-(4.2) to get the error equation

$$(\nabla\cdot(R_h\boldsymbol{u}^n-\boldsymbol{u}_h^n),w)=0,\quad \forall w\in W_h, \tag{4.10}$$
$$(\alpha(c_h^n)(R_h\boldsymbol{u}^n-\boldsymbol{u}_h^n),\boldsymbol{v})-(R_hp^n-p_h^n,\nabla\cdot\boldsymbol{v})=((\alpha(c^n)-\alpha(c_h^n))R_h\boldsymbol{u}^n,\boldsymbol{v}),\quad \forall\boldsymbol{v}\in\mathbf{V}_h. \tag{4.11}$$

We have already seen by Brezzi's proposition 2.1 in [1] that the solution operator of the error is bounded; hence [16],

$$\begin{aligned}\|\boldsymbol{u}_h^n-R_h\boldsymbol{u}^n\|_V+\|p_h^n-R_hp^n\|_W&\le C\{1+\|R_h\boldsymbol{u}^n\|_{L^\infty}\}\|c^n-c_h^n\|\\&\le C\|p^n\|_{L^\infty(J;H^3(\Omega))}\|c^n-c_h^n\|,\end{aligned} \tag{4.12}$$

with again only bounds on $a_i(c)$ being involved in the constant C.

Lemma 4.2. *Let $(\boldsymbol{u}_h^n,p_h^n)$ be the solution of* (3.15)- (3.16), *$(\boldsymbol{u}^n,p^n)$ be the solution of* (3.6)- (3.7) *at $t=t^n$. If we choose $p_h^0=R_hp_0$, then for $1\le n\le N$, $2\le q<\infty$ and $k\ge1$, we have*

$$\|p^n-p_h^n\|_W+\|\boldsymbol{u}^n-\boldsymbol{u}_h^n\|_V\le M\{h_p^{k+1}+\|c^n-c_h^n\|\}. \tag{4.13}$$

Next, let $(R_hc,R_h\boldsymbol{z}):J\to M_h\times\boldsymbol{V}_{h_c}$ be the projection of c defined by:

$$(\nabla\cdot(R_h\boldsymbol{z}-\boldsymbol{z}),\varphi)=0,\quad \varphi\in M_h. \tag{4.14}$$
$$(\kappa(R_h\boldsymbol{z}-\boldsymbol{z}),\chi)+(R_hc-c,\nabla\cdot\chi)=0,\quad \chi\in\boldsymbol{V}_{h_c}. \tag{4.15}$$

Similar to the proof of Lemma 4.1, we have

Lemma 4.3. *$\boldsymbol{z}^n$ is the exact solution of the concentration equation* (3.1)-(3.2) *at $t=t^n$ with $1\le n\le N$, $R_h\boldsymbol{z}_h^n$ is the the mixed finite element solution of the concentration equation* (4.14)-(4.15), *then, for $2\le q<\infty$, we have*

$$\|\boldsymbol{z}^n-R_h\boldsymbol{z}^n\|_q\le C\|\boldsymbol{z}^n\|_{l+1,q}h_c^{l+1}, \tag{4.16}$$
$$\|\nabla\cdot(\boldsymbol{z}^n-R_h\boldsymbol{z}^n)\|_q\le C\|\nabla\cdot\boldsymbol{z}^n\|_{l+1,q}h_c^{l+1}. \tag{4.17}$$

In order to achieve the main results, first we assume that the Dirichlet problem

$$\begin{aligned}-\nabla\cdot(D\nabla\omega)&=\psi,\quad x\in\Omega,\\ \omega&=0,\quad x\in\partial\Omega,\end{aligned}$$

is uniquely solvable for $\omega\in L^p(\Omega)$ and has the following regularity:

$$\|\omega\|_{l+2,p}\le C\|\psi\|_{l,p},$$

for all $\psi\in W^{l,p}(\Omega)$.

Set

$$d=Q_hc-R_hc,\quad \beta=c-R_hc,\quad \sigma=\boldsymbol{z}-R_h\boldsymbol{z}.$$

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Lemma 4.4. *$R_h c^n$ is the elliptic projection at $t=t^n$ with $1\le n\le N$ defined in* (4.14)-(4.15)*, then, for $2\le q\le\infty$, we have*

$$\|c^n-R_hc^n\|_q\le C\|c^n\|_{l+1,q}h_c^{l+1}. \tag{4.18}$$

Proof. From the elliptic mixed finite element projection (4.14), (4.15), we obtain

$$\begin{aligned}(c^n-R_hc^n,\psi)&=(\beta^n,-\nabla\cdot(D\nabla\omega))=(\nabla\beta^n,D\nabla\omega)=(\sigma^n,\nabla\omega)\\&=(\sigma^n,\nabla(\omega-I_h\omega))\le h_c\|\sigma^n\|_q\|\omega\|_{2,p}\\&\le h_c\|z^n-R_hz^n\|_q\|\psi\|_p\le h_c^{l+1}\|z^n\|_{l,q}\|\psi\|_p\\&\le h_c^{l+1}\|c^n\|_{l+1,q}\|\psi\|_p,\end{aligned}$$

where I_h is a linear interpolation operator.

$$\|c^n-R_hc^n\|_q\le C\|c^n\|_{l+1,q}h_c^{l+1}.$$

This completes the proof. □

Lemma 4.5. *for $1\le n\le N$ and $2\le q<\infty$, we have*

$$\|d^n\|_q\le Ch_c(\|\sigma^n\|_q+h_c\|\nabla\cdot\sigma^n\|_q). \tag{4.19}$$

Proof. From the elliptic mixed finite element projection (4.14), (4.15) and the definition of L^2 projection, we get that

$$(\nabla\cdot\sigma^n,\varphi)=0,\quad \varphi\in M_h, \tag{4.20}$$

$$(\kappa\sigma^n,\chi)+(d^n,\nabla\cdot\chi)=0,\quad \chi\in\boldsymbol{V}_{h_c}. \tag{4.21}$$

Then from (4.20), (4.21), we get that

$$\begin{aligned}(d^n,\psi)&=(d^n,-\nabla\cdot(D\nabla\omega))=(d^n,-\nabla\cdot\Pi_h(D\nabla\omega))\\&=(\sigma^n,\Pi_h(\nabla\omega))=(\sigma^n,\nabla\omega)+(\sigma^n,\Pi_h(\nabla\omega)-\nabla\omega)\\&=(\nabla\cdot\sigma^n,Q_h\omega-\omega)+(\sigma^n,\Pi_h(\nabla\omega)-\nabla\omega)\\&\le\|\nabla\cdot\sigma^n\|_q\|\omega-Q_h\omega^n\|_p+\|\sigma\|_q\|\nabla\omega-\Pi_h(\nabla\omega)\|_p\\&\le C(h_c^2\|\nabla\cdot\sigma^n\|_q+h_c\|\sigma^n\|_q)\|\omega\|_{2,p}\\&\le Ch_c(h_c\|\nabla\cdot\sigma^n\|_q+\|\sigma^n\|_q)\|\psi\|_p,\end{aligned}$$

where the fifth equality used integration by parts. The proof is completed. □

As in the argument above, we can also derive the estimates for the time derivatives of d and β:

$$(Q_hc-R_hc)_t\quad\text{and}\quad(c-R_hc)_t,$$

just by differentiating equations (4.14) and (4.15) with respect to time t. Therefore, combine the results in Lemmas 4.4 and 4.5, we obtain the following lemma.

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Lemma 4.6. *Let* $(c,\boldsymbol{z})\in M\times\boldsymbol{V}$ *be the solution of the differential problem* (3.1)-(3.2) *and let* $(R_hc,R_h\boldsymbol{z})\in M_h\times\boldsymbol{V}_{h_c}$ *be its elliptic-mixed projection define in* (4.14)-(4.15). *Then, for* $1\leq r\leq l+1$,

$$\|Q_hc-R_hc\|_{L^q}+\|(Q_hc-R_hc)_t\|_{L^q}\leq C\|c\|_{r+1,q}h_c^{r+1}, \tag{4.22}$$

$$\|c-R_hc\|_{L^q}+\|(c-R_hc)_t\|_{L^q}\leq C\|c\|_{l+1,q}h_c^{l+1}. \tag{4.23}$$

Next, we can also investigate another superconvergence phenomenon between the full discrete solution and the elliptic-mixed finite method projection.

Now, we have to introduce an useful result which plays an important role in the proof of superconvergence property [2].

Lemma 4.7. *Let* g *be a piecewise smooth function on the partition* T_h. *If* $\bar{g}(c)$ *is the average of* $g(c)$ *on each element* Ω_e *of the* T_h *and* $\|\nabla g\|_{0,\infty}\leq K$, *then*

$$|(g(c)\theta,\psi)-(\bar{g}(c)\theta,\psi)|\leq Kh_c\|\theta\|_0\|\psi\|_0. \tag{4.24}$$

In the following analysis, we first introduce the following lemma [20]:

Lemma 4.8. *If* $\eta\in L^2(\mathbb{R})$ *and* $\bar{\eta}=\eta(x-g(x)\Delta t)$, *where* g *and* g' *are bounded, then*

$$\|\eta-\bar{\eta}\|_{-1}\leq K\|\eta\|\Delta t.$$

Proof. Let $z=F(x)=x-g(x)\Delta t$. then F is invertible for Δt sufficiently small, and F' and $(F^{-1})'$ are both of the form $1+\mathcal{O}(\Delta t)$. Hence,

$$\begin{aligned}\|\eta-\bar{\eta}\|_{-1}&=\sup_{\varphi\in W^{1,2}}\left(\|\varphi\|_1^{-1}\int_{\mathbb{R}}[\eta(x)-\eta(x-g(x)\Delta t)]\varphi(x)dx\right)\\&=\sup_{\varphi\in W^{1,2}}\left(\|\varphi\|_1^{-1}[\int_{\mathbb{R}}\eta(x)\varphi(x)dx-\int_{\mathbb{R}}\eta(z)\varphi(F^{-1}(z))(1+\mathcal{O}(\Delta t))dz]\right)\\&\leq\sup_{\varphi\in W^{1,2}}\left(\|\varphi\|_1^{-1}\int_{\mathbb{R}}\eta(x)[\varphi(x)-\varphi(F^{-1}(x))]dx\right)\\&\quad+K\Delta t\sup_{\varphi\in W^{1,2}}\left(\|\varphi\|_1^{-1}\int_{\mathbb{R}}\eta(z)\varphi(F^{-1}(z))dx\right).\end{aligned} \tag{4.25}$$

Let $G(x)=x-F^{-1}(x)$; then $G(x)\leq K\Delta t$, and

$$\begin{aligned}\|\varphi(x)-\varphi(F^{-1}(x))\|^2&\leq\int_{\mathbb{R}}\left(\int_{F^{-1}(x)}^{x}|\frac{d\varphi}{dx}|\right)^2dydx\\&\leq K(\Delta t)^2\int_{\mathbb{R}}\left(\int_0^1|\frac{d\varphi}{dx}(x-G(x)y)|\right)^2dydx\\&\leq K(\Delta t)^2\|\varphi\|_1^2,\end{aligned} \tag{4.26}$$

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where the last step uses the change of variable $x-G(x)y$, which induces a factor of $1+\mathcal{O}(\Delta t)$. A similar change of variable demonstrates that

$$\|\varphi\circ F^{-1}\|^2=\|\varphi\|^2(1+dK\Delta t),\quad |d|\leq 1, \tag{4.27}$$

where K is the constant of (2.5); the same is true for $\varphi\circ F$. Combining (4.25)-(4.27), we obtain the lemma. Note that the argument is valid for $x\in\mathbb{R}^k, k\geq 1$. □

Now, we will obtain L^2 error estimate for $\|R_hc^n-c_h^n\|$.

Lemma 4.9. *Let $(c_h^n,z_h^n)\in M_h\times \boldsymbol{V}_{h_c}$ be the solution of the mixed finite element equations* (3.13)-(3.14) *and (R_hc^n,R_hz^n) be the elliptic projection solution of the equations* (4.14)-(4.15). *If the initial function $c_h^0=R_hc^0$, then for $1\leq n\leq N$, we have*

$$\|R_hc^n-c_h^n\|+\|R_hz^n-z_h^n\|_{(0,t^m;L^2)}\leq C(\|c^n\|_{l+1}h_c^{l+2}+\Delta t). \tag{4.28}$$

Proof. Add (4.14)-(4.15), (3.4)- (3.5)and subtract (3.13)-(3.14) at $t=t^n$ to get

$$\left(\psi\frac{\partial c^n}{\partial\tau}-\phi\frac{c_h^n-\bar{c}_h^{n-1}}{\Delta t},\varphi\right)-(\nabla\cdot(R_hz^n-z_h^n),\varphi)=(f(c^n)-f(c_h^n),\varphi), \tag{4.29}$$

$$(\kappa(R_hz^n-z_h^n),\chi)+(R_hc^n-c_h^n,\nabla\cdot\chi)=0. \tag{4.30}$$

Let $\rho^n=R_hc^n-c_h^n$, $e^n=c^n-Q_hc^n$, $\eta^n=Q_hc^n-R_hc^n$, $\sigma^n=R_hz^n-z_h^n$ and choose $\varphi=\rho^n$, $\chi=\sigma^n$, then, adding (4.29), (4.30), we have

$$\begin{aligned}\left(\phi\frac{\rho^n-\bar{\rho}^{n-1}}{\Delta t},\rho^n\right)+(\kappa\sigma^n,\sigma^n)=&-\left(\psi\frac{\partial c^n}{\partial\tau}-\phi\frac{c^n-\bar{c}^{n-1}}{\Delta t},\rho^n\right)-\left(\phi\frac{\eta^n-\bar{\eta}^{n-1}}{\Delta t},\rho^n\right)\\&-\left(\phi\frac{e^n-\bar{e}^{n-1}}{\Delta t},\rho^n\right)+(f(c^n)-f(c_h^n),\rho^n)\\&=(F_1+F_2+F_3+F_4,\rho^n).\end{aligned} \tag{4.31}$$

For each term on the right-hand side of (4.31), we estimate them as follows:

$$|(F_1,\rho^n)|=\left|\left(\psi\frac{\partial c^n}{\partial\tau}-\phi\frac{c^n-\bar{c}^{n-1}}{\Delta t},\rho^n\right)\right|\leq K\left\|\frac{\partial^2c^n}{\partial\tau^2}\right\|^2_{L^2(t^{n-1},t^n,L^2)}\Delta t+\|\rho^n\|^2, \tag{4.32}$$

where the calculation of the above inequality is presented in detail see [11, 20]

$$\begin{aligned}|(F_2,\rho^n)|&=\left|\left(\phi\frac{\eta^n-\bar{\eta}^{n-1}}{\Delta t},\rho^n\right)\right|\\&\leq\left|\left(\phi\frac{\eta^n-\eta^{n-1}}{\Delta t},\rho^n\right)\right|+\left|\left(\phi\frac{\eta^{n-1}-\bar{\eta}^{n-1}}{\Delta t},\rho^n\right)\right|,\\&\leq K\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}\left\|\frac{\partial\eta}{\partial t}\right\|_{-1}dt\cdot\|\rho^n\|_1+\left|\left(\phi\frac{\eta^{n-1}-\bar{\eta}^{n-1}}{\Delta t},\rho^n\right)\right|\\&\leq K\frac{1}{\Delta t}\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^2(t^{n-1},t^n,W^{-1,2})}+2\theta\|\rho^n\|_1^2+K\|\eta^{n-1}\|^2,\end{aligned} \tag{4.33}$$

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following from the definition of e^n and (3.17), we have

$$|(F_3,\rho^n)|=\left|\left(\phi\frac{e^n-\bar{e}^{n-1}}{\Delta t},\rho^n\right)\right|\leq\frac{\phi^*}{\Delta t}|(e^n-\bar{e}^{n-1},\rho^n)|=0, \tag{4.34}$$

note that

$$F_4=f(c^n)-f(Q_hc^n)+f(Q_hc^n)-f(R_hc^n)+f(R_hc^n)-f(c_h^n),$$

then, we have:

$$|(f(Q_hc^n)-f(R_hc^n),\rho^n)|\leq\|f\|_{1,\infty}\|Q_hc^n-R_hc^n\|\|\rho^n\|\leq Ch_c^{2l+4}+\|\rho^n\|^2, \tag{4.35}$$

$$|(f(R_hc^n)-f(c_h^n),\rho^n)|\leq\|f\|_{1,\infty}\|\rho^n\|^2, \tag{4.36}$$

$$|(f(c^n)-f(Q_hc^n),\rho^n)|\leq|(f_c'(c^n)(c^n-Q_hc^n),\rho^n)|+\left|\left(\frac{\|f\|_{2,\infty}}{2}(c^n-Q_hc^n)^2,\rho^n\right)\right|. \tag{4.37}$$

Now, by using Lemma 4.7 with $g(c)=f_c'(c^n)$, definition of (3.17) and its approximation property (3.20), we have

$$\begin{aligned}|(f(c^n)-f(Q_hc^n),\rho^n)|&\leq Ch_c\|f\|_{1,\infty}\|e^n\|\|\rho^n\|+C\|f\|_{2,\infty}\|e^n\|_{0,4}^2\|\rho^n\|\\&\leq Ch_c^{2l+4}+\|\rho^n\|^2.\end{aligned} \tag{4.38}$$

Hence, from (4.35)-(4.38), we conclude that

$$|(F_4,\rho^n)|\leq(2+\|f\|_{1,\infty})\|\rho^n\|^2+Ch_c^{2l+4}. \tag{4.39}$$

For the left hand side of (4.31), we have

$$\begin{aligned}\left(\phi\frac{\rho^n-\bar{\rho}^{n-1}}{\Delta t},\rho^n\right)+(\kappa\sigma^n,\sigma^n)&\geq\frac{\phi_*}{2\Delta t}[(\rho^n,\rho^n)-(\bar{\rho}^{n-1},\bar{\rho}^{n-1})]+(\kappa\sigma^n,\sigma^n)\\&=\frac{\phi_*}{2\Delta t}[(\rho^n,\rho^n)-(\rho^{n-1},\rho^{n-1})(1+\gamma K\Delta t)]+(\kappa\sigma^n,\sigma^n),\end{aligned} \tag{4.40}$$

where (4.27) have been used.

Then, from (4.31)-(4.40),

$$\begin{aligned}&\frac{\phi_*}{2\Delta t}[(\rho^n,\rho^n)-(\rho^{n-1},\rho^{n-1})]+\kappa\|\sigma^n\|^2\\\leq&C\left(\left\|\frac{\partial^2c^n}{\partial\tau^2}\right\|^2_{L^2(t^{n-1},t^n,L^2)}\Delta t+\frac{1}{\Delta t}\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^2(t^{n-1},t^n,W^{-1,2})}\right.\\&\left.+2\theta\|\rho^n\|_1^2+\|\eta^{n-1}\|^2+h_c^{2l+4}\right)+(3+\|f\|_{1,\infty})\|\rho^n\|^2,\end{aligned} \tag{4.41}$$

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and multiply $2\Delta t$ on both side of the last inequality and sum from $n=1$ to m and note that $R_h c_0 = c_0$, we get

$$\begin{aligned}\|\rho^m\|^2+2b_1\Delta t\sum_{n=1}^{m}\|\sigma^n\|^2 \le C\Bigg((\Delta t)^2\sum_{n=1}^{m}\left\|\frac{\partial^2 c^n}{\partial\tau^2}\right\|^2_{L^2(t^{n-1},t^n,L^2)}+\sum_{n=1}^{m}\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^2(t^{n-1},t^n,W^{-1,2})}\\ +h_c^{2l+4}\Bigg)+\Delta t\sum_{n=1}^{m}\|\rho^n\|^2\\ \le C((\Delta t)^2+h_c^{2l+4})+\Delta t\sum_{n=1}^{m}\|\rho^n\|^2, \qquad (4.42)\end{aligned}$$

where θ is sufficiently small, so that $b_1=\kappa-2\theta>0$. Then, we easily have

$$\|\rho^m\|+b_1\|\sigma^n\|_{(0,t^m;L^2)}\le C(\Delta t+h_c^{l+2}). \qquad (4.43)$$

This completes the proof. □

5 Two-grid algorithm and error estimate

In this section, we will present the two-grid algorithms and analyze the convergence accuracy of schemes. The fundamental ingredient in this scheme is another finite element space $M_H\times\boldsymbol{V}_{H_c}\times\boldsymbol{V}_{H_p}\times W_H\subset M_h\times\boldsymbol{V}_{h_c}\times\boldsymbol{V}_{h_p}\times W_h$ ($h\ll H<1$) defined on a coarser quasi-uniform triangulation or rectangulation of Ω. Now, we present the two grid method which has two steps as follows.

Algorithm 1:

Step1: On the coarse grid $\mathcal{T}_H$, solve the following nonlinear coupling system for $(c_H^n,\boldsymbol{z}_H^n,\boldsymbol{u}_H^n,p_H^n)\in M_H\times\boldsymbol{V}_{H_c}\times\boldsymbol{V}_{H_p}\times W_H$

$$(\phi\partial_\tau c_H^n,\varphi)-(\nabla\cdot\boldsymbol{z}_H^n,\varphi)=(f(c_H^n),\varphi),\quad\forall\varphi\in M_H, \qquad (5.1)$$
$$(\kappa\boldsymbol{z}_H^n,\chi)+(c_H^n,\nabla\cdot\chi)=0,\quad\forall\chi\in\boldsymbol{V}_{H_c}, \qquad (5.2)$$
$$(\nabla\cdot\boldsymbol{u}_H^n,w)=(q,w),\quad\forall w\in W_H, \qquad (5.3)$$
$$(\alpha(c_H^n)\boldsymbol{u}_H^n,\boldsymbol{v})-(p_H^n,\nabla\cdot\boldsymbol{v})=0,\quad\forall\boldsymbol{v}\in\boldsymbol{V}_{H_p}. \qquad (5.4)$$

Step2: On the fine grid $\mathcal{T}_h$, we compute $(C_h^n,\boldsymbol{Z}_h^n,\boldsymbol{U}_h^n,P_h^n)\in M_h\times\boldsymbol{V}_{h_c}\times\boldsymbol{V}_{h_p}\times W_h$ to satisfy the following linear decoupling system:

$$(\phi\partial_\tau C_h^n,\varphi)-(\nabla\cdot\boldsymbol{Z}_h^n,\varphi)=(f(c_H^n)+f'(c_H^n)(C_h^n-c_H^n),\varphi),\quad\varphi\in M_h, \qquad (5.5)$$
$$(\kappa\boldsymbol{Z}_h^n,\chi)+(C_h^n,\nabla\cdot\chi)=0,\quad\forall\chi\in\boldsymbol{V}_{h_c}, \qquad (5.6)$$
$$(\nabla\cdot\boldsymbol{U}_h^n,w)=(q,w),\quad\forall w\in W_h, \qquad (5.7)$$
$$(\alpha'(c_H^n)\boldsymbol{u}_H^n(C_h^n-c_H^n)+\alpha(c_H^n)\boldsymbol{U}_h^n,\boldsymbol{v})-(P_h^n,\nabla\cdot\boldsymbol{v})=0,\quad\forall\boldsymbol{v}\in\boldsymbol{V}_h. \qquad (5.8)$$

In order to make an analysis of the two-grid solution $(C_h^n,\boldsymbol{Z}_h^n,\boldsymbol{U}_h^n,P_h^n)$, we first need to analyze the L^q error estimate of $\|c^n-c_H^n\|_q,\|\boldsymbol{u}^n-\boldsymbol{u}_H^n\|_q$.

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Lemma 5.1. *c^n is the exact solution satisfy Eq.* (3.4) *at $t=t^n$, and c_H^n is the two-grid solution satisfy Eq.* (5.1), *and choose $R_H c_0 = c_H^0$, time step Δt selected satisfy $2\Delta t < \phi_*$, then, for $1 \le n \le N$, we have*

$$\|c^n - c_H^n\|_q \le C(\Delta t + H_c^{l+1}). \tag{5.9}$$

Proof. Using approximation property (3.20), Lemmas 4.4, 4.9 and the inverse inequality

$$\begin{aligned}\|c^n - c_H^n\|_{L^q} &\le \|c^n - R_H c^n\|_{L^q} + \|R_H c^n - c_H^n\|_{L^q} \\ &\le C(\|c^n\|_{l+1,q} H_c^{l+1} + H_c^{2(\frac{1}{q}-\frac{1}{2})} \|R_H c^n - c_H^n\|) \\ &\le C(H_c^{l+1} + H^{(\frac{2}{q}-1)}(H_c^{l+2} + \Delta t)) \\ &\le C(H_c^{l+1} + \Delta t). \end{aligned} \tag{5.10}$$

This completes the proof. □

For (4.10)-(4.11), we choose $v \in \bar{V}_h$ and $\frac{1}{p} + \frac{1}{q} = 1$, then, we obtain

$$\begin{aligned}\alpha_* \|R_h \boldsymbol{u}^n - \boldsymbol{u}_h^n\|_{L^q} &= \sup_{0 \ne v \in \bar{V}_h} \alpha_* \frac{|(R_h \boldsymbol{u}^n - \boldsymbol{u}_h^n, \boldsymbol{v})|}{\|\boldsymbol{v}\|_{L^p}} \\ &\le \sup_{0 \ne v \in \bar{V}_h} \frac{|((\alpha(c^n) - \alpha(c_h^n)) R_h \boldsymbol{u}, \boldsymbol{v})|}{\|\boldsymbol{v}\|_{L^p}} \\ &\le \sup_{0 \ne v \in \bar{V}_h} (\|\alpha\|_{1,\infty} \|R_h \boldsymbol{u}\|_\infty \|c^n - c_h^n\|_{L^q}) \\ &\le C\|c^n - c_h^n\|_{L^q} \le C(h_c^{l+1} + \Delta t). \end{aligned} \tag{5.11}$$

From (4.8) and (5.11), we get the L^q error estimate of the Darcy velocity

$$\|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_{L^q} \le \|\boldsymbol{u}^n - R_h \boldsymbol{u}^n\|_{L^q} + \|R_h \boldsymbol{u} - \boldsymbol{u}_h^n\|_{L^q} \le C(h_p^{k+1} + h_c^{l+1} + \Delta t). \tag{5.12}$$

Thus, we have

$$\|\boldsymbol{u}^n - \boldsymbol{u}_H^n\|_{L^q} \le C(H_p^{k+1} + H_c^{l+1} + \Delta t). \tag{5.13}$$

Now, we start to analyze the error estimation of two-grid solution. First make analysis of the Darcy velocity and pressure variables.

Lemma 5.2. *$(R_h \boldsymbol{u}^n, R_h p^n)$ is the elliptic projection defined in* (4.1)-(4.2), *$(\boldsymbol{U}_h^n, P_h^n)$ is the two-grid solution defined in* (5.7)-(5.8), *then, we can have the following estimate:*

$$\|R_h \boldsymbol{u}^n - \boldsymbol{U}_h^n\|_V + \|R_h p^n - P_h^n\|_W \le C(h_p^{2k+2} + H_c^{2l+2} + H_p^{2k+2} + (\Delta t)^2) + M_\alpha M \|c^n - C_h^n\|. \tag{5.14}$$

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Proof. From (4.1)-(4.2) and (5.7)-(5.8), we have:

$$(\nabla\cdot(R_h\boldsymbol{u}^n-\boldsymbol{U}_h^n),w)=0, \tag{5.15}$$
$$(\alpha(c^n)R_h\boldsymbol{u}^n-\alpha'(c_H^n)\boldsymbol{u}_H^n(C_h^n-c_H^n)-\alpha(c_H^n)\boldsymbol{U}_h^n,\boldsymbol{v})-(R_hp^n-P_h^n,\nabla\cdot\boldsymbol{v})=0. \tag{5.16}$$

Note that

$$\alpha(c^n)=\alpha(c_H^n)+\alpha'(c_H^n)(c^n-c_H^n)+\frac{1}{2}\alpha''(c^*)(c^n-c_H^n)^2, \tag{5.17}$$

and a direct computation leads to

$$(\nabla\cdot(R_h\boldsymbol{u}^n-\boldsymbol{U}_h^n),w)=0 \tag{5.18}$$
$$(\alpha(c_H^n)(R_h\boldsymbol{u}^n-\boldsymbol{U}_h^n),\boldsymbol{v})-(R_hp^n-P_h^n,\nabla\cdot\boldsymbol{v})=(G_1+G_2+G_3+G_4,\boldsymbol{v}), \tag{5.19}$$

where

$$G_1=\alpha'(c_H^n)(c^n-c_H^n)(\boldsymbol{u}^n-R_h\boldsymbol{u}^n), \tag{5.20}$$
$$G_2=-\alpha'(c_H^n)(c^n-c_H^n)(\boldsymbol{u}^n-\boldsymbol{u}_H^n), \tag{5.21}$$
$$G_3=-\alpha'(c_H^n)\boldsymbol{u}_H^n(c^n-C_h^n), \tag{5.22}$$
$$G_4=-\frac{1}{2}\alpha''(c^*)(c^n-c_H^n)^2R_h\boldsymbol{u}^n. \tag{5.23}$$

We now analyze each term on the right-hand side of (5.19). Let $\mu^n=R_h\boldsymbol{u}^n-\boldsymbol{U}_h^n$ and $v=\mu^n$, then, we have

$$\begin{aligned}|(G_1,\mu^n)|&\le C\|(c^n-c_H^n)(\boldsymbol{u}^n-R_h\boldsymbol{u}^n)\|\ \|\mu^n\|\\&\le C\|(c^n-c_H^n)\|_{L^4}\|\boldsymbol{u}^n-R_h\boldsymbol{u}^n\|_{L^4}\|\mu^n\|\\&\le C(H_c^{2l+2}+h_p^{2k+2}+(\Delta t)^2)\|\mu^n\|.\end{aligned} \tag{5.24}$$

As to G_2, we have

$$\begin{aligned}|(G_2,\mu^n)|&\le C\|(c^n-c_H^n)(\boldsymbol{u}^n-\boldsymbol{u}_H^n)\|\ \|\mu^n\|\\&\le C\|(c^n-c_H^n)\|_{L^4}\|\boldsymbol{u}^n-\boldsymbol{u}_H^n\|_{L^4}\|\mu^n\|\\&\le C(H_p^{2k+2}+H_c^{2l+2}+(\Delta t)^2)\|\mu^n\|.\end{aligned} \tag{5.25}$$

We easily get the analysis of G_3 as follows:

$$|(G_3,\mu^n)|\le\|\alpha' u_H^n\|_\infty\|(c^n-C_h^n)\|\|\mu^n\|\le M_\alpha M\|c^n-C_h^n\|\|\mu^n\|. \tag{5.26}$$

Related to G_4, we have

$$\begin{aligned}|(G_4,\mu^n)|&\le\|\alpha''R_h\boldsymbol{u}^n\|_\infty\|c^n-c_H^n\|_{L^4}^2\|\mu^n\|\\&\le C(H_c^{2l+2}+(\Delta t)^2)\|\mu^n\|.\end{aligned} \tag{5.27}$$

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We have already seen by [1] that the solution operator for (5.16) is bounded, hence

$$\|R_h\boldsymbol{u}^n-\boldsymbol{U}_h^n\|_V+\|R_hp^n-P_h^n\|_W\leq C(h_p^{2k+2}+H_c^{2l+2}+H_p^{2k+2}+(\Delta t)^2)+M_\alpha M\|c^n-C_h^n\|. \tag{5.28}$$

This completes the proof. □

In the following analysis, we need to prove $\|c^n-C_h^n\|$, we have the following result.

Lemma 5.3. *c^n is the exact solution satisfy Eq. (3.4) at $t=t^n$, and C_h^n is the two-grid solution satisfy Eq. (5.4), then, choose $R_hc_0=C_h^0$, time step Δt selected satisfy $2\Delta t<\phi_*$, then, for $1\leq n\leq N$, we have*

$$\|c^n-C_h^n\|\leq C(h_c^{l+1}+H_c^{2l+2}+\Delta t). \tag{5.29}$$

Proof. Add (4.14)-(4.15), (3.4)-(3.5)and subtract (5.5)-(5.6) at $t=t^n$ to get

$$\Big(\psi\frac{\partial c^n}{\partial\tau}-\phi\partial_\tau C_h^n\Big),\varphi)-(\nabla\cdot(R_h\boldsymbol{z}^n-\boldsymbol{Z}_h^n),\varphi)=(f(c^n)-f(c_H^n)-f'(c_H^n)(C_h^n-c_H^n),\varphi), \tag{5.30}$$

$$(\kappa(R_h\boldsymbol{z}^n-\boldsymbol{Z}_h^n),\chi)+(R_hc^n-C_h^n,\nabla\cdot\chi)=0. \tag{5.31}$$

Let $\xi^n=R_hc^n-C_h^n$, $e^n=c^n-Q_hc^n$, $\eta^n=Q_hc^n-R_hc^n$, $\zeta^n=R_h\boldsymbol{z}^n-\boldsymbol{Z}_h^n$ and choose $\varphi=\xi^n$, $\chi=\zeta^n$, then, adding (5.30) and (5.31), we have

$$\begin{aligned}
&\Big(\phi\frac{\xi^n-\bar\xi^{n-1}}{\Delta t},\xi^n\Big)+(\kappa\zeta^n,\zeta^n)\\
=&-\Big(\psi\frac{\partial c^n}{\partial\tau}-\phi\frac{c^n-\bar c^{n-1}}{\Delta t},\xi^n\Big)-\phi\Big(\frac{\eta^n-\bar\eta^{n-1}}{\Delta t},\xi^n\Big)\\
&-\phi\Big(\frac{e^n-\bar e^{n-1}}{\Delta t},\xi^n\Big)+\Big(f'(c_H^n)(c^n-C_h^n)+\frac{1}{2}f''(c^*)(c^n-c_H^n)^2,\xi^n\Big)\\
=&(Q_1+Q_2+Q_3+Q_4,\xi^n),
\end{aligned} \tag{5.32}$$

where we use two order Taylor'expansion at a point c_H^n for $f(c^n)$. Then we have

$$|(Q_1,\xi^n)|\leq K\Big\|\frac{\partial^2c^n}{\partial\tau^2}\Big\|^2_{L^2(t^{n-1},t^n,L^2)}\Delta t+\|\xi^n\|^2, \tag{5.33}$$

$$|(Q_2,\xi^n)|\leq\frac{K}{\Delta t}\Big\|\frac{\partial\eta^n}{\partial t}\Big\|^2_{L^2(t^{n-1},t^n,W^{-1,2})}+2\epsilon\|\xi^n\|_1^2+K\|\eta^{n-1}\|^2, \tag{5.34}$$

$$|Q_3,\xi^n)|=\Big|\Big(\phi\frac{e^n-\bar e^{n-1}}{\Delta t},\xi^n\Big)\Big|\leq\frac{\phi^*}{\Delta t}|(e^n-\bar e^{n-1},\xi^n)|=0, \tag{5.35}$$

$$\begin{aligned}
|(Q_4,\xi^n)|\leq&\Big|(f'(c_H^n)(c^n-C_h^n),\xi^n)+\Big(\frac{1}{2}f''(c^*)(c^n-c_H^n)^2,\xi^n\Big)\Big|\\
&\leq\|f\|_{1,\infty}\|c^n-C_h^n\|\|\xi^n\|+C\|f\|_{2,\infty}\|c^n-c_H^n\|^4_{L^4}+\|\xi^n\|^2\\
&\leq\|f\|_{1,\infty}\|c^n-R_hc^n+R_hc^n-C_h^n\|\cdot\|\xi^n\|\\
&+C\|f\|_{2,\infty}\|c^n-c_H^n\|^4_{L^4}+\|\xi^n\|^2\\
&\leq C(h_c^{2l+2}+H_c^{4l+4})+3\|\xi^n\|^2,
\end{aligned} \tag{5.36}$$

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where it is proved that the first three inequalities similar to Lemma 4.9.

For the left hand side of (5.32), we obtain

$$\begin{aligned}\left(\phi\frac{\xi^n-\bar{\xi}^{n-1}}{\Delta t},\xi^n\right)+(\kappa\zeta^n,\zeta^n)&\geq\frac{\phi_*}{2\Delta t}[(\xi^n,\xi^n)-(\bar{\xi}^{n-1},\bar{\xi}^{n-1})]+(\kappa\zeta^n,\zeta^n)\\&=\frac{\phi_*}{2\Delta t}[(\xi^n,\xi^n)-(\xi^{n-1},\xi^{n-1})(1+\gamma K\Delta t)]+(\kappa\zeta^n,\zeta^n),\end{aligned}\tag{5.37}$$

then, from (5.32)-(5.37),

$$\begin{aligned}\frac{\phi_*}{2\Delta t}[(\xi^n,\xi^n)-(\xi^{n-1},\xi^{n-1})]+\kappa\|\zeta^n\|^2\leq C\Bigg(&\left\|\frac{\partial^2 c^n}{\partial\tau^2}\right\|^2_{L^2(t^{n-1},t^n,L^2)}\Delta t+\frac{1}{\Delta t}\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^2(t^{n-1},t^n,W^{-1,2})}\\&+2\epsilon\|\xi^n\|_1^2+\|\eta^{n-1}\|^2+\|\eta^n\|^2\\&+h_c^{2l+2}+H_c^{4l+4}\Bigg)+4\|\xi^n\|^2,\end{aligned}\tag{5.38}$$

and multiply $2\Delta t$ on both side of the last inequality and sum from $n=1$ to m and note that $R_h c_0=c_0$, we get

$$\begin{aligned}\|\xi^m\|^2+2b_1\Delta t\sum_{n=1}^m\|\zeta^n\|^2\leq C\Bigg(&(\Delta t)^2\sum_{n=1}^m\left\|\frac{\partial^2 c^n}{\partial\tau^2}\right\|^2_{L^2(t^{n-1},t^n,L^2)}+\sum_{n=1}^m\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^2(t^{n-1},t^n,W^{-1,2})}\\&+h_c^{2l+2}+H_c^{4l+4}\Bigg)+\Delta t\sum_{n=1}^m\|\xi^n\|^2\\\leq C((\Delta t)^2+h_c^{2l+2}+H_c^{4l+4})&+\Delta t\sum_{n=1}^m\|\xi^n\|^2,\end{aligned}\tag{5.39}$$

where ϵ is sufficiently small, so that $b_1=\kappa-2\epsilon>0$. Then, we easily have

$$\|\xi^m\|+b_1\|\zeta^n\|_{(0,t^m;L^2)}\leq C(\Delta t+h_c^{l+1}+H_c^{2l+2}).\tag{5.40}$$

We have

$$\|c^n-C_h^n\|\leq\|c^n-R_hc^n\|+\|R_hc^n-C_h^n\|\leq C(h_c^{l+1}+H_c^{2l+2}+\Delta t).$$

This completes the proof. □

Theorem 5.1. *$(\boldsymbol{u}^n,p^n)$ are the weak solution of Eqs. (3.2)-(3.3) at $t=t^n$, $(\boldsymbol{U}_h^n,P_h^n)$ be the solution of Eqs. (5.7)-(5.8), then, once the condition of Lemma 5.3 holds, we have*

$$\|\boldsymbol{u}^n-\boldsymbol{U}_h^n\|_V+\|p^n-P_h^n\|_W\leq C(h_c^{l+1}+H_c^{2l+2}+H_p^{2k+2}+\Delta t).\tag{5.41}$$

Proof. From (4.4), (5.14) and (5.29), we easily find that

$$\|\boldsymbol{u}^n-\boldsymbol{U}_h^n\|_V+\|p^n-P_h^n\|_W\leq C(h_c^{l+1}+H_c^{2l+2}+H_p^{2k+2}+\Delta t).$$

This completes the proof. □

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6 Numerical experiment

We construct two examples in the following to illustrate the efficiency of the algorithm discussed in Section 5. In both cases, we use piecewise constant for concentration c, the lowest Raviart-Thomas element for flux $\boldsymbol{z}$, $\boldsymbol{u}$ and piecewise constant for pressure p.

Example 6.1. we consider the following incompressible miscible displacement problem.

$$\phi\frac{\partial c}{\partial t}+\boldsymbol{u}\cdot\nabla c-\nabla\cdot(\nabla c)=f(c), \tag{6.1}$$

$$\nabla\cdot\boldsymbol{u}=q, \tag{6.2}$$

$$\alpha(c)\boldsymbol{u}=-\nabla p, \tag{6.3}$$

where $\Omega=[0,1]\times[0,1]$, $t\in[0,T]$, $f(c)$ is suitably chosen such that the exact solution is $c=\sin^2(\pi x)\sin^2(\pi y)e^{-t}$, $\alpha(c)=1$, $p=-c+\frac{1}{4}e^{-t}$.

For the sake of simplicity, the time step $\tau=1.0e-5$, $T=2.0e-4$. we assume $H_p=H_c=H$, $h_c=h_p=h$. We shall demonstrate the efficiency of our algorithm for incompressible miscible displacement problem.

As shown in Fig. 1 and Fig. 2, we can see that the exact solution c, p, respectively. The two grid solution C_h, P_h are shown in Fig. 3 and Fig. 4, respectively. The pressure distribution shows a low pressure in the center of the bottom of the cavity, similarly, a high pressure is visible around the cavity. From Table 1 for $\|c-C_h\|$, we can find that the two method are the same accuracy, but Two grid method have much less cost time than Newton iterative, since we just have to solve a small scale nonlinear problem. This phenomenon shows that two grid algorithm is a very effective algorithm when it comes to deal with the nonlinear problems. In Table 2, we can see that the convergence order of the error for $\|c-C_h\|$, $\|\boldsymbol{z}^n-\boldsymbol{Z}_h^n\|$, $\|\boldsymbol{u}^n-\boldsymbol{U}_h^n\|$, $\|p-P_h\|$ are first order accuracy, respectively. in Figs. 5-8, we can observe that error of Two grid solution C_h, $\boldsymbol{Z}_h$, $\boldsymbol{U}_h$, P_h varied with time. From the Figs. 9-12, we know the algorithms achieve asymptotically optimal approximation applying the two-grid methods when the mesh size between the coarse grid and fine grid satisfy $h=\mathcal{O}(H^2)$ for Algorithm 1.

Table 1: Error and CPU time of Two grid and Newton iterative.

H	h	Newton iterative		Two grid	
$\frac{1}{2}$	$\frac{1}{4}$	1.084e-1	0.12s	1.084e-1	0.18s
$\frac{1}{4}$	$\frac{1}{16}$	2.840e-2	0.89s	2.840e-2	0.69s
$\frac{1}{8}$	$\frac{1}{64}$	7.500e-3	31.94s	7.500e-3	11.63s
$\frac{1}{16}$	$\frac{1}{256}$	3.100e-3	331.88s	3.100e-3	185.84s

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Table 2: Error of Two grid.

H	h	$\|c^n-C_h^n\|$	rate	$\|z^n-Z_h^n\|$	rate	$\|u^n-U_h^n\|$	rate	$\|p^n-P_h^n\|$	rate
$\frac{1}{2}$	$\frac{1}{4}$	1.084e-1		8.100e-1		8.595e-1		1.141e-1	
$\frac{1}{4}$	$\frac{1}{16}$	2.840e-2	1	2.177e-1	1	2.179e-1	1	2.840e-2	1
$\frac{1}{8}$	$\frac{1}{64}$	7.500e-3	1	5.640e-2	1	5.450e-2	1	7.100e-3	1
$\frac{1}{16}$	$\frac{1}{256}$	3.100e-3	1	1.990e-2	1	1.360e-2	1	1.800e-3	1

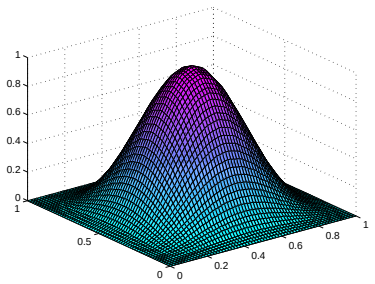


Figure 1: The exact solution c. $t=2.0e-4$, $h=1/64$, $\tau=1.0e-5$.

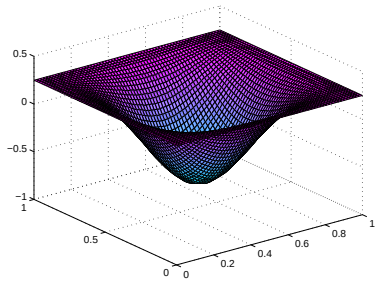


Figure 2: The exact solution p. $t=2.0e-4$, $h=1/64$, $\tau=1.0e-5$.

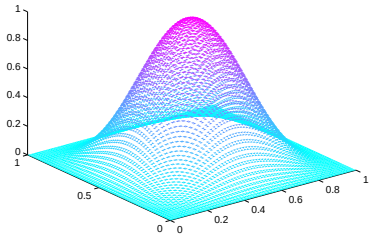


Figure 3: Two grid solution C_h. $t=2.0e-4$, $h=1/64$, $\tau=1.0e-5$.

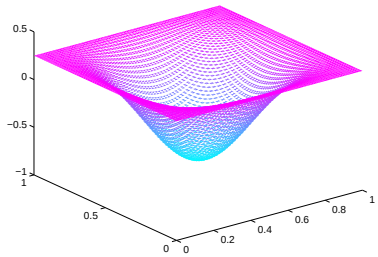


Figure 4: Two grid solution P_h. $t=2.0e-4$, $h=1/64$, $\tau=1.0e-5$.

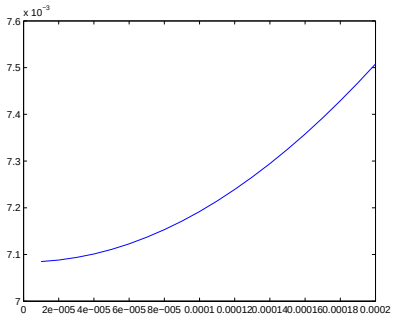


Figure 5: Error of Two grid solution C_h. $t=2.0e-4$, $h=1/64$, $\tau=1.0e-5$.

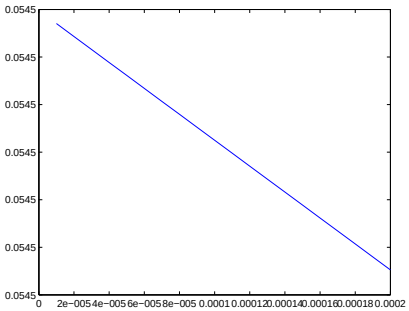


Figure 6: Error of Two grid solution P_h. $t=2.0e-4$, $h=1/64$, $\tau=1.0e-5$.

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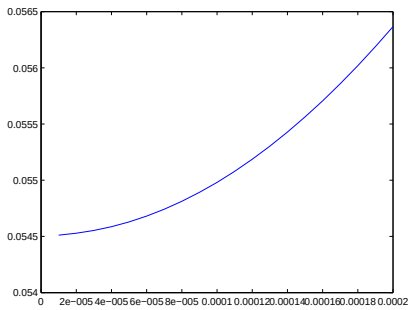


Figure 7: Error of Two grid solution $\boldsymbol{Z}_h$. $t=2.0e-4$, $h=1/64$, $\tau=1.0e-5$.

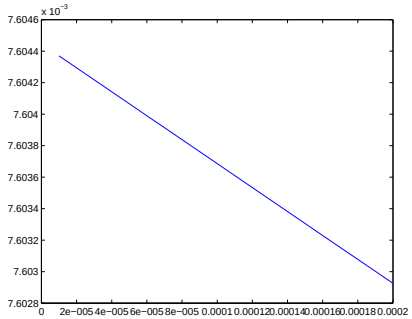


Figure 8: Error of Two grid solution $\boldsymbol{U}_h$. $t=2.0e-4$, $h=1/64$, $\tau=1.0e-5$.

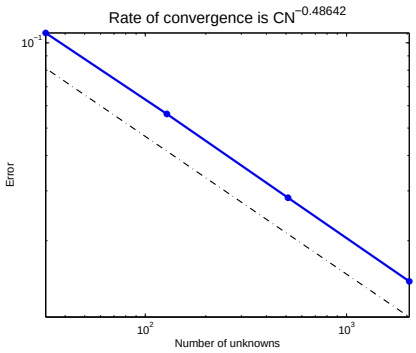


Figure 9: Order of Two grid solution $\boldsymbol{C}_h$. $t=2.0e-4$, $\tau=1.0e-5$.

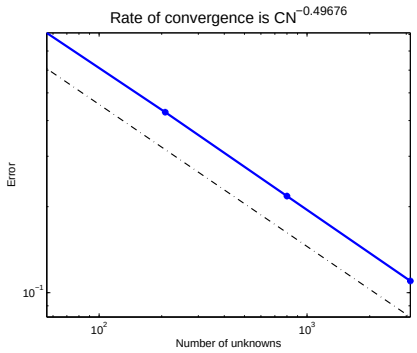


Figure 10: Order of Two grid solution $\boldsymbol{Z}_h$. $t=2.0e-4$, $\tau=1.0e-5$.

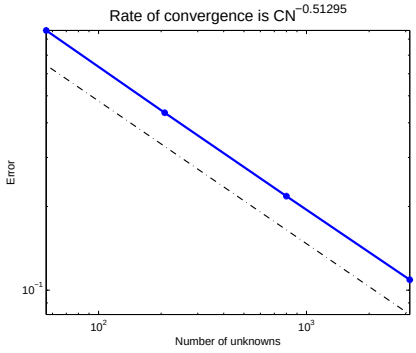


Figure 11: Order of Two grid solution $\boldsymbol{U}_h$. $t=2.0e-4$, $\tau=1.0e-5$.

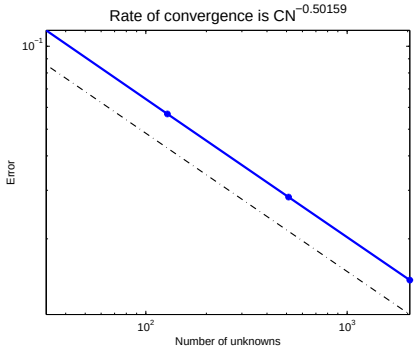


Figure 12: Order of Two grid solution $\boldsymbol{P}_h$. $t=2.0e-4$, $\tau=1.0e-5$.

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Example 6.2. we consider another incompressible miscible displacement problem, as follows:

$$\phi\frac{\partial c}{\partial t}+\boldsymbol{u}\cdot\nabla c-\nabla\cdot(\nabla c)=f(c), \tag{6.4}$$

$$\nabla\cdot\boldsymbol{u}=q, \tag{6.5}$$

$$\alpha(c)\boldsymbol{u}=-\nabla p, \tag{6.6}$$

where $\Omega=[0,1]\times[0,1]$, $t\in[0,T]$, $f(c)$ is suitably chosen such that the exact solution is $c=\sin^2(\pi x)\sin^2(\pi y)e^{-t}$, $\alpha(c)=c+2$, $p=-\frac{1}{2}c^2-2c+\frac{9}{128}e^{-2t}+\frac{1}{2}e^{-t}$.

The two grid solution C_h, P_h are shown in Fig. 13 and Fig. 14, respectively. The pressure distribution shows a low pressure in the center of the bottom of the cavity, Similarly, a high pressure is visible around the cavity. As indicated in Table 3 for $\|c-C_h\|$ and Table 4, we derive the same result similar to Example 6.1. We note that Table 3 have more cost time than Table 1, because Third equation of Example 6.2 is a nonlinear equation. in Figs. 15-18, we can observe that error of Two grid solution C_h, $\boldsymbol{Z}_h$, $\boldsymbol{U}_h$, P_h varied with time. From the Figs. 19-22, we know the algorithms achieve asymptotically optimal approximation applying the two-grid methods when the mesh size between the coarse grid and fine grid satisfy $h=\mathcal{O}(H^2)$ for Algorithm 1.

Table 3: Error and CPU time of Two grid and Newton iterative.

H	h	Newton iterative		Two grid	
$\frac{1}{2}$	$\frac{1}{4}$	1.084e-1	0.21s	1.084e-1	0.37s
$\frac{1}{4}$	$\frac{1}{16}$	2.840e-2	1.60s	2.840e-2	0.78s
$\frac{1}{8}$	$\frac{1}{64}$	7.500e-3	32.07s	7.500e-3	11.72s
$\frac{1}{16}$	$\frac{1}{128}$	4.300e-3	168.15s	4.300e-3	62.53s

Table 4: Error of Two grid.

H	h	$\|c^n-C_h^n\|$	rate	$\|\boldsymbol{z}^n-\boldsymbol{Z}_h^n\|$	rate	$\|\boldsymbol{u}^n-\boldsymbol{U}_h^n\|$	rate	$\|p^n-P_h^n\|$	rate
$\frac{1}{2}$	$\frac{1}{4}$	1.084e-1		8.100e-1		8.730e-1		2.856e-1	
$\frac{1}{4}$	$\frac{1}{16}$	2.840e-2	1	2.177e-1	1	2.179e-1	1	6.880e-2	1
$\frac{1}{8}$	$\frac{1}{64}$	7.500e-3	1	5.640e-2	1	5.450e-2	1	1.720e-2	1
$\frac{1}{16}$	$\frac{1}{128}$	4.300e-3	1	3.080e-2	1	2.730e-2	1	8.600e-3	1

7 Conclusion

In this paper, we present two-grid algorithms for coupled miscible displacement problems discretized by mixed finite element methods of characteristics and mixed finite element methods. The main ingredient of the two-grid method in this paper is that we use

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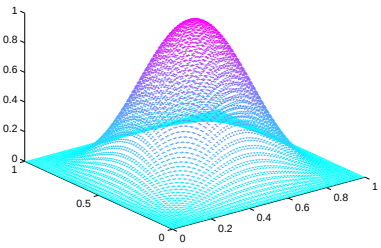


Figure 13: Two grid solution C_h. $t=2.0e-4$, $h=1/64$, $\tau=1.0e-5$.

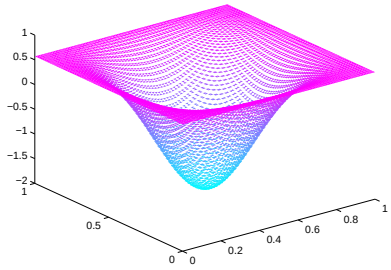


Figure 14: Two grid solution P_h. $t=2.0e-4$, $h=1/64$, $\tau=1.0e-5$.

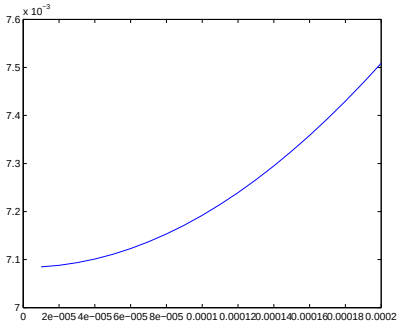


Figure 15: Error of Two grid solution C_h. $t=2.0e-4$, $h=1/64$, $\tau=1.0e-5$.

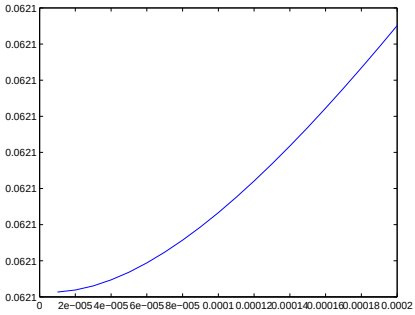


Figure 16: Error of Two grid solution P_h. $t=2.0e-4$, $h=1/64$, $\tau=1.0e-5$.

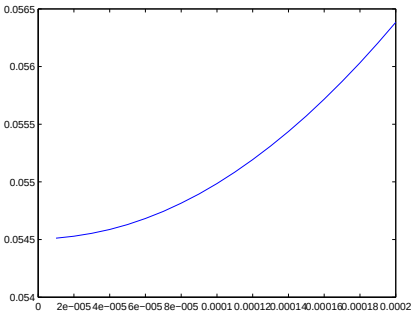


Figure 17: Error of Two grid solution $\boldsymbol{Z}_h$. $t=2.0e-4$, $h=1/64$, $\tau=1.0e-5$.

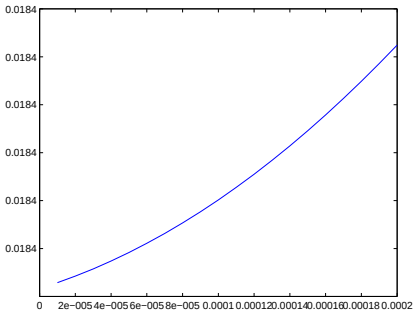


Figure 18: Error of Two grid solution $\boldsymbol{U}_h$. $t=2.0e-4$, $h=1/64$, $\tau=1.0e-5$.

the Newton iteration on the coarse grid and a correction technique on the fine grid. Assume $H_p=H_c=H$, $h_c=h_p=h$, and $l=k$, then, we know the algorithms achieve asymptotically optimal approximation applying the two-grid methods when the mesh size between

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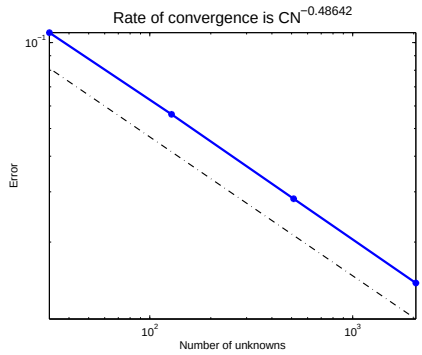


Figure 19: Order of Two grid solution C_h. $t = 2.0e-4$, $\tau = 1.0e-5$.

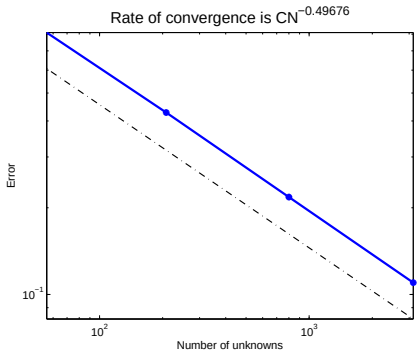


Figure 20: Order of Two grid solution Z_h. $t = 2.0e-4$, $\tau = 1.0e-5$.

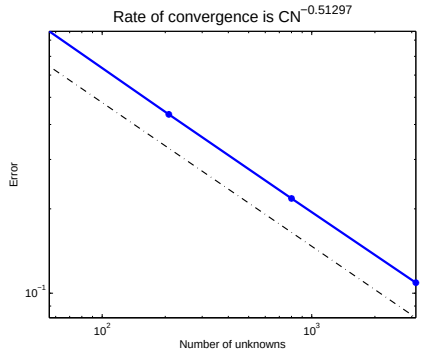


Figure 21: Order of Two grid solution U_h. $t = 2.0e-4$, $\tau = 1.0e-5$.

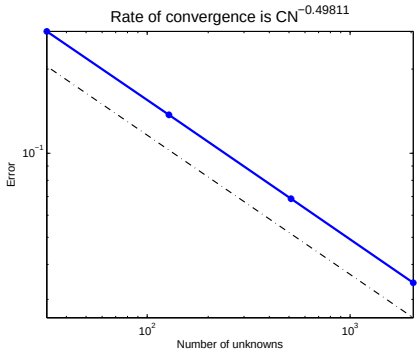


Figure 22: Order of Two grid solution P_h. $t = 2.0e-4$, $\tau = 1.0e-5$.

the coarse grid and fine grid satisfy $h = \mathcal{O}(H^2)$ for Algorithm 1. From the numerical results, we can achieve the same accuracy as the finite element method but with much less cost time since we just have to solve a small scale nonlinear problem. In general, different aspects of a complex problem can be treated by spaces of different scales. In the problem we studied in this paper, a very coarse grid space is sufficient for nonlinear problem that are dominated by linear part. The two-grid method studied in this paper provide a new approach to deal with many complex problem. Moreover, we will consider more complicated two-grid algorithms for such complex system and give numerical experiment for these algorithms.

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