The mean-square value of the divisor function

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1. *Introduction*

The behaviour of the divisor function $d(n)$ is rather tricky. For a prime p, we have $d(p) = 2$, but if n is the product of the first k primes then, by Chebyshev's estimate for the prime counting function [1, Theorem 414], we have $\frac{1}{2}k \log k < \log n < 2k \log k$ so that

$$d(n) = 2^k > 2^{\frac{\log n}{2 \log k}} > 2^{\frac{\log n}{4 \log \log n}};$$

for such n then, $d(n)$ is 'unusually large' — it can exceed any fixed power of $\log n$, for example.

In [2] Jameson gives, amongst other things, a derivation of Dirichlet's theorem, which shows that the mean-value of the divisor function in an interval containing n is $\log n$. However, the result is somewhat deceptive because, for most n, the value of $d(n)$ is substantially smaller than $\log n$. In fact [1, Theorem 432], for $\varepsilon > 0$, we have

$$2^{(1-\varepsilon) \log \log n} < d(n) < 2^{(1+\varepsilon) \log \log n},$$

for almost all n, so that $d(n)$ is 'usually' only about $2^{\log \log n} = (\log n)^{\log 2} = (\log n)^{0.69\cdots}$. There is no contradiction here — the result only highlights the fact that the mean value has been raised to the power of $\log n$ by the appearance of some unusually large $d(n)$. The phenomenon can be explained as follows: The average value of a collection of numbers can be 300, even though most of the numbers are near 10, because there are also a few which are in the millions.

Another way of detecting the phenomenon is to consider the higher moments of $d(n)$; this is because, for $k > 1$, the value for $d^k(n)$ will be accentuated when $d(n)$ is unusually large. Thus, if $d(n)$ were near the average value $\log n$ for most n, then the mean value of $d^k(n)$ would have the order $\log^k n$. It turns out that the mean value of $d^k(n)$ has the order $(\log n)^{2^k - 1}$; since $2^k - 1 > k$, the result is a reflection of the appearance of some unusually large $d(n)$.

For the mean-square value of the divisor function, we have the following theorem:

Theorem: (Ramanujan-Wilson).

There are constants A, B, C, D, K such that

$$\sum_{n \leqslant x} d^2(n) = Ax \log^3 x + Bx \log^2 x + Cx \log x + Dx + E(x),$$

where $E(x)$ satisfies $|E(x)| \leqslant Kx^{\frac{1}{2}} \log^6 x$ for all large x.

In 1916, S. Ramanujan [3] stated the theorem with the error term

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estimate $|E(x)| < x^{\frac{3}{5}+\varepsilon}$, but without proof. In 1922, B. M. Wilson [4] established the theorem with the estimate $|E(x)| < x^{\frac{1}{2}+\varepsilon}$. The current best estimate is $|E(x)| \leqslant Kx^{\frac{1}{2}}\log^5 x$, by Chaohua Jia and Ayyadurai Sankaranarayanan [5]. We use the letter K for a generic constant, so that its value is not necessarily the same at each of its appearances.

The proof of the theorem is based on the use of Dirichlet series as the generating function of $d^2(n)$. The argument then depends on properties of the Riemann zeta function $\zeta(s)$ and also contour integration, and the main purpose of the article is to let the reader have a glimpse of the action in the application of a powerful method in analytic number theory.

I wish to thank the referee for pointing out that there is also an elementary treatment of the problem, a sketch of which is given in the last section.

2. *The Riemann zeta function $\zeta(s)$, and the Dirichlet series associated with $d^2(n)$*

Following established tradition in analytic number theory, we write $s = \sigma + it$ for a complex number s, with real part σ and imaginary part t. The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \sigma > 1;$$

the series is absolutely convergent, and uniformly so in $\sigma \geqslant c > 1$ for any $c > 1$, so that it is an analytic function of s. Indeed $\zeta(s)$ has an analytic continuation into the whole complex plane, where the function has a simple pole at $s = 1$, but we need only consider $\zeta(s)$ in the half-plane $\sigma \geqslant \frac{1}{2}$. As Euler already discovered, by considering the unique decomposition of a number into its prime divisors, $\zeta(s)$ also has an 'Euler product' representation:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ \ldots\right), \qquad \sigma > 1, \qquad p \text{ prime}.$$

Note that the convergence of the product implies that $\zeta(s)$ has no zeros in $\sigma > 1$, but we shall also make use of the deeper result that it has no zeros on $\sigma = 1$ either. Other properties of $\zeta(s)$ will be stated later when they are needed; their proofs can be found in [6], the famous text on the Riemann zeta function by E. C. Titchmarsh.

Next, by a Dirichlet series we mean a series of the form $\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$, which converges in some half-plane $\sigma > c$. Thus, the zeta function, for which $a(n) = 1$ for all n, is the simplest example of such a series. If $a(n)$ is a multiplicative function of n, that is $a(mn) = a(m)a(n)$ when m, n are coprime, then the series also enjoys an Euler product representation:

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$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p \sum_{m=0}^{\infty} \frac{a(p^m)}{p^{ms}} = \prod_p \left(1 + \frac{a(p)}{p^s} + \frac{a(p^2)}{p^{2s}} + \dots\right);$$

we call the function of s here the generating function of $a(n)$, and $\sum_{n \leqslant x} a(n)$ its sum function.

The proof of the theorem is based on the fact that the generating function of $d^2(n)$ is representable in terms of zeta functions.

Lemma 1:

$$\sum_{n=1}^{\infty} \frac{d^2(n)}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)}, \qquad \sigma > 1.$$

Proof: By Theorem 315 in [1], $d(n) < n^{\varepsilon}$ for every $\varepsilon > 0$ and all large n, so that the Dirichlet series $\sum_{n=1}^{\infty} \frac{d^2(n)}{n^s}$ is absolutely convergent in $\sigma > 1$. Since $d^2(n)$ is a multiplicative function, the series has an Euler product representation:

$$\sum_{n=1}^{\infty} \frac{d^2(n)}{n^s} = \prod_p \sum_{m=0}^{\infty} \frac{d^2(p^m)}{p^{ms}} = \prod_p \sum_{m=0}^{\infty} \frac{(m+1)^2}{p^{ms}}, \qquad \sigma > 1.$$

On the other hand, we also have the following Euler product representation for $\frac{\zeta^4(s)}{\zeta(2s)}$:

$$\frac{\zeta^4(s)}{\zeta(2s)} = \prod_p \frac{1 - p^{-2s}}{(1 - p^{-s})^4} = \prod_p \frac{1 + p^{-s}}{(1 - p^{-s})^3}, \qquad \sigma > 1,$$

so that it remains to show that the two Euler products are the same. Now, on differentiating the geometric progression $\sum u^m$ twice, we find that

$$\frac{1+u}{(1-u)^3} = \frac{1}{2} \sum_{m=0}^{\infty} \big((m+1)(m+2) + m(m+1)\big) u^m = \sum_{m=0}^{\infty} (m+1)^2 u^m, \quad |u| < 1.$$

Setting $u = p^{-s}$, so that $|u| = p^{-\sigma} < 1$, the required result follows.

3. *Perron's formula, and an asymptotic formula for the sum function*

A useful method in analytic number theory enables us to represent the sum function of an arithmetic function as a contour integral involving its generating function. For generating functions with Dirichlet series, we have Perron's formula (see §7): If $A(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ is absolutely convergent in $\sigma \geqslant c > 0$, then

$$\sum_{n \leqslant x} a(n) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \frac{A(s) x^s}{s} \, ds;$$

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here, if x is an integer then the last term in the sum should be taken as $\frac{1}{2}a(x)$ instead of $a(x)$. We take x to be half an odd number to avoid the complication.

By itself, Perron's formula is of not much use. In practice, we only want an estimate for the sum function when x is large, so we set $T = T(x) \to \infty$ as $x \to \infty$, and replace the infinite line of integration by the finite segment

$$\Gamma_1 = \{c - iT, c + iT\} = \{s = c + it : |t| < T\}.$$

We choose T large enough to ensure that the incurred error is smaller than the expected dominating terms for the sum; see Lemma 3 below. Moreover, by exploiting the analytic properties of the generating function $A(s)$, the integral along Γ_1 can be estimated by applying Cauchy's residue theorem. After such manoeuvres, an asymptotic formula for the sum $\sum_{n \leqslant x} a(n)$ often emerges.

Let us now set $a(n) = d^2(n)$, so that, by Lemma 1, $A(s) = \dfrac{\zeta^4(s)}{\zeta(2s)}$, and we have

$$\sum_{n \leqslant x} d^2(n) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\zeta^4(s)}{\zeta(2s)} \cdot \frac{x^s}{s}\, ds + E_1, \qquad c > 1,$$

where $E_1 = E_1(x, c, T)$ is an error term if c is near 1, and T is large; as we shall see in Lemma 3 below, if $c = 1 + \varepsilon$ then $|E_1| \leqslant x^{\frac{1}{4} + \varepsilon}$ when $T = x^{\frac{3}{4}}$, say. We still have to estimate the integral along the vertical line segment $\Gamma_1 = \{c - iT, c + iT\}$, and it is time to examine the integrand. The following are pertinent remarks on the four factors concerned:

(i) The main contribution to the integral comes from the effect of the 4th-order pole at $s = 1$ arising from the factor $\zeta^4(s)$, so that the residue there is vitally relevant;

(ii) The factor $1/\zeta(2s)$ should not bother us too much, provided we stay clear of the zeros of $\zeta(2s)$. We already remarked in §2 that $\zeta(s)$ has no zeros in $\sigma \geqslant 1$, so that $1/\zeta(2s)$ is analytic in $\sigma \geqslant \frac{1}{2}$, but we need to estimate $1/\zeta(2s)$;

(iii) The simple pole at $s = 0$ is sufficiently far away from the action to be ignored. However, we may need to make good use of the fact that $1/s$ is small in magnitude when t is large.

(iv) The factor $x^s = x^\sigma x^{it} = x^\sigma \exp(it \log x)$ has magnitude x^σ, and an oscillating part $\exp(it \log x)$ as t varies. However, there is no real hope in making use of the cancellation effect from the oscillation, so we can only apply $|x^s| = x^\sigma$ in an estimate.

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We now form the rectangular contour $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$, by setting

$$\Gamma_2 = \{c + iT, \sigma + iT\}, \quad \Gamma_3 = \{\sigma + iT, \sigma - iT\},$$

$$\Gamma_4 = \{\sigma - iT, c - iT\}; \qquad c = 1 + \varepsilon, \qquad \sigma = \tfrac{1}{2}.$$

Since the integrand is analytic inside Γ, apart from the 4th-order pole at $s = 1$, we may apply Cauchy's residue theorem: $I_1 + I_2 + I_3 + I_4 = 2\pi i \alpha_1$, where

$$I_j = \int_{\Gamma_j} \frac{\zeta^4(s)}{\zeta(2s)} \cdot \frac{x^s}{s}\, ds, \qquad \alpha_1 = \operatorname*{Res}_{s=1} \left\{ \frac{\zeta^4(s)}{\zeta(2s)} \cdot \frac{x^s}{s} \right\},$$

so that

$$\sum_{n \leqslant x} d^2(n) = \alpha_1 + E(x), \qquad \text{where } |E(x)| \leqslant |E_1| + |I_2 + I_3 + I_4|.$$

It remains to show that $\alpha_1 = xP(\log x)$, where $P(u)$ is a certain cubic, and that, for a suitable choice of T, the contributions from E_1 and the integrals I_2, I_3, I_4 are less than $Kx^{\frac{1}{2}} \log^6 x$.

4. *The residue of* $\dfrac{\zeta^4(s) x^s}{\zeta(2s)\, s}$ *at the pole* $s = 1$

We consider the Laurent expansion of the integrand at its 4th-order pole $s = 1$ by writing

$$\frac{\zeta^4(s)}{\zeta(2s)} \cdot \frac{x^s}{s} = G(s) + \frac{\alpha_1}{s-1} + \frac{\alpha_2}{(s-1)^2} + \frac{\alpha_3}{(s-1)^3} + \frac{\alpha_4}{(s-1)^4},$$

where $G(s)$ is analytic at $s = 1$, and $\alpha_4 \neq 0$. Then

$$\frac{(s-1)^4 \zeta^4(s)}{\zeta(2s)} \cdot \frac{x^s}{s} = G(s)(s-1)^4 + \alpha_1(s-1)^3 + \alpha_2(s-1)^2 + \alpha_3(s-1) + \alpha_4,$$

so that the required residue is given by

$$\alpha_1 = \frac{1}{6} \left(\frac{d}{ds} \right)^3 \{F(s) x^s\} \Big|_{s=1}, \qquad \text{where } F(s) = \frac{(s-1)^4 \zeta^4(s)}{\zeta(2s)\, s}.$$

Since $x^s = \exp(s \log x)$, so that $\left(\frac{d}{ds}\right)^k \{x^s\} = x^s \log^k x$ for $k = 0, 1, 2, 3$, it follows that $\alpha_1 = xP(\log x)$, with $P(u)$ being a certain cubic (see §8), as required.

5. *An estimate for the integral along* Γ_3

The difficult part of the proof of the theorem is to estimate the contribution from the integral I_3. On the vertical path Γ_3, we have $s = \frac{1}{2} + it$, so that $|x^s| = x^{\frac{1}{2}}$. The mean value of the fourth moment of $\zeta(s)$ on $\zeta = \frac{1}{2}$, $1 \leqslant t \leqslant T$ has the order $\log^4 T$; more precisely, it is known

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((7.6.1) in [6]) that

$$\int_1^T |\zeta(\tfrac{1}{2} + it)|^4\, dt \leqslant KT \log^4 T;$$

we also make use of (3.11.8) in [6]:

$$\frac{1}{|\zeta(\sigma + it)|} \leqslant K \log(t+1), \qquad \sigma \geqslant 1, \qquad t \geqslant 1.$$

The same estimates apply in $-T \leqslant t \leqslant -1$, and the contribution from $|t| < 1$ to I_3 is at most $Kx^{\frac{1}{2}}$, so that

$$|I_3| \leqslant Kx^{\frac{1}{2}} \log T \int_1^T \frac{|\zeta(\sigma + it)|^4\, dt}{t}.$$

Lemma 2:

$$\int_1^T \frac{|\zeta(\sigma + it)|^4\, dt}{t} \leqslant K \log^5 T.$$

Proof: We need to annihilate the factor T arising from the length of integration, by making use of the factor $1/t$. The technique is to divide $1 < t \leqslant T$ into subintervals of the form $u < t \leqslant 2u$, inside which we can use $1/u$ as an upper bound for the factor $1/t$ in the integrand. Thus, on dividing the interval $1 < t \leqslant T$ into subintervals $2^{h-1} < t \leqslant 2^h$, with $h = 1, 2, \ldots, H$, where $2^{H-1} < T \leqslant 2^H$, so that $H \approx (\log T)/(\log 2)$, we find that

$$\int_1^T \frac{\left|\zeta(\frac{1}{2}+it)^4\right|}{t}\, dt \leqslant \sum_{h=1}^{H} \int_{2^{h-1}}^{2^h} \frac{\left|\zeta(\frac{1}{2}+it)^4\right|}{t}\, dt \leqslant \sum_{h=1}^{H} \frac{1}{2^{h-1}} \int_{2^{h-1}}^{2^h} \left|\zeta(\tfrac{1}{2}+it)^4\right| dt$$

$$\leqslant K \sum_{h=1}^{H} \frac{2^h \log^4(2^h)}{2^{h-1}} \leqslant K \sum_{h=1}^{H} h^4 \leqslant KH^5 \leqslant K \log^5 T.$$

We have established that $|I_3| \leqslant Kx^{\frac{1}{2}} \log^6 T \leqslant Kx^{\frac{1}{2}} \log^6 x$, for $T \leqslant x$.

6. *Completion of the proof*

On Γ_2, we have $s = \sigma + iT$, with $\frac{1}{2} \leqslant \sigma \leqslant c = 1 + \varepsilon$, so that $|x^s| = x^\sigma$ varies accordingly. Next, it follows from Theorem 5.12 in [6] that

$$|\zeta(\sigma + iT)| \leqslant KT^{\frac{1}{3}(1-\sigma)+\varepsilon},$$

so that

$$|I_2| \leqslant K \max_{\frac{1}{2} \leqslant \sigma \leqslant 1+\varepsilon} \frac{T^{\frac{4}{3}(1-\sigma)+4\varepsilon} (\log T) x^\sigma}{T} \leqslant KT^{\frac{4}{3}+\varepsilon} \log T \max_{\frac{1}{2} \leqslant \sigma \leqslant 1+\varepsilon} T^{-\frac{4\pi}{3}} x^\sigma.$$

Setting $T = x^{\frac{3}{4}}$, we then find that $|I_2| \leqslant Kx^{\frac{1}{4}+5\varepsilon}$, and the same estimate applies for I_4. Thus, collecting all our estimates with $T = x^{\frac{3}{4}}$, the theorem is proved.

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7. *Remarks on Perron's formula*

Perron's formula is not that mysterious. Since the series for $A(s)$ converges absolutely, we may perform term-by-term integration, and apply the rather handy formula

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{X^s}{s}\,ds = \begin{cases} 0, & 0 \leqslant X < 1, \\ \frac{1}{2}, & X = 1, \\ 1, & X > 1, \end{cases}$$

which can be established by the residue theorem, to deliver

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{A(s)x^s}{s}\,ds = \sum_{n=1}^{\infty}\frac{a(n)}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{x}{n}\right)^s\frac{ds}{s} = \sum_{n\leqslant x}a(n).$$

The following lemma [6, Lemma 3.12] gives a more precise statement for the error incurred when replacing the infinite contour $\{c - i\infty, c + i\infty\}$ in Perron's formula by the finite segment Γ_1.

Lemma 3: (Perron's truncated formula)

Let $A(s) = \sum_{n=1}^{\infty}\frac{a(n)}{n^s}$ be absolutely convergent in $\sigma > 1$. Suppose that $|a(n)| \leqslant K\psi(n)$, where $\psi(n)$ is increasing, and that, as $\sigma \to 1_+$,

$$\sum_{n=1}^{\infty}\frac{|a(n)|}{n^\sigma} \leqslant \frac{K}{(\sigma-1)^\alpha}.$$

Then, for $c > 1$ and x being half an odd integer,

$$\sum_{n<x}a(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{A(s)x^s}{s}\,ds + E_1 + E_2,$$

where

$$|E_1| \leqslant \frac{Kx^c}{T(c-1)^\alpha} \qquad \text{and} \qquad |E_2| \leqslant \frac{K\psi(2x)x\log x}{T}.$$

Thus, for $a(n) = d^2(n)$, we may take $\psi(n) = n^\varepsilon$ and, by Lemma 1, we may take $\alpha = 4$. We can ensure that E_1, E_2 have order at most $x^{\frac{1}{4}+2\varepsilon}$ if we fix $c = 1 + \varepsilon$ and $T = x^{\frac{3}{4}}$, for example.

8. *The values for A, B, C, D*

The values for the coefficients A, B, C, D for the cubic are not given in the theorem. They are, in effect, the first four coefficients of the Taylor series expansion for $F(s)$ at $s = 1$. Thus, from

$$\left(\frac{d}{ds}\right)^3\{F(s)x^s\} = F(s)x^s\log^3 x + 3F'(s)x^s\log^2 x + 3F''(s)x^s\log x + F'''(s)x^s,$$

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we find that

$$A = \frac{F(1)}{6}, \qquad B = \frac{F'(1)}{2}, \qquad C = \frac{F''(1)}{2}, \qquad D = \frac{F'''(1)}{6}.$$

Such expressions will involve the coefficients of the Laurent expansion of $\zeta(s)$ about its simple pole at $s = 1$, together with the coefficients of the expansion for $1/\zeta(2s)$ at $s = 1$. We have

$$\begin{aligned}\zeta(s) &= \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\gamma_n (s-1)^n \\ &= \frac{1}{s-1} + \gamma_0 - \gamma_1(s-1) + \tfrac{1}{2}\gamma_2(s-1)^2 + \dots,\end{aligned}$$

where

$$\gamma_n = \lim_{m\to\infty}\left\{\sum_{k=1}^{m}\frac{(\log k)^n}{k} - \frac{(\log m)^{n+1}}{n+1}\right\}, \qquad n = 0, 1, 2, \dots;$$

in particular, γ_0 is Euler's constant, which is usually denoted by γ. Thus, from

$$(s-1)^4\zeta^4(s) = (1+\gamma(s-1)+\dots)^4 = 1 + 4\gamma(s-1) + \dots,$$

$$\frac{1}{\zeta(2s)} = \frac{1}{\zeta(2)} - \frac{2\zeta'(2)}{\zeta^2(2)}(s-1) + \dots,$$

$$\frac{1}{s} = \frac{1}{1+(s-1)} = 1 - (s-1) + \dots,$$

we find that

$$F(s) = \frac{1}{\zeta(2)} + \left\{\frac{4\gamma - 1}{\zeta(2)} - \frac{2\zeta'(2)}{\zeta^2(2)}\right\}(s-1) + \dots,$$

so that

$$6A = F(1) = \frac{1}{\zeta(2)}, \qquad 2B = F'(1) = \frac{4\gamma-1}{\zeta(2)} - \frac{2\zeta'(2)}{\zeta^2(2)}.$$

With computer algebra, we also find that

$$2C = \frac{u_1}{\zeta(2)} + \frac{u_2}{\zeta^2(2)} + \frac{u_3}{\zeta^3(2)} \text{ and } 6D = \frac{u_1}{\zeta(2)} + \frac{u_2}{\zeta^2(2)} + \frac{u_3}{\zeta^3(2)} + \frac{u_4}{\zeta^4(2)},$$

where

$$\begin{aligned}u_1 &= 1, \qquad u_2 = (4\gamma-1)\zeta(2) - 2\zeta'(2), \\ u_3 &= (6\gamma^2 - 4\gamma - 4\gamma_1 + 1)\zeta^2(2) - (4\gamma-1)\zeta(2)\zeta'(2) + 4\zeta'(2)^2 - 2\zeta(2)\zeta''(2) \\ u_4 &= \frac{1}{3}\big(3(4\gamma^3 - 6\gamma^2 - 4(3\gamma-1)\gamma_1 + 4\gamma + 2\gamma_2 - 1)\zeta^3(2)\big) \\ &\quad - 6(6\gamma^2 - 4\gamma - 4\gamma_1 + 1)\zeta^2(2)\zeta'(2) + 12(4\gamma-1)\zeta(2)\zeta'(2)^2 - 24\zeta'(2)^3 \\ &\quad - 4\zeta(2)^2\zeta'''(2) - 6\big((4\gamma-1)\zeta(2)^2 - 4\zeta(2)\zeta'(2)\big)\zeta''(2).\end{aligned}$$

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9. *An elementary derivation of the asymptotic formula*

The arithmetic equivalent of Lemma 1 is the equation

$$d^2(n) = \sum_{ab^2 = n} d_4(a)\mu(b),$$

which can be established by considering n being a power of a prime; here $d_k(n)$ is the number of ways of expressing n as a product of k factors, and $\mu(n)$ is the Möbius function. Moreover, the asymptotic formula for $\sum_{n \leqslant x} d_k(n)$ was considered by Dirichlet, and the classical elementary theorem (see [6, Section 12.1]) is that

$$\sum_{n \leqslant x} d_k(n) = xP_k(\log x) + \Delta_k(x),$$

where $P_k(u)$ is a polynomial of degree $k - 1$, and $|\Delta_k(x)| \leqslant Kx^{1-1/k}\log^{k-2}x$; in particular $|\Delta_4(x)| \leqslant Kx^{\frac{3}{4}}\log^2 x$, and it can be shown that the leading coefficient of $P_4(u)$ is $\frac{1}{6}$.

It follows from Dirichlet's result that

$$\sum_{n \leqslant x} d^2(n) = \sum_{ab^2 \leqslant x} d_4(a)\mu(b) = \sum_{b \leqslant \sqrt{x}} \mu(b) \sum_{a \leqslant x/b^2} d_4(a)$$

$$= \sum_{b \leqslant \sqrt{x}} \mu(b)\left(\frac{x}{b^2}P_4\left(\log\frac{x}{b^2}\right) + \Delta_k\left(\frac{x}{b^2}\right)\right).$$

The asymptotic formula in the theorem can now be obtained by extending the sum to $b \geqslant 1$, allowing the terms with $b > \sqrt{x}$ to form another error term. The cubic $P(u)$ then emerges from the sum $\sum \mu(b) P_4\left(\frac{\log x}{b^2}\right)\frac{1}{b^2}$; indeed, the coefficient A in the theorem is now given by

$$A = \frac{1}{6}\sum_{b=1}^{\infty}\frac{\mu(b)}{b^2} = \frac{1}{6\zeta(2)},$$

and the values for B, C, D can also be evaluated accordingly.

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These *Nemo* quotations are to be identified by reference to author and work. Solutions are invited to the Editor by 30th September 2016.

1. There were other fashionable spheres above them, and other fashionable spheres below, and none of the stars in any one of these spheres had anything to say to the stars in any other of these spheres.

2. Now, now, you stars, that move in your right spheres,
Where be your powers?

3. Mr Casson's person was by no means of that common type which can be allowed to pass without description. On a front view it appeared to consist principally of two spheres, bearing about the same relation to each other as the earth and the moon: that is to say, the lower sphere might be said, at a rough guess, to be thirteen times larger than the upper, which naturally performed the function of a mere satellite and tributary.

4. When I survey the bright
Celestial sphere;
So rich with jewels hung, that Night
Doth like an Ethiop bride appear –

5. While circling Time moves round in an eternal sphere –

6. The Grass so little has to do,
A Sphere of simple green
With only Butterflies to brood,
And Bees to entertain.

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