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Two more hypergeometric integral equations

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Summary. This paper is a sequel to one with a similar title to appear in the *Proceedings of the Edinburgh Mathematical Society*. Explicit solutions are found for two more integral equations of similar form; and also conditions necessary and sufficient for existence, and sufficient for uniqueness, of solutions. These theorems are preceded by several preparatory theorems on fractional integrals with origin ∞, including integrals of purely imaginary order.

1. Introduction

In (12) I gave theorems on solutions of the two integral equations

$$\int_0^x \frac{(x-t)^{c-1}}{\Gamma(c)} F\left(a,b,c,1-\frac{x}{t}\right) f(t)\,dt =^0 g(x) \quad \text{for} \quad 0<x<d, \tag{A}$$

$$\int_0^x \frac{(x-t)^{c-1}}{\Gamma(c)} F\left(a,b,c,1-\frac{t}{x}\right) f(t)\,dt =^0 g(x) \quad \text{for} \quad 0<x<d, \tag{B}$$

where F is the hypergeometric function, f is the unknown function, $=^0$ means 'equals almost everywhere' and $0<d\leqslant\infty$. These were accompanied by auxiliary theorems on the equation

$$\int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)} t^a f(t)\,dt =^0 x^a \int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)} g(t)\,dt \quad \text{for} \quad 0<x<d.$$

Here I give corresponding theorems on the two integral equations

$$\int_x^\infty \frac{(t-x)^{c-1}}{\Gamma(c)} F\left(a,b,c,1-\frac{x}{t}\right) f(t)\,dt =^0 g(x) \quad \text{for} \quad d<x<\infty, \tag{C}$$

$$\int_x^\infty \frac{(t-x)^{c-1}}{\Gamma(c)} F\left(a,b,c,1-\frac{t}{x}\right) f(t)\,dt =^0 g(x) \quad \text{for} \quad d<x<\infty, \tag{D}$$

where $0\leqslant d<\infty$. Special cases of these have received attention—see Erdélyi (5) and Love (12)—from Ta Li (1), Buschman (2), Higgins (3) and (6), Srivastava (4) and Wimp (7). I give conditions for unique solutions (Theorems 4 C and 4 D), necessary conditions for existence of solutions (Theorems 7 C and 7 D), and sufficient conditions for validity of explicit solutions (Theorems 8 C–10 C and 8 D–10 D). These theorems are found in sections 5 and 7, and they are numbered to correspond with the numbering in (12).

The essence of the method used is to transform (C) into (B) by a reciprocal transformation $x=1/x'$ and $t=1/t'$, and thus to translate the theorems about (B) in (12) into theorems about (C). This simple idea seems to be rather less trivial to implement

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than might be expected. In fact I have had to fall back on more fundamental methods in section 6, in order to obtain certain extreme cases of the theorems. Theorems on (D) are obtained in section 7 from those on (C) using one of Kummer's relations. I have also checked most of these using reciprocal transformations instead, but these checks are too long to include here.

The solutions are expressed in terms of fractional integrals with origin ∞, that is, the transform J^c given by

$$J^c f(x) = \int_x^\infty \frac{(t-x)^{c-1}}{\Gamma(c)} f(t)\,dt.$$

In preparation for this, section 2 contains several theorems on this transform and section 3 on its inverse: a sort of miniature, suited to the present context, of the works of Tamarkin (8) and Kober (9, 11).

2. Fractional integrals with origin ∞

Integrability definition. *R_r is the class of complex-valued 'functions' f on (d,∞) such that $x^r f(x)$ is L-integrable on (l,∞) for each $l > d$.*

Here 'function' means 'class of equivalent functions', that is, class of functions defined almost everywhere and agreeing with one another almost everywhere. Also r is real, and $d \geqslant 0$.

R_r is contracting: If $p < q$ then $R_p \supset R_q$, and the inclusion is proper.

This follows from the inequality, for $l > 0$ and measurable functions f,

$$\int_l^\infty x^p |f(x)|\,dx \leqslant l^{p-q} \int_l^\infty x^q |f(x)|\,dx;$$

and from the instance $f(x) = 1/x^{q+1}\log x$ if $x > e$, $f(x) = 0$ otherwise.

Fractional integral definition. *For complex c with $\Re c > 0$, $J^c f$ is the function whose values are*

$$J^c f(x) = \int_x^\infty \frac{(t-x)^{c-1}}{\Gamma(c)} f(t)\,dt. \tag{1}$$

Theorem 1. *If $\gamma = \Re c > 0$ and $f \in R_{\gamma-1}$, then $J^c f(x)$ is defined for almost all x in (d,∞), and is locally integrable therein.*

If $\lambda > \gamma > 0$ and $f \in R_{\lambda-1}$, then $J^c f$ exists in $R_{\lambda-\gamma-1}$.

(i) Suppose $f \in R_{\gamma-1}$. It will be enough to prove that $J^c f(x)$ is defined for almost all x in $(l, l+1)$, and that it is integrable on $(l, l+1)$, for each $l > d$. For this we have, remembering that $\gamma > 0$ and $l > 0$,

$$\int_l^{l+2} dx \int_x^{l+2} (t-x)^{\gamma-1} |f(t)|\,dt = \int_l^{l+2} |f(t)|\,dt \int_l^t (t-x)^{\gamma-1}\,dx$$

$$= \int_l^{l+2} \frac{(t-l)^\gamma}{\gamma} |f(t)|\,dt \leqslant \frac{2^\gamma}{\gamma} \int_l^{l+2} |f(t)|\,dt,$$

$$\leqslant \frac{2^\gamma}{\gamma l^{\gamma-1}} \int_l^{l+2} t^{\gamma-1} |f(t)|\,dt$$

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if $\gamma \geqslant 1$, while if $\gamma < 1$ this is true if the denominator factor $l^{\gamma-1}$ is replaced by $(l+2)^{\gamma-1}$. Since $f \in R_{\gamma-1}$, we have shown that

$$\int_x^{l+2} (t-x)^{\gamma-1} |f(t)|\, dt \tag{2}$$

is integrable on $(l, l+2)$; consequently it is finite for almost all x in $(l, l+1)$, and integrable thereon.

Further, for all x in $(l, l+1)$ we have, if $\gamma \geqslant 1$,

$$\int_{l+2}^{\infty} (t-x)^{\gamma-1} |f(t)|\, dt \leqslant \int_{l+2}^{\infty} t^{\gamma-1} |f(t)|\, dt < \infty;$$

while if $\gamma < 1$ the left side is shown to be finite by the inequality

$$\begin{aligned}\int_{l+2}^{\infty} (t-x)^{\gamma-1} |f(t)|\, dt &= \int_{l+2}^{\infty} \left(1-\frac{x}{t}\right)^{\gamma-1} t^{\gamma-1} |f(t)|\, dt \\ &\leqslant \left(1-\frac{l+1}{l+2}\right)^{\gamma-1} \int_{l+2}^{\infty} t^{\gamma-1} |f(t)|\, dt.\end{aligned}$$

Combined with (2), this shows that (1) is absolutely convergent for almost all x in $(l, l+1)$, and is integrable thereon; from which the first conclusion follows.

(ii) Suppose $f \in R_{\lambda-1}$. For each $l > d$,

$$\begin{aligned}\int_l^{\infty} x^{\lambda-\gamma-1}\, dx \int_x^{\infty} (t-x)^{\gamma-1} |f(t)|\, dt &= \int_l^{\infty} |f(t)|\, dt \int_l^t x^{\lambda-\gamma-1}(t-x)^{\gamma-1}\, dx \\ &= \int_l^{\infty} |f(t)|\, t^{\lambda-1}\, dt \int_{l/t}^1 s^{\lambda-\gamma-1}(1-s)^{\gamma-1}\, ds \\ &\leqslant \frac{\Gamma(\lambda-\gamma)\,\Gamma(\gamma)}{\Gamma(\lambda)} \int_l^{\infty} |f(t)|\, t^{\lambda-1}\, dt < \infty.\end{aligned}$$

Thus $J^c f(x)$ exists for almost all $x > l$, and is measurable; and

$$\int_l^{\infty} x^{\lambda-\gamma-1} |J^c f(x)|\, dx < \infty$$

for each $l > d$, so that $J^c f \in R_{\lambda-\gamma-1}$ as required.

Remark. For real values of c, Theorem 1 *is best possible in the following respects*:

(i) *if* $\lambda < \gamma$, *there is* $f \in R_{\lambda-1}$ *such that* $J^c f(x)$ *exists nowhere*;

(ii) *if* $\lambda = \gamma$, *there is* $f \in R_{\lambda-1}$ *such that* $J^c f$ *is not in* R_{-1};

(iii) *if* $\lambda > \gamma$, *there is* $f \in R_{\lambda-1}$ *such that* $J^c f$ *is not in* $R_{\mu-1}$ *for any* $\mu > \lambda-\gamma$.

(i) Let $\lambda < \mu < \gamma$ and $f(x) = \Gamma(\gamma) x^{-\mu}$; then $f \in R_{\lambda-1}$ but

$$J^c f(x) = J^{\gamma} f(x) = \int_x^{\infty} \frac{(t-x)^{\gamma-1}}{t^{\mu}}\, dt$$

is clearly divergent for every $x \geqslant d$.

(ii) and (iii) Let $\gamma > 0$, $f(x) = \Gamma(\gamma)/x^{\lambda} (\log x)^2$ for all $x > e$, and $f(x) = 0$ otherwise; then $f \in R_{\lambda-1}$. First notice that

$$(1-s)^{\gamma-1} \geqslant \delta = \min(1, 2^{1-\gamma}) \quad \text{if} \quad 0 \leqslant s \leqslant \tfrac{1}{2},$$

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and that δ so defined is positive. Now, l and the integrand being positive,

$$\int_l^\infty x^{\mu-1}\,dx\int_x^\infty \frac{(t-x)^{\gamma-1}}{t^\lambda(\log t)^2}\,dt = \int_l^\infty \frac{1}{t^\lambda(\log t)^2}\,dt\int_l^t x^{\mu-1}(t-x)^{\gamma-1}\,dx$$
$$= \int_l^\infty \frac{t^{\mu+\gamma-1}}{t^\lambda(\log t)^2}\,dt\int_{l/t}^1 s^{\mu-1}(1-s)^{\gamma-1}\,ds;$$

substituting $x = ts$ in the inner integral. So, if $l > e$,

$$\int_l^\infty x^{\mu-1}J^\gamma f(x)\,dx \geqslant \int_{2l}^\infty \frac{t^{\mu-\lambda+\gamma}}{t(\log t)^2}\,dt\int_{l/t}^{\frac{1}{2}} s^{\mu-1}(1-s)^{\gamma-1}\,ds$$
$$\geqslant \delta\int_{2l}^\infty \frac{t^{\mu-\lambda+\gamma}}{t(\log t)^2}\,dt\int_{l/t}^{\frac{1}{2}} s^{\mu-1}\,ds. \tag{3}$$

(ii) Let $\lambda = \gamma$ and $\mu = 0$. From (3) we have, if $l > e$,

$$\int_l^\infty x^{-1}J^\gamma f(x)\,dx \geqslant \delta\int_{2l}^\infty \frac{1}{t(\log t)^2}\log\frac{t}{2l}\,dt = \infty,$$

and so $J^\gamma f$ is not in R_{-1}.

(iii) Let $\mu > \lambda - \gamma > 0$. Again (3) applies, giving, if $l > e$,

$$\int_l^\infty x^{\mu-1}J^\gamma f(x)\,dx \geqslant \delta\int_{4l}^\infty \frac{t^{\mu-\lambda+\gamma}}{t(\log t)^2}\,dt\int_{\frac{1}{4}}^{\frac{1}{2}} s^{\mu-1}\,ds = \infty,$$

the inner integral being independent of t, and $\mu-\lambda+\gamma$ being positive. Thus $J^\gamma f$ is not in $R_{\mu-1}$.

THEOREM 2. *If $\alpha = \Re a > 0$, $\beta = \Re b > 0$, and $f \in R_{\alpha+\beta-1}$, then*

$$J^bJ^af(x) = {}^0J^{a+b}f(x) = {}^0J^aJ^bf(x).$$

It is sufficient to justify, for almost all $x > d$, the inversion of order of integration involved in

$$\int_x^\infty \frac{(s-x)^{b-1}}{\Gamma(b)}\,ds\int_s^\infty \frac{(t-s)^{a-1}}{\Gamma(a)}f(t)\,dt = {}^0\int_x^\infty f(t)\,dt\int_x^t \frac{(t-s)^{a-1}}{\Gamma(a)}\frac{(s-x)^{b-1}}{\Gamma(b)}\,ds$$
$$= \int_x^\infty \frac{(t-x)^{a+b-1}}{\Gamma(a+b)}f(t)\,dt = J^{a+b}f(x). \tag{4}$$

If we replace f, a and b by g, α and β where $g(t) = |f(t)|$, these equations become true since the integrands become real and non-negative; and the common value of the resulting integral is $J^{\alpha+\beta}g(x)$, finite or infinite. If this value is finite, the original integrals with f, a and b are absolutely convergent, and so the original equations (4) are justified. This is the case for almost all x, because $g \in R_{\alpha+\beta-1}$ and so $J^{\alpha+\beta}g(x)$ is finite for almost all x by Theorem 1.

THEOREM 3. *If $\Re a > 0$, $\Re b > 0$, $q > 0$ and $f \in R_{q-1}$, then $J^ax^{-a}f(x)$, that is*

$$\int_x^\infty \frac{(t-x)^{a-1}}{\Gamma(a)}\,t^{-a}f(t)\,dt,$$

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exists in R_{q-1} *also; and*

$$J^b x^{-b} J^a x^{-a} f(x) = {}^0J^a x^{-a} J^b x^{-b} f(x).$$

Let $\alpha = \Re a$ and $\beta = \Re b$. Since $x^{q+\alpha-1}|x^{-a}f(x)| = x^{q-1}|f(x)|$, $x^{-a}f(x)$ is in $R_{q+\alpha-1}$; thus $J^a x^{-a} f(x)$ exists in R_{q-1}, by Theorem 1.

Similarly $J^a x^{-a} J^b x^{-b} f(x)$ exists in R_{q-1}. Subject to justification of the inversion of order of integration involved,

$$J^a x^{-a} J^b x^{-b} f(x) = \int_x^\infty \frac{(s-x)^{a-1}}{\Gamma(a)} s^{-a}\, ds \int_s^\infty \frac{(t-s)^{b-1}}{\Gamma(b)} t^{-b} f(t)\, dt \tag{5}$$

$$= {}^0\int_x^\infty t^{-b} f(t)\, dt \int_x^t \frac{(t-s)^{b-1}}{\Gamma(b)} \frac{(s-x)^{a-1}}{\Gamma(a)} s^{-a}\, ds. \tag{6}$$

The inner integral is reducible to Euler's integral [(10): 2·1, (10)] for a hypergeometric function F by the substitution $(t-s)/(t-x) = u$; it becomes, under the given conditions $\Re a > 0$ and $\Re b > 0$,

$$(t-x)^{a+b-1} t^{-a} \int_0^1 \frac{u^{b-1}}{\Gamma(b)} \frac{(1-u)^{a-1}}{\Gamma(a)} \left(1 - \frac{t-x}{t} u\right)^{-a} du = \frac{(t-x)^{a+b-1}}{\Gamma(a+b)} t^{-a} F\left(a, b, a+b, \frac{t-x}{t}\right); \tag{7}$$

alternatively this evaluation could be made using [(12): (3·8)]. Thus (6) gives

$$J^a x^{-a} J^b x^{-b} f(x) = {}^0\int_x^\infty \frac{(t-x)^{a+b-1}}{\Gamma(a+b)} F\left(a, b, a+b, 1 - \frac{x}{t}\right) t^{-a-b} f(t)\, dt.$$

The right side is symmetrical in a and b; so the proof will be complete when we have justified the step from (5) to (6).

For this justification we show that (5) is an absolutely convergent double integral for almost all x. Replacing the integrand by its modulus, and writing $g(t)$ for $|f(t)|$, (5) becomes

$$\int_x^\infty \frac{(s-x)^{\alpha-1}}{|\Gamma(a)|} s^{-\alpha}\, ds \int_s^\infty \frac{(t-s)^{\beta-1}}{|\Gamma(b)|} t^{-\beta} g(t)\, dt = \left|\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(a)\,\Gamma(b)}\right| J^\alpha x^{-\alpha} J^\beta x^{-\beta} g(x). \tag{8}$$

Now $g \in R_{q-1}$; hence as in the first paragraph $J^\beta x^{-\beta} g(x)$ and (8) are in R_{q-1}. Thus (8) is finite for almost all x, as required.

3. Fractional derivatives with origin ∞

Fractional derivative definition. *For complex* c *with* $\gamma = \Re c > 0$, $J^{-c}f$ *is a function* $\phi \in R_{\gamma-1}$ *such that* $J^c\phi = f$.

Definitions of fractional derivatives of purely imaginary order are given later, in connection with Theorem 6. Those definitions apply only to functions possessing a fractional derivative of some order c as above. This is no limitation in the present context, and in fact it provides a simpler treatment than could be given using Kober's more general definition (11).

In order to establish uniqueness of J^{-c}, and for other purposes, we obtain a relation connecting the operator J^c with the operator I^c of fractional integration with origin 0; this is the purpose of Lemma 1 below. Later, in section 4, we obtain a corresponding relation between J^{-c} and I^{-c}; this will be in continual use.

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AUXILIARY DEFINITIONS. As in (12), $I^c f$ is the function whose values are

$$I^c f(x) = \int_0^x \frac{(x-t)^{c-1}}{\Gamma(c)} f(t)\,dt \tag{9}$$

and $I^c f$ is locally integrable on $[0, d^{-1})$ whenever f is. Further, $I^{-c} f$ is a function ϕ locally integrable on $[0, d^{-1})$ such that $I^c \phi = f$. Also we shall consistently understand that

$$x' = 1/x \quad \text{and} \quad t' = 1/t. \tag{10}$$

LEMMA 1. *If $\gamma = \Re c > 0$, and f and f' are functions related by*

$$f'(x') = x^{1+c} f(x), \quad \textit{where} \quad x' = 1/x, \tag{11}$$

then f' is locally integrable on $[0, d^{-1})$ if and only if $f \in R_{\gamma-1}$; and in that case

$$I^c f'(x') = {}^0x^{1-c} J^c f(x). \tag{12}$$

For each $l > d$, and whether the integrals are finite or infinite,

$$\int_0^{1/l} |f'(x')|\,dx' = \int_l^\infty |x^{1+c} f(x)| \frac{1}{x^2} dx = \int_l^\infty x^{\gamma-1} |f(x)|\,dx;$$

so f' is locally integrable on $[0, d^{-1})$ if and only if $f \in R_{\gamma-1}$.

In these circumstances $I^c f'$ exists almost everywhere in $(0, d^{-1})$. For such values of x' we have, changing the variable by (10),

$$\begin{aligned} I^c f'(x') &= \int_0^{x'} \frac{(x'-t')^{c-1}}{\Gamma(c)} f'(t')\,dt' \\ &= \int_x^\infty \left(\frac{1}{x} - \frac{1}{t}\right)^{c-1} \frac{t^{1+c}}{\Gamma(c)} \frac{f(t)}{t^2}\,dt = x^{1-c} \int_x^\infty \frac{(t-x)^{c-1}}{\Gamma(c)} f(t)\,dt. \end{aligned}$$

This gives (12); the excepted set of x-values being the map by (10) of a null set of x'-values and therefore a null set.

THEOREM 4. *If $\Re c > 0$ and $J^{-c} f$ exists then it is unique.*

Suppose there are ϕ_1 and ϕ_2 in $R_{\gamma-1}$ such that $J^c \phi_1 = f = J^c \phi_2$, where $\gamma = \Re c$. Then $\phi = \phi_1 - \phi_2$ is such that

$$\phi \in R_{\gamma-1} \quad \text{and} \quad J^c \phi = 0.$$

Let $\phi'(x') = x^{1+c}\phi(x)$ where $x' = 1/x$. By Lemma 1, ϕ' is locally integrable on $[0, d^{-1})$ and $I^c \phi'(x') = {}^0x^{1-c} J^c \phi(x) = {}^0 0$. By the familiar uniqueness theorem for I^c these two things imply that $\phi' = 0$, that is, $\phi'(x') = {}^0 0$; whence $\phi = 0$ and the theorem is established.

THEOREM 5. *If $\alpha = \Re a > 0$, $\beta = \Re b > 0$ and $J^{-a} f$ exists in $R_{\alpha+\beta-1}$, then*

$$J^b J^{-a} f = J^{b-a} f = J^{-a} J^b f,$$

all three functions existing in $R_{\alpha-1}$.

By definition of $J^{-a} f$, there is $\phi \in R_{\alpha-1}$ such that $J^a \phi = f$; and also $\phi \in R_{\alpha+\beta-1}$ by data. Actually $R_{\alpha+\beta-1} \subset R_{\alpha-1}$ since $\beta > 0$; so the information is simply that $\phi \in R_{\alpha+\beta-1}$.

By Theorem 1, $f = J^a \phi \in R_{\beta-1}$. And by Theorem 2,

$$J^b f = J^b J^a \phi = J^{a+b} \phi = J^a J^b \phi.$$

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Also by Theorem 1, $J^b\phi \in R_{\alpha-1}$; so $J^{-a}J^bf$ exists and is $J^b\phi$. Thus

$$J^{-a}J^bf = J^bJ^{-a}f \in R_{\alpha-1}.$$

It remains only to consider $J^{b-a}f$.

If $\beta > \alpha$ we have, since $f \in R_{\beta-1}$ as already shown, that $J^{b-a}f$ exists in $R_{\alpha-1}$ by Theorem 1, and that $J^aJ^{b-a}f = J^bf$ by Theorem 2. So by definition of J^{-a},

$$J^{b-a}f = J^{-a}J^bf.$$

If $\beta < \alpha$ we have, since $\phi \in R_{\alpha-1}$, that $f = J^a\phi = J^{a-b}J^b\phi$ by Theorem 2; and that $J^b\phi \in R_{(\alpha-\beta)-1}$ by Theorem 1. So by definition of J^{b-a},

$$J^{b-a}f = J^b\phi = J^bJ^{-a}f.$$

If $\beta = \alpha$, $J^{b-a}f = J^bJ^{-a}f$ by the definition immediately following, since $J^{-a}f$ exists by data.

DERIVATIVE (OR INTEGRAL) OF IMAGINARY ORDER. *If $\iota\theta$ is purely imaginary, $\Re c > 0$ and $J^{-c}f$ exists, then $J^{\iota\theta}f$ is defined as $J^{c+\iota\theta}J^{-c}f$.*

Thus defined, $J^{\iota\theta}f(x)$ exists for almost all x in (d,∞), by Theorem 1; for $J^{-c}f \in R_{\gamma-1}$ by definition, where $\gamma = \Re c = \Re(c+\iota\theta)$. However the definition of $J^{\iota\theta}f$ is unsatisfactory until it has been shown self-consistent for various values of c. The following theorem does this; and it gives other expressions for $J^{\iota\theta}f$ in certain circumstances.

THEOREM 6. *If $\iota\theta$ is purely imaginary, $\Re c > \Re b > 0$ and $J^{-c}f$ exists, then $J^{-b}f$ exists and*

$$J^{b+\iota\theta}J^{-b}f = J^{c+\iota\theta}J^{-c}f = J^{\iota\theta}f. \tag{13}$$

If $\gamma = \Re c > 0$, $\kappa = \Re k > 0$ and $J^{-c}f$ exists in $R_{\gamma+\kappa-1}$, then $J^{\iota\theta}f$ is in $R_{\kappa-1}$ and

$$J^{\iota\theta}f = J^{-k}J^{k+\iota\theta}f. \tag{14}$$

In particular (compare definition of $I^{\iota\theta}$ in [12: Lemma 5]), *if $J^{-c}f$ exists in R_γ then $J^{\iota\theta}f$ is in R_0 and*

$$J^{\iota\theta}f = J^{-1}J^{1+\iota\theta}f = -\frac{d}{dx}\int_x^\infty \frac{(t-x)^{\iota\theta}}{\Gamma(1+\iota\theta)}f(t)\,dt. \tag{15}$$

By definition, $J^{-c}f = \phi \in R_{\gamma-1}$ and $J^c\phi = f$. So by Theorem 2, $J^bJ^{c-b}\phi = J^c\phi = f$. But by Theorem 1, $J^{c-b}\phi \in R_{\beta-1}$ where $\beta = \Re b$; so by definition $J^{-b}f$ exists and

$$J^{-b}f = J^{c-b}\phi = J^{c-b}J^{-c}f.$$

Further, using Theorem 2 this equation gives

$$J^{b+\iota\theta}J^{-b}f = J^{b+\iota\theta}J^{c-b}\phi = J^{c+\iota\theta}\phi = J^{c+\iota\theta}J^{-c}f = J^{\iota\theta}f,$$

which proves (13).

Now suppose $J^{-c}f = \phi \in R_{\gamma+\kappa-1}$. By Theorem 2

$$J^{k+\iota\theta}f = J^{k+\iota\theta}J^c\phi = J^{k+c+\iota\theta}\phi = J^kJ^{c+\iota\theta}\phi;$$

and by Theorem 1 $J^{c+\iota\theta}\phi \in R_{\kappa-1}$. These together show that $J^{-k}J^{k+\iota\theta}f$ exists; and (14) follows, since

$$J^{-k}J^{k+\iota\theta}f = J^{c+\iota\theta}\phi = J^{c+\iota\theta}J^{-c}f = J^{\iota\theta}f.$$

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Finally (15) is the case $k = 1$ of (14); except the last expression for $J^{\iota 0}f$, which follows from the first part of Theorem 7 below.

THEOREM 7. *If $J^{-1}f$ exists then $J^{-1}f(x) =^0 -f'(x)$, where f' denotes the derivative of f.† In order that $J^{-1}f$ exist, it is necessary and sufficient that*:

(i) *$f(x) \to 0$ as $x \to \infty$;*

(ii) *f be locally absolutely continuous in (d, ∞);*

(iii) *the total variation $V(f; l, \infty)$ be finite for some $l > d$.*

Suppose $J^{-1}f$ exists, so that $f = J^1\phi$ where $\phi \in R_0$. Then

$$f(x) = \int_x^\infty \phi(t)\,dt \quad \text{and} \quad \int_x^\infty |\phi(t)|\,dt < \infty \tag{16}$$

for all $x > d$. From these we infer that $\phi(x) =^0 -f'(x)$ as stated, and also that (i), (ii) and (iii) hold.

Conversely suppose that (i), (ii) and (iii) hold. By (ii) there is ϕ locally integrable in (d, ∞) such that, if $d < x < \xi < \infty$,

$$f(x) - f(\xi) = \int_x^\xi \phi(t)\,dt \quad \text{and} \quad V(f; x, \xi) = \int_x^\xi |\phi(t)|\,dt.$$

Letting $\xi \to \infty$ this gives, using (iii), that ϕ is integrable on (l, ∞); and so ϕ is integrable on (x, ∞) for each $x > d$. Using (i) also, we obtain (16), and so ϕ is the required function $J^{-1}f$.

4. OTHER PRELIMINARIES

In translating the later theorems of (12) into theorems about (C) and (D) we shall refer continually to the integrability classes Q_r used in that paper. These are the analogues, in that context, of the classes R_r defined in section 2 above.

AUXILIARY DEFINITIONS. *Q_r is the class of complex-valued functions f on $(0, d^{-1})$ such that $x^r f(x)$ is L-integrable on $(0, l^{-1})$ for each $l^{-1} < d^{-1}$.*

The glosses used in defining R_r in section 2 apply here also, including $d^{-1} \leqslant \infty$. Except for replacing d by d^{-1} this definition of Q_r is the same as (5·4) of (12).

The class Q_r evidently expands as r increases.

If $\mathrm{rec} > 0$, $I^{-c}f$ is a function $\phi \in Q_0$, unique if existent, such that $I^c\phi = f$; as at (3·3) of (12).

We now obtain a companion to Lemma 1, related in part to it by the mere formal substitution of $-c$ for c. Lemma 5 is a similar result for $\Re c = 0$.

LEMMA 2. *If $q + q' = \Re c > 0$, and g and g' are functions related by*

$$g'(x') = x^{1-c}g(x), \quad \text{where} \quad x' = 1/x, \tag{17}$$

then $I^{-c}g'$ exists in $Q_{q'}$ if and only if $J^{-c}g$ exists in R_{q-1}; and in that case

$$I^{-c}g'(x') =^0 x^{1+c}J^{-c}g(x). \tag{18}$$

† Unlike the meaning of f' everywhere else in this paper, which is essentially that given by (11) in Lemma 1.

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Suppose that $I^{-c}g'$ exists in $Q_{q'}$; this means that $g' = I^c\phi$ where $\phi \in Q_0 \cap Q_{q'}$. Let $f(x) = x^{-1-c}\phi(1/x)$; then

$$g(x) = x^{c-1}g'(x') = x^{c-1}I^c\phi(x') = {}^0J^cf(x)$$

using Lemma 1 at the third step and noticing that ϕ, by its relation to f, is the function f' defined therein. The local integrability of f' required for this application of Lemma 1 is assured by the knowledge that $\phi \in Q_0$; and we infer also that $f \in R_{\gamma-1}$. Thus $J^{-c}g = f$.

Further, for each $l > d$,

$$\int_l^\infty x^{q-1}|f(x)|\,dx = \int_l^\infty x^{q-\gamma-2}|\phi(1/x)|\,dx = \int_0^{1/l} x'^{\gamma-q}|\phi(x')|\,dx'; \tag{19}$$

so from the fact that $\phi \in Q_{q'}$ we infer that $f \in R_{q-1}$.

Conversely, suppose that $J^{-c}g$ exists in R_{q-1}. Then $g = J^cf$ where f (independent of f above) is in $R_{q-1} \cap R_{\gamma-1}$. Defining f' from this f by (11), Lemma 1 gives that $f' \in Q_0$ and that

$$I^cf'(x') = {}^0x^{1-c}J^cf(x) = {}^0x^{1-c}g(x) = g'(x');$$

thus $f' = I^{-c}g'$. Also $f' \in Q_{q'}$, by (19) with ϕ replaced by f'.

Finally, (18) follows from the equations

$$I^{-c}g'(x') = {}^0f'(x') = x^{1+c}f(x) = {}^0x^{1+c}J^{-c}g(x).$$

The next lemma translates Theorem 5 of (12) into terms of J.

LEMMA 3. *If $q > \Re c > 0$ and $q > \Re(a+c)$, then the equation*

$$J^cx^af(x) = {}^0x^aJ^cg(x) \quad \text{in} \quad d < x < \infty \tag{20}$$

has, for each f in R_{q-1} a solution g in R_{q-1}, and for each g in R_{q-1} a solution f in R_{q-1}.

As in Lemma 1, let $x' = 1/x$,

$$f'(x') = x^{1+c}f(x) \quad \text{and} \quad g'(x') = x^{1+c}g(x); \tag{21}$$

let α and γ denote $\Re a$ and $\Re c$, and let $q' = \gamma - q$.

(i) Suppose $f \in R_{q-1}$ is given. As in (19), for each $l > d$

$$\int_l^\infty x^{q-1}|f(x)|\,dx = \int_l^\infty x^{q-\gamma}|x^{1+c}f(x)|\,x^{-2}\,dx = \int_0^{1/l} x'^{q'}|f'(x')|\,dx', \tag{22}$$

so $f' \in Q_{q'}$. Hence $x'^{-a}f'(x') \in Q_{q'+\alpha} \subset Q_0$ since $q' + \alpha = \alpha + \gamma - q < 0$. Lemma 1 is thus applicable to the relation $x'^{-a}f'(x') = x^{1+c}.x^af(x)$, giving

$$I^cx'^{-a}f'(x') = {}^0x^{1-c}J^cx^af(x).$$

Since $q' < 0 < \Re c$ and $q' < \Re(-a)$, Theorem 5 (i) of (12) shows that there is $g' \in Q_{q'}$ such that

$$I^cx'^{-a}f'(x') = {}^0x'^{-a}I^cg'(x'); \tag{23}$$

and then g, defined from g' by (21), is in R_{q-1} as in (22). Further $R_{q-1} \subset R_{\gamma-1}$ since $q > \gamma$; so $g \in R_{\gamma-1}$ and Lemma 1 gives

$$I^cg'(x') = {}^0x^{1-c}J^cg(x).$$

Putting the last three equations together, (20) is satisfied by $g \in R_{q-1}$.

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(ii) If $g \in R_{q-1}$ is given we have only to reverse the above argument, using Theorem 5(ii) of (12) at the appropriate point.

LEMMA 4. *If $q > \Re c > 0$ and $q > \Re(a+c)$, then $x^{-a}J^{-c}x^a g(x)$ exists in R_{q-1} if and only if $J^{-c}g(x)$ exists in R_{q-1}.*

(i) Suppose that $J^{-c}x^a g(x)$ exists, that $x^{-a}J^{-c}x^a g(x) = \phi(x)$ and that $\phi \in R_{q-1}$. Then $x^a g(x) = {}^0 J^c x^a \phi(x)$. By Lemma 3(i) there is $\psi \in R_{q-1}$ such that $J^c x^a \phi(x) = {}^0 x^a J^c \psi(x)$. Thus

$$x^a g(x) = {}^0 x^a J^c \psi(x).$$

Also $\psi \in R_{\gamma-1}$ since $q > \gamma$; so $J^{-c}g$ exists and is $\psi \in R_{q-1}$.

(ii) Suppose that $\psi = J^{-c}g \in R_{q-1}$. By Lemma 3(ii) there is $\phi \in R_{q-1}$ such that

$$J^c x^a \phi(x) = {}^0 x^a J^c \psi(x) = {}^0 x^a g(x).$$

Since $q - \alpha > \gamma$, $x^a \phi(x) \in R_{q-\alpha-1} \subset R_{\gamma-1}$; so $J^{-c}x^a g(x)$ exists and is $x^a \phi(x)$. Thus $x^{-a}J^{-c}x^a g(x) = {}^0 \phi(x) \in R_{q-1}$.

The remaining lemma justifies, for integrals of purely imaginary order, the formula of Lemmas 1 and 2 connecting I- and J-integrals.

LEMMA 5. *If $\iota\theta$ is purely imaginary, $q > \gamma = \Re c > 0$, $J^{-c}f$ exists in $R_{q+\gamma-1}$, and*

$$f'(x') = x^{1+\iota\theta} f(x) \quad \text{where} \quad x' = 1/x, \tag{24}$$

then $I^{\iota\theta}f'(x')$ exists in Q_{-q}, $J^{\iota\theta}f$ exists in R_{q-1}, and

$$I^{\iota\theta}f'(x') = {}^0 x^{1-\iota\theta} J^{\iota\theta} f(x). \tag{25}$$

Let $k = c + \iota\theta$. By data, $x^k J^{-c}\{x^{-k} . x^k f(x)\}$ exists in R_{q-1}; so Lemma 4, with a and $g(x)$ replaced by $-k$ and $x^k f(x)$ respectively, shows that $J^{-c}x^k f(x)$ exists in R_{q-1}.

Since $f'(x') = x^{1-c} . x^k f(x)$ and $J^{-c}x^k f(x)$ exists in R_{q-1}, we have by Lemma 2 that $I^{-c}f'$ exists in $Q_{\gamma-q}$ and

$$\begin{aligned} I^{-c}f'(x') &= {}^0 x^{1+c} J^{-c} x^k f(x), \\ &= x^{1+k} . x^{c-k} J^{-c} x^k f(x). \end{aligned}$$

Since $I^{-c}f' \in Q_{\gamma-q} \subset Q_0$, Lemma 1 with c and f' replaced by k and $I^{-c}f'$ now shows that

$$I^k I^{-c} f'(x') = {}^0 x^{1-k} J^k x^{c-k} J^{-c} x^k f(x). \tag{26}$$

By ((12): Lemma 5) the left side is $I^{\iota\theta}f'$; and by ((12): (2·12)), replacing a by 0, q by $\gamma - q$ and f by $I^{-c}f'$ in that result, $I^{\iota\theta}f' \in Q_{-q}$.

Since $x^c J^{-c} x^k f(x) \in R_{q-\gamma-1}$ and $q - \gamma > 0$, Theorem 3 gives

$$\begin{aligned} J^c x^{-c} J^k x^{-k}\{x^c J^{-c} x^k f(x)\} &= {}^0 J^k x^{-k} J^c x^{-c}\{x^c J^{-c} x^k f(x)\}, \\ &= J^k f(x). \end{aligned} \tag{27}$$

Also by Theorem 3, $J^k x^{-k}\{x^c J^{-c} x^k f(x)\} \in R_{q-\gamma-1}$, and consequently

$$x^{-c} J^k x^{-k}\{x^c J^{-c} x^k f(x)\} \in R_{q-1} \subset R_{\gamma-1}.$$

By definition of J^{-c} this gives, with (27),

$$x^{-c} J^k x^{c-k} J^{-c} x^k f(x) = {}^0 J^{-c} J^k f(x).$$

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This with (26) gives

$$I^{i\theta}f'(x') = {}^0 x^{1-k+c}J^{-c}J^k f(x). \tag{28}$$

Replacing the k in Theorem 6 by c, we obtain from (14) that $J^{-c}J^{c+i\theta}f = J^{i\theta}f$, since $R_{q+\gamma-1} \subset R_{2\gamma-1}$ and $J^{-c}f$ exists in both. Thus (28) becomes (25).

Lastly $J^{i\theta}f = J^{c+i\theta}J^{-c}f$; so by Theorem 1 $J^{i\theta}f \in R_{(q+\gamma-1)-\gamma} = R_{q-1}$.

5. The hypergeometric integral equation (C)

We now obtain theorems on solutions of equation (C) of section 1, namely

$$\int_x^\infty \frac{(t-x)^{c-1}}{\Gamma(c)} F\left(a, b, c, 1-\frac{x}{t}\right) f(t)\,dt = {}^0 g(x) \quad \text{for} \quad d < x < \infty, \tag{C}$$

where $0 \leqslant d < \infty$. The theorems are applicable also to this equation with a finite upper terminal replacing ∞, as shown in Theorem 4C, Corollary, below; hence in particular to the equation considered by Wimp (7).

To convert (C) into the form of (B) of section 1 we write

$$\left.\begin{aligned} x' = 1/x, \qquad t' = 1/t, \qquad q' = \Re c - q, \\ f'(t') = t^{1+c}f(t), \qquad g'(x') = x^{1-c}g(x). \end{aligned}\right\} \tag{29}$$

This gives the equation, equivalent to (C):

$$\int_0^{x'} \frac{(x'-t')^{c-1}}{\Gamma(c)} F\left(a, b, c, 1-\frac{t'}{x'}\right) f'(t')\,dt' = {}^0 g'(x') \quad \text{for} \quad 0 < x' < d', \tag{30}$$

where $d' = \infty$ if $d = 0$, and $d' = 1/d$ otherwise. This has the form of ((12): (10·1)); and we apply Theorems 4B, 7B, ... of ((12): section 10) to it to obtain corresponding Theorems 4C, 7C, ... below.

Theorem 4C. *If* $\Re c > 0$, $q \geqslant \max(\Re a, \Re b)$ *and* $q > \max(\Re c, \Re(a+b))$, *then* (C) *cannot have more than one solution* f *in* R_{q-1}.

If f is the difference of two solutions in R_{q-1}, then $f \in R_{q-1}$ and (C) holds with $g(x) = 0$. So (30) holds with $g'(x') = 0$, and $f' \in Q_{q'}$ by equations (22). Theorem 4B of (12) now shows that $f'(x') = {}^0 0$, because, by data and (29),

$$q' \leqslant \min(\Re(c-a), \Re(c-b)), \quad q' < \min(0, \Re(c-a-b)).$$

So by (29) $f(x) = {}^0 0$, as required.

Corollary. *If* $\Re c > 0$ *and* $0 \leqslant d < \delta < \infty$, *then the solutions* f *of*

$$\int_x^\delta \frac{(t-x)^{c-1}}{\Gamma(c)} F\left(a, b, c, 1-\frac{x}{t}\right) f(t)\,dt = {}^0 g(x) \quad \textit{for} \quad d < x < \delta \tag{31}$$

which are locally integrable on $(d, \delta]$ *are the truncations, omitting* (δ, ∞), *of the solutions* f *in* R_{q-1} *of* (C), *provided that* $g(x) = 0$ *in* $x > \delta$ *and* q *satisfies the hypotheses of Theorem* 4C.

The locally integrable solutions f of (31) clearly satisfy (C) if $f(x) = 0 = g(x)$ for $x > \delta$; and $f \in R_{q-1}$ because $x^{q-1}f(x)$, considered on (l, ∞) where $l > d \geqslant 0$, is the product of $f(x)$ which is integrable with a truncated power which is bounded.

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Conversely, solutions f in R_{q-1} of (C), with the truncated g, satisfy

$$\int_x^\infty \frac{(t-x)^{c-1}}{\Gamma(c)} F\left(a, b, c, 1-\frac{x}{t}\right) f(t)\, dt =^0 0 \quad \text{for} \quad \delta < x < \infty;$$

so by Theorem 4 C they vanish almost everywhere in (δ, ∞). So they satisfy (31) when truncated; and they are integrable on (l, δ) whenever $d < l < \delta$, because they are products of $x^{q-1}f(x)$ with a truncated x^{1-q} which is bounded.

THEOREM 7 C. *If* $\Re c > 0$, $q \geqslant \Re(c+a)$, $q > \max(\Re c, \Re(a+b))$, f *is in* R_{q-1} *and* (C) *holds, then* $J^{-c}g$ *exists in* R_{q-1}.

Further, $x^{-a}J^{-c}x^a g(x)$ *and* $x^{c-a-b}J^{-c}x^{a+b-c}g(x)$ *exist in* R_{q-1}; *so also does*

$$\frac{1}{x^{c+a}} \int_x^\infty t^{c-1} J^{-c}\{t^a g(t)\}\, dt. \tag{32}$$

By data, $q' = \Re c - q$ satisfies

$$q' \leqslant -\Re a, \quad q' < \min(0, \Re(c-a-b)).$$

And since $f \in R_{q-1}$, $f' \in Q_{q'}$ using (22). Thus Theorem 7 B of (12) applies to (30), giving that $I^{-c}g'$ and three other functions exist in $Q_{q'}$. Noticing (17) and (29), Lemma 2 gives that $J^{-c}g$ exists in R_{q-1}, the primary conclusion.

Since $x'^a I^{-c}x'^{-a}g'(x')$ is in $Q_{q'}$ by Theorem 7 B, $I^{-c}x'^{-a}g'(x') \in Q_{q'+\alpha}$ where $\alpha = \Re a$. By (29), $x'^{-a}g'(x') = x^{1-c}.x^a g(x)$; so by Lemma 2 with $g(x)$, q and q' replaced by $x^a g(x)$, $q-\alpha$ and $q'+\alpha$ respectively, $J^{-c}x^a g(x)$ exists in $R_{q-\alpha-1}$. So $x^{-a}J^{-c}x^a g(x)$ exists in R_{q-1}, as stated.

Similarly, replacing a by $a+b-c$ in the preceding paragraph, we prove that $x^{c-a-b}J^{-c}x^{a+b-c}g(x)$ exists in R_{q-1}.

Completing the application of Lemma 2 made five lines above,

$$I^{-c}x'^{-a}g'(x') =^0 x^{1+c}J^{-c}x^a g(x).$$

Also completing the application of Theorem 7 B made above, $Q_{q'}$ contains, as a function of x',

$$x'^{a-1}\int_0^{x'} I^{-c}t'^{-a}g'(t')\, dt'.$$

So for each $l > d$ the following integrals are equal, using the change of variables (29), and finite:

$$\int_0^{1/l} x'^{q'+\alpha-1}\left|\int_0^x I^{-c}t'^{-a}g'(t')\, dt'\right| dx' = \int_l^\infty x^{-q'-\alpha-1}\left|\int_x^\infty t^{c-1}J^{-c}t^a g(t)\, dt\right| dx.$$

This shows that (32) is in R_{q-1}, and completes the proof.

THEOREM 8 C will be found *following* the proof of Theorem 9 C below. Either of these theorems can be deduced from the other; but to prove Theorem 8 C from Theorem 8 B seems to be much less direct than to prove Theorem 9 C from Theorem 9 B, if one wishes to include the cases of equality $q = \Re(c+a)$ and $q = \Re(c+b)$. These cases are quite troublesome, and I have deferred completion of their proof until section 6; some lemmas needed only for this purpose are given there.

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THEOREM 9C. *If* $\Re c > 0$, $\Re b > 0$, $q \geqslant \Re(c+b)$, $q > \Re(a+b)$ *and* $J^{-c}g$ *exists in* R_{q-1}, *then*

$$\int_x^\infty \frac{(t-x)^{c-1}}{\Gamma(c)} F\left(a, b, c, 1-\frac{x}{t}\right) f(t)\, dt = {}^0 g'(x') \quad \text{for} \quad d < x < \infty \tag{C}$$

has a solution f *in* R_{q-1}, *expressible by*

$$f(x) = x^{-a} J^{-b} x^a J^{b-c} g(x). \tag{33}$$

The assumption that $J^{-c}g$ *exists in* R_{q-1} *may be replaced by either of the following, which are equivalent to it in this context*:

(i) $x^{c-a-b} J^{-c} x^{a+b-c} g(x)$ *exists in* R_{q-1};

(ii) $x^{-b} J^{-c} x^b g(x)$ *exists in* R_{q-1}, *supplemented if* $q = \Re(c+b)$ *by*

$$\frac{1}{x^{c+b}} \int_x^\infty t^{c-1} J^{-c}\{t^b g(t)\}\, dt \quad \textit{being in } R_{q-1}.$$

By (29), $q' \leqslant -\Re b$ and $q' < \Re(c-a-b)$; and by Lemma 2 $I^{-c}g'$ exists in $Q_{q'}$. So Theorem 9B shows that equation (30) has a solution f' in $Q_{q'}$, expressible by

$$f'(x') = x'^{a+b-c} I^{-b} x'^{c-a} h'(x'), \quad \text{where} \quad h'(x') = I^{b-c} x'^{-b} g'(x'). \tag{34}$$

So (C) has a solution f in R_{q-1}, by (22).

Since $q \geqslant \Re(c+b) > \Re c > 0$ and $q > \Re((a+b-c)+c)$, Lemma 4 with a replaced by $a+b-c$ shows that $x^{c-a-b} J^{-c} x^{a+b-c} g(x)$ exists in R_{q-1} if and only if $J^{-c}g(x)$ exists in R_{q-1}. This proves that (i) is an equivalent alternative hypothesis as stated.

Excluding the case $q = \Re(c+b)$, and writing β for $\Re b$, we have that $q-\beta > \Re c > 0$ and $q-\beta > \Re(c-b)$. So Lemma 4, with q, a and $g(x)$ replaced by $q-\beta$, $-b$ and $x^b g(x)$ respectively, shows that $x^b J^{-c}\{x^{-b}.x^b g(x)\}$ exists in $R_{q-\beta-1}$ if and only if $J^{-c}\{x^b g(x)\}$ does. Thus $J^{-c}g(x)$ exists in R_{q-1} if and only if $x^{-b} J^{-c} x^b g(x)$ does. This proves that (ii) is an equivalent alternative hypothesis, except perhaps if $q = \Re(c+b)$; the exception will be dealt with in section 6.

We now translate (34) into (33), using (29). We write α, β, γ for $\Re a$, $\Re b$, $\Re c$ when convenient. From (34) and (29),

$$x'^{c-a} h'(x') = I^b x'^{c-a-b} f'(x') \quad \text{and} \quad x'^{c-a-b} f'(x') = x^{1+b}.x^a f(x),$$

the former since $\Re b > 0$. And by Lemma 1 with c replaced by b,

$$I^b x'^{c-a-b} f'(x') = {}^0 x^{1-b} J^b x^a f(x),$$

since $f \in R_{q-1}$ and so $x^a f(x) \in R_{q-\alpha-1} \subset R_{\beta-1}$. Thus

$$x'^{c-a} h'(x') = {}^0 x^{1-b} J^b x^a f(x),$$

that is,

$$x^{a+b-c-1} h'(x') = {}^0 J^b x^a f(x).$$

Since $x^a f(x) \in R_{\beta-1}$ as just remarked, this relation can be inverted:

$$x^a f(x) = {}^0 J^{-b} x^{a+b-c-1} h(x), \tag{35}$$

$h(x)$ denoting $h'(x')$ expressed as a function of x. To make this expression explicit and in terms of g, we consider separately three cases.

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If $\Re b > \Re c$, we have by Theorem 1 that $g = J^c J^{-c} g \in R_{q-\gamma-1}$ since $q > \gamma > 0$. Also $q > \beta$, so that $g \in R_{\beta-\gamma-1}$. Thus Lemma 1 with c replaced by $b-c$ and (11) by $x'^{-b} g'(x') = x^{1+b-c} g(x)$, shows that

$$I^{b-c} x'^{-b} g'(x') = {}^0 x^{1-b+c} J^{b-c} g(x). \tag{36}$$

The left side is $h'(x')$ by (34), so the right is $h(x)$ in (35). Substituting it in (35) we obtain (33) as required.

If $0 < \Re b < \Re c$, we observe that $R_{q-1} \subset R_{\gamma+\beta-1}$ since $q \geqslant \gamma+\beta$, and so $J^{-c} g$ exists in $R_{\gamma+\beta-1}$. By Theorem 5 with a and f replaced by c and g we have that $J^{b-c} g$ exists in $R_{\gamma-1}$. Thus Lemma 2, with c replaced by $c-b$ and (17) by $x'^{-b} g'(x') = x^{1-c+b} g(x)$, shows that (36) again holds. This with (34) gives (33) as before.

Finally, if $\Re b = \Re c$ we exclude the case $q = \Re(c+b)$. Then $q-\gamma > \beta = \gamma > 0$, and Lemma 5 applies with $\iota\theta$, q and f replaced by $b-c$, $q-\gamma$ and g. This again gives (36); and hence (33) as before.

Theorem 9C has now been proved except for (ii) if $q = \Re(c+b)$ and (33) if $q = \Re(c+b) = 2\Re c$. These exceptions will be assumed for the present and proved in section 6.

THEOREM 8C. *If $\Re c > 0$, $\Re b < \Re c$, $q \geqslant \Re(c+a)$, $q > \Re c$ and $J^{-c} g$ exists in R_{q-1}, then* (C) *has a solution f in R_{q-1}, expressible by*

$$f(x) = x^{-b} J^{b-c} x^{c-a} J^{-b} x^{a+b-c} g(x). \tag{37}$$

The assumption that $J^{-c} g$ exists in R_{q-1} may be replaced by either of the following, which are equivalent to it in this context:

(i) $x^{c-a-b} J^{-c} x^{a+b-c} g(x)$ *exists in* R_{q-1};

(ii) $x^{-a} J^{-c} x^a g(x)$ *exists in* R_{q-1}, *supplemented if* $q = \Re(c+a)$ *by*

$$\frac{1}{x^{c+a}} \int_x^\infty t^{c-1} J^{-c}\{t^a g(t)\}\, dt \quad \textit{being in } R_{q-1}.$$

We deduce this from Theorem 9C, using

$$\left.\begin{aligned} f'(x) &= x^{a+b-c} f(x), & g'(x) &= x^{a+b-c} g(x), \\ a' &= c-a, \qquad b' = c-b, & q' &= q+\gamma-\alpha-\beta, \end{aligned}\right\} \tag{38}$$

where α, β, γ denote $\Re a, \Re b, \Re c$ as usual. This, with Kummer's relation ((10): 2·9(2)), transforms (C) into the equivalent equation

$$\int_x^\infty \frac{(t-x)^{c-1}}{\Gamma(c)} F\left(a', b', c, 1-\frac{x}{t}\right) f'(t)\, dt = {}^0 g'(x) \quad \text{for} \quad d < x < \infty \tag{39}$$

which also has the form of (C). Noticing that the accents now have different meanings from those in the previous proofs and in (29) and (30), we verify that Theorem 9C applies to (39).

Our present data gives

$$\Re c > 0, \quad \Re b' > 0, \quad q' \geqslant 2\gamma-\beta = \Re(c+b') \quad \text{and} \quad q' > 2\gamma-\alpha-\beta = \Re(a'+b');$$

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also the existence of $J^{-c}g$ in R_{q-1} implies that of $J^{-c}x^{a'+b'-c}g'(x)$, and so

$$x^{c-a'-b'}J^{-c}x^{a'+b'-c}g'(x) \quad \text{exists in } R_{q'-1}. \tag{40}$$

So Theorem 9C, with its alternative hypothesis (i), applies to (39), showing that it has a solution f' in $R_{q'-1}$, expressible by

$$f'(x) = x^{-a'}J^{-b'}x^{a'}J^{b'-c}g'(x).$$

Using (38) this shows that f given by (37) is in R_{q-1} and satisfies (C).

Also by Theorem 9C the statement (40) is equivalent in the present context to '$J^{-c}g'(x)$ exists in $R_{q'-1}$', and consequently to (i) of the present theorem. And by reversing the two steps above which led to it, (40) is also equivalent to '$J^{-c}g$ exists in R_{q-1}'. This proves the equivalence of alternative hypothesis (i).

Finally by Theorem 9C the statement (40) is also equivalent to:

'$x^{-b'}J^{-c}x^{b'}g'(x)$ exists in $R_{q'-1}$; supplemented if $q' = \Re(c+b')$ by

$$\frac{1}{x^{c+b'}}\int_x^\infty t^{c-1}J^{-c}\{t^{b'}g'(t)\}\,dt \quad \text{being in } R_{q'-1}.'$$

Using (38) this statement is equivalent to alternative hypothesis (ii) of the present theorem, and this establishes the required equivalence.

THEOREM 10C. *If* $\Re c > 0$, $q \geqslant \max(\Re(c+a), \Re(c+b))$, $q > \max(\Re c, \Re(a+b))$ *and* $J^{-c}g$ *exists in* R_{q-1}, *then the equation*

$$\int_x^\infty \frac{(t-x)^{c-1}}{\Gamma(c)} F\left(a, b, c, 1-\frac{x}{t}\right) f(t)\,dt = {}^0g(x) \quad \text{for} \quad d < x < \infty \tag{C}$$

has a unique solution f *in* R_{q-1}, *given by* (33) *if* $\Re b > 0$ *and by* (37) *if* $\Re b < \Re c$.

The assumption that $J^{-c}g$ *exists in* R_{q-1} *may be replaced by the assumption, equivalent to it in this context, that* $x^{c-a-b}J^{-c}x^{a+b-c}g(x)$ *exists in* R_{q-1}.

The uniqueness follows from Theorem 4C; the rest from Theorem 9C when $\Re b > 0$ and from Theorem 8C when $\Re b < \Re c$.

Referring to Theorem 7C and 10C we also obtain:

COROLLARY. *If* $\Re c > 0$, $q \geqslant \max(\Re(c+a), \Re(c+b))$ *and* $q > \max(\Re c, \Re(a+b))$, *then a necessary and sufficient condition that equation* (C) *should have a solution* f *in* R_{q-1} *is that* $J^{-c}g$ *should exist in* R_{q-1}.

6. COMPLETION OF PROOF OF THEOREM 9C

It remains to prove two parts of Theorem 9C when $q = \Re(c+b)$, as indicated in the last paragraph of the proof so far given. These gaps occur because of the lack of Lemma 4 and Lemma 5 when $q = \Re c$ (this q being replaced by $q - \Re b$ when proving Theorem 9C). Lemma 4 is false when $q = \Re c$, as shown below. I have not been able to prove or disprove Lemma 5 when $q = \Re c$.

Lemmas 3 and 4 depend on Theorem 5 of (12); they are replaceable when $q = \Re c$ by Lemmas 6 and 7 below, which depend similarly on Theorem 6 of (12). These lemmas provide the means of proving the missing part of (ii) of Theorem 9C.

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For lack of a substitute for Lemma 5, translation from Theorem 9 B by reciprocals (using (10)) fails as a means of proving (33) in the remaining case. Instead I return to the methods of (12). Lemmas 8 and 9 below are analogous to Theorems 1 and 2 of (12). Lemma 8 is admittedly only the translation of Theorem 1 of (12) by reciprocals; but Lemma 9 seems to be independent of Theorems 2 and 3 of (12), although closely analogous to Theorem 2. There is a fourth result of this kind, analogous to Theorem 3 of (12); it is omitted since it is not needed here. Four more results of this kind may also be obtained by applying the reciprocal transformation to these four, but they are not needed here either.

Failure of Lemma 4 when $q = \Re c$. Let $q = c = 1$, $a < 0, d = 1$ and $g(x) = 1/\log x$. By Theorem 7, $J^{-1}g(x)$ and $J^{-1}x^a g(x)$ exist and

$$J^{-c}g(x) = \frac{1}{x(\log x)^2}, \quad x^{-a}J^{-c}x^a g(x) = \frac{-a}{x\log x} + \frac{1}{x(\log x)^2}.$$

The former of these is in R_0 but the latter is not.

AN INTEGRABILITY CLASS. *For complex* c *with* $\gamma = \Re c > 0$, R^*_{c-1} *is the class of complex-valued functions* g *such that both* g *itself and*

$$\frac{1}{x^c}\int_x^\infty t^{c-1}g(t)\,dt$$

are in $R_{\gamma-1}$.

This definition is made in order to simplify the statements of Lemmas 6 and 7. Notice in particular that R^*_{c-1} is a subclass of $R_{\gamma-1}$.

LEMMA 6. *If* $\Re a < 0 < \Re c = \gamma$ *then the equation*

$$J^c x^a f(x) = {}^0 x^a J^c g(x) \quad in \quad d < x < \infty \tag{20}$$

has, for each f *in* $R_{\gamma-1}$ *a solution* g *in* R^*_{c-1}, *and for each* g *in* R^*_{c-1} *a solution* f *in* $R_{\gamma-1}$.

As in Lemma 3 let $x' = 1/x$ and $g'(x') = x^{1+c}g(x)$. As in (5.13) of (12) let

$$\bar{g}'(x') = \frac{1}{x'}\int_0^{x'} g'(t')\,dt'.$$

The proof of Lemma 3, with q replaced by γ and q' by 0, is applicable here if Theorem 6 of (12) is used instead of Theorem 5 of (12). The following additional remarks then complete the proofs of the two parts.

(i) Supposing $f \in R_{\gamma-1}$ is given, we thus obtain $g \in R_{\gamma-1}$ satisfying (20), but Theorem 6 (i) of (12) also gives that $\bar{g}' \in Q_0$. Consequently

$$\int_l^\infty x^{\gamma-1}\left|\frac{1}{x^c}\int_x^\infty t^{c-1}g(t)\,dt\right|dx = \int_l^\infty \frac{1}{x}\left|\int_x^\infty t^{c+1}g(t)\,\frac{1}{t^2}\,dt\right|dx$$
$$= \int_0^{1/l}\frac{1}{x'}\left|\int_0^{x'} g'(t')\,dt'\right|dx' = \int_0^{1/l}|\bar{g}'(x')|\,dx'$$

is finite for each $l > d$; and so $g \in R^*_{c-1}$, as required.

(ii) If $g \in R^*_{c-1}$ is given, the above equations show that g' and $\bar{g}'$ are in Q_0, and so Theorem 6 (ii) of (12) applies to (23) in the proof of Lemma 3 (ii).

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LEMMA 7. *If* $\Re a < 0 < \Re c = \gamma$, *then* $x^{-a}J^{-c}x^{a}g(x)$ *exists in* $R_{\gamma-1}$ *if and only if* $J^{-c}g(x)$ *exists in* R^{*}_{c-1}.

The proof of Lemma 4, with q replaced by γ, is applicable here if Lemma 6 is used instead of Lemma 3, and if '$\psi \in R_{q-1}$' is replaced by '$\psi \in R^{*}_{c-1}$'. The latter replacement is needed twice in (i) and once in (ii), and $\psi \in R^{*}_{c-1} \subset R_{\gamma-1}$ is used in the last line of (i).

Proof of (ii) *of Theorem* 9C *when* $q = \Re(c+b)$. Since $\Re c > 0$ and $\Re b > 0$ in Theorem 9C, Lemma 7 with a and $g(x)$ replaced by $-b$ and $x^{b}g(x)$ shows that $x^{b}J^{-c}\{x^{-b}.x^{b}g(x)\}$ exists in $R_{\gamma-1}$ if and only if $J^{-c}\{x^{b}g(x)\}$ exists in R^{*}_{c-1}. Thus, writing β for $\Re b$, $J^{-c}g(x)$ exists in $R_{\beta+\gamma-1}$ if and only if $J^{-c}\{x^{b}g(x)\}$ exists and

$$J^{-c}\{x^{b}g(x)\} \quad \text{and} \quad \frac{1}{x^{c}}\int_{x}^{\infty} t^{c-1}J^{-c}\{t^{b}g(t)\}\,dt$$

are both in $R_{\gamma-1}$.

Since $q = \beta+\gamma$ the last sentence states that $J^{-c}g$ exists in R_{q-1} if and only if $x^{-b}J^{-c}\{x^{b}g(x)\}$ exists in R_{q-1} and

$$\frac{1}{x^{c+b}}\int_{x}^{\infty} t^{c-1}J^{-c}\{t^{b}g(t)\}\,dt$$

is also in R_{q-1}. This is what is required about (ii) of Theorem 9C.

LEMMA 8. *If* $\kappa = \Re k > 0$, $q-\kappa > \max(0, \Re(a+b-c))$, $c \neq 0, -1, -2, \ldots$ *and* $f \in R_{q-1}$ *then*

$$\int_{x}^{\infty} \left| \frac{(t-x)^{k-1}}{\Gamma(k)} F\left(a, b, c, 1-\frac{x}{t}\right) f(t) \right| dt \tag{41}$$

is in $R_{q-\kappa-1}$. (An extension of Theorem 1.)

Denote the value of (41) by $\phi(x)$, whether finite or infinite: ϕ is a measurable non-negative function. Choose $l > d$, and let $l' = 1/l$. Putting $x' = 1/x$ and $t' = 1/t$, and using Kummer's relation ((10): 2·9 (3))

$$\begin{aligned}
\int_{l}^{\infty} x^{q-\kappa-1}\phi(x)\,dx &= \int_{0}^{l'} x'^{\kappa-q-1}\phi(x)\,dx' \\
&= \int_{0}^{l'} x'^{\kappa-q-1}\,dx' \int_{0}^{x'} \left| \frac{(x'-t')^{k-1}}{x'^{k-1}t'^{k+1}} \frac{f(t)}{\Gamma(k)} F\left(a, b, c, 1-\frac{t'}{x'}\right) \right| dt' \\
&= \int_{0}^{l'} x'^{-q}\,dx' \int_{0}^{x'} \left| \frac{(x'-t')^{k-1}}{\Gamma(k)} \frac{f(t)}{t'^{k+1}} \left(\frac{t'}{x'}\right)^{-a} F\left(a, c-b, c, 1-\frac{x'}{t'}\right) \right| dt' \\
&= \int_{0}^{l'} x'^{\alpha-q}\,dx' \int_{0}^{x'} \left| \frac{(x'-t')^{k-1}}{\Gamma(k)} F\left(a, c-b, c, 1-\frac{x'}{t'}\right) g(t') \right| dt'
\end{aligned}$$

where $\alpha = \Re a$ and $g(t') = t^{a+k+1}f(t)$.

Since $f \in R_{q-1}$,

$$\infty > \int_{l}^{\infty} t^{q-1}|f(t)|\,dt = \int_{l}^{\infty} t^{q-\alpha-\kappa-2}|g(t')|\,dt = \int_{0}^{l'} t'^{\kappa+\alpha-q}|g(t')|\,dt',$$

so that, putting $\kappa+\alpha-q = q'$, $t'^{q'}g(t')$ is integrable on $(0, l')$. Theorem 1 of (12), with q, h, b, f replaced by $q', a-q, c-b, g$ respectively, now applies to show that

$$|x'^{a-q}| \int_{0}^{x'} \left| \frac{(x'-t')^{k-1}}{\Gamma(k)} F\left(a, c-b, c, 1-\frac{x'}{t'}\right) g(t') \right| dt'$$

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is finite almost everywhere and integrable on $(0, l')$. Hence

$$\int_l^\infty x^{q-\kappa-1}\phi(x)\,dx < \infty$$

for each $l > d$, so that $\phi \in R_{q-\kappa-1}$ as required.

LEMMA 9. *If* $\Re\lambda > 0$, $\Re c > 0$, $q \geqslant \Re(c+\lambda)$, $q > \Re(a+b)$ *and* $f \in R_{q-1}$, *then*

$$\int_x^\infty \frac{(s-x)^{\lambda-1}}{\Gamma(\lambda)}\,ds \int_s^\infty \frac{(t-s)^{c-1}}{\Gamma(c)} F\left(a, b, c, 1-\frac{s}{t}\right) f(t)\,dt$$
$$= {}^0\int_x^\infty \frac{(t-x)^{c+\lambda-1}}{\Gamma(c+\lambda)} F\left(a, b, c+\lambda, 1-\frac{x}{t}\right) f(t)\,dt, \qquad (42)$$

that is, the two sides exist and are equal for almost all $x > d$. (Compare Theorem 2 of (12), of which this is, however, not a corollary.)

Provided the order of integration in the double integral on the left can be reversed, the left side is equal to

$$\int_x^\infty f(t)\,dt \int_x^t \frac{(t-s)^{c-1}}{\Gamma(c)} \frac{(s-x)^{\lambda-1}}{\Gamma(\lambda)} F\left(a, b, c, 1-\frac{s}{t}\right) ds;$$

and this is equal to the right side, evaluating the inner integral by Lemma 1 of (12) with x and t interchanged in (3·8). It is therefore enough to prove the absolute convergence of the double integral in (42) for amost all $x > d$.

Let $$\phi(s) = \int_s^\infty \left| \frac{(t-s)^{c-1}}{\Gamma(c)} F\left(a, b, c, 1-\frac{s}{t}\right) f(t) \right| dt.$$

Putting $k = c$ in Lemma 8, we obtain that $\phi \in R_{q-\gamma-1}$ where $\gamma = \Re c$. Writing μ for $\Re\lambda$, we have by data that $q-\gamma \geqslant \mu$, so that $\phi \in R_{\mu-1}$. Theorem 1 with c and f replaced by μ and ϕ now shows that $J^\mu\phi(x)$ is finite for almost all $x > d$; whence

$$\int_x^\infty \left| \frac{(s-x)^{\lambda-1}}{\Gamma(\lambda)} \right| \phi(s)\,ds < \infty$$

for the same values of x. This proves the required absolute convergence of the double integral in (42).

Remark. To deduce Lemma 9 merely by a reciprocal transformation from a theorem like Theorem 2 of (12), we should need, not Theorem 2 or 3 of (12), but the following formula, proved under suitable conditions:

$$\int_0^x \frac{(x-s)^{\lambda-1}}{\Gamma(\lambda)} s^{-c-\lambda}\,ds \int_0^s \frac{(s-t)^{c-1}}{\Gamma(c)} F\left(a, b, c, 1-\frac{t}{s}\right) f(t)\,dt$$
$$= {}^0 x^{-c} \int_0^x \frac{(x-t)^{c+\lambda-1}}{\Gamma(c+\lambda)} F\left(a, b, c+\lambda, 1-\frac{t}{x}\right) t^{-\lambda} f(t)\,dt.$$

Proof of (33) *of Theorem* 9 C *when* $\Re b = \Re c$. We have already proved this in section 5 when $q > \Re(c+b)$. The following proof uses that result to prove the case $q = \Re(c+b)$.

Suppose $\Re b = \Re c$, and introduce k such that $0 < \Re k < \Re c$. Write γ, γ and κ for $\Re b$, $\Re c$ and $\Re k$; and assume as in Theorem 9 C that $J^{-c}g$ exists in $R_{2\gamma-1}$. Applying

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Theorem 5 with f, a and b replaced by g, c and k respectively, as we may since $J^{-c}g \in R_{2\gamma-1} \subset R_{\gamma+\kappa-1}$, we obtain that $J^k J^{-c}g = J^{k-c}g$. Writing h for $J^{k-c}g$, and noticing that $J^{-c}g \in R_{\gamma+\kappa-1} \subset R_{\kappa-1}$, it follows that

$$J^{-c}g = J^{-k}h, \quad \text{where} \quad h = J^{k-c}g. \tag{43}$$

Applying Theorem 6 with f, b, c and $\iota\theta$ replaced by g, $c-k$, c and $b-c$, we obtain also that

$$J^{b-k}h = J^{b-k}J^{k-c}g = J^{b-c}g. \tag{44}$$

Since $\Re b > \Re k > 0$, $q > \Re(k+b)$, $q > \Re(a+b)$ and $J^{-k}h$ exists in R_{q-1} by (43), Theorem 9 C as already proved, with c and g replaced by k and h, shows that

$$f(x) = x^{-a}J^{-b}x^a J^{b-k}h(x) \tag{45}$$

exists in R_{q-1} and satisfies the equation

$$\int_x^\infty \frac{(t-x)^{k-1}}{\Gamma(k)} F\left(a, b, k, 1-\frac{x}{t}\right) f(t)\, dt = {}^0 h(x) \quad \text{for} \quad d < x < \infty.$$

Applying J^{c-k} to both sides of this, f given by (45) satisfies

$$\int_x^\infty \frac{(s-x)^{c-k-1}}{\Gamma(c-k)}\, ds \int_s^\infty \frac{(t-s)^{k-1}}{\Gamma(k)} F\left(a, b, k, 1-\frac{s}{t}\right) f(t)\, dt = {}^0 J^{c-k}h(x),$$

both sides existing for almost all $x > d$ because the right side does, being by (43) simply $g(x)$. The left side can be re-written using Lemma 9 with λ and c replaced by $c-k$ and k, obtaining

$$\int_x^\infty \frac{(t-x)^{c-1}}{\Gamma(c)} F\left(a, b, c, 1-\frac{x}{t}\right) f(t)\, dt = {}^0 g(x);$$

that is, f given by (45) satisfies equation (C). But (44) shows that (45) is the same as (33). Thus f given by (33) is in R_{q-1} and satisfies (C) when $q = \Re(c+b) = 2\Re c$. This completes the proof of Theorem 9 C at last.

7. The hypergeometric integral equation (D)

We now discuss equation (D) of section 1, namely

$$\int_x^\infty \frac{(t-x)^{c-1}}{\Gamma(c)} F\left(a, b, c, 1-\frac{t}{x}\right) f(t)\, dt = {}^0 g(x) \quad \text{for} \quad d < x < \infty, \tag{D}$$

where $0 \leqslant d < \infty$. The theorems obtained are applicable also to this equation with a finite upper terminal in place of ∞, as mentioned in Theorem 4 D Corollary below; in particular, to the equations considered by Ta Li (1), Buschman (2), Higgins (3, 6), Srivastava (4) and Wimp (7).

Our theorems can be obtained by converting (D) into the form of (A) of section 1 by means of (29). They were first obtained in that way, by work analogous to section 5 (without the finishing touches, which would need something like section 6). That work is not included, but I have used it to check the theorems obtained below.

Instead we use a more economical procedure: we convert (D) into the form of (C) using Kummer's relation ((10): 2·9 (3)) and

$$\left.\begin{aligned} f'(x) &= x^{-a}f(x), & g'(x) &= x^{-a}g(x), \\ b' &= c-b, & q' &= q+\Re a. \end{aligned}\right\} \tag{46}$$

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This gives the following equation, of the form of (C), equivalent to (D):

$$\int_x^\infty \frac{(t-x)^{c-1}}{\Gamma(c)} F\left(a, b', c, 1-\frac{x}{t}\right) f'(t)\,dt = {}^0g'(x) \quad \text{for} \quad d < x < \infty; \tag{47}$$

and we apply the theorems of sections 5 and 6 to this.

THEOREM 4 D. *If* $\Re c > 0$, $q \geqslant \max(0, \Re(c-a-b))$ *and* $q > \max(\Re(c-a), \Re(c-b))$, *then* (D) *cannot have more than one solution* f *in* R_{q-1}.

This is proved in the same simple manner as Theorem 4 C, replacing C by D, (29) and (30) by (46) and (47), $Q_{q'}$ by $R_{q'-1}$ and Theorem 4 B by Theorem 4 C.

COROLLARY. *If* $\Re c > 0$ *and* $0 \leqslant d < \delta < \infty$, *then the solutions* f *of*

$$\int_x^\delta \frac{(t-x)^{c-1}}{\Gamma(c)} F\left(a, b, c, 1-\frac{t}{x}\right) f(t)\,dt = {}^0g(x) \quad \text{for} \quad d < x < \delta \tag{48}$$

which are locally integrable in $(d, \delta]$ *are the truncations, omitting* (δ, ∞), *of the solutions* f *in* R_{q-1} *of* (D), *provided that* $g(x) = 0$ *in* $x > \delta$ *and* q *satisfies the hypotheses of Theorem* 4 D.

The proof of Theorem 4 C Corollary applies also to this with very few changes.

THEOREM 7 D. *If* $\Re c > 0$, $q \geqslant \Re c$, $q > \max(\Re(c-a), \Re(c-b))$, f *is in* R_{q-1} *and* (D) *holds, then* $J^{-c}g$ *exists in* R_{q-1}.

Further, $x^a J^{-c} x^{-a} g(x)$ *and* $x^b J^{-c} x^{-b} g(x)$ *exist in* R_{q-1}; *and so also does*

$$\frac{1}{x^c}\int_x^\infty t^{c-1} J^{-c} g(t)\,dt.$$

Taken with (46), the data imply that $\Re c > 0, q' \geqslant \Re(c+a), q' > \max(\Re c, \Re(a+b'))$, f' is in $R_{q'-1}$ and (47) holds. So Theorem 7 C gives that the following functions exist in $R_{q'-1}$:

$$J^{-c}g'(x), \quad x^{-a}J^{-c}x^a g'(x), \quad x^{c-a-b'}J^{-c}x^{a+b'-c}g'(x),$$

and

$$\frac{1}{x^{c+a}}\int_x^\infty t^{c-1} J^{-c}\{t^a g'(t)\}\,dt.$$

The functions obtained from these by multiplying by x^a are in R_{q-1}, and this with (46) gives the stated conclusions.

THEOREM 8 D. *If* $\Re c > 0$, $\Re b > 0$, $q \geqslant \Re c$, $q > \Re(c-a)$ *and* $x^a J^{-c} x^{-a} g(x)$ *exists in* R_{q-1}, *then*

$$\int_x^\infty \frac{(t-x)^{c-1}}{\Gamma(c)} F\left(a, b, c, 1-\frac{t}{x}\right) f(t)\,dt = {}^0g(x) \quad \text{for} \quad d < x < \infty \tag{D}$$

has a solution f *in* R_{q-1}, *expressible by*

$$f(x) = x^{a+b-c} J^{-b} x^{c-a} J^{b-c} x^{-b} g(x). \tag{49}$$

The assumption that $x^a J^{-c} x^{-a} g(x)$ *exists in* R_{q-1} *may be replaced by either of the following, which are equivalent to it in this context*:

(i) $x^b J^{-c} x^{-b} g(x)$ *exists in* R_{q-1};

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(ii) $J^{-c}g(x)$ *exists in* R_{q-1}; *supplemented if* $q = \Re c$ *by*

$$\frac{1}{x^c}\int_x^\infty t^{c-1}J^{-c}g(t)\,dt \quad \textit{being in } R_{q-1}.$$

Taken with (46), the data are equivalent to: $\Re c > 0$, $\Re b' < \Re c$, $q' \geqslant \Re(c+a)$, $q' > \Re c$ and $J^{-c}g'(x)$ exists in $R_{q'-1}$. It follows by Theorem 8 C that (47) has a solution f' in $R_{q'-1}$, expressible by

$$f'(x) = x^{-b'}J^{b'-c}x^{c-a}J^{-b'}x^{a+b'-c}g'(x);$$

with (46) this gives that (D) has a solution f in R_{q-1}, expressible by (49). This proves the main part of the theorem.

Theorem 8 C also gives that the existence of $J^{-c}g'(x)$ in $R_{q'-1}$ is equivalent to the existence of $x^{c-a-b'}J^{-c}x^{a+b'-c}g'(x)$ in $R_{q'-1}$. Translating these equivalent statements by (46) we obtain the required equivalence of (i) to the assumption that $x^aJ^{-c}x^{-a}g(x)$ exists in R_{q-1}. Similarly for (ii).

THEOREM 9 D. *If* $\Re c > 0$, $\Re b < \Re c$, $q \geqslant \Re(2c-a-b)$, $q > \Re(c-b)$ *and* $x^aJ^{-c}x^{-a}g(x)$ *exists in* R_{q-1}, *then* (D) *has a solution* f *in* R_{q-1}, *expressible by*

$$f(x) = J^{b-c}x^aJ^{-b}x^{-a}g(x). \tag{50}$$

The assumption that $x^aJ^{-c}x^{-a}g(x)$ *exists in* R_{q-1} *may be replaced by either of the following, which are equivalent to it in this context*:

(i) $x^bJ^{-c}x^{-b}g(x)$ *exists in* R_{q-1};

(ii) $x^{a+b-c}J^{-c}x^{c-a-b}g(x)$ *exists in* R_{q-1}, *supplemented if* $q = \Re(2c-a-b)$ *by*

$$\frac{1}{x^{2c-a-b}}\int_x^\infty t^{c-1}J^{-c}\{t^{c-a-b}g(t)\}\,dt \quad \textit{being in } R_{q-1}.$$

Taken with (46), the data are equivalent to: $\Re c > 0$, $\Re b' > 0$, $q' \geqslant \Re(c+b')$, $q' > \Re(a+b')$ and $J^{-c}g'(x)$ exists in $R_{q'-1}$. So, by Theorem 9 C, (47) has a solution f' in $R_{q'-1}$, expressible by $f'(x) = x^{-a}J^{-b'}x^aJ^{b'-c}g'(x)$;

with (46) this gives that (D) has a solution f in R_{q-1}, expressible by (50). This proves the first paragraph of the theorem.

Alternative hypothesis (i) is established from Theorem 9 C exactly as in the previous proof. For (ii) we have by Theorem 9 C that existence of $J^{-c}g'$ in $R_{q'-1}$ is equivalent to that of $x^{-b'}J^{-c}x^{b'}g'(x)$ in $R_{q'-1}$, supplemented if $q' = \Re(c+b')$ by

$$\frac{1}{x^{c+b'}}\int_x^\infty t^{c-1}J^{-c}\{t^{b'}g'(t)\}\,dt$$

being in $R_{q'-1}$. Translating these equivalent statements by (46) we obtain the required equivalence of (ii) to the assumption that $x^aJ^{-c}x^{-a}g(x)$ exists in R_{q-1}.

Remarks. Higgins (6) discusses equation (48) with $0 < d < \delta = 1$ and obtains the solution to which (50) reduces in the special case he considers. The present theory shows that (50), with $g(x) = 0$ for $x > \delta$, is the unique solution of (48) locally integrable in $(d, \delta]$ if $\Re c > 0$, $\Re b < \Re c$ and $J^{-c}x^{-a}g(x)$ is locally integrable in $(d, \delta]$. This follows from Theorem 4 D Corollary and from Theorems 4 D and 9 D, all of which are applicable with a sufficiently large value of q. Similar remarks apply to (49) when $\Re b > 0$;

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Higgins's operators are, however, so defined that the form (50) still serves then (compare Theorems 11 and 12 of (12)).

Comparing the explicit solutions (37) and (49), (33) and (50), we notice that formal reversal of the order of operations in the solution formulae for (C) gives the solution formulae for (D). This might be expected from the similar event regarding (A) and (B) in (12).

THEOREM 10 D. *If* $\Re c > 0$, $q \geqslant \max(\Re c, \Re(2c-a-b))$, $q > \max(\Re(c-a), \Re(c-b))$ *and* $x^a J^{-c} x^{-a} g(x)$ *exists in* R_{q-1}, *then*

$$\int_x^\infty \frac{(t-x)^{c-1}}{\Gamma(c)} F\left(a, b, c, 1-\frac{t}{x}\right) f(t)\, dt = {}^0 g(x) \quad \text{for} \quad d < x < \infty \tag{D}$$

has a unique solution f *in* R_{q-1}, *given by* (49) *if* $\Re b > 0$ *and by* (50) *if* $\Re b < \Re c$.

The assumption that $x^a J^{-c} x^{-a} g(x)$ *exists in* R_{q-1} *may be replaced by the assumption, equivalent to it in this context, that* $x^b J^{-c} x^{-b} g(x)$ *exists in* R_{q-1}.

The uniqueness follows from Theorem 4 D; the rest from Theorem 8 D when $\Re b > 0$ and from Theorem 9 D when $\Re b < \Re c$. In the second paragraph the replaceability is obvious from the symmetry between a and b of the other hypotheses and of (D), but the equivalence is not.

Referring in Theorems 7 D and 10 D we also obtain:

COROLLARY. *If* $\Re c > 0$, $q \geqslant \max(\Re c, \Re(2c-a-b))$ *and* $q > \max(\Re(c-a), \Re(c-b))$, *then a necessary and sufficient condition that equation* (D) *should have a solution* f *in* R_{q-1} *is that* $x^a J^{-c} x^{-a} g(x)$ *should exist in* R_{q-1}.

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