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ARITHMETIC PROPERTIES SATISFIED BY A RECENT INTEGER PARTITION FUNCTION OF DOMBOS

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Abstract

The set of integer partitions of n, wherein the parts are either divisible by 4 or congruent to $\pm 1 \pmod 6$, arose in a natural way in work of Dombos ['A new grounded partition identity of type $D_4^{(3)}$', *SIGMA* **22** (2026), Article no. 042]. Denote the function that counts the number of such partitions of n by $dp(n)$. Using elementary generating function manipulations and classical q-series results, we prove several congruences satisfied by $dp(n)$. As an example, we prove that, for all $\alpha \geq 1$ and all $n \geq 0$,

$$dp\left(3^{2\alpha+1}n + \frac{7 \cdot 9^{\alpha} + 1}{4}\right) \equiv 0 \pmod 3.$$

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1. Introduction

The arithmetic properties of partition functions have been a central theme in the theory of partitions since the pioneering work of Ramanujan [6]. His celebrated congruences for the ordinary partition function (see below) initiated a vast literature devoted to the congruence properties of both the ordinary partition function and numerous families of restricted partition functions. More recently, many new partition functions arising from Rogers–Ramanujan-type identities and related q-series have also been shown to satisfy remarkable arithmetic properties.

We recall that a partition of a positive integer n is a nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $n = \lambda_1 + \lambda_2 + \cdots + \lambda_r$. Each λ_i, $1 \leq i \leq r$, is called a part of the partition. We denote the number of partitions of n by $p(n)$, the ordinary partition function.

In the late 1910s, Ramanujan [6] proved the following congruences satisfied by $p(n)$ for the moduli 5, 7 and 11 using generating function manipulations: for all $n \geq 0$,

$$p(5n + 4) \equiv 0 \pmod 5, \quad p(7n + 5) \equiv 0 \pmod 7 \quad \text{and} \quad p(11n + 6) \equiv 0 \pmod{11}.$$

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These congruences, and others that have since been proven for numerous restricted partition functions, serve as the motivation for this work.

Recently, Dombos [4] introduced a new Rogers–Ramanujan-type identity involving grounded partitions. One family of partitions arising naturally from this identity consists of partitions whose parts are either divisible by 4 or congruent to $\pm 1 \pmod 6$. We refer the reader to [4, Theorem 1.6] for more details. Let $dp(n)$ denote the number of such partitions of n. In light of this definition, the generating function for $dp(n)$ is given by

$$G(q) := \sum_{n=0}^{\infty} dp(n)q^n = \frac{1}{(q;q^6)_\infty (q^5;q^6)_\infty (q^4;q^4)_\infty}, \tag{1.1}$$

where $(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$ is the usual q-Pochhammer symbol.

Since the generating function of $dp(n)$ has a simple infinite product representation, it is natural to investigate whether $dp(n)$ satisfies Ramanujan-type congruences analogous to those enjoyed by many other restricted partition functions. To the best of our knowledge, no arithmetic properties of $dp(n)$ have previously been established.

The purpose of this paper is to initiate the study of the arithmetic properties of the partition function $dp(n)$ by establishing several infinite families of Ramanujan-type congruences. Our proofs rely entirely on elementary generating function manipulations together with classical 2- and 3-dissection identities, and several well-known q-series identities, avoiding the use of modular forms. In addition to congruences modulo powers of 2 and modulo 3, we obtain an infinite family of congruences modulo 4 indexed by primes in two residue classes. These results are summarised in the following theorems.

THEOREM 1.1. *For all $n \geq 0$,*

$$dp(6n+4) \equiv 0 \pmod 2, \tag{1.2}$$

$$dp(18n+10) \equiv 0 \pmod 4, \tag{1.3}$$

$$dp(54n+52) \equiv 0 \pmod 8. \tag{1.4}$$

THEOREM 1.2. *Let p be a prime such that $p \equiv 13, 17, 19$ or $23 \pmod{24}$. Then, for all $k, m \geq 0$ with $p \nmid m$,*

$$dp\left(6p^{2k+1}m + \frac{15p^{2k+2}+1}{4}\right) \equiv 0 \pmod 4.$$

THEOREM 1.3. *For all $n \geq 0$,*

$$dp(27n+16) \equiv 0 \pmod 3. \tag{1.5}$$

THEOREM 1.4. *For all $n \geq 0$,*

$$dp(27n+7) \equiv dp(3n+1) \pmod 3.$$

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THEOREM 1.5. *For all $\alpha \geq 1$ and all $n \geq 0$,*

$$dp\left(3^{2\alpha+1}n + \frac{7 \cdot 9^{\alpha} + 1}{4}\right) \equiv 0 \pmod 3.$$

2. Preliminaries

Throughout the remainder of this paper, we define

$$f_k := (q^k; q^k)_\infty$$

to shorten the notation. We begin by rewriting the generating function (1.1) in terms of the f_k.

LEMMA 2.1. *We have*

$$\sum_{n=0}^{\infty} dp(n)q^n = \frac{f_2 f_3}{f_1 f_4 f_6}. \tag{2.1}$$

PROOF. Initially, we note that

$$\begin{aligned}
\frac{1}{(q;q^6)_\infty (q^5;q^6)_\infty} &= \frac{(q^2;q^6)_\infty (q^3;q^6)_\infty (q^4;q^6)_\infty (q^6;q^6)_\infty}{(q;q^6)_\infty (q^2;q^6)_\infty (q^3;q^6)_\infty (q^4;q^6)_\infty (q^5;q^6)_\infty (q^6;q^6)_\infty} \\
&= \frac{(q^3;q^3)_\infty (q^2;q^6)_\infty (q^4;q^6)_\infty}{(q;q^3)_\infty (q^2;q^3)_\infty (q^3;q^3)_\infty} \\
&= \frac{(q^2;q^6)_\infty (q^4;q^6)_\infty}{(q;q^3)_\infty (q^2;q^3)_\infty} \\
&= \frac{(q^2;q^2)_\infty}{(q^6;q^6)_\infty} \cdot \frac{(q^3;q^3)_\infty}{(q;q)_\infty} = \frac{f_2 f_3}{f_1 f_6}.
\end{aligned}$$

Thus, it follows from (1.1) that

$$\sum_{n=0}^{\infty} dp(n)q^n = \frac{f_2 f_3}{f_1 f_4 f_6},$$

which is (2.1). □

The proofs of the above-mentioned main results require a few well-known 2- and 3-dissections.

LEMMA 2.2 [3, Lemma 1]. *We have*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{2.2}$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}. \tag{2.3}$$

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LEMMA 2.3 [7, (2.1c)]. *We have*

$$\frac{f_2}{f_1 f_4} = \frac{f_{18}^9}{f_3^2 f_9^3 f_{12}^2 f_{36}^3} + q\frac{f_6^2 f_{18}^3}{f_3^3 f_{12}^3} + q^2\frac{f_6^4 f_9^3 f_{36}^3}{f_3^4 f_{12}^4 f_{18}^3}. \tag{2.4}$$

LEMMA 2.4 [7, (2.1b)]. *We have*

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}. \tag{2.5}$$

LEMMA 2.5 [5, (14.3.2)]. *We have*

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}. \tag{2.6}$$

LEMMA 2.6 [7, (3.1)]. *We have*

$$\frac{f_2^3}{f_1^3} = \frac{f_6}{f_3} + 3q\frac{f_6^4 f_9^5}{f_3^8 f_{18}} + 6q^2\frac{f_6^3 f_9^2 f_{18}^2}{f_3^7} + 12q^3\frac{f_6^2 f_{18}^5}{f_3^6 f_9}. \tag{2.7}$$

We recall Ramanujan's theta functions

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \quad \text{for } |ab| < 1,$$

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2},$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{f_2^2}{f_1}.$$

With the above definitions, we can state the following 3-dissection of $1/\psi(q)$.

LEMMA 2.7 ([2, Lemma 2.2]). *We have*

$$\frac{1}{\psi(q)} = \frac{f_3^2 f_9^3}{f_6^6} - q\frac{f_3^3 f_{18}^3}{f_6^7} + q^2\frac{f_3^4 f_{18}^6}{f_6^8 f_9^3}. \tag{2.8}$$

Lastly, we require the classical identity of Jacobi (see [1, Theorem 1.3.9]):

$$f_1^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}. \tag{2.9}$$

3. Proof of Theorem 1.1

Let

$$H(q) = \sum_{n=0}^{\infty} h(n) q^n = G(-q) = \sum_{n=0}^{\infty} dp(n)(-q)^n.$$

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Thus, $h(n) = (-1)^n dp(n)$ for all n. We note that

$$(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4}.$$

Using this identity and (2.1),

$$H(q) = \frac{f_1 f_6^2}{f_2^2 f_3 f_{12}} = \frac{f_6^2}{f_3 f_{12}} \frac{1}{\psi(q)}.$$

Thanks to (2.8),

$$\sum_{n=0}^{\infty} h(3n+1) q^{3n+1} = -q \frac{f_3^2 f_{18}^3}{f_6^5 f_{12}},$$

which, after dividing by q and replacing q^3 by q, yields

$$\sum_{n=0}^{\infty} h(3n+1) q^n = -\frac{f_1^2 f_6^3}{f_2^5 f_4}.$$

Using (2.2), we can 2-dissect the expression above to obtain

$$\sum_{n=0}^{\infty} h(6n+4) q^{2n+1} = 2q \frac{f_6^3 f_{16}^2}{f_2^4 f_4 f_8}.$$

Thus,

$$\sum_{n=0}^{\infty} h(6n+4) q^n = 2 \frac{f_3^3 f_8^2}{f_1^4 f_2 f_4},$$

which proves (1.2) since $dp(6n+4) = h(6n+4) \equiv 0 \pmod 2$.

Working modulo 8,

$$\sum_{n=0}^{\infty} h(6n+4) q^n = 2 \frac{f_3^3 f_8^2}{f_1^4 f_2 f_4} \equiv 2 f_3^3 \frac{f_4^3}{f_2^3} \pmod 8. \tag{3.1}$$

Thanks to (2.7),

$$\sum_{n=0}^{\infty} h(18n+10) q^{3n+1} \equiv 4q^4 \frac{f_3^3 f_{12}^3 f_{18}^2 f_{36}^2}{f_6^7} \pmod 8,$$

which simplifies to

$$\sum_{n=0}^{\infty} h(18n+10) q^n \equiv 4q \frac{f_1^3 f_4^3 f_6^2 f_{12}^2}{f_2^7} \pmod 8.$$

It follows that $h(18n+10) \equiv 0 \pmod 4$, which completes the proof of (1.3) since $h(n) = (-1)^n dp(n)$.

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To prove (1.4), we use (2.7) to extract the terms of the form q^{3n+2} from (3.1). The resulting congruence is

$$\begin{aligned}\sum_{n=0}^{\infty} h(18n+16)q^n &\equiv 6\frac{f_1^3 f_4^4 f_6^5}{f_2^8 f_{12}} \pmod 8\\ &\equiv 6\frac{f_6^5}{f_{12}} f_1^3 \pmod 8\\ &\equiv 6\frac{f_6^5}{f_{12}} \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} \pmod 8 \end{aligned} \tag{3.2}$$

by (2.9). We note that f_6^5/f_{12} is a function of q^3 and $n(n+1)/2$ is never congruent to 2 modulo 3. Hence, once we expand (3.2) as a power series in q, there will be no powers of the form q^{3n+2}. Therefore, for all $n \geq 0$,

$$h(18(3n+2)+16) = h(54n+52) \equiv 0 \pmod 8,$$

which completes the proof of (1.4).

4. Proof of Theorem 1.2

Initially, we use (2.4) to extract the terms of the form q^{3n+1} from (2.1), which gives

$$\sum_{n=0}^{\infty} dp(3n+1)q^{3n+1} = q\frac{f_6 f_{18}^3}{f_3^2 f_{12}^3}.$$

After dividing by q and replacing q^3 by q, we are left with

$$\sum_{n=0}^{\infty} dp(3n+1)q^n = \frac{f_2 f_6^3}{f_1^2 f_4^3}. \tag{4.1}$$

Now, we use (2.3) to extract the odd parts of (4.1), which gives

$$\sum_{n=0}^{\infty} dp(6n+4)q^n = 2\frac{f_3^3 f_8^2}{f_1^4 f_2 f_4} \equiv 2f_2^3 f_3^3 \pmod 4. \tag{4.2}$$

Using (2.9), it follows from (4.2) that

$$\begin{aligned}\sum_{n=0}^{\infty} dp(6n+4)q^n &\equiv 2f_2^3 f_3^3 \pmod 4\\ &= 2\sum_{k,l=0}^{\infty}(-1)^{k+l}(2k+1)(2l+1)q^{k(k+1)+3l(l+1)/2}\\ &\equiv 2\sum_{k,l=0}^{\infty} q^{k(k+1)+3l(l+1)/2} \pmod 4.\end{aligned}$$

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Thus,

$$\sum_{n=0}^{\infty} dp(6n+4)q^{24n+15} \equiv 2\sum_{k,l=0}^{\infty} q^{6(2k+1)^2+(6l+3)^2} \pmod 4.$$

It follows that $dp(6n+4) \equiv 0 \pmod 4$ if $24n+15$ is not of the form $6x^2+y^2$.

Since $p \equiv 13, 17, 19$ or $23 \pmod{24}$, it follows that $(-6/p) = -1$, which implies that $\nu_p(N)$ is even if N is of the form $6x^2+y^2$. Taking $n = p^{2k+1}m + 5(p^{2k+2}-1)/8$, we have

$$24n+15 = 24p^{2k+1}m + 15p^{2k+2} = p^{2k+1}(24m+15p).$$

Therefore, $\nu_p(24n+15)$ is odd and the result follows by the fact that

$$6\left(p^{2k+1}m + \frac{5(p^{2k+2}-1)}{8}\right) + 4 = 6p^{2k+1}m + \frac{15p^{2k+2}+1}{4}.$$

5. Proofs of Theorems 1.3–1.5

We now transition to proving Theorems 1.3–1.5 where the focus is on arithmetic properties modulo 3. Let us begin by proving (1.5). It follows from (4.1) that

$$\sum_{n=0}^{\infty} dp(3n+1)q^n \equiv \frac{f_2 f_6^3}{f_1^2 f_{12}} \pmod 3.$$

Thanks to (2.5),

$$\sum_{n=0}^{\infty} dp(9n+7)q^{3n+2} \equiv 4q^2 \frac{f_6^5 f_{18}^3}{f_3^6 f_{12}} \pmod 3.$$

Dividing by q and replacing q^3 by q yields

$$\sum_{n=0}^{\infty} dp(9n+7)q^n \equiv \frac{f_2^5 f_6^3}{f_1^6 f_4} \equiv \frac{f_6^4 f_2^2}{f_3^2 f_4} \pmod 3.$$

Finally, we use (2.6) to get

$$\sum_{n=0}^{\infty} dp(9n+7)q^n \equiv \frac{f_6^4}{f_3^2}\left(\frac{f_{18}^2}{f_{36}} - 2q^2\frac{f_6 f_{36}^2}{f_{12} f_{18}}\right) \pmod 3, \tag{5.1}$$

from which (1.5) follows.

We next consider the 'internal congruence' given in Theorem 1.4. It follows from (5.1) that

$$\sum_{n=0}^{\infty} dp(27n+7)q^{3n} \equiv \frac{f_6^4 f_{18}^2}{f_3^2 f_{36}} \pmod 3.$$

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Replacing q^3 by q, we are left with

$$\sum_{n=0}^{\infty} dp(27n+7)q^n \equiv \frac{f_2^4 f_6^2}{f_1^2 f_{12}} \equiv \frac{f_2 f_6^3}{f_1^2 f_4^3} \pmod 3.$$

Thus, it follows by (4.1) that

$$\sum_{n=0}^{\infty} dp(27n+7)q^n \equiv \sum_{n=0}^{\infty} dp(3n+1)q^n \pmod 3,$$

which completes the proof.

Lastly, we use the previous two theorems to prove Theorem 1.5. We prove this theorem by induction on α. Note that the case $\alpha = 1$ has already been established earlier in Theorem 1.3. We now use it as our base case in a proof by induction. Thus, we assume that, for some $\alpha \geq 1$ and all $n \geq 0$,

$$dp\left(3^{2\alpha+1}n + \frac{7\cdot 9^{\alpha}+1}{4}\right) \equiv 0 \pmod 3.$$

We then want to prove that

$$dp\left(3^{2\alpha+3}n + \frac{7\cdot 9^{\alpha+1}+1}{4}\right) \equiv 0 \pmod 3.$$

We note that

$$\begin{aligned} dp\left(3\left(3^{2\alpha}n + \frac{3\cdot 7\cdot 9^{\alpha-1}-1}{4}\right)+1\right) &= dp\left(3^{2\alpha+1}n + \frac{7\cdot 9^{\alpha}-3}{4}+1\right) \\ &= dp\left(3^{2\alpha+1}n + \frac{7\cdot 9^{\alpha}+1}{4}\right) \equiv 0 \pmod 3, \end{aligned}$$

by the induction hypothesis. Finally, by Theorem 1.4,

$$\begin{aligned} dp\left(3^{2\alpha+3}n + \frac{7\cdot 9^{\alpha+1}+1}{4}\right) &= dp\left(3^{2\alpha+3}n + \frac{7\cdot 9^{\alpha+1}-27}{4}+7\right) \\ &= dp\left(27\left(3^{2\alpha}n + \frac{3\cdot 7\cdot 9^{\alpha-1}-1}{4}\right)+7\right) \\ &\equiv dp\left(3\left(3^{2\alpha}n + \frac{3\cdot 7\cdot 9^{\alpha-1}-1}{4}\right)+1\right) \equiv 0 \pmod 3, \end{aligned}$$

which concludes the inductive proof.

References

[1] B. C. Berndt, *Ramanujan's Notebooks*, Part III (Springer, New York, 1991).

[2] S. Chern and L.-J. Hao, 'Congruences for partition functions related to mock theta functions', *Ramanujan J.* **48** (2019), 369–384.

[3] R. da Silva and J. A. Sellers, 'New congruences for 3-regular partitions with designated summands', *Integers* **20A** (2020), Article no. A6.

[4] B. Dombos, 'A new grounded partition identity of type $D_4^{(3)}$', *SIGMA* **22** (2026), Article no. 042.

[5] M. D. Hirschhorn, *The Power of q: A Personal Journey* (Springer, Cham, 2017).

https://doi.org/10.1017/S0004972726102226 Published online by Cambridge University Press

[6] S. Ramanujan, 'Some properties of $p(n)$, the number of partitions of n', *Proc. Camb. Phil. Soc.* **19** (1919), 207–210.
[7] P. C. Toh, 'Ramanujan type identities and congruences for partition pairs', *Discrete Math.* **312** (2012), 1244–1250.

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