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Kernel convergence of hyperbolic components

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Abstract. We study families $G(\lambda, \cdot)$ of entire functions that are approximated by a sequence of families $G_n(\lambda, \cdot)$ of entire functions, where $\lambda \in \mathbb{C}$ is a parameter. In order to control the dynamics, the families are assumed to be of the same constant finite type. In this setting we prove the convergence of the hyperbolic components in parameter space as kernels in the sense of Carathéodory.

1. *Introduction*

In iteration theory, founded by Fatou and Julia [**11, 13**], there has been much progress on the iteration of rational functions in the last few decades; for an overview see [**4, 21**]. However, even though Fatou studied the iteration of entire *transcendental* functions, the transcendental case has only recently received much attention; see the expositions [**1, 3, 8, 9**]. The theory develops along the lines of the rational case, but often other and generally more complicated methods of proof need to be used. There is also a variety of phenomena that do not occur in the rational case. One question is: which results carry over from the rational case and which do not?

An interesting approach to this question was suggested in [**6**] and is illustrated in [**5, 15, 16**]. The polynomials $P_n(\lambda, z) = \lambda(1+(z/n))^n$ converge uniformly on compact sets of the complex plane $\mathbb{C}$ to the exponentials $E(\lambda, z) = \lambda e^z$ as n tends to infinity. In [**6**] a combinatorial description is given how external rays to the connectedness loci of the $P_n(\lambda, \cdot)$ converge to certain 'hairs' in the *parameter plane* for $E(\lambda, \cdot)$. Furthermore, the pointwise convergence of hyperbolic components is shown in [**5, 6, 15**]. In the *dynamical plane* the convergence of Julia sets with respect to the Hausdorff metric is shown in [**15**] for the above families for suitable values of the parameter λ. The convergence of Julia sets was obtained in [**7**] for polynomials of constant degree, in [**19**] for rational functions, and has now been generalized to larger classes of functions; see [**14, 16, 17, 20**].

In this paper we are interested in convergence of hyperbolic components in *parameter space* for entire functions of the *same constant finite type* (for the exact definition see §3).

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MAIN THEOREM. *Let $G : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $G_n : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be holomorphic families of functions of the same constant finite type, such that the G_n converge to G uniformly on compact subsets of $\mathbb{C} \times \mathbb{C}$ as n tends to ∞. Then every hyperbolic component H of the family G is the kernel of a sequence $\{H_n\}$ of hyperbolic components of the families G_n.*

The convergence as kernels in the sense of Carathéodory is a stronger notion than the pointwise convergence that has been used up to now. We show in Example 2 that, in general, the hyperbolic components cannot be expected to converge in the Hausdorff metric. Note that the methods in [**5, 6, 15**] cannot be used in this general setting since they make use of explicit properties of the exponentials and the approximating polynomials.

The paper is organized as follows. First we recall in §2 some basic facts from iteration theory. Then we introduce families of constant finite type, their hyperbolic components and the notion of kernel convergence in §3. Furthermore, the Main Theorem is illustrated with examples. §4 is dedicated to the proof. The convergence of singular values is shown in §4.1 and is then used to prove the first part of the Main Theorem in §4.2. The proof is completed in §4.3.

2. *Background and Motivation*

We begin by recalling some basic notions and refer to [**3, 4, 8, 21**] for details and proofs. Let $\mathcal{O}(D)$ denote the set of *holomorphic* functions $f : D \to \mathbb{C}$, where $D \subset \mathbb{C}$ is a domain. A function $f \in \mathcal{O}(\mathbb{C})$ is called *entire*, and it is either a polynomial or an entire transcendental function. Consider an entire function f and let $f^{\circ n}$ denote its nth iterate.

Let ζ be a point on the Riemann sphere $\widehat{\mathbb{C}}$. If f is transcendental and ζ is not equal to the essential singularity ∞ or f is a polynomial, then $f^{\circ n}(\zeta)$ is defined for all $n \in \mathbb{N}$ and we call $O^+(\zeta) := \{f^{\circ n}(\zeta) \mid n \in \mathbb{N}\}$ the *(forward) orbit* of ζ. The point ζ is called *k-periodic* if all iterates are defined and $f^{\circ k}(\zeta) = \zeta$. It is called attracting (or repelling) if $|(f^{\circ k})'(\zeta)|$ is smaller (respectively larger) than one. To every attracting periodic orbit $O^+(\zeta)$ there belongs a *basin* $A(O^+(\zeta))$, defined as the set of all points $z \in \mathbb{C}$ whose orbits accumulate on ζ.

The *Julia set* $\mathcal{J}(f)$ of f is the closure (with respect to $\widehat{\mathbb{C}}$) of the set of repelling periodic points of f. The *Fatou set* $\mathcal{F}(f)$ of f is its complement in $\widehat{\mathbb{C}}$, and clearly any basin $A(O^+(\zeta))$ is a subset of the Fatou set $\mathcal{F}(f)$. The Julia set is a non-empty perfect set, and both $\mathcal{J}(f)$ and $\mathcal{F}(f)$ are completely invariant under f. Note that we regard both these sets as subsets of $\widehat{\mathbb{C}}$.

A point $v \in \widehat{\mathbb{C}}$ is called a *singular value* of f if, for every neighborhood U of v, there exists a branch of the inverse of f that is not holomorphic on U. Let $\operatorname{sing}(f)$ denote the set of all *finite* singular values of f. This set consists of the *finite critical values* of f and, if f is transcendental, also of the *finite asymptotic values* of f. It is an important property that each basin of an attracting periodic orbit contains at least one singular value.

It is not clear how one should define hyperbolicity in the general case of entire transcendental functions. To avoid this difficulty we work in the class of functions f of *finite type* for which $\operatorname{sing}(f)$ is finite. In this class it makes sense to call f *hyperbolic* if the complete forward orbit of all (finitely many) singular values is relatively compact

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in the Fatou set, in short, if $O^+(\mathrm{sing}(f)) \subseteq \mathcal{F}(f)$. The interesting question of how one should define hyperbolicity in the general case is beyond the scope of this paper.

Functions of finite type have other nice properties; see [**3, 9, 12**]. As for polynomials, Sullivan's classification of the components of the Fatou set holds, and in particular there are no wandering domains or Baker domains (where $\{f^{\circ n}\}$ converges uniformly to ∞).

3. *Families of functions*

Consider a family $G : \mathbb{C} \times \mathbb{C} \to \mathbb{C}; (\lambda, z) \mapsto G(\lambda, z)$ of entire functions $G(\lambda, \cdot)$ depending holomorphically on a complex parameter λ. To keep things simple we usually take $\lambda \in \mathbb{C}$, but any domain $M \subset \mathbb{C}$ or $M \subset \mathbb{C}^m$ may in fact occur. We require that G is of finite type independently of the parameter λ. More precisely, we say that the family G is of *constant finite type* if, firstly, $|\mathrm{sing}(G(\lambda, \cdot))| = N(G) < \infty$ (counted without multiplicity) independently of λ and, secondly, the set $\mathrm{sing}(G(\lambda, \cdot))$ has a *holomorphic parametrization* on $\mathbb{C}$. This means that there are analytic functions $s_\nu : \mathbb{C} \to \mathbb{C}$, $\nu = 1, \ldots, N(G)$, such that $\mathrm{sing}(G(\lambda, \cdot)) = \{s_1(\lambda), \ldots, s_{N(G)}(\lambda)\}$. Throughout this paper we assume that all families are of constant finite type. For such families it makes sense to define a *hyperbolic component* of G as a connected component of $\mathcal{H}(G) := \{\lambda \in \mathbb{C} \mid G(\lambda, \cdot) \text{ is hyperbolic}\}$. Note that $\mathcal{H}(G)$ is open; see Proposition 3.

Now consider a sequence $G_n : \mathbb{C} \times \mathbb{C} \to \mathbb{C}; (\lambda, z) \mapsto G_n(\lambda, z)$ of families of entire functions $G_n(\lambda, \cdot)$ of the *same* constant finite type as G, that is, $|\mathrm{sing}(G_n(\lambda, \cdot))| = |\mathrm{sing}(G(\lambda, \cdot))| = N(G)$, converging to G uniformly on compact subsets of $\mathbb{C} \times \mathbb{C}$. In order to avoid the pathological case of approximating polynomials by transcendental functions (compare [**17**]) we assume that the G_n are families of polynomials if G is a family of polynomials.

The notion of kernel convergence in the sense of Carathéodory is the following. A domain $H \subset \mathbb{C}$ is the *kernel* of a sequence $\{H_n\}$ of domains if every compact subset of H is contained in almost every H_n, and H is maximal with this property, that is, this is not true for any domain $\widetilde{H}$ with $H \subsetneqq \widetilde{H}$. Clearly kernel convergence is a stronger notion than pointwise convergence, but weaker than convergence in the Hausdorff metric; compare [**18**].

It follows from the definition of kernel convergence that the Main Theorem consists of two statements, which we will prove separately.

MAIN THEOREM. *Let G and G_n be as above.*

(i) *Any hyperbolic component H of G is contained in the kernel of hyperbolic components $\{H_n\}$ of the G_n.*

(ii) *Any bigger set $\widetilde{H} \supsetneqq H$ is* not *contained in the kernel of the $\{H_n\}$.*

Example 1. As an illustration we discuss the classical case of the $P_n(\lambda, z) := \lambda(1+(z/n))^n$ converging to $E(\lambda, z) := \lambda e^z$, where $\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, as suggested in [**6**].

Consider the parameter plane $\mathbb{C}$ of the polynomials $P_n(\lambda, \cdot)$. Since the origin is the sole singular value of the P_n (and of E) one can define the sets $C_k(P_n) := \{\lambda \in \mathbb{C} \mid P_n(\lambda, \cdot)$ has an attracting k-periodic orbit$\}$ which are open and mutually disjoint; see Figure 1. The generic hyperbolicity conjecture asserts that $\mathcal{H}(P_n) = \bigcup_{k \in \mathbb{N}} C_k(P_n)$ is dense in the connectedness locus $B_n := \{\lambda \in \mathbb{C} \mid \mathcal{J}(P_n(\lambda, \cdot))$ is connected$\}$. Note that B_2 is the

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well-known Mandelbrot set and that the B_k essentially look like Mandelbrot sets with more limbs attached; see Figure 1 and compare [**5, 6, 15**].

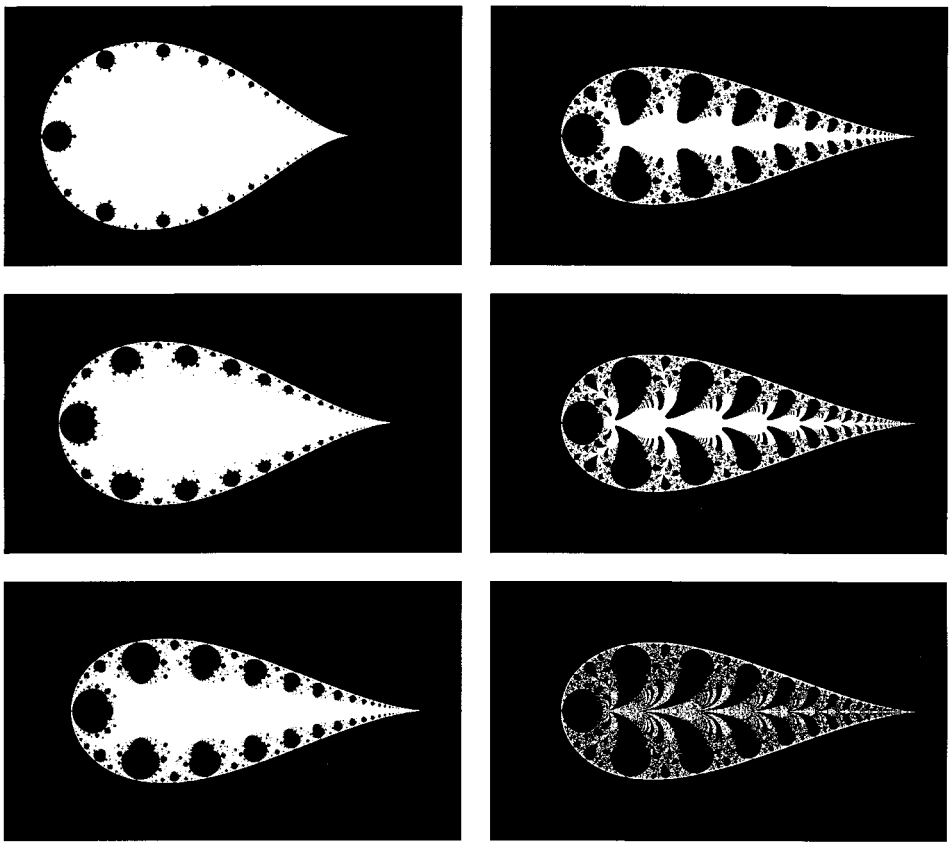


FIGURE 1. The hyperbolic components (dark domains) of P_2, P_4, P_{16}, P_{256}, P_{65536} and of E (lower right), shown in a chart near infinity.

In complete analogy one can define the sets $C_k(E) := \{\lambda \in \mathbb{C} \mid E(\lambda, \cdot)$ has an attracting k-periodic orbit$\}$. The set $C_1(E)$ is the cardioid-shaped region $\{\xi e^{-\xi} : |\xi| < 1\}$, the sets $C_k(E)$ are unbounded for $k \geq 2$ and have infinitely many components for $k \geq 3$. The generic hyperbolicity conjecture for the exponential family asserts that $\mathcal{H}(E) = \bigcup_{k\in\mathbb{N}} C_k(E)$ is dense in $\mathbb{C}$. On any hyperbolic component the unique attracting periodic orbit depends analytically on λ; see [**2**] and Figure 1.

In [**6**] it is shown that the hyperbolic components converge pointwise. Furthermore, symbolic dynamics is set up to show that certain external rays to the B_n converge to 'hairs' in $\mathcal{M}(E) := \{\lambda \in \mathbb{C} \mid \mathcal{J}(E(\lambda, \cdot)) = \widehat{\mathbb{C}}\}$. This gives a very nice connection between the Mandelbrot set B_2 and the parameter plane of E.

The Main Theorem states that the hyperbolic components of the P_n converge as kernels to those of E. In fact we conjecture that in this example they actually converge in the

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Hausdorff metric; compare Figure 1 and see also [**18**].

We get a large class of examples for the application of the Main Theorem as follows. Take polynomials p and q and consider the families $G_n(\lambda, z) = \lambda \cdot p(z) \cdot P_n(\lambda, q(z))$ converging uniformly on compact sets to $G(\lambda, z) = \lambda \cdot p(z) \cdot E(\lambda, q(z))$, where $\lambda \in \mathbb{C}^*$. It is not difficult to check that G and G_n are of the same constant finite type and the Main Theorem can be applied. Note that the case $p(z) \equiv q(z) \equiv z$ is considered in [**10**] and see [**18**] for figures and more examples.

We proceed by showing that, in general, the hyperbolic components do not converge in the Hausdorff metric.

Example 2. Consider the families

$$G(\lambda, z) = (1 + \lambda^2)z + z^2 \quad \text{and} \quad G_n(\lambda, z) = \left(1 + \lambda^2 - \frac{1}{n}\right) z + z^2.$$

For every $n \in \mathbb{N}$ the polynomial $G_n(0, \cdot)$ is hyperbolic, but $G(0, \cdot)$ is not. The set of parameter values for which the origin is an attracting fixed point of G_n forms a hyperbolic component H_n. Its boundary is given by the parameter values for which the origin is an indifferent fixed point, that is, the boundary is $\{\lambda \in \mathbb{C} \mid |1 + \lambda^2 - (1/n)| = 1\}$. For every $n \in \mathbb{N}$ this is a Cassini-type curve, and for the limit family G one obtains the lemniscate $\{\lambda \in \mathbb{C} \mid |1 + \lambda^2| = 1\}$; see Figure 2. Hence, there are hyperbolic components H^1 and H^2 of G that are not a limit in the Hausdorff metric of hyperbolic components H_n of the families G_n.

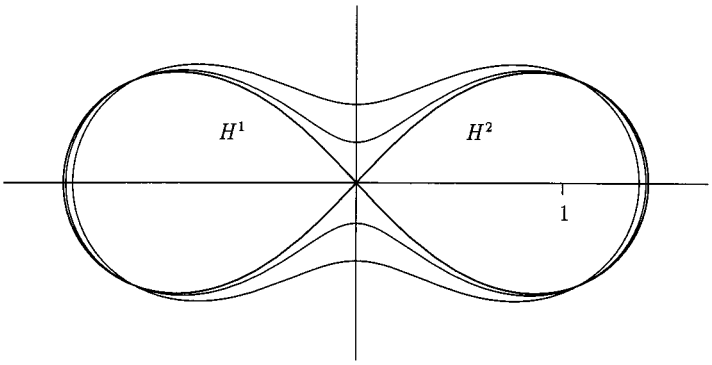


FIGURE 2. The limit of the hyperbolic components of the attracting fixed point 0 of the functions G_n of Example 2 is the union of the two kernels H^1 and H^2.

4. *Proof of the Main Theorem*

It is no coincidence that the Main Theorem works only for families of the *same constant finite type*. The reason is that one needs to have good control of the singular values. An important ingredient is the proof of the convergence of singular values in the dynamical plane given in §4.1, which works for entire functions in general. By applying this result

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to the class of functions of the same finite type in §4.2 we get robustness of hyperbolicity, which gives statement (i) of the Main Theorem as a corollary. Statement (ii) also uses the convergence of singular values and is proved in §4.3.

4.1. *Convergence of singular values.* For the following one does not even need that the functions f_n and g be of finite type—uniform convergence is sufficient. Lemma 1 has been obtained independently by Kisaka in [**14**].

LEMMA 1. *Let $\{f_n\}$ converge to g uniformly on compact sets and let $w \in \mathbb{C}$ be a singular value of g. Then there exists a sequence $\{w_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} w_n = w$ and w_n is a singular value of f_n for almost every $n \in \mathbb{N}$.*

Proof. Suppose w is a critical value. Then there exists some $c \in \mathbb{C}$ such that $g(c) = w$ and $g'(c) = 0$. It follows from Rouché's theorem that there is a sequence $\{c_n\}_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty} c_n = c$, $\lim_{n\to\infty} f_n(c_n) = w$ and $f_n'(c_n) = 0$ for almost every $n \in \mathbb{N}$. We choose $w_n := f_n(c_n)$ and are done.

Assume now that w is an asymptotic value (this implies that g is transcendental and consequently $\infty \in \mathcal{J}(g)$) and that the statement is false. Then there exists a simply connected neighborhood $W \subset \mathbb{C}$ of w such that W does not contain any singular value of f_n for any $n \in \mathbb{N}$. As w is an asymptotic value, there exists an unbounded domain U such that $w \in \partial(g(U))$ but $w \notin g(\partial U \cap \mathbb{C})$. We may assume that $\widetilde{U} := g(U) \subseteq W$. Then $\widetilde{U}$ is the kernel of the sequence $\{f_n(U)\}_{n\in\mathbb{N}}$. We write $\widetilde{U}_n := f_n(U)$. Now, for every $n \in \mathbb{N}$ there exist functions $h_n \in \mathcal{O}(\widetilde{U}_n)$ satisfying $h_n \circ f_n = \mathrm{id}\,|_U$. Clearly, $(\lim_{n\to\infty} h_n) \circ g = id|_U$, hence, $h = \lim_{n\to\infty} h_n \in \mathcal{O}(U)$ is not constant. W does not contain any singularity of f_n^{-1}, thus h_n extends holomorphically to W.

Since $W \subseteq \mathbb{C}$ we may choose two points in $\mathbb{C}$, say ζ_1 and ζ_2, such that $\zeta_1 \neq \zeta_2$, $\{\zeta_1, \zeta_2\} \subset \mathbb{C}$ and $\{g(\zeta_1), g(\zeta_2)\} \cap W = \emptyset$. We may also assume that $\{f_n(\zeta_1), f_n(\zeta_2)\} \cap W = \emptyset$ for every $n \in \mathbb{N}$.

By construction, we have $h_n(W) \cap \{\zeta_1, \zeta_2, \infty\} = \emptyset$. By Montel's theorem $\{h_n\}_{n\in\mathbb{N}}$ is a normal family on W. On $\widetilde{U} \subset W$ we have $h := \lim_{n\to\infty} h_n$, which yields $h = \lim_{n\to\infty} h_n$ uniformly on compact subsets of W and $h \in \mathcal{O}(W)$. The function $h|_{\widetilde{U}}$ is not constant, hence, h is a non-constant holomorphic function on W. The property $h_n \circ f_n = \mathrm{id}$ carries over to h: $h \circ g = \mathrm{id}\,|_U$. Now $\widetilde{U} \subseteq W$ implies $U = h(\widetilde{U}) \subseteq \mathbb{C}$. But U was supposed to be unbounded, a contradiction. □

The following lemma is well-known for rational maps on the Riemann sphere, but is more complicated to show in the current situation; see [**14**, **16**, **17**].

LEMMA 2. *Let g be a hyperbolic entire function and let $\{f_n\}$ be a sequence of entire functions converging to g uniformly on compact subsets of $\mathbb{C}$. Then for every set $K \subseteq \mathcal{F}(g)$ there exists some number $n_0 \in \mathbb{N}$ such that $K \subseteq \mathcal{F}(f_n)$ for every $n \geq n_0$.*

4.2. *Proof of (i).* We return to functions of constant finite type. Combining the two lemmas from §4.1 we obtain the following.

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PROPOSITION 3. *Hyperbolicity is a structurally stable property in the class of functions with the same number of singular values, that is, the set of hyperbolic functions is open in this class.*

As a consequence we obtain part (i) of the Main Theorem.

COROLLARY 4. *For families of the same constant finite type every hyperbolic component H of the limit family G is contained in a kernel of a sequence $\{H_n\}$ of hyperbolic components of the families G_n.*

It is for this part of the proof that the families need to be of the *same* constant finite type.

Example 3. Consider the families

$$G_n(\lambda, z) = z^2\left(\lambda + \left(\frac{3-2\lambda n}{n^2}\right)z + \left(\frac{\lambda n - 2}{n^3}\right)z^2\right)$$

converging uniformly on compact subsets to $G(\lambda, z) = \lambda z^2$, where $\lambda \in \mathbb{C}^*$. Clearly the limit family is not of the same constant type as the approximating families. Every $G_n(\lambda)$ has the point n as a rationally indifferent fixed point and, hence, is not hyperbolic. Consequently, the hyperbolic component $\mathbb{C}^*$ of G cannot be contained in the kernel of a sequence of hyperbolic components of the G_n.

4.3. *Proof of (ii).* The essential step is to show the following.

THEOREM 5. *Let $H \subset \mathbb{C}$ be the kernel of a sequence $\{H_n\}_{n\in\mathbb{N}}$ of hyperbolic components of the families G_n. Then either:*
(i) *$G(\lambda, \cdot)$ is not hyperbolic for any $\lambda \in H$; or*
(ii) *$H \subset \widetilde{H}$ for some hyperbolic component $\widetilde{H}$ of the family G.*

Remark. In this theorem it is not required that the families are of the *same* constant finite type. In fact, $|\operatorname{sing}(G_n)| \geq |\operatorname{sing}(G)|$ is allowed.

Proof. The outline of the proof is as follows. We assume that the statement of the theorem is false. Then there exists a hyperbolic component $\widetilde{H}$ of the family G such that $H \cap \partial\widetilde{H} \neq \emptyset$. We choose a fixed $\lambda_0 \in H \cap \partial\widetilde{H}$ and construct an open neighborhood B of λ_0 for which we prove that in fact $B \subset \widetilde{H}$. This is done by repeatedly using Montel's theorem to show the normality of certain families of holomorphic functions on B. We conclude that $\lambda_0 \in \widetilde{H}$, which is a contradiction.

From now on let $\lambda_0 \in H \cap \partial\widetilde{H}$ be fixed. Note that there is an open neighborhood U_1 of λ_0 such that $U_1 \subseteq H \cap H_n$ for (almost) every $n \in \mathbb{N}$. Since a Julia set is nonempty, perfect, and the closure of the set of repelling periodic points, there are two distinct repelling periodic points $p(\lambda_0), q(\lambda_0) \in \mathcal{J}(G(\lambda_0, \cdot)) \setminus \{\infty\}$. By the implicit function theorem each repelling periodic point has a holomorphic parametrization on some open neighborhood of λ_0. Consequently, there exists an open neighborhood $U_2 \subseteq U_1$ of λ_0 and functions p and q holomorphic on U_2 such that for each $\lambda \in U_2$ the points

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$p(\lambda)$ and $q(\lambda)$ are distinct repelling periodic points of $G(\lambda, \cdot)$ with the property that $p(\lambda), q(\lambda) \in \mathcal{J}(G(\lambda, \cdot)) \setminus \{\infty\}$.

By Rouché's theorem repelling periodic points are persistent. Hence, we can choose a fixed open neighborhood $B \subseteq U_2$ of λ_0 such that for (almost) every $n \in \mathbb{N}$ the following is true. For each $\lambda \in B$ there exist unique repelling periodic points $p_n(\lambda)$ and $q_n(\lambda)$ of $G_n(\lambda, \cdot)$ with the same periods as $p(\lambda)$ and $q(\lambda)$, and, furthermore, the maps p_n and q_n are holomorphic on B and converge there uniformly to p and q, respectively. In particular, $p_n(\lambda), q_n(\lambda) \in \mathcal{J}(G_n(\lambda, \cdot)) \setminus \{\infty\}$. Since $B \subseteq H_n$ for (almost) every $n \in \mathbb{N}$, and since H_n is a hyperbolic component of G_n, we conclude that $p_n(\lambda), q_n(\lambda) \in \mathbb{C} \setminus \operatorname{sing}(G_n(\lambda, \cdot))$. Let

$$T(\lambda, z) := \frac{z - p(\lambda)}{q(\lambda) - p(\lambda)} \quad \text{and} \quad T_n(\lambda, z) := \frac{z - p_n(\lambda)}{q_n(\lambda) - p_n(\lambda)}.$$

After conjugating $G(\lambda, \cdot)$ and $G_n(\lambda, \cdot)$ with $T(\lambda, \cdot)$ and $T_n(\lambda, \cdot)$, respectively, we may and will assume that $\operatorname{sing}(G_n(\lambda, \cdot)) \in \mathbb{C} \setminus \{0, 1\}$ for each $\lambda \in B$. Note that—after this conjugation—0 and 1 are repelling periodic points of all functions in question and, hence, we may and will assume that $\{0, 1\} \subset \mathcal{J}(G(\lambda, \cdot)), \mathcal{J}(G_n(\lambda, \cdot))$.

Let $s_1, \ldots, s_m \in \mathcal{O}(\mathbb{C})$ be the parametrizations of the $m := |\operatorname{sing}(G)|$ singular values of $G(\lambda, \cdot)$ and let $s_{n,1}, \ldots, s_{n,m}, \ldots \in \mathcal{O}(\mathbb{C})$ be the parametrizations of the $N(G_n) \geq m$ singular values of $G_n(\lambda, \cdot)$. From now on we fix $\mu \in \{1, \ldots, m\}$. Note that due to Lemma 1 and Montel's theorem we have $s_{n,\mu} \to s_\mu$ uniformly on compact sets of B (after renumbering the $s_{n,\mu}$ if necessary) since we have $s_{n,\mu} \subset \mathbb{C} \setminus \{0, 1\}$.

For $\lambda \in B$ we know that $s_{n,\mu} \notin \mathcal{J}(G_n(\lambda, \cdot))$, and the invariance of the Julia sets yields $G_n^{\circ\nu}(\lambda, s_{n,\mu}) \notin \mathcal{J}(G_n(\lambda, \cdot))$ and, in particular, $G_n^{\circ\nu}(\lambda, s_{n,\mu}) \in \mathbb{C} \setminus \{0, 1\}$ for all $\lambda \in B$. In other words, $\{G_n^{\circ\nu}(\lambda, s_{n,\mu}) \mid n \in \mathbb{N}\}$ is a normal family for any fixed ν. We call the limit

$$S_{\nu,\mu} : B \to \widehat{\mathbb{C}}; \lambda \mapsto S_{\nu,\mu}(\lambda) := \lim_{n \to \infty} G_n^{\circ\nu}(\lambda, s_{n,\mu}(\lambda))$$

and conclude that in fact $S_{\nu,\mu}(\lambda) = G^{\circ\nu}(\lambda, s_\mu(\lambda))$ because of the uniform convergence on compact sets of the $G_n(\lambda, \cdot)$ to $G(\lambda, \cdot)$ and of the $s_{n,\mu}$ to s_μ. $G^{\circ\nu}(\lambda, \cdot)$ is an entire function, hence $G^{\circ\nu}(\lambda, s_\mu(\lambda)) \neq \infty$ for each $\lambda \in B$. If $G^{\circ\nu}(\lambda, s_\mu(\lambda)) \equiv 0, 1$ for some ν then G would not be hyperbolic on $B \cap \widetilde{H}$, a contradiction. We conclude that indeed

$$S_{\nu,\mu}(\lambda) = G^{\circ\nu}(\lambda, s_\mu(\lambda)) \in \mathbb{C} \setminus \{0, 1\}$$

for all $\lambda \in B$ and $\nu \in \mathbb{N}$. In other words, $\{S_{\nu,\mu}(\lambda) \mid \nu \in \mathbb{N}\}$ is a normal family so that there exists a limit function, which we denote by S_μ.

For this limit function we have either $S_\mu \equiv 0, 1, \infty$ or $S_\mu(B) \in \mathbb{C} \setminus \{0, 1\}$. Since $\widetilde{H}$ is a hyperbolic component of $G(\lambda, \cdot)$ we have on $B \cap \widetilde{H}$ that $S_\mu(\lambda) = a(\lambda)$, where $a(\lambda)$ is an attracting periodic point of, say, period p of $G(\lambda, \cdot)$. Consequently, $S_\mu \not\equiv 0, 1$.

In the case that $S_\mu \equiv \infty$, the identity theorem shows that ∞ is an attracting periodic point of $G(\lambda, \cdot)$ for every $\lambda \in B$. This implies that $G(\lambda, \cdot)$ is a polynomial and that ∞ is a superattracting fixed point. Hence, the basin of ∞ contains the singular value $s_\mu(\lambda)$.

We consider the case that $S_\mu \not\equiv \infty$. It follows from Rouché's theorem that, for $\lambda \in B \cap \widetilde{H}$, there are attracting periodic points $a_n(\lambda)$ of $G_n(\lambda, \cdot)$ with $a_n(\lambda) \to a(\lambda)$. By the implicit function theorem attracting periodic points have a holomorphic parametrization, hence, we may assume that a_n is a holomorphic function on $B \cap \widetilde{H}$. But since $B \subseteq H_n$

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we conclude that a_n has a holomorphic extension on B such that $a_n(\lambda)$ is an attracting periodic point of period p of $G_n(\lambda, \cdot)$ for all $\lambda \in B$. In particular, $a_n(B) \subset \mathbb{C} \setminus \{0, 1\}$ and $\{a_n \mid n \in \mathbb{N}\}$ is a normal family which converges on $B \cap \widetilde{H}$ to the well-defined function a. This means that we can extend a to the whole of B. Due to the uniform convergence of the $G_n(\lambda, \cdot)$ to $G(\lambda, \cdot)$ the point $a(\lambda)$ is a non-repelling point of $G(\lambda, \cdot)$ for all $\lambda \in B$. Since $a(\lambda)$ is attracting on $B \cap \widetilde{H}$ we conclude from the maximum modulus principle that it is even attracting on the whole of $B \subseteq H_n$.

Summing up the discussion, we conclude that

$$S_\mu(\lambda) \in \mathcal{F}(G(\lambda, \cdot))$$

for all $\lambda \in B$. Since μ was arbitrary we have shown that the forward orbits of the singular values are relatively compact in $\mathcal{F}(G(\lambda, \cdot))$. This means that B is contained in a hyperbolic component of G and, hence, $B \subset \widetilde{H}$. □

Proof of (ii). According to Corollary 4 every compact set in H is contained in some hyperbolic components H_n of the G_n. Let $\widetilde{H}$ be the kernel of the sequence $\{H_n\}$, in particular, $H \subset \widetilde{H}$. We have to show that $H = \widetilde{H}$, that is, that $\widetilde{H} \subset H$. According to Theorem 5 this is the case or $G(\lambda, \cdot)$ is not hyperbolic for all $\lambda \in \widetilde{H}$. The latter contradicts the assumption that the hyperbolic component H is a subset of $\widetilde{H}$. □

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