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STEIN'S METHOD FOR NEGATIVELY ASSOCIATED RANDOM VARIABLES WITH APPLICATIONS TO SECOND-ORDER STATIONARY RANDOM FIELDS

NATHAKHUN WIROONSRI,* *University of Southern California*

Abstract

Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m)$ be a negatively associated mean-zero random vector with components that obey the bound $|\xi_i| \le B$, $i = 1, \dots, m$, and whose sum $W = \sum_{i=1}^m \xi_i$ has variance 1. The bound $d_1(\mathcal{L}(W), \mathcal{L}(Z)) \le 5B - 5.2\sum_{i \ne j} \sigma_{ij}$ is obtained, where Z has the standard normal distribution and $d_1(\cdot, \cdot)$ is the L^1 metric. The result is extended to the multidimensional case with the L^1 metric replaced by a smooth functions metric. Applications to second-order stationary random fields with exponential decreasing covariance are also presented.

Keywords: Correlation inequality; block dependence; negative dependence; random field

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1. Introduction

In this paper we extend the work of Goldstein and Wiroonsri [11] on positive association. In particular, we provide nonasymptotic L^1 bounds to the normal for negatively associated random fields in $\mathbb{Z}^d$ using the same technique developed in [11]. We recall that the L^1 (or Wasserstein) distance between the distributions $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ of real-valued random variables X and Y is given by

$$d_1(\mathcal{L}(X), \mathcal{L}(Y)) = \int_{-\infty}^{\infty} |\mathbb{P}(X \le t) - \mathbb{P}(Y \le t)| \, \mathrm{d}t.$$

A random vector $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$ is said to be *positively associated* whenever

$$\operatorname{cov}(f(\boldsymbol{\xi}), g(\boldsymbol{\xi})) \ge 0$$

for all real-valued coordinatewise nondecreasing functions f and g on $\mathbb{R}^m$ such that $f(\boldsymbol{\xi})$ and $g(\boldsymbol{\xi})$ have finite second moments. A random vector $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$ is said to be *negatively associated* if, for any disjoint subsets A, B of $[m] := \{1, 2, \dots, m\}$,

$$\operatorname{cov}(f(\xi_i,\ i \in A), g(\xi_j,\ j \in B)) \le 0 \tag{1.1}$$

for all coordinatewise increasing functions $f\colon \mathbb{R}^{|A|} \to \mathbb{R}$ and $g\colon \mathbb{R}^{|B|} \to \mathbb{R}$ such that $f(\xi_i,\ i \in A)$ and $g(\xi_j,\ j \in B)$ have finite second moments. In general, a collection $\{\xi_\alpha : \alpha \in I\}$ of real-valued random variables indexed by a set I is said to be positively associated (respectively, negatively associated) if all finite subcollections are positively associated (respectively,

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* Postal address: Department of Mathematics, University of Southern California, Los Angeles, CA 90089, USA. Email address: wiroonsr@usc.edu

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negatively associated). Positive association was introduced by Esary *et al.* [10] and can be found frequently in probabilistic models in several areas, especially statistical physics. In some of the literature, positive association is termed the 'FKG-inequality' or simply 'association' (see, e.g. [8] and [14]). Negative association was later introduced in [12] and has well-known applications related to permutation distributions.

Over the last few decades, central limit theorems and rates of convergence have been established for sums of positively associated random variables (see [4], [5], and [14]) and negatively associated random variables (see [6] and [13]) under different assumptions. Recently, Goldstein and Wiroonsri [11] developed an L^1 version of Stein's method adapted to sums of positively associated random variables with applications to statistical physics. Stein's method [18] is nowadays one of the most powerful methods used to prove convergence in distribution, with the main advantages that it provides nonasymptotic bounds on the distance between distributions, and that it can handle various situations involving dependence. Thus far, many applications in several areas such as statistics, statistical physics, and applied sciences have been developed using this method. For more detail about the method in general, see [7] and [16].

The one-dimensional result of [11] is stated below.

Theorem 1.1. (Goldstein and Wiroonsri [11].) *Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m)$ be a positively associated mean-zero random vector with components obeying the bound $|\xi_i| \le B$ for some $B > 0$, and whose sum $W = \sum_{i=1}^m \xi_i$ has variance* 1. *Let Z be a standard normal random variable. Then*

$$d_1(\mathcal{L}(W), \mathcal{L}(Z)) \le 5B + \sqrt{\frac{8}{\pi}} \sum_{i \ne j} \sigma_{ij},$$

where $\sigma_{ij} = \mathbb{E}[\xi_i \xi_j]$.

The multidimensional result was also obtained in [11] with the L^1 metric replaced by a smooth functions metric, following the development of Chen *et al.* [7, Chapter 12].

For $\boldsymbol{x} \in \mathbb{R}^p$, let

$$|\boldsymbol{x}|_1 = \sum_{i=1}^p |x_i|$$

be the L^1 vector norm, and for a real-valued function $\varphi(u)$ defined on the domain $\mathcal{D}$, let $|\varphi|_\infty = \sup_{x \in \mathcal{D}} |\varphi(x)|$. We include in this definition the $|\cdot|_\infty$ norm of vectors and matrices, e.g. by considering them as real-valued functions of their indices. Also, from this point, we denote $\mathbb{N}_k = [k, \infty) \cap \mathbb{Z}$ for $k \in \mathbb{Z}$.

For $m \in \mathbb{N}_0$, let $L_m^\infty(\mathbb{R}^p)$ be the collection of all functions $h \colon \mathbb{R}^p \to \mathbb{R}$ such that, for all $\boldsymbol{k} = (k_1, \dots, k_p) \in \mathbb{N}_0^p$ with $|\boldsymbol{k}|_1 \le m$, the partial derivative

$$h^{(\boldsymbol{k})}(\boldsymbol{x}) = \frac{\partial^{|\boldsymbol{k}|_1} h}{\partial^{k_1} x_1 \cdots \partial^{k_p} x_p}$$

exists, and

$$|h|_{L_m^\infty(\mathbb{R}^p)} := \max_{0 \le |\boldsymbol{k}|_1 \le m} |h^{(\boldsymbol{k})}|_\infty$$

is finite. For $f \in L_m^\infty(\mathbb{R}^p)$, let

$$\mathcal{H}_{m,\infty,p} = \{h \in L_m^\infty(\mathbb{R}^p) \colon |h|_{L_m^\infty(\mathbb{R}^p)} \le 1\},$$

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and for random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $\mathbb{R}^p$, define the smooth functions metric

$$d_{\mathcal{H}_{m,\infty,p}}(\mathcal{L}(\boldsymbol{X}), \mathcal{L}(\boldsymbol{Y})) = \sup_{h \in \mathcal{H}_{m,\infty,p}} |\mathbb{E}h(\boldsymbol{X}) - \mathbb{E}h(\boldsymbol{Y})|. \tag{1.2}$$

For a positive semidefinite matrix $\boldsymbol{H}$, we let $\boldsymbol{H}^{1/2}$ denote the unique positive semidefinite square root of $\boldsymbol{H}$. When $\boldsymbol{H}$ is positive definite, we write $\boldsymbol{H}^{-1/2} = (\boldsymbol{H}^{1/2})^{-1}$. In the following theorem we state the multidimensional result of [11].

Theorem 1.2. (Goldstein and Wiroonsri [11].) *With m, $p \in \mathbb{N}_1$, let $\{\xi_{i,j} : i \in [m], j \in [p]\}$ be positively associated mean-zero random variables bounded in absolute value by some positive constant B. Let $\boldsymbol{S} = (S_1, S_2, \ldots, S_p)$, where $S_j = \sum_{1 \le i \le m} \xi_{i,j}$ for $j \in [p]$ and assume that $\boldsymbol{\Sigma} = \mathrm{var}(\boldsymbol{S})$ is positive definite. Then*

$$\begin{aligned}
&d_{\mathcal{H}_{3,\infty,p}}(\mathcal{L}(\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{S} - \mathbb{E}\boldsymbol{S})), \mathcal{L}(\boldsymbol{Z})) \\
&\quad \le \left(\frac{1}{6} + 2\sqrt{2}\right) p^3 B |\boldsymbol{\Sigma}^{-1/2}|_\infty^3 \sum_{j=1}^{p} \boldsymbol{\Sigma}_{j,j} \\
&\qquad + \left(\frac{3}{\sqrt{2}} + \frac{1}{2}\right) p^2 |\boldsymbol{\Sigma}^{-1/2}|_\infty^2 \sum_{j=1}^{p} \sum_{i,k \in [m],\, i \ne k} \mathrm{cov}(\xi_{i,j}, \xi_{k,j}) \\
&\qquad + \left(2\sqrt{2} p^3 B |\boldsymbol{\Sigma}^{-1/2}|_\infty^3 + \left(\frac{3}{\sqrt{2}} + \frac{1}{2}\right) p^2 |\boldsymbol{\Sigma}^{-1/2}|_\infty^2\right) \sum_{j,l \in [p],\, j \ne l} \boldsymbol{\Sigma}_{j,l},
\end{aligned}$$

where $\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{0}, I_p)$ is a standard normal vector in $\mathbb{R}^p$.

The results above have been applied to second-order stationary random fields, assuming exponentially decreasing covariance, and to four models in statistical physics: Ising and voter models, bond percolation, and contact process. In this paper we prove similar results adapted to negative association.

Stein's method has been used previously for negative association and some related concepts. In [13], Stein's method was used in normal approximation for sums of pairwise negative quadrant-dependent random variables which allows the derivation of a central limit theorem for pairwise negative quadrant-dependent random variables with Lindeberg's condition. We note that when $m = 2$ in (1.1), negative association and negative quadrant dependence are equivalent; see [12]. In [9], Stein's method was used to obtain the total variation distance between a compound Poisson distribution and sums of positively associated or negatively associated random variables.

Our main results in the one-dimensional and multidimensional cases are stated in Theorems 1.3 and 1.4, respectively.

Theorem 1.3. *Let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m)$ be a negatively associated mean-zero random vector with components obeying the bound $|\xi_i| \le B$ for some $B > 0$, and whose sum $W = \sum_{i=1}^m \xi_i$ has variance* 1. *Let Z be a standard normal random variable. Then with $\sigma_{ij} = \mathbb{E}[\xi_i \xi_j]$,*

$$d_1(\mathcal{L}(W), \mathcal{L}(Z)) \le 5B - 5.2 \sum_{i \ne j} \sigma_{ij}. \tag{1.3}$$

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Theorem 1.4. *With $m, p \in \mathbb{N}_1$, let $\{\xi_{i,j} : i \in [m], j \in [p]\}$ be negatively associated mean-zero random variables satisfying $|\xi_{i,j}| \le B$ for some $B > 0$. Let $\boldsymbol{S} = (S_1, S_2, \ldots, S_p)$, where $S_j = \sum_{1 \le i \le m} \xi_{i,j}$ for $j \in [p]$ and assume that $\boldsymbol{\Sigma} = \operatorname{var}(\boldsymbol{S})$ is positive definite. Then*

$$\begin{aligned} d_{\mathcal{H}_{3,\infty,p}}(\mathcal{L}(\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{S} - \mathbb{E}\boldsymbol{S})), \mathcal{L}(\boldsymbol{Z})) \\ \le \frac{5}{6} p^3 B |\boldsymbol{\Sigma}^{-1/2}|_\infty^3 \sum_{j=1}^{p} \boldsymbol{\Sigma}_{j,j} \\ - \left(\frac{3}{2} p^3 B |\boldsymbol{\Sigma}^{-1/2}|_\infty^3 + p^2 |\boldsymbol{\Sigma}^{-1/2}|_\infty^2 \right) \sum_{j=1}^{p} \sum_{i,k \in [m],\, i \ne k} \operatorname{cov}(\xi_{i,j}, \xi_{k,j}) \\ - \left(\frac{2}{3} p^3 B |\boldsymbol{\Sigma}^{-1/2}|_\infty^3 + p^2 |\boldsymbol{\Sigma}^{-1/2}|_\infty^2 \right) \sum_{j,l \in [p],\, j \ne l} \boldsymbol{\Sigma}_{j,l}, \end{aligned} \tag{1.4}$$

where $\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{0}, I_p)$ is a standard normal random vector in $\mathbb{R}^p$.

Remark 1.1. We note that the differences between our results, in Theorems 1.3 and 1.4 for negative association and those in Theorems 1.1 and 1.2 of [11] for positive association are that the signs of the covariance terms are reversed and that the constants are different. Nevertheless, in Section 3 we have an example where these changes do not contribute to the rates of convergence. We also note that the bounds in the four theorems above are particularly useful when the variables are bounded and (positively or negatively) associated. However, let us compare the one-dimensional results in Theorems 1.1 and 1.3 with the classical result for independent and identically distributed variables X_i, $i \in [n]$, with $\mathbb{E}X_i = 0$, $\operatorname{var} X_i = \sigma^2 > 0$, and $|X_i| \le K$ for some $K > 0$. With $W = (\sum_i X_i)/(\sigma\sqrt{n})$, both Theorems 1.1 and 1.3 yield $d_1(\mathcal{L}(W), \mathcal{L}(Z)) \le 5K\sigma^{-1}n^{-1/2}$. The classical Berry–Esseen theorem provides the bound $\sup_{x \in \mathbb{R}} |\mathbb{P}(W \le x) - \mathbb{P}(Z \le x)| \le CK^3\sigma^{-3}n^{-1/2}$ with the smallest constant $C = 0.4748$ obtained by Shevtsova [17]. Though the rates of convergence in n are the same, the constants are different and actually the two distances are not comparable as it is known that $\sup_{x \in \mathbb{R}} |\mathbb{P}(X \le x) - \mathbb{P}(Y \le x)| \le \sqrt{2cd_1(\mathcal{L}(X), \mathcal{L}(Y))}$ for some $c > 0$ but not conversely; see Proposition 1.2 of [16].

The remainder of the paper is organized as follows. In Section 2 we use Stein's method to prove the two main results, Theorems 1.3 and 1.4. We state our results for negatively associated random fields whose covariance decays exponentially, in Section 3. One similar advantage of the four theorems stated in this section is that, unlike many results based on Stein's method, they may be applied without the need for coupling constructions.

2. Proofs of main theorems

In this section we prove our main results, Theorems 1.3 and 1.4, using similar techniques as in [11]. For this purpose, we first state the following two lemmas proved in [12] and [6], respectively. In Lemma 2.1 we state that two disjoint sums of negatively associated random variables are negatively associated. This will be used throughout the remainder of the paper. Lemma 2.2 allows us to bound $|\operatorname{cov}(f(\xi_i : i \in A), g(\xi_j : j \in B))|$ by a linear combination of $-\operatorname{cov}(\xi_i, \xi_j)$ with $i \in A$, $j \in B$ when A and B are disjoint.

Lemma 2.1. (Joag-Dev and Proschan [12].) *Increasing functions defined on disjoint subsets of a set of negatively associated random variables are negatively associated.*

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Lemma 2.2. (Cai and Wang [6].) *Let A and B be disjoint finite sets, and let ξ_j, $j \in A \cup B$, be negatively associated random variables. If $f\colon \mathbb{R}^{|A|} \to \mathbb{R}$, $g\colon \mathbb{R}^{|B|} \to \mathbb{R}$ are partially differentiable with bounded partial derivatives then*

$$|\operatorname{cov}(f(\xi_i : i \in A), g(\xi_j : j \in B))| \le -\sum_{i\in A}\sum_{j\in B}\left|\frac{\partial f}{\partial \xi_i}\right|_\infty \left|\frac{\partial g}{\partial \xi_j}\right|_\infty \operatorname{cov}(\xi_i, \xi_j).$$

We note that the difference between the proofs below and the ones in [11] results from the fact that Lemma 2.2 requires A and B to be disjoint, unlike the one for positive association. Therefore, we add a few more steps in the proofs to handle this situation. In the proof that follows, we use the alternate form of the L^1 distance (see, e.g. [15]):

$$d_1(\mathcal{L}(X), \mathcal{L}(Y)) = \sup_{h\in\mathbb{L}} |\mathbb{E}h(X) - \mathbb{E}h(Y)|, \tag{2.1}$$

where $\mathbb{L} = \{h\colon |h(y) - h(x)| \le |y - x|\}$.

Proof of Theorem 1.3. For given $h \in \mathbb{L}$, let f be the unique bounded solution to the Stein equation

$$f'(w) - wf(w) = h(w) - Nh, \tag{2.2}$$

where $Nh = \mathbb{E}h(Z)$ with $\mathcal{L}(Z)$ the standard normal distribution. Then (see, e.g. Lemma 2.4 of [7])

$$|f'|_\infty \le \sqrt{\frac{2}{\pi}} \quad \text{and} \quad |f''|_\infty \le 2. \tag{2.3}$$

In the proof below we will use the notation $\sigma_{ij} = \mathbb{E}[\xi_i\xi_j]$ and $\sigma_i^2 = \operatorname{var}(\xi_i)$ for $i \ne j \in [n]$. As $\operatorname{var}(W) = \sum_{i=1}^n \sigma_i^2 + \sum_{i\ne j}\sigma_{ij} = 1$, we obtain

$$\begin{aligned}\mathbb{E}[f'(W)] &= \mathbb{E}\left[\sum_{i=1}^m \sigma_i^2 f'(W) + \sum_{i\ne j}\sigma_{ij} f'(W)\right]\\ &= \mathbb{E}\left[\sum_{i=1}^m \xi_i^2 f'(W) + \sum_{i\ne j}\sigma_{ij} f'(W) + \sum_{i=1}^m(\sigma_i^2 - \xi_i^2) f'(W)\right].\end{aligned}$$

Now letting $W^i = W - \xi_i$, write

$$\begin{aligned}\mathbb{E}[Wf(W)] &= \mathbb{E}\sum_{i=1}^m \xi_i f(W)\\ &= \mathbb{E}\sum_{i=1}^m \xi_i f(W^i + \xi_i)\\ &= \mathbb{E}\sum_{i=1}^m\left[\xi_i f(W^i) + \xi_i^2\int_0^1 f'(W^i + u\xi_i)\,\mathrm{d}u\right].\end{aligned}$$

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Recalling the Stein equation (2.2) and subtracting the two equations above, we obtain

$$\begin{aligned}\mathbb{E}[h(W) - Nh] &= \mathbb{E}[f'(W) - Wf(W)] \\ &= \mathbb{E}\Bigg[\sum_{i=1}^{m} \xi_i^2 \bigg(\int_0^1 (f'(W) - f'(W^i + u\xi_i))\,\mathrm{d}u\bigg) + \sum_{i=1}^{m} (\sigma_i^2 - \xi_i^2) f'(W) \\ &\quad + \sum_{i\neq j} \sigma_{ij} f'(W) - \sum_{i=1}^{m} \xi_i f(W^i)\Bigg]. \end{aligned} \tag{2.4}$$

Using the second inequality in (2.3), we bound the first term in (2.4) by

$$\begin{aligned}\bigg|\mathbb{E}\sum_{i=1}^{m} \xi_i^2 \int_0^1 (f'(W) - f'(W^i + u\xi_i))\,\mathrm{d}u\bigg| &= \bigg|\mathbb{E}\sum_{i=1}^{m} \xi_i^2 \int_0^1 \int_{u\xi_i}^{\xi_i} f''(W^i + t)\,\mathrm{d}t\,\mathrm{d}u\bigg| \\ &\leq 2\mathbb{E}\sum_{i=1}^{m} \xi_i^2 \bigg(\int_0^1 \int_{u|\xi_i|}^{|\xi_i|} \mathrm{d}t\,\mathrm{d}u\bigg) \\ &= \mathbb{E}\sum_{i=1}^{m} |\xi_i|^3 \\ &\leq B\mathbb{E}\sum_{i=1}^{m} \xi_i^2 \\ &= B\bigg(1 - \sum_{i\neq j} \sigma_{ij}\bigg). \end{aligned} \tag{2.5}$$

To handle the second term in (2.4), using the triangle inequality, we first bound it by the three terms denoted by I_1, I_2, and I_3:

$$\begin{aligned}\bigg|\mathbb{E}\sum_{i=1}^{m} f'(W)(\sigma_i^2 - \xi_i^2)\bigg| &\leq \bigg|\mathbb{E}\sum_{i=1}^{m} f'(W^i)(\sigma_i^2 - \xi_i^2)\bigg| + \bigg|\mathbb{E}\sum_{i=1}^{m} (f'(W) - f'(W^i))(\sigma_i^2 - \xi_i^2)\bigg| \\ &\leq \bigg|\mathbb{E}\sum_{i=1}^{m} f'(W^i)(\sigma_i^2 - \xi_i^2)\bigg| + \mathbb{E}\sum_{i=1}^{m} |f'(W) - f'(W^i)||\sigma_i^2 - \xi_i^2| \\ &\leq \bigg|\mathbb{E}\sum_{i=1}^{m} f'(W^i)(\sigma_i^2 - \xi_i^2)\bigg| + \mathbb{E}\sum_{i=1}^{m} |f'(W) - f'(W^i)|\sigma_i^2 \\ &\quad + \mathbb{E}\sum_{i=1}^{m} |f'(W) - f'(W^i)|\xi_i^2 \\ &\coloneqq I_1 + I_2 + I_3. \end{aligned}$$

Note that W^i and ξ_i are coordinatewise increasing functions defined on disjoint subsets of $\boldsymbol{\xi}$, and, hence, negatively associated by Lemma 2.1. Now, for I_1, applying Lemma 2.2 with

$$g(x) = \begin{cases} x^2, & |x| \leq B, \\ B^2, & |x| > B, \end{cases}$$

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and using the second inequality in (2.3), we have

$$I_1 = \left|\sum_{i=1}^{m} \operatorname{cov}(f'(W^i), g(\xi_i))\right| \le -4B \sum_{i=1}^{m} \operatorname{cov}(W^i, \xi_i) = -4B \sum_{i \ne j} \sigma_{ij}. \tag{2.6}$$

For I_2 and I_3, applying again the second inequality in (2.3), we obtain

$$I_2 \le |f''|_\infty \sum_{i=1}^{m} \sigma_i^2 \mathbb{E}|W - W^i| \le 2B \sum_{i=1}^{m} \sigma_i^2 = 2B\left(1 - \sum_{i \ne j} \sigma_{ij}\right) \tag{2.7}$$

and

$$I_3 \le |f''|_\infty \sum_{i=1}^{m} \mathbb{E}\xi_i^2 |W - W^i| \le 2B \sum_{i=1}^{m} \mathbb{E}\xi_i^2 = 2B\left(1 - \sum_{i \ne j} \sigma_{ij}\right). \tag{2.8}$$

For the third term in (2.4), using the negativity of the covariances σ_{ij}, $i \ne j$, and the first inequality in (2.3), we obtain

$$\left|\mathbb{E}\sum_{i \ne j} \sigma_{ij} f'(W)\right| \le -|\mathbb{E}f'(W)| \sum_{i \ne j} \sigma_{ij} \le -\sqrt{\frac{2}{\pi}} \sum_{i \ne j} \sigma_{ij}. \tag{2.9}$$

The final term in (2.4), using again the fact that the pair (W^i, ξ_i) is negatively associated and applying Lemma 2.2 and the first inequality in (2.3), now yields

$$\left|\mathbb{E}\sum_{i=1}^{m} \xi_i f(W^i)\right| = \left|\sum_{i=1}^{m} \operatorname{cov}(\xi_i, f(W^i))\right| \le -\sqrt{\frac{2}{\pi}} \sum_{i=1}^{m} \operatorname{cov}(\xi_i, W^i) = -\sqrt{\frac{2}{\pi}} \sum_{i \ne j} \sigma_{ij}. \tag{2.10}$$

Summing the bounds (2.5)–(2.10), taking supremum over $h \in \mathbb{L}$, and using the form of the L^1-distance given in (2.1), we obtain

$$d_1(\mathcal{L}(W), \mathcal{L}(Z)) \le 5B - 9B \sum_{i \ne j} \sigma_{ij} - \sqrt{\frac{8}{\pi}} \sum_{i \ne j} \sigma_{ij}.$$

Using the fact that $d_1(\cdot, \cdot) \le 2$ and that $\sigma_{i,j}$ are negative, we can assume that $B \le 0.4$ and, thus, the last expression is bounded by the right-hand side of (1.3). □

Next we use the following result, which is slightly different from Lemma 2.6 of [7] due to [3], to prove Theorem 1.4. Let $\boldsymbol{Z}$ be a standard normal random vector in $\mathbb{R}^p$. For $h\colon \mathbb{R}^p \to \mathbb{R}$, let $Nh = \mathbb{E}h(\boldsymbol{Z})$ and, for $u \ge 0$, define

$$(T_u h)(\boldsymbol{s}) = \mathbb{E}h\big(\boldsymbol{s}\mathrm{e}^{-u} + \sqrt{1 - \mathrm{e}^{-2u}}\boldsymbol{Z}\big).$$

We write $\boldsymbol{D}^2 h$ for the Hessian matrix of h when it exists.

Lemma 2.3. *For $m \ge 3$ and $h \in L_m^\infty(\mathbb{R}^p)$, the function*

$$g(\boldsymbol{s}) = -\int_0^\infty [T_u h(\boldsymbol{s}) - Nh]\,\mathrm{d}u$$

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solves

$$\operatorname{tr} \boldsymbol{D}^2 g(\boldsymbol{s}) - \boldsymbol{s} \cdot \nabla g(\boldsymbol{s}) = h(\boldsymbol{s}) - Nh,$$

and, for any $0 \le |\boldsymbol{k}|_1 \le m$,

$$|g^{(\boldsymbol{k})}|_\infty \le \frac{1}{|\boldsymbol{k}|_1} |h^{(\boldsymbol{k})}|_\infty.$$

Furthermore, for any $\boldsymbol{\lambda} \in \mathbb{R}^p$ *and positive definite* $p \times p$ *matrix* $\boldsymbol{\Sigma}$, f *defined by the change of variable*

$$f(\boldsymbol{s}) = g(\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{s} - \boldsymbol{\lambda})) \tag{2.11}$$

solves

$$\operatorname{tr} \boldsymbol{\Sigma} \boldsymbol{D}^2 f(\boldsymbol{s}) - (\boldsymbol{s} - \boldsymbol{\lambda}) \cdot \nabla f(\boldsymbol{s}) = h(\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{s} - \boldsymbol{\lambda})) - Nh, \tag{2.12}$$

and satisfies

$$|f^{(\boldsymbol{k})}|_\infty \le \frac{p^{|\boldsymbol{k}|_1}}{|\boldsymbol{k}|_1} |\boldsymbol{\Sigma}^{-1/2}|_\infty^{|\boldsymbol{k}|_1} |h^{(\boldsymbol{k})}|_\infty.$$

In particular, if $h \in \mathcal{H}_{m,\infty,p}$ *then*

$$|f^{(\boldsymbol{k})}|_\infty \le \frac{p^{|\boldsymbol{k}|_1}}{|\boldsymbol{k}|_1} |\boldsymbol{\Sigma}^{-1/2}|_\infty^{|\boldsymbol{k}|_1} \quad \textit{for all } 0 \le |\boldsymbol{k}|_1 \le m. \tag{2.13}$$

We apply the same technique as in the univariate case, along with Lemmas 2.2 and 2.3, to prove our main multivariate theorem below.

Proof of Theorem 1.4. Given $h \in \mathcal{H}_{3,\infty,p}$, let f be the solution of (2.12) given by (2.11) with $\boldsymbol{\lambda} = \boldsymbol{0}$. Writing out the expressions in (2.12) yields

$$\begin{aligned}
&\mathbb{E}[h(\boldsymbol{\Sigma}^{-1/2}\boldsymbol{S}) - Nh] \\
&\quad = \mathbb{E}\Bigg[\sum_{j=1}^p \sum_{l=1}^p \boldsymbol{\Sigma}_{j,l} \frac{\partial^2}{\partial s_j \partial s_l} f(\boldsymbol{S}) - \sum_{j=1}^p S_j \frac{\partial}{\partial s_j} f(\boldsymbol{S})\Bigg] \\
&\quad = \mathbb{E} \sum_{j=1}^p \boldsymbol{\Sigma}_{j,j} \frac{\partial^2}{\partial s_j^2} f(\boldsymbol{S}) + \mathbb{E} \sum_{j,l \in [p],\, j \ne l} \boldsymbol{\Sigma}_{j,l} \frac{\partial^2}{\partial s_j \partial s_l} f(\boldsymbol{S}) - \mathbb{E} \sum_{j=1}^p S_j \frac{\partial}{\partial s_j} f(\boldsymbol{S}).
\end{aligned} \tag{2.14}$$

We consider the first term in (2.14) and deal with each term under the sum separately for $j = 1, \ldots, p$. Letting $\sigma_{i,j}^2 := \operatorname{var}(\xi_{i,j})$ and $\sigma_{i,j;k,l} := \operatorname{cov}(\xi_{i,j}, \xi_{k,l})$, we have

$$\begin{aligned}
&\boldsymbol{\Sigma}_{j,j} \frac{\partial^2}{\partial s_j^2} f(\boldsymbol{S}) \\
&\quad = \sum_{i=1}^m \sigma_{i,j}^2 \frac{\partial^2}{\partial s_j^2} f(\boldsymbol{S}) + \sum_{i,k \in [m],\, i \ne k} \sigma_{i,j;\,k,j} \frac{\partial^2}{\partial s_j^2} f(\boldsymbol{S}) \\
&\quad = \sum_{i=1}^m \xi_{i,j}^2 \frac{\partial^2}{\partial s_j^2} f(\boldsymbol{S}) + \sum_{i,k \in [m],\, i \ne k} \sigma_{i,j;\,k,j} \frac{\partial^2}{\partial s_j^2} f(\boldsymbol{S}) + \sum_{i=1}^m (\sigma_{i,j}^2 - \xi_{i,j}^2) \frac{\partial^2}{\partial s_j^2} f(\boldsymbol{S}).
\end{aligned} \tag{2.15}$$

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Now, with $S_{j*i} \coloneqq S_j - \xi_{i,j}$, we write the summands of the third term on the right-hand side of (2.14) as

$$\begin{aligned} S_j \frac{\partial}{\partial s_j} f(\boldsymbol{S}) &= \sum_{i=1}^{m} \xi_{i,j} \frac{\partial}{\partial s_j} f(\boldsymbol{S}) \\ &= \sum_{i=1}^{m} \xi_{i,j} \frac{\partial}{\partial s_j} f(S_1, \dots, S_{j*i}, \dots, S_p) \\ &\quad + \sum_{i=1}^{m} \xi_{i,j}^2 \int_0^1 \frac{\partial^2}{\partial s_j^2} f(S_1, \dots, S_{j*i} + u\xi_{i,j}, \dots, S_p)\, \mathrm{d}u. \end{aligned} \tag{2.16}$$

Substituting (2.15) and (2.16) into (2.14) and letting

$$\boldsymbol{S}^{j*i} =: (S_1, \dots, S_{j-1}, S_{j*i}, S_{j+1}, \dots, S_p),$$

we obtain

$$\begin{aligned} &\mathbb{E}[h(\boldsymbol{\Sigma}^{-1/2}\boldsymbol{S}) - Nh] \\ &\quad = \mathbb{E} \sum_{j=1}^{p} \sum_{i=1}^{m} \xi_{i,j}^2 \int_0^1 \left(\frac{\partial^2}{\partial s_j^2} f(\boldsymbol{S}) - \frac{\partial^2}{\partial s_j^2} f(S_1, \dots, S_{j*i} + u\xi_{i,j}, \dots, S_p) \right) \mathrm{d}u \\ &\qquad + \mathbb{E} \sum_{j=1}^{p} \sum_{i=1}^{m} (\sigma_{i,j}^2 - \xi_{i,j}^2) \frac{\partial^2}{\partial s_j^2} f(\boldsymbol{S}) - \mathbb{E} \sum_{j=1}^{p} \sum_{i=1}^{m} \xi_{i,j} \frac{\partial}{\partial s_j} f(\boldsymbol{S}^{j*i}) \\ &\qquad + \mathbb{E} \sum_{j=1}^{p} \sum_{i,k \in [m],\, i \neq k} \sigma_{i,j;\,k,j} \frac{\partial^2}{\partial s_j^2} f(\boldsymbol{S}) + \mathbb{E} \sum_{j,l \in [p],\, j \neq l} \boldsymbol{\Sigma}_{j,l} \frac{\partial^2}{\partial s_j \partial s_l} f(\boldsymbol{S}). \end{aligned} \tag{2.17}$$

Now we handle the five terms in (2.17) separately. For the first term, using (2.13) we have

$$\begin{aligned} &\left| \mathbb{E} \sum_{j=1}^{p} \sum_{i=1}^{m} \xi_{i,j}^2 \int_0^1 \left(\frac{\partial^2}{\partial s_j^2} f(\boldsymbol{S}) - \frac{\partial^2}{\partial s_j^2} f(S_1, \dots, S_{j*i} + u\xi_{i,j}, \dots, S_p) \right) \mathrm{d}u \right| \\ &\quad = \left| \mathbb{E} \sum_{j=1}^{p} \sum_{i=1}^{m} \xi_{i,j}^2 \int_0^1 \int_{u\xi_{i,j}}^{\xi_{i,j}} \frac{\partial^3}{\partial s_j^3} f(S_1, \dots, S_{j*i} + t, \dots, S_p)\, \mathrm{d}t\, \mathrm{d}u \right| \\ &\quad \leq \frac{p^3}{3} |\boldsymbol{\Sigma}^{-1/2}|_\infty^3 \mathbb{E} \sum_{j=1}^{p} \sum_{i=1}^{m} \xi_{i,j}^2 \int_0^1 \int_{u|\xi_{i,j}|}^{|\xi_{i,j}|} \mathrm{d}t\, \mathrm{d}u \\ &\quad = \frac{p^3}{6} |\boldsymbol{\Sigma}^{-1/2}|_\infty^3 \sum_{j=1}^{p} \sum_{i=1}^{m} \mathbb{E}|\xi_{i,j}|^3 \\ &\quad \leq \frac{p^3}{6} |\boldsymbol{\Sigma}^{-1/2}|_\infty^3 B \sum_{j=1}^{p} \sum_{i=1}^{m} \mathbb{E}\xi_{i,j}^2 \\ &\quad = \frac{p^3}{6} |\boldsymbol{\Sigma}^{-1/2}|_\infty^3 B \sum_{j=1}^{p} \left(\boldsymbol{\Sigma}_{j,j} - \sum_{i,k \in [m],\, i \neq k} \sigma_{i,j;k,j} \right), \end{aligned} \tag{2.18}$$

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where we have used the almost sure bound on the variables $\xi_{i,j}$, and that their sum S_j over i from 1 to m has zero mean in the last two inequalities, respectively.

For the second term in (2.17), we first bound it by the three terms denoted by I_1, I_2, and I_3:

$$\begin{aligned}
&\left|\mathbb{E}\sum_{j=1}^{p}\sum_{i=1}^{m}(\sigma_{i,j}^2-\xi_{i,j}^2)\frac{\partial^2}{\partial s_j^2}f(\boldsymbol{S})\right| \\
&\quad\le\left|\mathbb{E}\sum_{j=1}^{p}\sum_{i=1}^{m}(\sigma_{i,j}^2-\xi_{i,j}^2)\frac{\partial^2}{\partial s_j^2}f(\boldsymbol{S}^{j*i})\right| \\
&\qquad+\left|\mathbb{E}\sum_{j=1}^{p}\sum_{i=1}^{m}(\sigma_{i,j}^2-\xi_{i,j}^2)\left(\frac{\partial^2}{\partial s_j^2}f(\boldsymbol{S}^{j*i})-\frac{\partial^2}{\partial s_j^2}f(\boldsymbol{S})\right)\right| \\
&\quad\le\left|\sum_{j=1}^{p}\sum_{i=1}^{m}\operatorname{cov}\left(\frac{\partial^2}{\partial s_j^2}f(\boldsymbol{S}^{j*i}),\xi_{i,j}^2\right)\right|+\mathbb{E}\sum_{j=1}^{p}\sum_{i=1}^{m}\sigma_{i,j}^2\left|\left(\frac{\partial^2}{\partial s_j^2}f(\boldsymbol{S}^{j*i})-\frac{\partial^2}{\partial s_j^2}f(\boldsymbol{S})\right)\right| \\
&\qquad+\mathbb{E}\sum_{j=1}^{p}\sum_{i=1}^{m}\xi_{i,j}^2\left|\left(\frac{\partial^2}{\partial s_j^2}f(\boldsymbol{S}^{j*i})-\frac{\partial^2}{\partial s_j^2}f(\boldsymbol{S})\right)\right| \\
&\quad:=I_1+I_2+I_3.
\end{aligned}$$

Then we write I_1 as

$$I_1=\left|\sum_{j=1}^{p}\sum_{i=1}^{m}\operatorname{cov}\left(\frac{\partial^2}{\partial s_j^2}f(\boldsymbol{S}^{j*i}),g(\xi_{i,j})\right)\right|,$$

where

$$g(x)=\begin{cases}x^2, & |x|\le B,\\ B^2, & |x|>B.\end{cases}$$

As $\boldsymbol{S}^{j*i}$ and $\xi_{i,j}$ are increasing functions defined on disjoint subsets of $\boldsymbol{\xi}$, by Lemma 2.1, $(\boldsymbol{S}^{j*i},\xi_{i,j})$ are negatively associated for all i, j. Applying Lemma 2.2 and using bound (2.13), we obtain

$$\begin{aligned}
I_1&\le\left|\sum_{j=1}^{p}\sum_{i=1}^{m}\left(\sum_{l\in[p],\,l\ne j}\left|\frac{\partial^3}{\partial s_l\partial s_j^2}f\right|_\infty\left|\frac{\partial g}{\partial x}\right|_\infty\operatorname{cov}(S_l,\xi_{i,j})+\left|\frac{\partial^3}{\partial s_j^3}f\right|_\infty\left|\frac{\partial g}{\partial x}\right|_\infty\operatorname{cov}(S_{j*i},\xi_{i,j})\right)\right| \\
&\le\frac{2}{3}p^3|\boldsymbol{\Sigma}^{-1/2}|_\infty^3B\left|\sum_{j=1}^{p}\sum_{i=1}^{m}\left(\sum_{l\in[p],\,l\ne j}\operatorname{cov}(S_l,\xi_{i,j})+\operatorname{cov}(S_{j*i},\xi_{i,j})\right)\right| \\
&=\frac{2}{3}p^3|\boldsymbol{\Sigma}^{-1/2}|_\infty^3B\left|\sum_{j,l\in[p],\,j\ne l}\sum_{i,k=1}^{m}\sigma_{i,j;\,k,l}+\sum_{j=1}^{p}\sum_{i,k\in[m],\,i\ne k}\sigma_{i,j;\,k,j}\right| \\
&=-\frac{2}{3}p^3|\boldsymbol{\Sigma}^{-1/2}|_\infty^3B\left(\sum_{j,l\in[p],\,j\ne l}\boldsymbol{\Sigma}_{j,l}+\sum_{j=1}^{p}\sum_{i,k\in[m],\,i\ne k}\sigma_{i,j;\,k,j}\right).
\end{aligned}\tag{2.19}$$

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Again using (2.13), we have

$$\begin{aligned} I_2 &\le \sum_{j=1}^{p}\sum_{i=1}^{m}\sigma_{i,j}^2\left|\frac{\partial^3}{\partial s_j^3}f\right|_\infty \mathbb{E}|S_{j*i}-S_j| \\ &= \sum_{j=1}^{p}\sum_{i=1}^{m}\sigma_{i,j}^2\left|\frac{\partial^3}{\partial s_j^3}f\right|_\infty \mathbb{E}|\xi_{i,j}| \\ &\le \frac{p^3}{3}|\boldsymbol{\Sigma}^{-1/2}|_\infty^3 B\sum_{j=1}^{p}\left(\boldsymbol{\Sigma}_{j,j}-\sum_{i,k\in[m],\, i\neq k}\sigma_{i,j;\,k,j}\right) \end{aligned} \tag{2.20}$$

and

$$I_3 \le \sum_{j=1}^{p}\sum_{i=1}^{m}\left|\frac{\partial^3}{\partial s_j^3}f\right|_\infty \mathbb{E}\xi_{i,j}^2|S_{j*i}-S_j| \tag{2.21}$$

$$\le \frac{p^3}{3}|\boldsymbol{\Sigma}^{-1/2}|_\infty^3 B\sum_{j=1}^{p}\sum_{i=1}^{m}\mathbb{E}\xi_{i,j}^2$$

$$= \frac{p^3}{3}|\boldsymbol{\Sigma}^{-1/2}|_\infty^3 B\sum_{j=1}^{p}\left(\boldsymbol{\Sigma}_{j,j}-\sum_{i,k\in[m],\, i\neq k}\sigma_{i,j;\,k,j}\right). \tag{2.22}$$

For the third term in (2.17), again applying Lemma 2.2 and arguing as for I_1 in (2.19), we have

$$\left|\mathbb{E}\sum_{j=1}^{p}\sum_{i=1}^{m}\xi_{i,j}\frac{\partial}{\partial s_j}f(\boldsymbol{S}^{j*i})\right| \tag{2.23}$$

$$= \left|\sum_{j=1}^{p}\sum_{i=1}^{m}\mathrm{cov}\left(\xi_{i,j},\,\frac{\partial}{\partial s_j}f(\boldsymbol{S}^{j*i})\right)\right| \tag{2.24}$$

$$\le \left|\sum_{j=1}^{p}\sum_{i=1}^{m}\left(\sum_{l\in[p]\setminus\{j\}}\left|\frac{\partial^2}{\partial s_l\partial s_j}f\right|_\infty \mathrm{cov}(\xi_{i,j},S_l)+\left|\frac{\partial^2}{\partial s_j^2}f\right|_\infty \mathrm{cov}(\xi_{i,j},S_{j*i})\right)\right|$$

$$\le -\frac{1}{2}p^2|\boldsymbol{\Sigma}^{-1/2}|_\infty^2\left(\sum_{j,l\in[p],\, j\neq l}\sum_{i=1}^{m}\mathrm{cov}(\xi_{i,j},S_l)+\sum_{j=1}^{p}\sum_{i=1}^{m}\mathrm{cov}(\xi_{i,j},S_{j*i})\right)$$

$$= -\frac{1}{2}p^2|\boldsymbol{\Sigma}^{-1/2}|_\infty^2\left(\sum_{j,l\in[p],\, j\neq l}\sum_{i,k=1}^{m}\sigma_{i,j;\,k,l}+\sum_{j=1}^{p}\sum_{i,k\in[m],\, i\neq k}\sigma_{i,j;\,k,j}\right)$$

$$= -\frac{1}{2}p^2|\boldsymbol{\Sigma}^{-1/2}|_\infty^2\left(\sum_{j,l\in[p],\, j\neq l}\boldsymbol{\Sigma}_{j,l}+\sum_{j=1}^{p}\sum_{i,k\in[m],\, i\neq k}\sigma_{i,j;\,k,j}\right). \tag{2.25}$$

For the fourth and the fifth terms in (2.17), again using (2.13), we obtain

$$\left|\sum_{j=1}^{p}\sum_{i,k\in[m],\, i\neq k}\sigma_{i,j;\,k,j}\frac{\partial^2}{\partial s_j^2}f(\boldsymbol{S})\right| \le -\frac{p^2}{2}|\boldsymbol{\Sigma}^{-1/2}|_\infty^2\sum_{j=1}^{p}\sum_{i,k\in[m],\, i\neq k}\sigma_{i,j;\,k,j} \tag{2.26}$$

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and

$$\left|\mathbb{E}\sum_{j,l\in[p],\ j\neq l}\boldsymbol{\Sigma}_{j,l}\frac{\partial^2}{\partial s_j\partial s_l}f(\boldsymbol{S})\right| \le -\frac{p^2}{2}|\boldsymbol{\Sigma}^{-1/2}|_\infty^2\sum_{j,l\in[p],\ j\neq l}\boldsymbol{\Sigma}_{j,l}. \tag{2.27}$$

Summing the bounds (2.18)–(2.27), we find that $|\mathbb{E}[h(\boldsymbol{\Sigma}^{-1/2}\boldsymbol{S}) - Nh]|$ is bounded by the right-hand side of (1.4). Taking the supremum over $h \in \mathcal{H}_{3,\infty,p}$ and using definition (1.2) of $d_{\mathcal{H}_{m,\infty,p}}$ yields the claim. □

3. Applications

In this section we follow the same structure as in [11, Section 2.1] with positive association replaced by negative association. In particular, we apply our main theorems in Section 1 to second-order stationary negatively associated random fields with exponential covariance decay.

First we introduce the definitions and notation used in [11] that will also be used here. Let $\{X_{\boldsymbol{j}}\colon \boldsymbol{j} \in \mathbb{Z}^d\}$ be a negatively associated random field on the d-dimensional integer lattice $\mathbb{Z}^d$ and assume that the field is second-order stationary. We recall that a random field $\{X_{\boldsymbol{j}}\colon \boldsymbol{j} \in \mathbb{Z}^d\}$ is called *second-order stationary* when $\mathbb{E}X_{\boldsymbol{j}}^2 < \infty$ for all $\boldsymbol{j} \in \mathbb{Z}^d$, and the covariance $\operatorname{cov}(X_{\boldsymbol{i}}, X_{\boldsymbol{j}}) = R(\boldsymbol{j}-\boldsymbol{i})$ for all $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{R}^d$ with $R(\cdot)$ given by

$$R(\boldsymbol{k}) = \operatorname{cov}(X_{\boldsymbol{0}}, X_{\boldsymbol{k}}). \tag{3.1}$$

We let $\boldsymbol{1} \in \mathbb{Z}^d$ denote the vector of all 1s, and write inequalities such as $\boldsymbol{a} < \boldsymbol{b}$ for vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ when they hold componentwise. For $\boldsymbol{k} \in \mathbb{Z}^d, n \in \mathbb{N}_1$, define the 'block sum' variables, over a block with side length n, by

$$S_{\boldsymbol{k}}^n = \sum_{\boldsymbol{j}\in B_{\boldsymbol{k}}^n} X_{\boldsymbol{j}}, \tag{3.2}$$

where $B_{\boldsymbol{k}}^n = \{\boldsymbol{j} \in \mathbb{Z}^d\colon \boldsymbol{k} \le \boldsymbol{j} < \boldsymbol{k} + n\boldsymbol{1}\}$. Note that $B_{\boldsymbol{k}}^n = B_{\boldsymbol{0}}^n + \boldsymbol{k}$.

For $R(\cdot)$ given in (3.1), we have

$$\operatorname{var}(S_{\boldsymbol{k}}^n) = \sum_{\boldsymbol{i},\boldsymbol{j}\in B_{\boldsymbol{k}}^n}\operatorname{cov}(X_{\boldsymbol{i}}, X_{\boldsymbol{j}}) = n^d A_n, \tag{3.3}$$

where $A_n = (1/n^d)\sum_{\boldsymbol{i},\boldsymbol{j}\in B_{\boldsymbol{1}}^n} R(\boldsymbol{i}-\boldsymbol{j})$. Since the field is negatively associated, $R(\boldsymbol{k}) \le 0$ for all $\boldsymbol{k} \neq \boldsymbol{0}$, which implies that $0 \le A_n \le R(\boldsymbol{0})$ for all $n \in \mathbb{N}_1$. For simplicity, in this paper we assume that $\inf_n A_n > 0$, which implies that A_n is of constant order. With this assumption, we may include A_n in our bounds without affecting the rate of convergence.

With $S_{\boldsymbol{k}}^n$ defined in (3.2), we consider the standardized variables

$$W_{\boldsymbol{k}}^n = \frac{S_{\boldsymbol{k}}^n - \mathbb{E}S_{\boldsymbol{k}}^n}{\sqrt{n^d A_n}}, \qquad \boldsymbol{k} \in \mathbb{Z}^d,\ n \in \mathbb{N}_1, \tag{3.4}$$

that have zero mean and variance 1. In the following theorem we provide a bound of order $n^{-d/(2d+2)}$ with an explicit constant on the L^1-distance between the distribution of $W_{\boldsymbol{k}}^n$ and the normal, under the assumption that the covariance function $R(\cdot)$ decays at an exponential rate in the L^1-norm in $\mathbb{R}^d$. Since all norms in $\mathbb{R}^d$ are equivalent, we use the L^1-norm in order to simplify our calculation.

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Theorem 3.1. *Let $d \in \mathbb{N}_1$ and $\{X_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbb{Z}^d\}$ be a negatively associated second-order stationary random field with covariance function $R(\boldsymbol{k}) = \operatorname{cov}(X_{\boldsymbol{j}}, X_{\boldsymbol{j}+\boldsymbol{k}})$ for all $\boldsymbol{j}, \boldsymbol{k} \in \mathbb{Z}^d$, and suppose that, for some $K > 0$, $|X_{\boldsymbol{j}}| \le K$ almost surely for all $\boldsymbol{j} \in \mathbb{Z}^d$. Assume that there exist $\lambda > 0$ and $\kappa_0 > 0$ such that*

$$-R(\boldsymbol{k}) \le \kappa_0 \mathrm{e}^{-\lambda|\boldsymbol{k}|_1} \quad \text{for all } \boldsymbol{k} \in \mathbb{Z}^d/\{\boldsymbol{0}\} \tag{3.5}$$

and $\inf_n A_n > 0$, where A_n is given in (3.3). Let

$$\mu_\lambda = \frac{\mathrm{e}^{\lambda}}{(\mathrm{e}^{\lambda}-1)^2}, \qquad \nu_\lambda = \frac{\mathrm{e}^{2\lambda}}{(\mathrm{e}^{\lambda}-1)^2}, \qquad \gamma_{\lambda,d} = (4\mu_\lambda + 2\nu_\lambda)^d - (2\nu_\lambda)^d, \tag{3.6}$$

and

$$C_{\lambda,\kappa_0,d} = \frac{10Kd\sqrt{A_n}}{5.2\kappa_0\gamma_{\lambda,d}}. \tag{3.7}$$

Then, for any $\boldsymbol{k} \in \mathbb{Z}^d$ with $W_{\boldsymbol{k}}^n$ as given in (3.4) and Z a standard normal random variable,

$$d_1(\mathcal{L}(W_{\boldsymbol{k}}^n), \mathcal{L}(Z)) \le \frac{\kappa_1}{n^{d/(2d+2)}} \quad \text{for all } n \ge \max\{C_{\lambda,\kappa_0,d}^{2/d}, C_{\lambda,\kappa_0,d}^{-2/(d+2)}\}, \tag{3.8}$$

where

$$\kappa_1 = \left(\frac{10K(5.2\kappa_0\gamma_{\lambda,d})^d}{A_n^{d+1/2}}\right)^{1/(d+1)}\left(\frac{1}{d^{d/(d+1)}} + 2d^{1/(d+1)}\right).$$

The bound in (3.8) is of order $n^{-d/(2d+2)}$ recalling that A_n is bounded away from 0 and ∞ and, hence, does not contribute to the rate. We also extend Theorem 3.1 to the multidimensional case. For any $p \in \mathbb{N}_1$ and indices $\boldsymbol{k}_1, \ldots, \boldsymbol{k}_p$ in $\mathbb{Z}^d$ such that $B_{\boldsymbol{k}_i}^n$, $i \in [p]$ are disjoint, Theorem 3.2 provides a bound in the metric $d_{\mathcal{H}_{3,\infty,p}}$ to the multivariate normal for $\boldsymbol{S}^n = (S_{\boldsymbol{k}_1}^n, \ldots, S_{\boldsymbol{k}_p}^n)$ under exponential decay of the covariance function. We note the difference between assumption (3.9) below and the one of Equation (30) of [11]: we require $B_{\boldsymbol{k}_i}^n$, $i \in [p]$ here to be disjoint, otherwise there exist $\boldsymbol{j} \in \mathbb{Z}^d$ that belong to both $B_{\boldsymbol{k}_i}^n$ and $B_{\boldsymbol{k}_j}^n$ for some i, j, and $(X_{\boldsymbol{j}}, X_{\boldsymbol{j}})$ is positively associated to which Theorem 1.4 is not applicable. The same issue does not arise in the positively associated case as a pair of the same variables has positive covariance. In the following result and its proof, constants will not be tracked with precision, but will be indexed by the set of variables on which it depends.

Theorem 3.2. *For $d \in \mathbb{N}_1$, let $\{X_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbb{Z}^d\}$ be a negatively associated second-order stationary random field with covariance function $R(\boldsymbol{k}) = \operatorname{cov}(X_{\boldsymbol{j}}, X_{\boldsymbol{j}+\boldsymbol{k}})$ for all $\boldsymbol{j}, \boldsymbol{k} \in \mathbb{Z}^d$, and suppose that there exist constants $K > 0$, $\kappa_0 > 0$, $\lambda > 0$ such that $|X_{\boldsymbol{j}}| \le K$ almost surely for all $\boldsymbol{j} \in \mathbb{Z}^d$,*

$$-R(\boldsymbol{k}) \le \kappa_0 \mathrm{e}^{-\lambda|\boldsymbol{k}|_1} \quad \text{for all } \boldsymbol{k} \in \mathbb{Z}^d/\{\boldsymbol{0}\},$$

and $\inf_n A_n > 0$, where A_n is given in (3.3). For $p \in \mathbb{N}_1$ let $\boldsymbol{k}_1, \ldots, \boldsymbol{k}_p \in \mathbb{Z}^d$ be such that

$$\min_{q,s\in[p],\, q\ne s} |\boldsymbol{k}_q - \boldsymbol{k}_s|_\infty \ge n. \tag{3.9}$$

Let $\boldsymbol{S}^n = (S_{\boldsymbol{k}_1}^n, \ldots, S_{\boldsymbol{k}_p}^n)$, where $S_{\boldsymbol{k}}^n$ is defined as in (3.2) and $\boldsymbol{\Sigma}$ be the covariance matrix of $\boldsymbol{S}^n$. Then, for $n > (p-1)\kappa_0\nu_\lambda^d \mathrm{e}^{-\lambda} A_n^{-1}$ with ν_λ as in (3.6), $\boldsymbol{\Sigma}$ is invertible and

$$|\boldsymbol{\Sigma}^{-1}|_\infty \le \frac{1}{n^{d-1}(nA_n - (p-1)\kappa_0\nu_\lambda^d \mathrm{e}^{-\lambda})}. \tag{3.10}$$

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Furthermore, with

$$\psi_n = n^{d/2}|\boldsymbol{\Sigma}^{-1/2}|_\infty \quad \text{for } n > (p-1)\kappa_0 \nu_\lambda^d \mathrm{e}^{-\lambda} A_n^{-1} \quad \text{and} \quad B_{n,d} = d\psi_n A_n,$$

there exists a constant $C_{\lambda,\kappa_0,d,p,K}$ *such that, for*

$$n > \max\{B_{n,d}^{2/d}, B_{n,d}^{-2/(d+2)}, (p-1)\kappa_0 \nu_\lambda^d \mathrm{e}^{-\lambda} A_n^{-1}\}, \tag{3.11}$$

we have

$$\begin{aligned} &d_{\mathcal{H}_{3,\infty,p}}(\mathcal{L}(\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{S}^n - \mathbb{E}\boldsymbol{S}^n)), \mathcal{L}(\boldsymbol{Z})) \\ &\quad \le C_{\lambda,\kappa_0,d,p,K}\bigg(\frac{\psi_n^{(2d+4)/(d+1)}}{A_n^{(d-1)/(d+1)} n^{d/(d+1)}} + \frac{\psi_n^{(2d+3)/(d+1)}}{A_n^{d/(d+1)} n^{(3d+2)/(2d+2)}} \\ &\qquad + \frac{\psi_n^2}{n} + \frac{A_n^{1/(d+1)} \psi_n^{(2d+3)/(d+1)}}{n^{d/(2d+2)}}\bigg), \end{aligned} \tag{3.12}$$

where $\boldsymbol{Z}$ *is a standard normal random vector in* $\mathbb{R}^p$.

Since A_n is of constant order, $|\boldsymbol{\Sigma}^{-1/2}|_\infty$ is of order at most $n^{-d/2}$ by (3.10). This implies that ψ_n is of, at most, constant order and, thus, so is $B_{n,d}$. Therefore, the last term on the right-hand side of (3.12) is the only one that contributes to the rate of convergence of order $n^{-d/(2d+2)}$ as the other terms converge to 0 at much faster rates. We note that the bounds in Theorems 3.1 and 3.2 have the same order as the ones of Theorems 2.1 and 2.2 of [11], respectively. However, compared to the results of [11], the constant of bound (3.8) is greater, and bound (3.12) has extra terms that do not contribute to the rate.

To prove Theorems 3.1 and 3.2, we use the same technique as in [11], decomposing the sum $S_{\boldsymbol{k}}^n$ over the block $B_{\boldsymbol{k}}^n$ into sums over smaller, disjoint blocks whose side lengths are at most some integer l. That is, for $1 \le l \le n$, we uniquely write $n = (m-1)l + r$ with $m \ge 1$, $1 \le r \le l$ and correspondingly decompose $B_{\boldsymbol{k}}^n$ into m^d disjoint blocks $D_{\boldsymbol{i},\boldsymbol{k}}^l$, $\boldsymbol{i} \in [m]^d$, where there are $(m-1)^d$ 'main' blocks having all sides of length l, and $m^d - (m-1)^d$ remainder blocks having all sides of length r or l with at least one side of length r.

To be more precise, for $\boldsymbol{k} \in \mathbb{Z}^d$ and $\boldsymbol{i} \in [m]^d$, set $D_{\boldsymbol{i},\boldsymbol{k}}^l = D_{\boldsymbol{i}}^l + \boldsymbol{k} - \boldsymbol{1}$, where

$$\begin{aligned} D_{\boldsymbol{i}}^l = \{\boldsymbol{j} \in \mathbb{Z}^d : &(i_s - 1)l + 1 \le j_s \le i_s l \text{ for } i_s \ne m, \\ &(m-1)l + 1 \le j_s \le (m-1)l + r \text{ for } i_s = m\}. \end{aligned}$$

It is easy to see that, for $\boldsymbol{i} \in [m-1]^d$, the vectors indexing the 'main blocks', we have

$$D_{\boldsymbol{i}}^l = B_{(\boldsymbol{i}-\boldsymbol{1})l+\boldsymbol{1}}^l \quad \text{for } \boldsymbol{i} \in [m-1]^d, \tag{3.13}$$

and if $r = l$ then $D_{\boldsymbol{i}}^l$ is given by (3.13) for all $i \in [m]^d$. Furthermore, it is straightforward to verify that the elements of the collection $\{D_{\boldsymbol{i},\boldsymbol{k}}^l, \boldsymbol{i} \in [m]^d\}$ is the partition of $B_{\boldsymbol{k}}^n$.

Letting

$$\xi_{\boldsymbol{i},\boldsymbol{k}}^l = \sum_{\boldsymbol{t} \in D_{\boldsymbol{i},\boldsymbol{k}}^l} (X_{\boldsymbol{t}} - \mathbb{E}X_{\boldsymbol{t}}) \quad \text{for } \boldsymbol{i} \in [m]^d, \qquad W_{\boldsymbol{k}}^n = \sum_{\boldsymbol{i} \in [m]^d} \frac{\xi_{\boldsymbol{i},\boldsymbol{k}}^l}{\sqrt{n^d A_n}}, \tag{3.14}$$

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we see that $\xi_{i,k}^n$ has zero mean, and W_k^n as in (3.14) agrees with its representation as given in (3.4), and has zero mean and variance 1. For simplicity, we will drop the index k in $\xi_{i,k}$ when $k = \mathbf{1}$, as we do for $D_{i,k}$, and also suppress n in $\xi_{i,k}^n$.

As the elements of $\{\xi_{i,k} : i \in [m]^d\}$ are increasing functions of disjoint subsets of $\{X_j : j \in \mathbb{Z}^d\}$, they are negatively associated by Lemma 2.1. We prove Theorems 3.1 and 3.2 with the help of the following three lemmas. The first, Lemma 3.1, bounds the sum of the covariances between $\xi_{i,k}^l$ and $\xi_{j,k}^l$, defined in (3.14), over $i, j \in [m]^d$. Next, we state Lemma 3.2, proved in [11], which is used in the proof of Lemma 3.3, that bounds the covariance between two nonoverlapped block sums of size n^d.

Lemma 3.1. *Let $\{X_j : j \in \mathbb{Z}^d\}$ be a second-order, negatively associated stationary random field with covariance function $R(k) = \operatorname{cov}(X_j, X_{j+k})$ for all $j \in \mathbb{Z}^d$ and $k \in \mathbb{Z}^d/\{\mathbf{0}\}$, where $R(\cdot)$ satisfies the exponential decay condition (3.5). For $n \geq 2$ and $1 \leq l \leq n$, let $n = (m-1)l + r$ for integers $m \in \mathbb{N}_1$ and $1 \leq r \leq l$. Then, for $i \in [m]$ and $k \in \mathbb{Z}^d$ with $\xi_{i,k}$ given by (3.14), we have*

$$\sum_{i,j \in [m]^d,\, i \neq j} -\mathbb{E}[\xi_{i,k}^l \xi_{j,k}^l] \leq \frac{\kappa_0 \gamma_{\lambda,d} n^d}{l},$$

where κ_0 is given in (3.5), and $\gamma_{\lambda,d}$ in (3.6).

Proof. We note that the difference between (3.5) of this paper and Equation (22) of [11] is only the sign on the left-hand side of the inequality. Thus, the proof follows immediately from the proof of Lemma 2.4 of [11] with $\mathbb{E}[\xi_{i,k}^l \xi_{j,k}^l]$ and $R(k)$ replaced by $-\mathbb{E}[\xi_{i,k}^l \xi_{j,k}^l]$ and $-R(k)$, respectively. □

Lemma 3.2. (Goldstein and Wiroonsri [11].) *For all $n \in \mathbb{N}_2$ and $\lambda > 0$,*

$$\sum_{a=-n+1}^{n-1} (n - |a|)\mathrm{e}^{-\lambda|q+a|} \quad \text{is decreasing as a function of } |q| \in \mathbb{N}_0.$$

In the following, we will use the identities

$$\sum_{k=1}^{n-1}(n-k)w^k = \frac{w((n-1) - nw + w^n)}{(w-1)^2} \quad \text{for } w \neq 1 \tag{3.15}$$

and

$$n + 2\sum_{b=1}^{n-1}(n-b)u^b = \frac{(1-u^2)n - 2u + 2u^{n+1}}{(u-1)^2} \quad \text{for } u \neq 1. \tag{3.16}$$

Lemma 3.3. *Let $\{X_j : j \in \mathbb{Z}^d\}$ be a second-order stationary random field with covariance function $R(k) = \operatorname{cov}(X_j, X_{j+k})$ for all $j \in \mathbb{Z}^d$ and $k \in \mathbb{Z}^d/\{\mathbf{0}\}$, where $R(\cdot)$ satisfies (3.5). Let $n \in \mathbb{N}_2$, and k_1 and k_2 be vectors in $\mathbb{Z}^d$ such that*

$$|k_1 - k_2|_\infty \geq n.$$

Then with λ and κ_0 as in (3.5), and ν_λ as in (3.6),

$$-\operatorname{cov}(S_{k_1}^n, S_{k_2}^n) \leq \kappa_0 \nu_\lambda^d \mathrm{e}^{-\lambda} n^{d-1}.$$

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Proof. Using the definition of $S_{\boldsymbol{k}}^n$ and the fact that $X_{\boldsymbol{j}}$, $\boldsymbol{j} \in \mathbb{Z}^d$ are second-order stationary, we have

$$\begin{aligned}
-\operatorname{cov}(S_{\boldsymbol{k}_1}^n, S_{\boldsymbol{k}_2}^n) &= -\sum_{p_1,\dots,p_d=0,\, q_1,\dots,q_d=0}^{n-1} R\left(\begin{bmatrix} (p_1+k_1^2)-(q_1+k_1^1) \\ \vdots \\ (p_d+k_d^2)-(q_d+k_d^1) \end{bmatrix}\right) \\
&\le \kappa_0 \sum_{a_1,\dots,a_d=-n+1}^{n-1} (n-|a_1|)\cdots(n-|a_d|) \exp\left(-\lambda \left\| \begin{bmatrix} a_1+(k_1^2-k_1^1) \\ \vdots \\ a_d+(k_d^2-k_d^1) \end{bmatrix} \right\|_1\right) \\
&= \kappa_0 \prod_{i=1}^{d} \sum_{a_i=-n+1}^{n-1} (n-|a_i|)\mathrm{e}^{-\lambda|(k_i^2-k_i^1)+a_i|}. \qquad (3.17)
\end{aligned}$$

Applying Lemma 3.2, we have

$$\sum_{a_i=-n+1}^{n-1} (n-|a_i|)\mathrm{e}^{-\lambda|(k_i^2-k_i^1)+a_i|} \quad \text{is a decreasing function of } |k_i^1-k_i^2|. \qquad (3.18)$$

Hence, the ith sum appearing in the product (3.17) is maximized when $k_i^1 = k_i^2$. As $|\boldsymbol{k}_1-\boldsymbol{k}_2|_\infty \ge n$, there must exist at least one i for which $|k_i^2-k_i^1| \ge n$, and whose corresponding sum is bounded by its value when $|k_i^2-k_i^1|$ is exactly n, using (3.18). The product of these sums, by (3.18) again, is maximized when there is just a single coordinate achieving n as its absolute difference, and where this difference in all other terms achieves equality at 0. Therefore, by symmetry, (3.17) is bounded by the case where $k_i^1 = k_i^2$ for $i \in [d-1]$ and $k_d^2 - k_d^1 = n$, and, thus,

$$\begin{aligned}
-\operatorname{cov}(S_{\boldsymbol{k}_1}^n, S_{\boldsymbol{k}_2}^n) &\le \kappa_0 \prod_{i=1}^{d-1} \sum_{a_i=-n+1}^{n-1} (n-|a_i|)\mathrm{e}^{-\lambda|a_i|} \sum_{a_d=-n+1}^{n-1} (n-|a_d|)\mathrm{e}^{-\lambda|a_d+n|} \\
&\le \kappa_0 (n\nu_\lambda)^{d-1} \sum_{a_d=-n+1}^{n-1} (n-|a_d|)\mathrm{e}^{-\lambda|a_d+n|}, \qquad (3.19)
\end{aligned}$$

where we have applied (3.16) in the final inequality and ν_λ is given in (3.6).

Now considering the sum in (3.19), we obtain

$$\begin{aligned}
\sum_{a=-n+1}^{n-1} (n-|a|)\mathrm{e}^{-\lambda|a+n|} &= \sum_{a=-n+1}^{n-1} (n-|a|)\mathrm{e}^{-\lambda(a+n)} \\
&= \sum_{a=1}^{n-1} (n-a)\mathrm{e}^{-\lambda(-a+n)} + \sum_{a=0}^{n-1} (n-a)\mathrm{e}^{-\lambda(a+n)}.
\end{aligned}$$

For each sum, making a change of variable and applying (3.15), we obtain

$$\sum_{a=1}^{n-1} (n-a)\mathrm{e}^{-\lambda(-a+n)} = \mathrm{e}^{-\lambda n} \sum_{a=1}^{n-1} (n-a)\mathrm{e}^{\lambda a}$$

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$$= \frac{e^{-\lambda n}e^{\lambda}(n-1-ne^{\lambda}+e^{\lambda n})}{(e^{\lambda}-1)^2}$$
$$= \frac{e^{\lambda}+n(1-e^{\lambda})e^{\lambda(1-n)}-e^{\lambda(1-n)}}{(e^{\lambda}-1)^2}$$

and

$$\sum_{a=0}^{n-1}(n-a)e^{-\lambda(a+n)} = ne^{-\lambda n}+e^{-\lambda n}\sum_{a=1}^{n-1}(n-a)e^{-\lambda a}$$
$$= ne^{-\lambda n}+\frac{e^{-\lambda n}e^{-\lambda}(n-1-ne^{-\lambda}+e^{-\lambda n})}{(e^{-\lambda}-1)^2}$$
$$= ne^{-\lambda n}+\frac{e^{-\lambda n}e^{\lambda}(n-1-ne^{-\lambda}+e^{-\lambda n})}{(e^{\lambda}-1)^2}$$
$$= \frac{e^{\lambda(1-2n)}-e^{\lambda(1-n)}+n(e^{\lambda}-1)e^{\lambda(1-n)}}{(e^{\lambda}-1)^2}.$$

Summing these two terms yields

$$\sum_{a=-n+1}^{n-1}(n-|a|)e^{-\lambda|a+n|} = \frac{e^{\lambda(1-2n)}+e^{\lambda}-2e^{\lambda(1-n)}}{(e^{\lambda}-1)^2} \le \frac{e^{\lambda}}{(e^{\lambda}-1)^2} = \nu_{\lambda}e^{-\lambda}.$$

Substituting the last bound into (3.19) yields the claim. □

Now we have all the ingredients to prove Theorems 3.1 and 3.2. In the following, we use the same technique as in Equation (44) of [11], that is, for any positive real numbers a and b, the minimum of $al^d + b/l$ over real numbers l is achieved at $l_0 = (b/ad)^{1/(d+1)}$. Taking $l = \lfloor l_0 \rfloor$ when $l_0 \ge 1$ and using the fact that $l_0/2 \le l \le l_0$ yields

$$\min_{l\in\mathbb{N}_1}\left(al^d+\frac{b}{l}\right) \le a\left(\frac{b}{ad}\right)^{d/(d+1)}+2b\left(\frac{ad}{b}\right)^{1/(d+1)}$$
$$= a^{1/(d+1)}b^{d/(d+1)}\left(\frac{1}{d^{d/(d+1)}}+2d^{1/(d+1)}\right). \tag{3.20}$$

Proof of Theorem 3.1. By second-order stationarity, it suffices to prove the $\boldsymbol{k} = \mathbf{1}$ case. Let $n \ge 2$, $B_{\mathbf{1}}^n$ be the block of size n^d as given in (3.2), and $W_{\mathbf{1}}^n$ the standardized sum over that block, as in (3.4). For any $1 \le l \le n$, write $n = (m-1)l + r$, $1 \le r \le l$, and decompose $W_{\mathbf{1}}^n$ as the sum of $\xi_{\boldsymbol{i}}/\sqrt{n^d A_n}$ over $\boldsymbol{i} \in [m]$, as in (3.14).

We apply Theorem 1.3, dealing with the two terms on the right-hand side of (1.3). For the first term, using $|X_j| \le K$, the definition (3.14) of $\xi_{\boldsymbol{i}}$, and the fact that the side lengths of all blocks $D_{\boldsymbol{i}}^l$ are at most l, we have

$$\left|\frac{\xi_{\boldsymbol{i}}}{\sqrt{n^d A_n}}\right| \le B$$

with $B = 2Kl^d/\sqrt{n^d A_n}$ for all $\boldsymbol{i} \in [m]^d$.

Applying Lemma 3.1 for the last term and invoking Theorem 1.3 now yields

$$d_1(\mathcal{L}(W_{\mathbf{1}}), \mathcal{L}(Z)) \le \frac{10Kl^d}{n^{d/2}A_n^{1/2}}+\frac{5.2\kappa_0\gamma_{\lambda,d}}{lA_n}. \tag{3.21}$$

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Applying bound (3.20) to the last expression, with $l_0 = C_{\lambda,\kappa_0,d}^{-1/(d+1)} n^{d/(2d+2)}$ and $C_{\lambda,\kappa_0,d}$ as in (3.7), and substituting $l = \lfloor l_0 \rfloor$ into the right-hand side of (3.21) yields the result. It is easy to check that $1 \le l_0 \le n$ for $n \ge \max\{C_{\lambda,\kappa_0,d}^{2/d}, C_{\lambda,\kappa_0,d}^{-2/(d+2)}\}$. □

To prove Theorem 3.2, we apply Theorem 1.4 and use the same techniques as in Theorem 3.1. We remind the reader that for this result we do not explicitly compute the constants, but index them by the parameters on which they depend.

Proof of Theorem 3.2. First we prove the claims that $\mathbf{\Sigma}$ is invertible and $|\mathbf{\Sigma}^{-1}|_\infty$ is bounded by (3.10) when $n > (p-1)\kappa_0 \nu_\lambda^d \mathrm{e}^{-\lambda}/A_n$. Applying Lemma 3.3, for all $q \in [p]$, we have

$$\mathbf{\Sigma}_{q,q} - \sum_{1 \le s \le p,\, s \ne q} |\mathbf{\Sigma}_{q,s}| \ge n^{d-1}(nA_n - (p-1)\kappa_0 \nu_\lambda^d \mathrm{e}^{-\lambda}) > 0 \quad \text{if } n > \frac{(p-1)\kappa_0 \nu_\lambda^d \mathrm{e}^{-\lambda}}{A_n},$$

which implies that $\mathbf{\Sigma}$ is a strictly diagonally dominant matrix, and is therefore invertible by the Gershgorin circle theorem; see, e.g. Theorem 15.10 of [2]. Another claim of (3.10) follows from [1], where it was shown that bound (3.10) holds for the norm $\|C\|_\infty = \max_i \sum_{j=1}^p |c_{ij}|$, which dominates $|C|_\infty$.

Next we proceed as in the one-dimensional case. For $n \ge 2$ and $1 \le l \le n$, we write $n = (m-1)l + r$ with $m \ge 1$ and $1 \le r \le l$, and decompose $S_{\boldsymbol{k}_q}^n - \mathbb{E}S_{\boldsymbol{k}_q}^n$ for $q \in [p]$ as the sum over $\boldsymbol{i} \in [m]^d$ of the variables $\xi_{\boldsymbol{i},\boldsymbol{k}_q}$ given in (3.14).

Applying Theorem 1.4, we handle the three terms on the right-hand side of (1.4). Using definition (3.14) of $\xi_{\boldsymbol{i},\boldsymbol{k}_q}$ and the fact that $|X_{\boldsymbol{t}}| \le K$, we have

$$|\xi_{\boldsymbol{i},\boldsymbol{k}_q}| \le B,$$

where $B = 2Kl^d$ for all $\boldsymbol{i} \in [m]^d, q \in [p]$ and, thus, using $|\mathbf{\Sigma}^{-1/2}|_\infty = n^{-d/2}\psi_n$ and $\mathbf{\Sigma}_{jj} = n^d A_n$, we may bound the first term as

$$\frac{5}{6} p^3 B n^{-3d/2} \psi_n^3 \sum_{q=1}^p \mathbf{\Sigma}_{q,q} \le \frac{C_{p,K} l^d A_n \psi_n^3}{n^{d/2}}. \tag{3.22}$$

For the second term, by Lemma 3.1, we have

$$\begin{aligned} &-\left(\frac{3}{2} p^3 B n^{-3d/2} \psi_n^3 + p^2 n^{-d} \psi_n^2\right) \sum_{q=1}^p \sum_{\boldsymbol{i},\boldsymbol{j} \in [m]^d,\, \boldsymbol{i} \ne \boldsymbol{j}} \mathbb{E}[\xi_{\boldsymbol{i},\boldsymbol{k}_q} \xi_{\boldsymbol{j},\boldsymbol{k}_q}] \\ &\quad = -\left(\frac{C_{p,K} l^d \psi_n^3}{n^{d/2}} + C_p \psi_n^2\right) \sum_{q=1}^p \sum_{\boldsymbol{i},\boldsymbol{j} \in [m]^d,\, \boldsymbol{i} \ne \boldsymbol{j}} \mathbb{E}\left[\frac{\xi_{\boldsymbol{i},\boldsymbol{k}_q} \xi_{\boldsymbol{j},\boldsymbol{k}_q}}{n^d}\right] \\ &\quad \le \frac{C_{\lambda,\kappa_0,p,K,d} l^{d-1} \psi_n^3}{n^{d/2}} + \frac{C_{\lambda,\kappa_0,p,d} \psi_n^2}{l}. \end{aligned} \tag{3.23}$$

Next, invoking Lemma 3.3 and assumption (3.9), we have

$$-\mathbf{\Sigma}_{q,s} = -\operatorname{cov}(S_{\boldsymbol{k}_q}^n, S_{\boldsymbol{k}_s}^n) \le \kappa_0 \nu_\lambda^d \mathrm{e}^{-\lambda} n^{d-1} \quad \text{for } q \ne s \in [p],$$

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and, hence, we may bound the last term as

$$-\left(\frac{2}{3}p^3Bn^{-3d/2}\psi_n^3+p^2n^{-d}\psi_n^2\right)\sum_{q,s\in[p],\,q\neq s}\mathbf{\Sigma}_{q,s}\leq\left(\frac{C_{p,K}l^d\psi_n^3}{n^{3d/2}}+\frac{C_p\psi_n^2}{n^d}\right)\kappa_0\nu_\lambda^d\mathrm{e}^{-\lambda}n^{d-1}$$
$$\leq\frac{C_{\lambda,\kappa_0 d,p,K}l^d\psi_n^3}{n^{(d+2)/2}}+\frac{C_{\lambda,\kappa_0,d,p}\psi_n^2}{n}. \tag{3.24}$$

By Theorem 1.4 and (3.22)–(3.24), we have

$$d_{\mathcal{H}_{3,\infty,p}}(\mathcal{L}(\mathbf{\Sigma}^{-1/2}\boldsymbol{W}),\mathcal{L}(\boldsymbol{Z}))\leq C_{\lambda,\kappa_0,K,p,d}\left(\frac{l^{d-1}\psi_n^3}{n^{d/2}}+\frac{l^d\psi_n^3}{n^{(d+2)/2}}+\frac{\psi_n^2}{n}+\frac{l^dA_n\psi_n^3}{n^{d/2}}+\frac{\psi_n^2}{l}\right).$$

Since the first three terms do not contribute to the rate, applying (3.20) to the last two terms in the parentheses, we obtain

$$l_0=\left(\frac{1}{d\psi_nA_n}\right)^{1/(d+1)}n^{d/(2d+2)},$$

which satisfies $1\leq l_0\leq n$ for the range of n given in (3.11), and substituting $l=\lfloor l_0\rfloor$ back into the bound yields the result. □

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References

[1] Ahlberg, J. H. and Nilson, E. N. (1963). Convergence properties of the spline fit. *J. Soc. Indust. Appl. Math.* **11,** 95–104.
[2] Banerjee, S. and Roy, A. (2014). *Linear Algebra and Matrix Analysis for Statistics*. CRC, Boca Raton, FL.
[3] Barbour, A. D. (1990). Stein's method for diffusion approximations. *Prob. Theory Relat. Fields* **84,** 297–332.
[4] Birkel, T. (1988). On the convergence rate in the central limit theorem for associated processes. *Ann. Prob.* **16,** 1685–1698.
[5] Bulinskii, A. (1996). Rate of convergence in the central limit theorem for fields of associated random variables. *Theory Prob. Appl.* **40,** 136–144.
[6] Cai, G.-H. and Wang, J.-F. (2009). Uniform bounds in normal approximation under negatively associated random fields. *Statist. Prob. Lett.* **79,** 215–222.
[7] Chen, L. H. Y., Goldstein, L. and Shao, Q.-M. (2011). *Normal Approximation by Stein's Method*. Springer, Heidelberg.
[8] Cox, J. T. and Grimmett, G. (1984). Central limit theorems for associated random variables and the percolation model. *Ann. Prob.* **12,** 514–528.
[9] Daly, F. (2013). Compound Poisson approximation with association or negative association via Stein's method. *Electron. Commun. Prob.* **18,** 30.
[10] Esary, J. D., Proschan F. and Walkup, D. W. (1967). Association of random variables, with applications. *Ann. Math. Statist.* **38,** 1466–1474.
[11] Goldstein, L. and Wiroonsri, N. (2018). Stein's method for positively associated random variables with applications to the Ising and voter models, bond percolation, and contact process. *Ann. Inst. H. Poincaré*. **54,** 385–421.
[12] Joag-Dev, K. and Proschan, F. (1983). Negative association of random variables, with applications. *Ann. Statist.* **11,** 286–295.

https://doi.org/10.1017/jpr.2018.13 Published online by Cambridge University Press

[13] Li, Y.-X. and Wang, J.-F. (2008). An application of Stein's method to limit theorems for pairwise negative quadrant-dependent random variables. *Metrika* **67,** 1–10.
[14] Newman, C. M. (1980). Normal fluctuations and the FKG inequalities. *Commun. Math. Phys.* **74,** 119–128.
[15] Rachev, S. T. (1985). The Monge–Kantorovich mass transference problem and its stochastic applications. *Theory Prob. Appl.* **29,** 647–676.
[16] Ross, N. (2011). Fundamentals of Stein's method. *Prob. Surv.* **8,** 210–293.
[17] Shevtsova, I. (2011). On the absolute constants in the Berry–Esseen type inequalities for identically distributed summands. Preprint. Available at https://arxiv.org/abs/1111.6554.
[18] Stein, C. (1972). A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proc. Sixth Berkeley Symp. Math. Statist. Prob.*, Vol. II, University of California Press, Berkeley, CA, pp. 583–602.

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