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TAIL INDEX OF AN AR(1) MODEL WITH ARCH(1) ERRORS

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Relevant sample quantities such as the sample autocorrelation function and extremes contain useful information about autoregressive time series with heteroskedastic errors. As these quantities usually depend on the tail index of the underlying heteroskedastic time series, estimating the tail index becomes an important task. Since the tail index of such a model is determined by a moment equation, one can estimate the underlying tail index by solving the sample moment equation with the unknown parameters being replaced by their quasi-maximum likelihood estimates. To construct a confidence interval for the tail index, one needs to estimate the complicated asymptotic variance of the tail index estimator, however. In this paper the asymptotic normality of the tail index estimator is first derived, and a profile empirical likelihood method to construct a confidence interval for the tail index is then proposed. A simulation study shows that the proposed empirical likelihood method works better than the bootstrap method in terms of coverage accuracy, especially when the process is nearly nonstationary.

1. INTRODUCTION

Consider a first-order autoregressive model (AR(1)) with autoregressive conditional heteroskedastic errors of order one (ARCH(1)) given by

$$Y_t = aY_{t-1} + \sqrt{\omega + bY_{t-1}^2}\,\varepsilon_t, \tag{1.1}$$

for $t = 1, \ldots, n$, where $\{\varepsilon_t\}$ is a sequence of independent and identically distributed random variables with zero mean and unit variance, $a \in \mathbb{R}$, $\omega > 0$ and

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$b > 0$. Because financial and econometric time series often exhibit certain stylized facts, such as heteroskedasticity and heavy tails (see Cont and Tankov (2004) for more details), model (1.1) was introduced by Weiss (1984) to capture the time-varying risk and fatter tails in modeling financial time series. Some particular studies on this model include Borkovec and Klüppelberg (2001), which showed that the sequence has heavy tails under Conditions H1 and H2, given below; Borkovec (2000, 2001), which studied the properties of extremes and sample autocovariance and sample autocorrelation functions; parameter estimation, pursued by Guo and Phillips (1998), Ling (2004), and Chan and Peng (2005); and Klüppelberg and Peng (2006), which proposed an empirical likelihood method to construct a confidence region for the parameters.

Assuming that $\{Y_t\}$ follows model (1.1), which satisfies two conditions,

Condition H1: $\mathbb{E}\log(|a+\sqrt{b}\varepsilon_1|) < 0$, and

Condition H2: ε_1 has a symmetric, positive and continuous Lebesgue density on $\mathbb{R}$.

Proposition 2 of Borkovec and Klüppelberg (2001) showed that there exists a unique constant $\alpha > 0$ such that

$$\mathbb{E}\left(\left|a+\sqrt{b}\varepsilon_1\right|^{\alpha}\right) = 1. \tag{1.2}$$

More technical explanations on Condition H2 can be found in Borkovec and Klüppelberg. Under some additional conditions on the density and the distribution tail of ε_i, according to Theorem 3 of Borkovec and Klüppelberg, there exists a constant $c > 0$ such that

$$P(Y_1 > x) = cx^{-\alpha}\{1+o(1)\} \quad \text{as } x \longrightarrow \infty, \tag{1.3}$$

that is, Y_1 has a regularly varying tail with index α.

The tail index α plays an important role in understanding properties of $\{Y_t\}$ such as extremes, sample autocovariance, and autocorrelation functions. Although α can be estimated by the Hill estimator for dependent data as shown in Drees (2000), it can only achieve the $\sqrt{k}$ convergence rate instead of $\sqrt{n}$, where $k = k(n) \longrightarrow \infty$, $k/n \longrightarrow 0$, and both the choice of k and the asymptotic variance depend on the underlying dependence structure of the sequence $\{Y_t\}$. Therefore, it is difficult to construct a confidence interval for the tail index based on the Hill estimator. In view of (1.2), one can simply estimate α by solving the equation

$$\frac{1}{n}\sum_{t=1}^{n}\left|\hat{a}+\sqrt{\hat{b}}\hat{\varepsilon}_t\right|^{\alpha} = 1, \tag{1.4}$$

where $\hat{\varepsilon}_t = (Y_t - \hat{a}Y_{t-1})/\sqrt{\hat{\omega}+\hat{b}Y_{t-1}^2}$ and $(\hat{\omega},\hat{a},\hat{b})^T$ is the quasi-maximum likelihood estimator (QMLE) of $(\omega,a,b)^T$; see Section 2 for details. Denote such an

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estimate by $\hat{\alpha}$. A similar estimator for the tail index of a generalized autoregressive conditional heteroskedasticity (GARCH(1,1)) sequence has been studied by Mikosch and Stărică (2000) and Berkes, Horváth, and Kokoszka (2003).

In this paper we first derive the asymptotic limit of $\hat{\alpha}$ with the $\sqrt{n}$ convergence rate under some regularity conditions. It turns out that the asymptotic variance is rather complicated. Based on the asymptotic distribution of $\hat{\alpha}$, an interval estimate of α can be constructed via substituting the asymptotic variance of $\hat{\alpha}$ by its estimate. To bypass estimating the complicated asymptotic variance, one can simply employ a bootstrap method to obtain critical points. It is known that the bootstrap method for time series is computationally intensive, however. Here we propose to employ the profile empirical likelihood method to construct a confidence interval for α, which avoids estimating additional quantities such as the asymptotic variance. Using the empirical likelihood for confidence interval is a popular procedure in nonparametric statistics. For an overview of empirical likelihood method and related studies, see Owen (2001) and, more recently, Chen and Van Keilegom (2009).

The paper is organized as follows. Section 2 presents the asymptotic limit of the proposed tail index estimator and the profile empirical likelihood method for constructing a confidence interval for the tail index. A simulation study is conducted in Section 3 to examine the finite sample behavior of the proposed method. All proofs are presented in Section 4.

2. METHODOLOGY AND MAIN RESULTS

To estimate the parameter $\theta = (\omega, a, b)^T$ in model (1.1), one finds θ that maximizes

$$L_n(\theta) = \sum_{t=1}^{n} \left\{ -\frac{\left(Y_t - aY_{t-1}\right)^2}{2\left(\omega + bY_{t-1}^2\right)} - \frac{1}{2}\log\left(\omega + bY_{t-1}^2\right) \right\},$$

where $L_n(\theta)$ takes the form of conditional Gaussian log-likelihood. Because ε_t may not be normal, the estimator $\hat{\theta}$ is usually known as the QMLE. Therefore, based on (1.2), the estimator $\hat{\alpha}$ for the tail index α is the solution to equation (1.4). Throughout, we use θ_0 and α_0 to denote the true value of θ and α, respectively.

Applying Taylor expansion to the difference between $n^{-1}\sum_{t=1}^{n} |\hat{a} + \sqrt{\hat{b}}\hat{\epsilon}_t|^{\alpha_0}$ and $n^{-1}\sum_{t=1}^{n} |a_0 + \sqrt{b_0}\epsilon_t|^{\alpha_0}$, we obtain the term $\mathbb{E}\frac{\partial}{\partial\theta}\left|a_0 + \frac{\sqrt{b_0}(Y_i - a_0 Y_{i-1})}{\sqrt{w_0 + b_0 Y_{i-1}^2}}\right|^{\alpha_0}$. In order to derive the asymptotic normality of $\hat{\alpha}$, we have to ensure that $\mathbb{E}||\frac{\partial}{\partial\theta}|a_0 + \frac{\sqrt{b_0}(Y_i - a_0 Y_{i-1})}{\sqrt{w_0 + b_0 Y_{i-1}^2}}|^{\alpha_0}|| < \infty$, which can be fulfilled by assuming that $\alpha_0 > 1$. The condition $\alpha_0 > 1$ also ensures that $P(|Y_1| > x) \sim 2cx^{-\alpha_0}$, as argued by Borkovec and Klüppelbrg (2001, p. 1232).

The following theorem shows the asymptotic normality of the proposed tail index estimator $\hat{\alpha}$.

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THEOREM 2.1. *Under Conditions H1, H2, and*

Condition H3: $\alpha_0 > 1$, $\mathbb{E}|\varepsilon_1|^{4\delta} < \infty$ *for some* $\delta > \max\{\alpha_0/2, 1\}$, $\sqrt{n}(\hat{\alpha} - \alpha_0) \xrightarrow{d} \mathrm{N}(0, d_0)$,

where d_0 equals the variance of the limit distribution given in (4.31) in Section 4.

Remark 2.1. Conditions H1 and H2 ensure the existence of a unique strict stationary solution of model (1.1); see, for example, Borkovec and Klüppelberg (2001). Condition $\alpha_0 > 1$ ensures $\mathbb{E}||\frac{\partial}{\partial\theta} Z_{t,1}(\theta_0, \alpha_0)|| < \infty$, where $Z_{t,1}(\theta_0, \alpha_0)$ is defined below. The requirement of $\mathbb{E}|\varepsilon_1|^{4\delta} < \infty$ is used to ensure that the joint asymptotic distribution of the QMLE and the estimator of α_0 is normal.

To construct a confidence interval for the tail index α, an obvious approach is to estimate the asymptotic variance d_0. Due to the complexity, one can simply employ a bootstrap method. Alternatively, we seek an empirical likelihood method to bypass estimating the asymptotic variance. A direct application of the empirical likelihood method to equation (1.4) cannot capture the variance of the plug-in estimator $\hat{\theta}$. However, the asymptotic variance d_0 of the tail index estimator $\hat{\alpha}$ really depends on the asymptotic variance of $\hat{\theta}$ as shown in the above theorem. Hence, the Wilks' theorem fails for the empirical likelihood method. Here we propose the profile empirical likelihood method.

Note that the QMLE $\hat{\theta}$ is a solution to the score equations

$$\sum_{t=1}^{n} Z_{t,j}(\theta) = 0 \quad \text{for } j = 2, 3, 4,$$

where

$$Z_{t,2}(\theta) = \tfrac{\partial}{\partial\omega}\left\{\left(Y_t - aY_{t-1}\right)^2 / \left(\omega + bY_{t-1}^2\right) + \log\left(\omega + bY_{t-1}^2\right)\right\},$$
$$Z_{t,3}(\theta) = \tfrac{\partial}{\partial a}\left\{\left(Y_t - aY_{t-1}\right)^2 / \left(\omega + bY_{t-1}^2\right) + \log\left(\omega + bY_{t-1}^2\right)\right\},$$
$$Z_{t,4}(\theta) = \tfrac{\partial}{\partial b}\left\{\left(Y_t - aY_{t-1}\right)^2 / \left(\omega + bY_{t-1}^2\right) + \log\left(\omega + bY_{t-1}^2\right)\right\}.$$

Hence, using (1.2), we can estimate θ and α simultaneously by solving the equations

$$\sum_{t=1}^{n} Z_{t,j} = 0 \quad \text{for } j = 1, 2, 3, 4, \tag{2.1}$$

where

$$Z_{t,1}(\theta, \alpha) = \left| a + \frac{\sqrt{b}(Y_t - aY_{t-1})}{\sqrt{\omega + bY_{t-1}^2}} \right|^{\alpha} - 1.$$

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This simultaneous estimation procedure motivates us to apply the empirical likelihood method to the above four equations and then profile the nuisance parameter θ. This is the so-called profile empirical likelihood method based on estimating equations proposed by Qin and Lawless (1994).

Specifically, based on equations (2.1), define the empirical likelihood function of $(\theta^T, \alpha)^T$ as

$$L(\theta,\alpha) = \sup\left\{\prod_{t=1}^{n}(np_t) : p_1 \geq 0, \ldots, p_n \geq 0, \sum_{t=1}^{n} p_t = 1, \sum_{t=1}^{n} p_t Z_t(\theta,\alpha) = 0\right\},$$

where $Z_t(\theta,\alpha) = (Z_{t,1}(\theta,\alpha), Z_{t,2}(\theta), Z_{t,3}(\theta), Z_{t,4}(\theta))^T$ for $t = 1,\ldots,n$. By virtue of the Lagrange multipliers, it is clear that $p_t = n^{-1}\{1+\lambda^T Z_t(\theta,\alpha)\}^{-1}$ for $t = 1,\ldots,n$ and

$$l(\theta,\alpha) := -2\log L(\theta,\alpha) = 2\sum_{t=1}^{n}\log\left\{1+\lambda^T Z_t(\theta,\alpha)\right\},$$

where $\lambda = \lambda(\theta,\alpha)$ satisfies

$$\sum_{t=1}^{n}\frac{Z_t(\theta,\alpha)}{1+\lambda^T Z_t(\theta,\alpha)} = 0. \tag{2.2}$$

Since we are interested in the tail index α, consider the profile empirical likelihood function $l_p(\alpha) = l(\tilde{\theta}(\alpha),\alpha)$, where $\tilde{\theta} = \tilde{\theta}(\alpha) := \arg\min_\theta l(\theta,\alpha)$. First we show the existence and consistency of $\tilde{\theta}(\alpha_0)$.

PROPOSITION 2.1. *Under conditions of Theorem 2.1, with probability tending to 1, $l(\theta,\alpha_0)$ attains its minimum value at some point $\tilde{\theta}$ in the interior of the ball $\Theta = \{\theta : ||\theta-\theta_0|| \leq Cn^{-1/(2\gamma)}\}$ for any given $\gamma \in (1,\delta)$ and $C > 0$, where δ is given in Theorem 2.1. Moreover $\tilde{\theta}$ and $\tilde{\lambda} = \tilde{\lambda}(\tilde{\theta},\alpha_0)$ satisfy*

$$Q_{1n}(\tilde{\theta},\tilde{\lambda}) = 0 \quad \textit{and} \quad Q_{2n}(\tilde{\theta},\tilde{\lambda}) = 0, \tag{2.3}$$

where

$$Q_{1n}(\theta,\lambda) = \frac{1}{n}\sum_{t=1}^{n}\frac{Z_t(\theta,\alpha_0)}{1+\lambda^T Z_t(\theta,\alpha_0)}$$

and

$$Q_{2n}(\theta,\lambda) = \frac{1}{n}\sum_{t=1}^{n}\frac{1}{1+\lambda^T Z_t(\theta,\alpha_0)}\left\{\frac{\partial}{\partial\theta}Z_t(\theta,\alpha_0)\right\}^T\lambda.$$

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The next theorem, Theorem 2.2, establishes the Wilks theorem for the proposed profile empirical likelihood method, which can then be used for interval estimation of the tail index as stipulated in Corollary 2.1.

THEOREM 2.2. *Under conditions of Theorem 2.1, the random variable* $l_p(\alpha_0)$, *with* $\tilde{\theta}$ *given in Proposition 2.1, converges in distribution to* $\chi^2(1)$ *as* $n \longrightarrow \infty$.

COROLLARY 2.1. *For any* $0 < \varsigma < 1$, *let* μ_ς *denote the* ς*th quantile of a* $\chi^2(1)$ *random variable and define the empirical likelihood confidence interval with level* ς *as* $I_\varsigma = \{\alpha | l_p(\alpha) \leq \mu_\varsigma\}$. *Then* $P(\alpha_0 \in I_\varsigma) \longrightarrow \varsigma$ *as* $n \longrightarrow \infty$.

3. SIMULATION STUDY

In this section we examine the finite sample behavior of the proposed profile empirical likelihood method and compare it with the naive bootstrap method in terms of coverage probabilities and interval lengths.

Consider model (1.1) with $(a, \omega, b) = (0, 1, 0.5)$, $(0, 1, 0.9)$, $(0.5, 1, 0.5)$, and $\varepsilon_t \sim N(0, 1)$ or $t(\nu)/\sqrt{\nu/(\nu-2)}$. Although the true value of the tail index α can be obtained via numerically solving the integration in (1.2), we draw 100,000 random samples of size $n = 10{,}000$ for the errors in (1.1) and solve equation (1.2) for each random sample with the right-hand side replaced by the sample average to obtain α, say $\tilde{\alpha}$. Then the true value α_0 is calculated as the average of these 100,000 $\tilde{\alpha}$'s.

Next we draw 1,000 random samples with size $n = 500$ or 2,000 from model (1.1). To compute the bootstrap confidence interval based on $\hat{\alpha}$, we draw $B = 200$ bootstrap samples with size n from $\hat{\varepsilon}_1, \ldots, \hat{\varepsilon}_n$ with replacement, say $\hat{\varepsilon}_1^*, \ldots, \hat{\varepsilon}_n^*$. Then we generate observations from model (1.1) with θ and ε_t's replaced by the QMLE and $\hat{\varepsilon}_t^*$'s, respectively. Based on these generated observations, the bootstrap version of $\hat{\alpha}$ is obtained. Hence, the percentile bootstrap confidence interval with level ς is obtained via the B bootstrapped estimators of α. Denote it by I_ς^b. In Table 1, we report the coverage probabilities of the proposed profile empirical likelihood interval I_ς and the percentile bootstrap interval I_ς^b at levels $\varsigma = 0.9$ and 0.95. The average interval lengths for both intervals are reported in Table 2. From these two tables, we observe that (i) the proposed profile empirical likelihood method performs much better than the percentile bootstrap method when $(a, \omega, b) = (0, 1, 0.9)$ and $(0.5, 1, 0.5)$, i.e., $a^2 + b$ is large; (ii) the proposed profile empirical likelihood method has a shorter interval length than the percentile bootstrap method when $n = 500$, but both methods have a similar interval length when $n = 2{,}000$; (iii) the bias of the tail index estimator $\hat{\alpha}$ becomes smaller when n is large. Moreover, the computation of the proposed empirical likelihood method is less intensive than the bootstrap method, which repeatedly estimates the model (1.1) and solves equation (1.4).

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TABLE 1. Empirical coverage probabilities of the proposed profile empirical likelihood interval I_ς and the percentile bootstrap interval I^b_ς at levels $\varsigma = 0.9$ and 0.95. The true value of the tail index α, the sample mean, and sample standard deviation of $\hat{\alpha}$ are, respectively, given in the last three columns of the table

$(n, a, \omega, b, \varepsilon_t)$	$I_{0.9}$	$I^b_{0.9}$	$I_{0.95}$	$I^b_{0.95}$	α_0	$\mathrm{E}(\hat{\alpha})$	$SD(\hat{\alpha})$
$(500, 0, 1, 0.5, \mathrm{N}(0, 1))$	0.891	0.896	0.934	0.925	4.735	5.175	1.572
$(500, 0, 1, 0.9, \mathrm{N}(0, 1))$	0.892	0.799	0.953	0.860	2.305	2.414	0.420
$(500, 0.5, 1, 0.5, \mathrm{N}(0, 1))$	0.893	0.867	0.956	0.896	2.989	3.129	0.599
$(500, 0, 1, 0.5, \frac{t(10)}{\sqrt{10/8}})$	0.850	0.892	0.908	0.921	4.013	4.444	1.237
$(500, 0, 1, 0.9, \frac{t(10)}{\sqrt{10/8}})$	0.874	0.810	0.925	0.866	2.255	2.380	0.413
$(500, 0.5, 1, 0.5, \frac{t(10)}{\sqrt{10/8}})$	0.894	0.863	0.937	0.880	2.881	3.029	0.593
(2,000,0,1,0.5,N(0,1))	0.915	0.919	0.952	0.952	4.735	4.803	0.513
(2,000,0,1,0.9,N(0,1))	0.916	0.820	0.953	0.860	2.305	2.321	0.183
(2,000,0.5,1,0.5, N(0,1))	0.905	0.875	0.947	0.929	2.989	3.025	0.271
$(2{,}000, 0, 1, 0.5, \frac{t(10)}{\sqrt{10/8}})$	0.877	0.906	0.932	0.940	4.013	4.106	0.485
$(2{,}000, 0, 1, 0.9, \frac{t(10)}{\sqrt{10/8}})$	0.878	0.768	0.936	0.820	2.255	2.281	0.200
$(2{,}000, 0.5, 1, 0.5, \frac{t(10)}{\sqrt{10/8}})$	0.868	0.859	0.923	0.908	2.881	2.917	0.305

TABLE 2. Average interval lengths of the proposed profile empirical likelihood interval I_ς and the percentile bootstrap interval I^b_ς at levels $\varsigma = 0.9$ and 0.95

$(n, a, \omega, b, \varepsilon_t)$	$I_{0.9}$	$I^b_{0.9}$	$I_{0.95}$	$I^b_{0.95}$
$(500, 0, 1, 0.5, \mathrm{N}(0, 1))$	3.974	4.876	4.615	6.468
$(500, 0, 1, 0.9, \mathrm{N}(0, 1))$	0.999	1.151	1.170	1.461
$(500, 0.5, 1, 0.5, \mathrm{N}(0, 1))$	1.917	1.924	2.308	2.419
$(500, 0, 1, 0.5, \frac{t(10)}{\sqrt{10/8}})$	3.345	4.596	3.896	6.136
$(500, 0, 1, 0.9, \frac{t(10)}{\sqrt{10/8}})$	0.930	1.207	1.086	1.524
$(500, 0.5, 1, 0.5, \frac{t(10)}{\sqrt{10/8}})$	1.895	2.028	2.273	2.543
(2,000,0,1,0.5,N(0,1))	1.701	1.666	2.041	2.065
(2,000,0,1,0.9,N(0,1))	0.506	0.517	0.576	0.635
(2,000,0.5,1,0.5, N(0,1))	0.871	0.795	1.045	1.007
$(2{,}000, 0, 1, 0.5, \frac{t(10)}{\sqrt{10/8}})$	1.442	1.446	1.722	1.804
$(2{,}000, 0, 1, 0.9, \frac{t(10)}{\sqrt{10/8}})$	0.465	0.506	0.524	0.632
$(2{,}000, 0.5, 1, 0.5, \frac{t(10)}{\sqrt{10/8}})$	0.895	0.819	1.072	1.036

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4. PROOFS

First, we establish several lemmas.

LEMMA 4.1. *Under conditions of Proposition 2.1, we have*

$$\sup_{1\le t\le n}\sup_{\theta\in\Theta}||Z_t(\theta,\alpha_0)|| = o_p\left(n^{1/2\gamma}\right). \tag{4.1}$$

Proof. Put

$$g(t,\theta) = (Z_{t,2}(\theta), Z_{t,3}(\theta), Z_{t,4}(\theta))^T.$$

Then it is easy to verify that

$$g(t,\theta) = \left(\omega + bY_{t-1}^2\right)^{-1}\left(1 - \frac{\left(Y_t - aY_{t-1}\right)^2}{\omega + bY_{t-1}^2}, -2\left(Y_t - aY_{t-1}\right)Y_{t-1},\right.$$
$$\left.\left\{1 - \frac{\left(Y_t - aY_{t-1}\right)^2}{\omega + bY_{t-1}^2}\right\}Y_{t-1}^2\right)^T. \tag{4.2}$$

Note that

$$\begin{aligned}
&1 - \frac{\left(Y_t - aY_{t-1}\right)^2}{\omega + bY_{t-1}^2}\\
&\quad = 1 - \frac{(Y_t - a_0Y_{t-1})^2}{\omega_0 + b_0Y_{t-1}^2} + \frac{(Y_t - a_0Y_{t-1})^2}{\omega_0 + b_0Y_{t-1}^2} - \frac{(Y_t - aY_{t-1})^2}{\omega + bY_{t-1}^2}\\
&\quad = 1 - \varepsilon_t^2 + \left[2(a-a_0)\sqrt{\omega_0 + b_0Y_{t-1}^2}\varepsilon_tY_{t-1} - (a-a_0)^2Y_{t-1}^2\right]\Big/\left(\omega_0 + b_0Y_{t-1}^2\right)\\
&\qquad + \left[(\omega - \omega_0) + (b - b_0)Y_{t-1}^2\right]\\
&\qquad \times \left[\sqrt{\omega_0 + b_0Y_{t-1}^2}\varepsilon_t + (a-a_0)Y_{t-1}\right]^2\Big/\left[\left(\omega + bY_{t-1}^2\right)\left(\omega_0 + b_0Y_{t-1}^2\right)\right]
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
(Y_t - aY_{t-1})Y_{t-1} &= \left[Y_t - a_0Y_{t-1} + (a_0 - a)Y_{t-1}\right]Y_{t-1}\\
&= \sqrt{\omega_0 + b_0Y_{t-1}^2}Y_{t-1}\varepsilon_t + (a_0 - a)Y_{t-1}^2.
\end{aligned} \tag{4.4}$$

By (4.3), we have for n large enough,

$$\sup_{\theta\in\Theta}|Z_{t,2}(\theta)| \le C_1\left(1 + \varepsilon_t^2 + n^{-1/2\gamma}|\varepsilon_t| + n^{-1/2\gamma}\right) \tag{4.5}$$

and

$$\sup_{\theta\in\Theta}|Z_{t,4}(\theta)| \le C_1\left(1 + \varepsilon_t^2 + n^{-1/2\gamma}|\varepsilon_t| + n^{-1/2\gamma}\right) \tag{4.6}$$

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for some $C_1 > 0$. Since $\mathbb{E}|\varepsilon_1|^{4\delta} < \infty$ and $\delta/\gamma > 1$, it follows from (4.5) and (4.6) that

$$\sup_{1\le t\le n}\sup_{\theta\in\Theta}|Z_{t,2}(\theta)| = o_p\left(n^{1/2\gamma}\right) \quad \text{and} \quad \sup_{1\le t\le n}\sup_{\theta\in\Theta}|Z_{t,4}(\theta)| = o_p\left(n^{1/2\gamma}\right).$$

Similarly, it follows from (4.4) that

$$\sup_{1\le t\le n}\sup_{\theta\in\Theta}|Z_{t,3}(\theta)| = o_p\left(n^{1/2\gamma}\right).$$

Hence,

$$\sup_{1\le t\le n}\sup_{\theta\in\Theta}||g(t,\theta)|| = o_p\left(n^{1/2\gamma}\right). \tag{4.7}$$

Observe that

$$\begin{aligned}
&\left|a+\sqrt{b}(Y_t-aY_{t-1})/\sqrt{\omega+bY_{t-1}^2}\right|^{\alpha_0}\\
&\quad=\left|a+\sqrt{b}\left(\frac{(Y_t-aY_{t-1})}{\sqrt{\omega+bY_{t-1}^2}}-\frac{(Y_t-a_0Y_{t-1})}{\sqrt{\omega_0+b_0Y_{t-1}^2}}\right)+\frac{\sqrt{b}(Y_t-a_0Y_{t-1})}{\sqrt{\omega_0+b_0Y_{t-1}^2}}\right|^{\alpha_0}\\
&\quad=\left|a+\sqrt{b}(Y_t-aY_{t-1})\left(\sqrt{\omega_0+b_0Y_{t-1}^2}-\sqrt{\omega+bY_{t-1}^2}\right)\Big/\left(\sqrt{\omega+bY_{t-1}^2}\sqrt{\omega_0+b_0Y_{t-1}^2}\right)\right.\\
&\qquad\left.+\sqrt{b}(a_0-a)Y_{t-1}/\sqrt{\omega_0+b_0Y_{t-1}^2}+\sqrt{b}\varepsilon_t\right|^{\alpha_0}\\
&\quad\le\left|a+\sqrt{b}\varepsilon_t\right|^{\alpha_0}+\left|\sqrt{b}\left(\sqrt{\omega_0+b_0Y_{t-1}^2}\varepsilon_t+(a_0-a)Y_{t-1}\right)\left[(\omega_0-\omega)+(b_0-b)Y_{t-1}^2\right]\Big/\right.\\
&\qquad\left.\times\left[\sqrt{\omega+bY_{t-1}^2}\sqrt{\omega_0+b_0Y_{t-1}^2}\left(\sqrt{\omega_0+b_0Y_{t-1}^2}+\sqrt{\omega+bY_{t-1}^2}\right)\right]\right|^{\alpha_0}\\
&\quad\le\left|a+\sqrt{b}\varepsilon_t\right|^{\alpha_0}+n^{-1/2\gamma}\left|\sqrt{b}\varepsilon_t/\left(\sqrt{\omega}\min\{\omega_0,b_0\}\right)|^{\alpha_0}+n^{-1/\gamma}\,|\varepsilon_t/\min\{\omega_0,b_0\}\right|^{\alpha_0}
\end{aligned} \tag{4.8}$$

uniformly for $t \ge 1$. Using (4.8) and $\mathbb{E}|\varepsilon_t|^{4\delta} < \infty$ for $\delta > \alpha_0$, we have

$$\sup_{1\le t\le n}\sup_{\theta\in\Theta}||a+\sqrt{b}\frac{Y_t-aY_{t-1}}{\sqrt{w+bY_{t-1}^2}\varepsilon_t}|^{\alpha_0}-1| = o_p\left(n^{1/2\gamma}\right). \tag{4.9}$$

Hence, the lemma follows from (4.7) and (4.9). ■

LEMMA 4.2. *Under conditions of Proposition 2.1, we have*

(i) $\sup_{\theta\in\Theta}\left\|\frac{1}{n}\sum_{t=1}^{n}\frac{\partial}{\partial\theta}Z_t(\theta,\alpha_0)-\mathbb{E}\left\{\frac{\partial}{\partial\theta}Z_1(\theta_0,\alpha_0)\right\}\right\| = o_p(1)$;

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(ii) $\sup_{\theta\in\Theta}\left\|\frac{1}{n}\sum_{t=1}^{n} Z_t(\theta,\alpha_0)Z_t^T(\theta,\alpha_0) - \mathbb{E}\{Z_1(\theta_0,\alpha_0)Z_1^T(\theta_0,\alpha_0)\}\right\| = o_p(1)$;

(iii) $\frac{1}{\sqrt{n}}\sum_{t=1}^{n} Z_t(\theta_0,\alpha_0) \xrightarrow{d} \mathrm{N}\left(0, \mathbb{E}\{Z_1(\theta_0,\alpha_0)Z_1^T(\theta_0,\alpha_0)\}\right)$.

Proof. Write

$$
\begin{aligned}
&\frac{\partial}{\partial\omega}Z_{t,1}(\theta,\alpha_0) - \frac{\partial}{\partial\omega}Z_{t,1}(\theta_0,\alpha_0)\\
&\quad= -\frac{\alpha_0\sqrt{b}(Y_t - aY_{t-1})}{2\left(\omega+bY_{t-1}^2\right)^{3/2}}\left|a+\frac{\sqrt{b}(Y_t-aY_{t-1})}{\sqrt{\omega+bY_{t-1}^2}}\right|^{\alpha_0-1}\operatorname{sgn}\left(a+\frac{\sqrt{b}(Y_t-aY_{t-1})}{\sqrt{\omega+bY_{t-1}^2}}\right)\\
&\qquad+\frac{\alpha_0\sqrt{b_0}(Y_t - a_0Y_{t-1})}{2\left(\omega_0+b_0Y_{t-1}^2\right)^{3/2}}\left|a_0+\frac{\sqrt{b_0}(Y_t-a_0Y_{t-1})}{\sqrt{\omega_0+b_0Y_{t-1}^2}}\right|^{\alpha_0-1}\operatorname{sgn}\left(a_0+\frac{\sqrt{b_0}(Y_t-a_0Y_{t-1})}{\sqrt{\omega_0+b_0Y_{t-1}^2}}\right)\\
&\quad= -\frac{\alpha_0\sqrt{b}(Y_t - aY_{t-1})}{2\left(\omega+bY_{t-1}^2\right)^{3/2}}\left\{\left|a+\frac{\sqrt{b}(Y_t-aY_{t-1})}{\sqrt{\omega+bY_{t-1}^2}}\right|^{\alpha_0-1}\operatorname{sgn}\left(a+\frac{\sqrt{b}(Y_t-aY_{t-1})}{\sqrt{\omega+bY_{t-1}^2}}\right)\right.\\
&\qquad\qquad\left.-\left|a_0+\frac{\sqrt{b_0}(Y_t-a_0Y_{t-1})}{\sqrt{\omega_0+b_0Y_{t-1}^2}}\right|^{\alpha_0-1}\operatorname{sgn}\left(a_0+\frac{\sqrt{b_0}(Y_t-a_0Y_{t-1})}{\sqrt{\omega_0+b_0Y_{t-1}^2}}\right)\right\}\\
&\qquad-\left(\frac{\alpha_0\sqrt{b}(Y_t - aY_{t-1})}{2\left(\omega+bY_{t-1}^2\right)^{3/2}}-\frac{\alpha_0\sqrt{b_0}(Y_t - a_0Y_{t-1})}{2\left(\omega_0+b_0Y_{t-1}^2\right)^{3/2}}\right)\\
&\qquad\times\left|a_0+\frac{\sqrt{b_0}(Y_t-a_0Y_{t-1})}{\sqrt{\omega_0+b_0Y_{t-1}^2}}\right|^{\alpha_0-1}\operatorname{sgn}\left(a_0+\frac{\sqrt{b_0}(Y_t-a_0Y_{t-1})}{\sqrt{\omega_0+b_0Y_{t-1}^2}}\right)\\
&\quad=: I_1(t,\theta)+I_2(t,\theta).
\end{aligned}
\tag{4.10}
$$

For $\theta=(\omega,a,b)^T$, define

$$
\begin{aligned}
d(\theta) &= (a-a_0)+\frac{\sqrt{b}(a_0-a)Y_{t-1}}{\sqrt{\omega+bY_{t-1}^2}}+\frac{(b-b_0)\varepsilon_t}{\sqrt{b}+\sqrt{b_0}}\\
&\quad+\frac{\sqrt{b}\varepsilon_t\left[\omega_0-\omega+(b_0-b)Y_{t-1}^2\right]}{\sqrt{\omega+bY_{t-1}^2}\left(\sqrt{\omega+bY_{t-1}^2}+\sqrt{\omega_0+b_0Y_{t-1}^2}\right)}.
\end{aligned}
$$

When $1<\alpha_0\le 2$, by the inequality that for any $x,y\in\mathbb{R}$

$$
\left||x+y|^{\alpha_0-1}\operatorname{sgn}(x+y)-|x|^{\alpha_0-1}\operatorname{sgn}(x)\right|\le 2|y|^{\alpha_0-1}
$$

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(see Lemma 4 of Arcones, 1996), we have for n large enough,

$$\begin{aligned}
\sup_{\theta\in\Theta}|I_1(t,\theta)| &\le \sup_{\theta\in\Theta}\frac{\alpha_0\sqrt{b}|Y_t-aY_{t-1}|}{\left(\omega+bY_{t-1}^2\right)^{3/2}}\left|d(\theta)\right|^{\alpha_0-1}\\
&= \sup_{\theta\in\Theta}\frac{\alpha_0\sqrt{b}\left(\omega_0+b_0Y_{t-1}^2\right)^{1/2}}{\left(\omega+bY_{t-1}^2\right)^{3/2}}\left|\varepsilon_t+\frac{(a_0-a)Y_{t-1}}{\sqrt{\omega_0+b_0Y_{t-1}^2}}\right|\\
&\quad\times\left|(a-a_0)+\frac{\sqrt{b}(a_0-a)Y_{t-1}}{\sqrt{\omega+bY_{t-1}^2}}+\frac{(b-b_0)\varepsilon_t}{\sqrt{b}+\sqrt{b_0}}\right.\\
&\qquad\left.+\frac{\sqrt{b}\varepsilon_t\left[\omega_0-\omega+(b_0-b)Y_{t-1}^2\right]}{\sqrt{\omega+bY_{t-1}^2}\left(\sqrt{\omega+bY_{t-1}^2}+\sqrt{\omega_0+b_0Y_{t-1}^2}\right)}\right|^{\alpha_0-1}\\
&\le C_2\left(|\varepsilon_t|+Cn^{-1/2\gamma}\right)\left[n^{-1/2\gamma}\left(1+|\varepsilon_t|\right)\right]^{\alpha_0-1}
\end{aligned}$$

for some $C_2>0$. When n is large and $\alpha_0>2$, we have

$$\begin{aligned}
\sup_{\theta\in\Theta}|I_1(t,\theta)| &\le \sup_{\theta\in\Theta}\frac{\alpha_0\sqrt{b}|Y_t-aY_{t-1}|}{2\left(\omega+bY_{t-1}^2\right)^{3/2}}\\
&\quad\times\left|a_0+\sqrt{b_0}\varepsilon_t+(a^*-a_0)+\frac{\sqrt{b^*}(a_0-a^*)Y_{t-1}}{\sqrt{\omega^*+b^*Y_{t-1}^2}}+\frac{(b^*-b_0)\varepsilon_t}{\sqrt{b^*}+\sqrt{b_0}}\right.\\
&\qquad\left.+\frac{\sqrt{b^*}\varepsilon_t\left[\omega_0-\omega^*+(b_0-b^*)Y_{t-1}^2\right]}{\sqrt{\omega^*+b^*Y_{t-1}^2}\left(\sqrt{\omega^*+b^*Y_{t-1}^2}+\sqrt{\omega_0+b_0Y_{t-1}^2}\right)}\right|^{\alpha_0-2}\\
&\quad\times\left|(a-a_0)+\frac{\sqrt{b}(a_0-a)Y_{t-1}}{\sqrt{\omega+bY_{t-1}^2}}+\frac{(b-b_0)\varepsilon_t}{\sqrt{b}+\sqrt{b_0}}\right.\\
&\qquad\left.+\frac{\sqrt{b}\varepsilon_t\left[\omega_0-\omega+(b_0-b)Y_{t-1}^2\right]}{\sqrt{\omega+bY_{t-1}^2}\left(\sqrt{\omega+bY_{t-1}^2}+\sqrt{\omega_0+b_0Y_{t-1}^2}\right)}\right|\\
&\le C_3\left(|\varepsilon_t|+Cn^{-1/2\gamma}\right)\left\{|a_0|+\sqrt{b_0}|\varepsilon_t|+n^{-1/2\gamma}\left(1+|\varepsilon_t|\right)\right\}^{\alpha_0-2}\\
&\quad\times n^{-1/2\gamma}\left(1+|\varepsilon_t|\right)
\end{aligned}$$

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for some $C_3 > 0$, where $\theta^* = (\omega^*, a^*, b^*)^T$ lies between θ_0 and θ. Hence, it follows from $\mathbb{E}|\varepsilon_1|^{2\alpha_0} < \infty$ that

$$\frac{1}{n}\sum_{t=1}^{n} \mathbb{E}\sup_{\theta\in\Theta} |I_1(t,\theta)| = o(1),$$

which implies that

$$\frac{1}{n}\sum_{t=1}^{n} \sup_{\theta\in\Theta} |I_1(t,\theta)| \xrightarrow{p} 0. \tag{4.11}$$

Similarly, when n is large, we have

$$\begin{aligned}
\sup_{\theta\in\Theta} |I_2(t,\theta)| &\le \sup_{\theta\in\Theta} \frac{\alpha_0}{2} \left| \frac{\sqrt{b}(a_0-a)Y_{t-1} + (\sqrt{b}-\sqrt{b_0})(Y_t - a_0 Y_{t-1})}{(\omega + bY_{t-1}^2)^{3/2}} \right. \\
&\qquad \left. - \frac{\sqrt{b_0}(Y_t - a_0 Y_{t-1})}{\sqrt{\omega_0 + b_0 Y_{t-1}^2}} \left(\frac{(\omega_0 + b_0 Y_{t-1}^2)^{1/2}}{(\omega + bY_{t-1}^2)^{3/2}} - \frac{1}{\omega_0 + b_0 Y_{t-1}^2} \right) \right| \\
&\qquad \times \left| a_0 + \frac{\sqrt{b_0}(Y_t - a_0 Y_{t-1})}{\sqrt{\omega_0 + b_0 Y_{t-1}^2}} \right|^{\alpha_0 - 1} \\
&\le \left[Cn^{-1/2\gamma} + Cn^{-1/2\gamma}|\varepsilon_t| + \sqrt{b_0}|\varepsilon_t| \sup_{\theta\in\Theta} (\omega_0 + b_0 Y_{t-1}^2)^{1/2} \right. \\
&\qquad \left. \times \left| \frac{1}{(\omega + bY_{t-1}^2)^{3/2}} - \frac{1}{(\omega_0 + b_0 Y_{t-1}^2)^{3/2}} \right| \right] \left| a_0 + \frac{\sqrt{b_0}(Y_t - a_0 Y_{t-1})}{\sqrt{\omega_0 + b_0 Y_{t-1}^2}} \right|^{\alpha_0 - 1} \\
&\le \left[Cn^{-1/2\gamma} + Cn^{-1/2\gamma}|\varepsilon_t| \right] \left| a_0 + \frac{\sqrt{b_0}(Y_t - a_0 Y_{t-1})}{\sqrt{\omega_0 + b_0 Y_{t-1}^2}} \right|^{\alpha_0 - 1},
\end{aligned}$$

which gives $\frac{1}{n}\sum_{t=1}^{n} \mathbb{E}\sup_{\theta\in\Theta} |I_2(t,\theta)| = o(1)$, i.e.,

$$\frac{1}{n}\sum_{t=1}^{n} \sup_{\theta\in\Theta} |I_2(t,\theta)| \xrightarrow{p} 0. \tag{4.12}$$

It follows from the law of large numbers that

$$\frac{1}{n}\sum_{t=1}^{n} \frac{\partial}{\partial\omega} Z_{t,1}(\theta_0, \alpha_0) \xrightarrow{p} \mathbb{E}\left\{ \frac{\partial}{\partial\omega} Z_{1,1}(\theta_0, \alpha_0) \right\}.$$

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Combining this with (4.10), (4.11), and (4.12) yields

$$\sup_{\theta\in\Theta}\left|\frac{1}{n}\sum_{t=1}^{n}\frac{\partial}{\partial\omega}Z_{t,1}(\theta,\alpha_0)-\mathbb{E}\left\{\frac{\partial}{\partial\omega}Z_{1,1}(\theta_0,\alpha_0)\right\}\right|=o_p(1). \tag{4.13}$$

Write

$$\begin{aligned}
&\frac{\partial}{\partial\omega}Z_{t,2}(\theta)-\frac{\partial}{\partial\omega}Z_{t,2}(\theta_0)\\
&\quad=3\left\{(Y_t-aY_{t-1})^2-(Y_t-a_0Y_{t-1})^2\right\}\left(\omega+bY_{t-1}^2\right)^{-4}\\
&\qquad+3(Y_t-a_0Y_{t-1})^2\left\{\left(\omega+bY_{t-1}^2\right)^{-4}-\left(\omega_0+b_0Y_{t-1}^2\right)^{-4}\right\}\\
&\qquad+\left(\omega_0+b_0Y_{t-1}^2\right)^{-2}-\left(\omega+bY_{t-1}^2\right)^{-2}\\
&\quad=:J_1(t,\theta)+J_2(t,\theta)+J_3(t,\theta).
\end{aligned} \tag{4.14}$$

It is easy to check that for n large enough,

$$\begin{aligned}
\sup_{\theta\in\Theta}|J_1(t,\theta)|&=\sup_{\theta\in\Theta}\frac{|(a_0-a)Y_{t-1}[2(Y_t-a_0Y_{t-1})+(a_0-a)Y_{t-1}]|}{\omega_0+b_0Y_{t-1}^2}\frac{\omega_0+b_0Y_{t-1}^2}{\left(\omega+bY_{t-1}^2\right)^4}\\
&\le Cn^{-1/2\gamma}\left(2b_0^{-1/2}|\varepsilon_t|+Cb_0^{-1}n^{-1/2\gamma}\right)\left(16\omega_0^{-3}\right).
\end{aligned} \tag{4.15}$$

Since $\mathbb{E}|\varepsilon_1|<\infty$, it follows from (4.15) that

$$\frac{1}{n}\sum_{t=1}^{n}\sup_{\theta\in\Theta}|J_1(t,\theta)|\xrightarrow{p}0. \tag{4.16}$$

For $J_2(t,\theta)$, there exists a constant $C_4>0$ such that for n large enough,

$$\begin{aligned}
\sup_{\theta\in\Theta}|J_2(t,\theta)|&=\sup_{\theta\in\Theta}\left|3\varepsilon_t^2\left(\omega+bY_{t-1}^2\right)^{-2}\left[\left(\omega+bY_{t-1}^2\right)^{-2}+(\omega_0+b_0Y_{t-1})^{-2}\right]\right.\\
&\qquad\left.\times\left[\omega+\omega_0+(b+b_0)Y_{t-1}^2\right]\left[\omega_0-\omega+(b_0-b)Y_{t-1}^2\right]\right|\\
&\le C_4n^{-1/2\gamma}\,\varepsilon_t^2.
\end{aligned} \tag{4.17}$$

Since $\mathbb{E}|\varepsilon_1|^2<\infty$, we have

$$\frac{1}{n}\sum_{t=1}^{n}\sup_{\theta\in\Theta}|J_2(t,\theta)|\xrightarrow{p}0. \tag{4.18}$$

Similarly,

$$\frac{1}{n}\sum_{t=1}^{n}\sup_{\theta\in\Theta}|J_3(t,\theta)|\xrightarrow{p}0. \tag{4.19}$$

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Thus, it follows from (4.16), (4.18), (4.19), and the law of large numbers that

$$\sup_{\theta\in\Theta}\left|\frac{1}{n}\sum_{t=1}^{n}\frac{\partial}{\partial\omega}Z_{t,2}(\theta)-\mathbb{E}\left\{\frac{\partial}{\partial\omega}Z_{1,2}(\theta_0)\right\}\right|=o_p(1). \tag{4.20}$$

Similar to the proofs of (4.13) and (4.20), we can show the convergence in probability of the other terms in $\frac{1}{n}\sum_{t=1}^{n}\frac{\partial}{\partial\theta}Z_t(\theta,\alpha_0)$, i.e., (i) holds.

In order to prove (ii), it suffices to show that

$$\sup_{\theta\in\Theta}\left|\frac{1}{n}\sum_{t=1}^{n}Z_{t,i}(\theta,\alpha_0)Z_{t,j}(\theta,\alpha_0)-\mathbb{E}\{Z_{1,i}(\theta_0,\alpha_0)Z_{1,j}(\theta_0,\alpha_0)\}\right|=o_p(1) \tag{4.21}$$

for $i,j=1,2,3,4$. Here we only show the case of $i=3$ and $j=4$, since the other cases can be proved in the same way. Write

$$\begin{aligned}
&Z_{t,3}(\theta,\alpha_0)Z_{t,4}(\theta,\alpha_0)-Z_{t,3}(\theta_0,\alpha_0)Z_{t,4}(\theta_0,\alpha_0)\\
&\quad=-2(a_0-a)Y_{t-1}^4\left\{1-(Y_t-aY_{t-1})^2/\left(\omega+bY_{t-1}^2\right)\right\}\left(\omega+bY_{t-1}^2\right)^{-2}\\
&\qquad-\frac{2Y_{t-1}^3(Y_t-a_0Y_{t-1})}{\left(\omega+bY_{t-1}^2\right)^2}\left\{\frac{(Y_t-a_0Y_{t-1})^2}{\omega_0+b_0Y_{t-1}^2}-\frac{(Y_t-aY_{t-1})^2}{\omega+bY_{t-1}^2}\right\}\\
&\qquad-2Y_{t-1}^3(Y_t-a_0Y_{t-1})\left(\varepsilon_t^2-1\right)\left\{\left(\omega+bY_{t-1}^2\right)^{-2}-\left(\omega_0+b_0Y_{t-1}^2\right)^{-2}\right\}\\
&\quad=:L_1(t,\theta)+L_2(t,\theta)+L_3(t,\theta).
\end{aligned} \tag{4.22}$$

For $L_1(t,\theta)$, there exists a constant $C_5>0$ such that for large n,

$$\begin{aligned}
&\sup_{\theta\in\Theta}|L_1(t,\theta)|\\
&\quad=\sup_{\theta\in\Theta}\left|\frac{2(a_0-a)Y_{t-1}^4}{\left(\omega+bY_{t-1}^2\right)^2}\left[1-\left(\varepsilon_t+\frac{(a_0-a)Y_{t-1}}{\sqrt{\omega_0+b_0Y_{t-1}^2}}\right)^2\left(\frac{\omega_0+b_0Y_{t-1}^2}{\omega+bY_{t-1}^2}\right)\right]\right|\\
&\quad\le C_5n^{-1/2\gamma}\left(1+2\varepsilon_t^2+2b_0^{-1}C^5n^{-1/\gamma}\right).
\end{aligned}$$

It follows from $\mathbb{E}\varepsilon_t^2<\infty$ that

$$\frac{1}{n}\sum_{t=1}^{n}\sup_{\theta\in\Theta}|L_1(t,\theta)|\xrightarrow{p}0. \tag{4.23}$$

Further, there exists a constant $C_6>0$ such that, for n large enough,

$$\begin{aligned}
&\sup_{\theta\in\Theta}|L_2(t,\theta)|\\
&\quad=\sup_{\theta\in\Theta}\left|\frac{2Y_{t-1}^3\varepsilon_t\sqrt{\omega_0+b_0Y_{t-1}^2}}{\left(\omega+bY_{t-1}^2\right)^2}\left\{\varepsilon_t^2-\left(\varepsilon_t+\frac{(a_0-a)Y_{t-1}}{\sqrt{\omega_0+b_0Y_{t-1}^2}}\right)^2\left(\frac{\omega_0+b_0Y_{t-1}^2}{\omega+bY_{t-1}^2}\right)\right\}\right|
\end{aligned}$$

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$$= \sup_{\theta\in\Theta}\left|\frac{2Y_{t-1}^3\varepsilon_t\sqrt{\omega_0+b_0Y_{t-1}^2}}{(\omega+bY_{t-1}^2)^2}\left\{\frac{\varepsilon_t^2\left(\omega-\omega_0+(b-b_0)Y_{t-1}^2\right)}{\omega+bY_{t-1}^2}\right.\right.$$
$$\left.\left.+\left(\frac{2(a_0-a)Y_{t-1}\varepsilon_t}{\sqrt{\omega_0+b_0Y_{t-1}^2}}-\frac{(a_0-a)^2Y_{t-1}^2}{\omega_0+b_0Y_{t-1}^2}\right)\frac{\omega_0+b_0Y_{t-1}^2}{\omega+bY_{t-1}^2}\right\}\right|$$
$$\le C_6n^{-1/2\gamma}\left(|\varepsilon_t|^3+\varepsilon_t^2+n^{-1/\gamma}|\varepsilon_t|\right),$$

which implies that $\frac{1}{n}\sum_{t=1}^n\mathbb{E}\{\sup_{\theta\in\Theta}|L_2(t,\theta)|\}\longrightarrow 0$. Hence,

$$\frac{1}{n}\sum_{t=1}^n\sup_{\theta\in\Theta}|L_2(t,\theta)|\xrightarrow{p}0. \tag{4.24}$$

Similarly, we can show that

$$\frac{1}{n}\sum_{t=1}^n\sup_{\theta\in\Theta}|L_3(t,\theta)|\xrightarrow{p}0. \tag{4.25}$$

It follows from (4.22)–(4.25) that (4.21) holds for the case of $i=3$ and $j=4$.

Next, we prove (iii). By the definition of $Z_{t,1}(\theta_0,\alpha_0)$ and (4.2), we have

$$Z_t(\theta_0,\alpha_0)=\left(\left|a_0+\sqrt{b_0}\varepsilon_t\right|^{\alpha_0}-1,\ \frac{1-\varepsilon_t^2}{\omega_0+b_0Y_{t-1}^2},\ \frac{-2\varepsilon_tY_{t-1}}{\sqrt{\omega_0+b_0Y_{t-1}^2}},\ \frac{\left(1-\varepsilon_t^2\right)Y_{t-1}^2}{\omega_0+b_0Y_{t-1}^2}\right)^T.$$

Since $Z_t(\theta_0,\alpha_0)$ is a martingale difference sequence, (iii) follows from the central limit theorem (CLT) for martingales (see Hall and Heyde, 1980). This completes the proof of Lemma 4.2. ■

Proof of Theorem 2.1. Let $g(t,\theta)$ be defined as in Lemma 4.1. Then $\frac{1}{n}\sum_{t=1}^n g\left(t,\hat{\theta}\right)=0$. By Taylor expansion, we have

$$0=\frac{1}{n}\sum_{t=1}^n g\left(t,\hat{\theta}\right)=\frac{1}{n}\sum_{t=1}^n g(t,\theta_0)+\frac{1}{n}\sum_{t=1}^n\frac{\partial}{\partial\theta}g(t,\theta^*)\left(\hat{\theta}-\theta_0\right),$$

where θ^* lies between θ_0 and $\hat{\theta}$. By Theorem 1 of Ling (2004), we have that under Conditions (H1)–(H3), $\hat{\theta}-\theta_0=O_p(n^{-1/2})$. Thus, it follows from Lemma 4.2 that

$$\hat{\theta}-\theta_0=\left\{\mathbb{E}\frac{\partial}{\partial\theta}g(1,\theta_0)\right\}^{-1}\frac{1}{n}\sum_{t=1}^n g(t,\theta_0)+o_p\left(n^{-1/2}\right). \tag{4.26}$$

By Taylor expansion, we have that for any $\alpha>1$,

$$\frac{1}{n}\sum_{t=1}^n\left\{\left|\hat{a}+\sqrt{\hat{b}}\hat{\varepsilon}_t\right|^\alpha-\left|a_0+\sqrt{b_0}\varepsilon_t\right|^\alpha\right\}$$
$$=\frac{1}{n}\sum_{t=1}^n\left\{\left|\hat{a}+\sqrt{\hat{b}}(Y_t-\hat{a}Y_{t-1})\Big/\sqrt{\hat{\omega}+\hat{b}Y_{t-1}^2}\right|^\alpha-\left|a_0+\sqrt{b_0}\varepsilon_t\right|^\alpha\right\}$$

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$$
\begin{aligned}
&= \frac{1}{n}\sum_{t=1}^{n} \alpha \left| a^* + \frac{\sqrt{b^*}(Y_t - a^* Y_{t-1})}{\sqrt{\omega^* + b^* Y_{t-1}^2}} \right|^{\alpha-1} \operatorname{sgn}\left(a^* + \frac{\sqrt{b^*}(Y_t - a^* Y_{t-1})}{\sqrt{\omega^* + b^* Y_{t-1}^2}} \right) \\
&\quad \times \left(\hat{a} + \frac{\sqrt{\hat{b}}(Y_t - \hat{a} Y_{t-1})}{\sqrt{\hat{\omega} + \hat{b} Y_{t-1}^2}} - a_0 - \sqrt{b_0}\varepsilon_t \right) \\
&= \frac{1}{n}\sum_{t=1}^{n} \alpha \left| a^* + \frac{\sqrt{b^*}(Y_t - a^* Y_{t-1})}{\sqrt{\omega^* + b^* Y_{t-1}^2}} \right|^{\alpha-1} \operatorname{sgn}\left(a^* + \frac{\sqrt{b^*}(Y_t - a^* Y_{t-1})}{\sqrt{\omega^* + b^* Y_{t-1}^2}} \right) \\
&\quad \times \left\{ \hat{a} - a_0 + \frac{(\hat{b} - b_0)\varepsilon_t}{\sqrt{\hat{b}} + \sqrt{b_0}} + \sqrt{\hat{b}}(Y_t - \hat{a} Y_{t-1}) \right. \\
&\qquad \left. \times \left(\frac{1}{\sqrt{\hat{\omega} + \hat{b} Y_{t-1}^2}} - \frac{1}{\sqrt{\omega_0 + b_0 Y_{t-1}^2}} \right) + \frac{\sqrt{\hat{b}}(a_0 - \hat{a}) Y_{t-1}}{\sqrt{\omega_0 + b_0 Y_{t-1}^2}} \right\} \\
&= \frac{1}{n}\sum_{t=1}^{n} \alpha \left| a^* + \frac{\sqrt{b^*}(Y_t - a^* Y_{t-1})}{\sqrt{\omega^* + b^* Y_{t-1}^2}} \right|^{\alpha-1} \operatorname{sgn}\left(a^* + \frac{\sqrt{b^*}(Y_t - a^* Y_{t-1})}{\sqrt{\omega^* + b^* Y_{t-1}^2}} \right) \\
&\quad \times \left[\hat{a} - a_0 + \frac{(\hat{b} - b_0)\varepsilon_t}{\sqrt{\hat{b}} + \sqrt{b_0}} + \frac{\sqrt{\hat{b}}(a_0 - \hat{a}) Y_{t-1}}{\sqrt{\omega_0 + b_0 Y_{t-1}^2}} \right. \\
&\qquad + \sqrt{\hat{b}}[Y_t - a_0 Y_{t-1} + (a_0 - \hat{a}) Y_{t-1}] \\
&\qquad \left. \times \left(\frac{(\omega_0 - \hat{\omega}) + (b_0 - \hat{b}) Y_{t-1}^2}{\sqrt{\hat{\omega} + \hat{b} Y_{t-1}^2}\sqrt{\omega_0 + b_0 Y_{t-1}^2}\left(\sqrt{\hat{\omega} + \hat{b} Y_{t-1}^2} + \sqrt{\omega_0 + b_0 Y_{t-1}^2} \right)} \right) \right],
\end{aligned}
\tag{4.27}
$$

where $(\omega^*, a^*, b^*)^T$ lies between $\hat{\theta}$ and θ_0. Since $\hat{\theta} - \theta_0 = O_p(n^{-1/2})$, using similar arguments in proving Lemma 4.2(i), uniformly for all $1 < \alpha \le 2\delta$, the right-hand side of (4.27) is

$$
\begin{aligned}
&\frac{1}{n}\sum_{t=1}^{n} \alpha \left| a_0 + \sqrt{b_0}\varepsilon_t \right|^{\alpha-1} \operatorname{sgn}\left(a_0 + \sqrt{b_0}\varepsilon_t \right) \\
&\quad \left\{ \hat{a} - a_0 + \frac{(\hat{b} - b_0)\varepsilon_t}{2\sqrt{b_0}} + \sqrt{b_0}(Y_t - a_0 Y_{t-1}) \left(\frac{(\omega_0 - \hat{\omega}) + (b_0 - \hat{b}) Y_{t-1}^2}{2(\omega_0 + b_0 Y_{t-1}^2)^{3/2}} \right) \right. \\
&\quad \left. + \frac{\sqrt{b_0}(a_0 - \hat{a}) Y_{t-1}}{\sqrt{\omega_0 + b_0 Y_{t-1}^2}} \right\} + o_p\left(n^{-1/2} \right)
\end{aligned}
$$

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$$= \frac{1}{n}\sum_{t=1}^{n} \alpha \left|a_0 + \sqrt{b_0}\varepsilon_t\right|^{\alpha-1} \operatorname{sgn}\left(a_0 + \sqrt{b_0}\varepsilon_t\right)$$
$$\times \left\{ \hat{a} - a_0 + \frac{\left(\hat{b} - b_0\right)\varepsilon_t}{2\sqrt{b_0}} + \sqrt{b_0}\varepsilon_t \left(\frac{(\omega_0 - \hat{\omega}) + \left(b_0 - \hat{b}\right) Y_{t-1}^2}{2\left(\omega_0 + b_0 Y_{t-1}^2\right)} \right) \right.$$
$$\left. + \frac{\sqrt{b_0}\left(a_0 - \hat{a}\right) Y_{t-1}}{\sqrt{\omega_0 + b_0 Y_{t-1}^2}} \right\} + o_p\left(n^{-1/2}\right). \tag{4.28}$$

Since $\mathbb{E}|\varepsilon_1|^{4\delta} < \infty$, it follows from the law of large numbers that

$$\frac{1}{n}\sum_{t=1}^{n} \alpha \left|a_0 + \sqrt{b_0}\varepsilon_t\right|^{\alpha-1} \operatorname{sgn}\left(a_0 + \sqrt{b_0}\varepsilon_t\right)$$
$$\times \left(1, \varepsilon_t, \frac{\sqrt{b_0}\varepsilon_t}{2\left(\omega_0 + b_0 Y_{t-1}^2\right)}, \frac{\sqrt{b_0}\varepsilon_t Y_{t-1}^2}{2\left(\omega_0 + b_0 Y_{t-1}^2\right)}, \frac{\sqrt{b_0} Y_{t-1}}{\sqrt{\omega_0 + b_0 Y_{t-1}^2}}\right)^T$$
$$\xrightarrow{p} \mathbb{E}\left[\left(\alpha \left|a_0 + \sqrt{b_0}\varepsilon_2\right|^{\alpha-1} \operatorname{sgn}\left(a_0 + \sqrt{b_0}\varepsilon_2\right)\right) \right.$$
$$\left. \times \left(1, \varepsilon_2, \frac{\sqrt{b_0}\varepsilon_2}{2\left(\omega_0 + b_0 Y_1^2\right)}, \frac{\sqrt{b_0}\varepsilon_2 Y_1^2}{2\left(\omega_0 + b_0 Y_1^2\right)}, \frac{\sqrt{b_0} Y_1}{\sqrt{\omega_0 + b_0 Y_1^2}}\right)^T \right]$$

holds uniformly for $1 < \alpha \le 2\delta$. Therefore,

$$\frac{1}{n}\sum_{t=1}^{n} \left(\left|\hat{a} + \sqrt{\hat{b}}\hat{\varepsilon}_t\right|^{\alpha} - \left|a_0 + \sqrt{b_0}\varepsilon_t\right|^{\alpha} \right)$$
$$= (\hat{a} - a_0)\mathbb{E}\left[\left(\alpha \left|a_0 + \sqrt{b_0}\varepsilon_2\right|^{\alpha-1} \operatorname{sgn}\left(a_0 + \sqrt{b_0}\varepsilon_2\right)\right) \left(1 - \frac{\sqrt{b_0} Y_1}{\sqrt{\omega_0 + b_0 Y_1^2}}\right) \right]$$
$$+ \left(\hat{b} - b_0\right)\mathbb{E}\left[\left(\alpha \left|a_0 + \sqrt{b_0}\varepsilon_2\right|^{\alpha-1} \operatorname{sgn}\left(a_0 + \sqrt{b_0}\varepsilon_2\right)\right) \left[\frac{\varepsilon_2}{2\sqrt{b_0}} - \frac{\sqrt{b_0}\varepsilon_2 Y_1^2}{2\left(\omega_0 + b_0 Y_1^2\right)} \right] \right]$$
$$+ (\omega_0 - \hat{\omega})\mathbb{E}\left[\frac{\sqrt{b_0}\left(\alpha \left|a_0 + \sqrt{b_0}\varepsilon_2\right|^{\alpha-1} \operatorname{sgn}\left(a_0 + \sqrt{b_0}\varepsilon_2\right)\right)\varepsilon_2}{2\left(\omega_0 + b_0 Y_1^2\right)} \right] + o_p\left(n^{-1/2}\right) \tag{4.29}$$

holds uniformly for $1 < \alpha \le 2\delta$. From (4.29), $|\hat{\theta} - \theta_0| = O_p\left(n^{-1/2}\right)$, and the law of large numbers, it follows that

$$\sup_{1<\alpha\le 2\delta} \left| \frac{1}{n}\sum_{t=1}^{n} \left|\hat{a} + \sqrt{\hat{b}}\hat{\varepsilon}_t\right|^{\alpha} - \mathbb{E}\left|a_0 + \sqrt{b_0}\varepsilon_1\right|^{\alpha} \right| = o_p(1).$$

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Since α_0 is the unique positive solution of $\mathbb{E}|a_0+b_0\varepsilon_1|^{\alpha}=1$, we have $\hat{\alpha}\xrightarrow{p}\alpha_0$. Thus, it follows from Taylor expansion and (4.29) that, under the set of $|\hat{\alpha}-\alpha_0|\le\nu$ for some $\nu>0$ small enough,

$$
\begin{aligned}
0&=\frac{1}{n}\sum_{t=1}^{n}\left|\hat{a}+\sqrt{\hat{b}}\hat{\varepsilon}_t\right|^{\hat{\alpha}}-1\\
&=\frac{1}{n}\sum_{t=1}^{n}\left(\left|\hat{a}+\sqrt{\hat{b}}\hat{\varepsilon}_t\right|^{\hat{\alpha}}-\left|a_0+\sqrt{b_0}\varepsilon_t\right|^{\hat{\alpha}}\right)+\frac{1}{n}\sum_{t=1}^{n}\left(\left|a_0+\sqrt{b_0}\varepsilon_t\right|^{\hat{\alpha}}-1\right)\\
&=\frac{1}{n}\sum_{t=1}^{n}\left(\left|\hat{a}+\sqrt{\hat{b}}\hat{\varepsilon}_t\right|^{\hat{\alpha}}-\left|a_0+\sqrt{b_0}\varepsilon_t\right|^{\hat{\alpha}}\right)+\frac{1}{n}\sum_{t=1}^{n}\left(\left|a_0+\sqrt{b_0}\varepsilon_t\right|^{\alpha_0}-1\right)\\
&\quad+\frac{1}{n}\sum_{t=1}^{n}\left|a_0+\sqrt{b_0}\varepsilon_t\right|^{\alpha_0}\left\{\log\left|a_0+\sqrt{b_0}\varepsilon_t\right|\right\}(\hat{\alpha}-\alpha_0)\\
&\quad+\frac{1}{2n}\sum_{t=1}^{n}\left|a_0+\sqrt{b_0}\varepsilon_t\right|^{\alpha^*}\left\{\log\left|a_0+\sqrt{b_0}\varepsilon_t\right|\right\}^2(\hat{\alpha}-\alpha_0)^2\\
&=(\hat{a}-a_0)\mathbb{E}\left[\left(\hat{\alpha}\left|a_0+\sqrt{b_0}\varepsilon_2\right|^{\hat{\alpha}-1}\operatorname{sgn}\left(a_0+\sqrt{b_0}\varepsilon_2\right)\right)\left(1-\frac{\sqrt{b_0}Y_1}{\sqrt{\omega_0+b_0Y_1^2}}\right)\right]\\
&\quad+(\hat{b}-b_0)\mathbb{E}\left[\left(\hat{\alpha}\left|a_0+\sqrt{b_0}\varepsilon_2\right|^{\hat{\alpha}-1}\operatorname{sgn}\left(a_0+\sqrt{b_0}\varepsilon_2\right)\right)\left[\frac{\varepsilon_2}{2\sqrt{b_0}}-\frac{\sqrt{b_0}\varepsilon_2Y_1^2}{2(\omega_0+b_0Y_1^2)}\right]\right]\\
&\quad+(\omega_0-\hat{\omega})\mathbb{E}\left[\frac{\sqrt{b_0}\left(\hat{\alpha}\left|a_0+\sqrt{b_0}\varepsilon_2\right|^{\hat{\alpha}-1}\operatorname{sgn}\left(a_0+\sqrt{b_0}\varepsilon_2\right)\right)\varepsilon_2}{2\left(\omega_0+b_0Y_1^2\right)}\right]\\
&\quad+\frac{1}{n}\sum_{t=1}^{n}\left(\left|a_0+\sqrt{b_0}\varepsilon_t\right|^{\alpha_0}-1\right)+\mathbb{E}\left(\left|a_0+\sqrt{b_0}\varepsilon_1\right|^{\alpha_0}\log\left|a_0+\sqrt{b_0}\varepsilon_1\right|\right)(\hat{\alpha}-\alpha_0)\\
&\quad+o_p\left(n^{-1/2}\right)\\
&=(\hat{a}-a_0)\mathbb{E}\left[\left(a_0|a_0+\sqrt{b_0}\varepsilon_2|^{\alpha_0-1}\operatorname{sgn}\left(a_0+\sqrt{b_0}\varepsilon_2\right)\right)\left(1-\frac{\sqrt{b_0}Y_1}{\sqrt{\omega_0+b_0Y_1^2}}\right)\right]\\
&\quad+(\hat{b}-b_0)\mathbb{E}\left[\left(a_0|a_0+\sqrt{b_0}\varepsilon_2|^{\alpha_0-1}\operatorname{sgn}(a_0+\sqrt{b_0}\varepsilon_2)\right)\left[\frac{\varepsilon_2}{2\sqrt{b_0}}-\frac{\sqrt{b_0}\varepsilon_2Y_1^2}{2(\omega_0+b_0Y_1^2)}\right]\right]\\
&\quad+(\omega_0-\hat{\omega})\mathbb{E}\left[\frac{\sqrt{b_0}(a_0|a_0+\sqrt{b_0}\varepsilon_2|^{\alpha_0-1}\operatorname{sgn}(a_0+\sqrt{b_0}\varepsilon_2))\varepsilon_2}{2(\omega_0+b_0Y_1^2)}\right]\\
&\quad+\frac{1}{n}\sum_{t=1}^{n}\left(|a_0+\sqrt{b_0}\varepsilon_t|^{\alpha_0}-1\right)+\mathbb{E}\left(|a_0+\sqrt{b_0}\varepsilon_1|^{\alpha_0}\log|a_0+\sqrt{b_0}\varepsilon_1|\right)\left(\hat{\alpha}-\alpha_0\right)\\
&\quad+o_p\left(n^{-1/2}\right),
\end{aligned}
\tag{4.30}
$$

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where α^* lies between $\hat{\alpha}$ and α_0. Put $e_0 = \mathbb{E}(|a_0+\sqrt{b_0}\varepsilon_1|^{\alpha_0}\log|a_0+\sqrt{b_0}\varepsilon_1|)$. By (4.30), we have, under the set of $|\hat{\alpha}-\alpha_0| \le \nu$ for some $\nu > 0$ small enough,

$$\begin{aligned}
&\sqrt{n}(\hat{\alpha}-\alpha_0)\\
&\quad= \sqrt{n}(\hat{\omega}-\omega_0)e_0^{-1}\mathbb{E}\left[\frac{\sqrt{b_0}\left(\alpha_0|a_0+\sqrt{b_0}\varepsilon_2|^{\alpha_0-1}\operatorname{sgn}\left(a_0+\sqrt{b_0}\varepsilon_2\right)\right)\varepsilon_2}{2\left(\omega_0+b_0Y_1^2\right)}\right]\\
&\qquad+\sqrt{n}(\hat{a}-a_0)e_0^{-1}\mathbb{E}\left[\left(\alpha_0\left|a_0+\sqrt{b_0}\varepsilon_2\right|^{\alpha_0-1}\operatorname{sgn}\left(a_0+\sqrt{b_0}\varepsilon_2\right)\right)\left(\frac{\sqrt{b_0}Y_1}{\sqrt{\omega_0+b_0Y_1^2}}-1\right)\right]\\
&\qquad+\sqrt{n}(\hat{b}-b_0)e_0^{-1}\mathbb{E}\left[\left(\alpha_0\left|a_0+\sqrt{b_0}\varepsilon_2\right|^{\alpha_0-1}\operatorname{sgn}\left(a_0+\sqrt{b_0}\varepsilon_2\right)\right)\left[\frac{\sqrt{b_0}\varepsilon_2Y_1^2}{2\left(\omega_0+b_0Y_1^2\right)}-\frac{\varepsilon_2}{2\sqrt{b_0}}\right]\right]\\
&\qquad-\frac{1}{e_0\sqrt{n}}\sum_{t=1}^{n}\left(\left|a_0+\sqrt{b_0}\varepsilon_t\right|^{\alpha_0}-\mathbb{E}\left|a_0+\sqrt{b_0}\varepsilon_1\right|^{\alpha_0}\right)+o_p(1). \qquad \textbf{(4.31)}
\end{aligned}$$

Thus, by (4.26), $\hat{\alpha} \xrightarrow{p} \alpha_0$, and the central limit theorem for martingales (see Hall and Heyde, 1980), the right-hand side of (4.31) converges in distribution to a Gaussian distribution with variance d_0, which is a function of θ_0. ■

Proof of Proposition 2.1. This can be shown in the same way as Lemma 1 of Qin and Lawless (1994) by using Lemmas 4.1 and 4.2. ■

Proof of Theorem 2.2. Using the same arguments in the proof of Theorem 1 of Qin and Lawless (1994), it follows from Lemmas 4.1 and 4.2 that

$$\begin{pmatrix}\tilde{\lambda}\\ \tilde{\theta}-\theta_0\end{pmatrix} = S_n^{-1}\begin{pmatrix}-Q_{1n}(\theta_0,0)+o_p\left(n^{-1/2}\right)\\ o_p\left(n^{-1/2}\right)\end{pmatrix},$$

where

$$\begin{aligned}
S_n &= \begin{pmatrix}\frac{\partial Q_{1n}(\theta_0,0)}{\partial\lambda} & \frac{\partial Q_{1n}(\theta_0,0)}{\partial\theta}\\ \frac{\partial Q_{2n}(\theta_0,0)}{\partial\lambda} & 0\end{pmatrix} \xrightarrow{p} \begin{pmatrix}S_{11} & S_{12}\\ S_{21} & 0\end{pmatrix}\\
&= \begin{pmatrix}-\mathbb{E}\left\{Z_1(\theta_0,\alpha_0)Z_1^T(\theta_0,\alpha_0)\right\} & \mathbb{E}\left\{\frac{\partial Z_1(\theta_0,\alpha_0)}{\partial\theta}\right\}\\ \mathbb{E}\left\{\frac{\partial Z_1(\theta_0,\alpha_0)}{\partial\theta}\right\}^T & 0\end{pmatrix}.
\end{aligned}$$

By the standard arguments of empirical likelihood method (see proof of Theorem 1 in Owen, 1990), it follows from Lemma 4.2 that

$$\begin{aligned}
l_p(\alpha_0) &= 2\sum_{t=1}^{n}\log\left\{1+\tilde{\lambda}^TZ_t\left(\tilde{\theta},\alpha_0\right)\right\}\\
&= 2n\left(\tilde{\lambda}^T,\tilde{\theta}^T-\theta_0^T\right)\left(Q_{1n}^T(\theta_0,0),0\right)^T\\
&\quad+n\left(\tilde{\lambda}^T,\tilde{\theta}^T-\theta_0^T\right)S_n\left(\tilde{\lambda}^T,\tilde{\theta}^T-\theta_0^T\right)^T+o_p(1)\\
&= -n\left(Q_{1n}^T(\theta_0,0),0\right)S_n^{-1}\left(Q_{1n}^T(\theta_0,0),0\right)^T+o_p(1)\\
&= -\left(W^T,0\right)\begin{pmatrix}S_{11} & S_{12}\\ S_{21} & 0\end{pmatrix}^{-1}\left(W^T,0\right)^T+o_p(1), \qquad \textbf{(4.32)}
\end{aligned}$$

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as $n \longrightarrow \infty$, where W is a multivariate normal random variable with mean zero and covariance matrix S_{11}. Since

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} S_{11}^{-1} - S_{11}^{-1} S_{12} \Delta^{-1} S_{21} S_{11}^{-1} & S_{11}^{-1} S_{12} \Delta^{-1} \\ \Delta^{-1} S_{21} S_{11}^{-1} & -\Delta^{-1} \end{pmatrix},$$

where $\Delta = S_{21} S_{11}^{-1} S_{12}$, we have

$$\begin{aligned} &-(W^T, 0) \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & 0 \end{pmatrix}^{-1} (W^T, 0)^T \\ &\quad = -W^T \left\{ S_{11}^{-1} - S_{11}^{-1} S_{12} \Delta^{-1} S_{21} S_{11}^{-1} \right\} W \\ &\quad = \left\{ (-S_{11})^{-1/2} W \right\}^T S_{11}^{1/2} \left\{ S_{11}^{-1} - S_{11}^{-1} S_{12} \Delta^{-1} S_{21} S_{11}^{-1} \right\} S_{11}^{1/2} \left\{ (-S_{11})^{-1/2} W \right\} \\ &\quad = \left\{ (-S_{11})^{-1/2} W \right\}^T \left\{ I - S_{11}^{-1/2} S_{12} \Delta^{-1} S_{21} S_{11}^{-1/2} \right\} \left\{ (-S_{11})^{-1/2} W \right\}. \end{aligned} \tag{4.33}$$

Since

$$\mathrm{tr}\left(S_{11}^{-1/2} S_{12} \Delta^{-1} S_{21} S_{11}^{-1/2} \right) = \mathrm{tr}\left(\Delta^{-1} S_{21} S_{11}^{-1} S_{12} \right) = 3,$$

we have $\mathrm{tr}\left(I - S_{11}^{-1/2} S_{12} \Delta^{-1} S_{21} S_{11}^{-1/2} \right) = 1$. Hence, it follows from (4.32) and (4.33) that $l_p(\alpha_0) \xrightarrow{d} \chi^2(1)$ as $n \longrightarrow \infty$. ■

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