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SOME REMARKS ON DISTRIBUTED SENSING WITH COSTS

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A recent paper by Pozdnyakov and Steele [6] is devoted to distributed sensor networks in which the sensors send summarized information to some remotely located fusion agents. Main focus is on sensors that send their information according to a so-called binary-plus-passive design. Some exact relations in a sequential approach, aimed at making decisions before a given budget is exhausted are presented. In this paper, we provide asymptotic results in the case of more general sensors for increasing budgets. The essential ingredient is the observation that the setting can be modeled within the framework of certain stopped two-dimensional random walks. We also provide asymptotics for the cost for large decision boundaries, and, finally, some comments on the probability of a positive/negative decision as well as on the duration of the process until a decision is made in the binary-plus-passive design.

1. INTRODUCTION

Following Pozdnyakov and Steele [6] we are concerned with distributed sensor networks in which the sensors, after having been interrogated by a remotely located fusion agent, return summarized information to the agent. This process continues until the fusion agent has sufficient evidence in order to support some given, positive or negative, decision.

Technically, let $X_1, X_2, \ldots$ be independent, identically distributed (i.i.d.) random variables that are interpreted as the signals or "information weights" coming from a sequence of queries that are put to the sensors, and $Y_1, Y_2, \ldots$, where $Y_k = c(X_k)$, $k \geq 1$, be the costs for collecting the queries, with X and Y as generic random variables. In Pozdnyakov and Steele [6], it is also assumed that the cost function $c(\cdot)$ is non-negative.

Finally, set $S_n = \sum_{k=1}^{n} X_k$ and $T_n = \sum_{k=1}^{n} Y_k$, $n \geq 1$, and let B be the budget at our disposal for the project. A decision is made as soon as the random walk $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$, hits either an upper level U or a lower level $-L$ (L, $U > 0$) — both of which may or may not depend on n — and a positive (negative) decision is made if the random walk hits the upper (lower) level.

Two central stopping times are

$$\tau_D(L, U) = \min\{n : S_n \notin (-L, U)\},$$
$$\tau_E(B) = \min\{n : T_n \geq B\},$$

that is, the decision time and the budget exhaustion time, respectively.

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In the first part of their paper Pozdnyakov and Steele [6] focus on a "binary-plus-passive design" in which case the sensor sends $+1$, -1 or nothing (0) to a fusion center, depending on the value of the weight of the observation data, and some exact relations are obtained. Section 5 contains (i.a.) some brief remarks on asymptotics for the budget exhaustion time for large budgets.

The main problems that are posed by Pozdnyakov and Steele [6] (in the binary-plus-passive setting) are

- $P(\tau_D(L,U) \le \tau_E(B))$, that is, the probability that a decision is made prior to exhaustion of the budget;
- $P(\tau_D(L,U) \le \tau_E(B)$ and $S_{\tau_D(L,U)} \ge U)$ and $P(\tau_D(L,U) \le \tau_E(B)$ and $S_{\tau_D(L,U)} \le -L)$, that is, the probability that the decision is positive or negative, respectively;
- $P(\tau_D(L,U) > \tau_E(B)$ and $S_{\tau_E(B)} \in A)$ for some $A \subset (-L,U)$, that is, "where the evidence stands when no decision has been made and yet the budget is exhausted" (Pozdnyakov and Steele [[6], page 528, formula (8)]).

The present paper is concerned with asymptotics for large budgets and for large decision levels. We weaken the restriction on the weights and costs in that we allow for arbitrary weights with finite expectation μ_X, and arbitrary costs with positive, finite, expectation μ_Y (which i.a. implies that costs may assume negative values with positive probability, and, hence, correspond to some kinds of profit or gain).

In Section 2, we present some key results from the theory of stopped two-dimensional random walks from Gut and Janson [5], Gut [2], after which, in Section 3, we "translate" the results into the current setting in order to provide limit theorems for the exhaustion time and the evidence at exhaustion time as the budget increases, and for the decision time and the cost at decision time for increasing boundaries.

The problem of making a decision before the budget is exhausted is dealt with in Section 4, after which we insert the parameters for the binary-plus-passive model into the strong laws in Section 5 — leaving the corresponding details for the central limit theorem to the reader(s). In the same section, we also include some remarks on the probability of a positive/negative decision as well as on the expected duration of the process, via an identification of our setting to that of a generalization of the gambler's ruin problem, more precisely, a *gambler's ruin problem with delays* (cf. Gut [4]).

We finally mention that we confine ourselves to strong laws and central limit theorems. Strong Marcinkiewicz–Zygmund laws, stable limits, and laws of the iterated logarithm can be stated and proved by the same approach by borrowing the appropriate tools from Gut and Janson [5], Gut [2].

2. STOPPED TWO-DIMENSIONAL RANDOM WALKS

In this section, we quote some results from Gut and Janson [5], Gut [2] that are the key to our proofs.

Let $\{(Z_n^{(1)}, Z_n^{(2)}), n \ge 1\}$ be a two-dimensional random walk with i.i.d. increments $(\xi_k^{(1)}, \xi_k^{(2)})$, $k \ge 1$, such that $\mu_2 = E\,\xi^{(2)} > 0$ and $\mu_1 = E\,\xi^{(1)}$ exist, finite. No independence assumptions are made between the components. Next, define $\mathcal{F}_n = \sigma\{(\xi_k^{(1)}, \xi_k^{(2)}) : k \le n\}$ for $n \ge 1$, the first passage time process

$$\tau(t) = \min\{n : Z_n^{(2)} > t\}, \quad t \ge 0,$$

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and the stopped random walk

$$\{Z^{(1)}_{\tau(t)}, \quad t \geq 0\}. \tag{2.1}$$

This process was introduced and examined in Gut and Janson [5]; see also Gut [[2], Chapter 4]. For our purpose we quote, for convenience, the strong law and the central limit theorem. We first note, however, that, since $\mu_2 > 0$, everything concerning renewal theory for random walks (cf. e.g. Gut [[2], Chapter 3]), applies to $\{\tau(t), t \geq 0\}$ as well as to $\{Z^{(2)}_{\tau(t)}, t \geq 0\}$.

THEOREM 2.1: (SLLN) $\dfrac{Z^{(1)}_{\tau(t)}}{t} \xrightarrow{a.s} \dfrac{\mu_1}{\mu_2}, \quad as \quad t \to \infty.$

THEOREM 2.2: (CLT) *Suppose, in addition, that* $\sigma_1^2 = \operatorname{Var} \xi^{(1)} < \infty$, $\sigma_2^2 = \operatorname{Var} \xi^{(2)} < \infty$ *and that*

$$v^2 = \operatorname{Var}(\mu_2 \xi^{(1)} - \mu_1 \xi^{(2)}) > 0.$$

Then

$$\frac{Z^{(1)}_{\tau(t)} - \frac{\mu_1}{\mu_2} t}{\sqrt{\mu_2^{-3} v^2 \, t}} \xrightarrow{d} N(0,1), \quad as \quad t \to \infty.$$

For proofs we refer to the above sources (or to Gut [3]).

3. ASYMPTOTICS FOR $\tau_E(B)$, $S_{\tau_E(B)}$, AND $T_{\tau_D(L,U)}$

The results in this section are immediate consequences of the known results from renewal theory for random walks (check Gut [[2], Chapter 3]) and the results in Section 2.

3.1. Asymptotics for the exhaustion time $\tau_E(B)$

As for the following results, cf. Gut [[2], Theorems 3.4.1 and 3.5.1].

THEOREM 3.1:

(i) $\dfrac{\tau_E(B)}{B} \xrightarrow{a.s} \dfrac{1}{\mu_Y} = \dfrac{1}{E\, c(X)}, \quad \text{as} \quad B \to \infty.$

(ii) *If, in addition,* $\operatorname{Var} Y = \sigma_Y^2 = \operatorname{Var} c(X) < \infty$, *then*

$$\frac{\tau_E(B) - B/\mu_Y}{\sqrt{\mu_Y^{-3} \sigma_Y^2 \, B}} \xrightarrow{d} N(0,1), \quad \text{as} \quad B \to \infty.$$

3.2. Asymptotics for the evidence $S_{\tau_E(B)}$ at exhaustion time

Here "evidence at exhaustion time" is to be interpreted as if there were no decision to be made, that is, the process is run until the budget is exhausted no matter what happens.

Adapting the model to the model of Section 2, we note that $\{(X_k, Y_k) = (X_k, c(X_k)), k \geq 1\}$ constitutes a sequence of i.i.d. two-dimensional random vectors with partial sums (S_n, T_n), $n \geq 1$. We also note that the individual components in each pair are *not* independent.

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With $\tau_E(B)$ corresponding to $\tau(t)$, the following result is immediate from Theorems 2.1, 2.2, and 3.1(i).

THEOREM 3.2:

(i) $\dfrac{S_{\tau_E(B)}}{B} \xrightarrow{a.s} \dfrac{\mu_X}{\mu_Y}, \quad \text{as} \quad B \to \infty.$

(ii) *If, in addition,* $0 < v^2 = \mathrm{Var}\,(\mu_Y X - \mu_X Y) = \mathrm{Var}\,\big(\mu_{c(X)} X - \mu_X c(X)\big) < \infty$, *then*

$$\frac{S_{\tau_E(B)} - B\mu_X/\mu_Y}{\sqrt{\mu_Y^{-3} v^2 B}} \xrightarrow{d} N(0,1), \quad \text{as} \quad B \to \infty.$$

3.3. Asymptotics for the cost $T_{\tau_D(L,U)}$ at decision time

The problem in this section might be viewed in the same way as in the previous one with the roles of cost and weight being interchanged, viz., we consider the two-dimensional random walk (T_n, S_n), $n \geq 1$, with i.i.d. increments $\{(Y_k, X_k),\ k \geq 1\}$.

There is, however, a technical complication here in that the stopping time $\tau_D(L,U)$ is two-sided. Also, the case with zero mean and non-zero mean have different behaviors. Here, we shall only consider the non-zero mean case, and assume, because of symmetry, that $\mu_X > 0$.

First of all, it is well known from the theory of sequential analysis that $\tau_D(L,U)$ has moments of all orders. Moreover, it follows from the strong law and the fact that $\tau_D(L,U) \to \infty$ as $L, U \to \infty$, that

$$\frac{T_{\tau_D(L,U)}}{\tau_D(L,U)} \xrightarrow{a.s} \mu_x \quad \text{and} \quad \frac{X_{\tau_D(L,U)}}{\tau_D(L,U)} \xrightarrow{a.s} 0, \quad \text{as} \quad L, U \to \infty,$$

(cf. e.g. Gut [[2], Theorem 1.2.3]). By arguing as in Gut [[2], Chapter 3], we then conclude from the latter that the overshoot (undershoot) is of the order $o(\tau_D(L,U))$, which, together with the fact that $\mu_X > 0$, implies that $U/\tau_D(L,U) \xrightarrow{a.s} \mu_X$ as $L, U \to \infty$. Turning this upside down proves (i) below. The other conclusions follow via Theorems 2.1 and 2.2.

THEOREM 3.3: *Suppose that* $\mu_X > 0$. *Then,*

(i) $\dfrac{\tau_D(L,U)}{U} \xrightarrow{a.s} \dfrac{1}{\mu_X}, \quad$ *as* $\quad L, U \to \infty.$

(ii) $\dfrac{T_{\tau_D(L,U)}}{U} \xrightarrow{a.s} \dfrac{\mu_Y}{\mu_X}, \quad$ *as* $\quad L, U \to \infty.$

(iii) *If, in addition,* $0 < v^2 = \mathrm{Var}\,(\mu_Y X - \mu_X Y) < \infty$, *then*

$$\frac{T_{\tau_D(L,U)} - U\mu_Y/\mu_X}{\sqrt{\mu_X^{-3} v^2 U}} \xrightarrow{d} N(0,1), \quad \text{as} \quad L, U \to \infty.$$

4. MAKING A DECISION BEFORE THE BUDGET IS OVERRUN

This happens whenever $\tau_D(L,U) < \tau_E(B)$. Combining Theorem 3.2 with the following asymptotics we are in the position to provide an asymptotic expression for the complementary event.

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The first observation is that, under the conditions of Theorem 3.2(ii), there exists a Donsker-type weak convergence analog (cf. Gut and Janson [[5], page 290] and/or Gut [[2], Theorem 5.4.1]), which, together with the continuous mapping theorem, tells us that

$$\max_{1\le k\le \tau_E(B)} \frac{S_k - k\mu_X/\mu_Y}{\sqrt{\mu_Y^{-3}v^2\,B}} \xrightarrow{d} \sup_{0\le t\le 1} W(t), \quad \text{as} \quad B\to\infty. \tag{4.1}$$

Secondly, we need the following well-known formulas for the supremum of the Wiener process:

PROPOSITION 4.1: *Let* $\{W(t),\, 0\le t\le 1\}$ *be a standard Wiener process. Then, for* $u<0<v$,

$$\begin{aligned} P(u<\inf_{0\le t\le 1} W(t) \le \sup_{0\le t\le 1} W(t) < v) \\ = \sum_{k=-\infty}^{\infty} (-1)^k P\big(u+k(v-u) \\ < W(1) < v+k(v-u)\big). \end{aligned}$$

In particular,

$$P\big(\sup_{0\le t\le 1} W(t) < v\big) = \sum_{k=-\infty}^{\infty} (-1)^k P\big((2k-1)v < W(1) < (2k+1)v\big).$$

For a proof, see for example Billingsley [[1], page 79].

Now, let $B,\, L,\, U\to\infty$, such that $L/\sqrt{B}\to c_*$ and $U/\sqrt{B}\to c^*$. Under the (simplifying) assumption that $\mu_X=0$ we then obtain, as $B\to\infty$, that

$$\begin{aligned} &P\big(\tau_E(B)<\tau_D(L,U)\big) = P\big(S_n\in(-L,U) \text{ for all } n\le\tau_E(B)\big) \\ &= P\big(-L<\min_{1\le n\le\tau_E(B)} S_n \le \max_{1\le n\le\tau_E(B)} S_n < U\big) \\ &= P\bigg(-\frac{L}{\sqrt{\mu_Y^{-1}\sigma_x^2\,B}} < \min_{1\le n\le\tau_E(B)} \frac{S_n}{\sqrt{\mu_Y^{-1}\sigma_x^2\,B}} \le \max_{1\le n\le\tau_E(B)} \frac{S_n}{\sqrt{\mu_Y^{-1}\sigma_x^2\,B}} < \frac{U}{\sqrt{\mu_Y^{-1}\sigma_x^2\,B}}\bigg) \\ &\to P\bigg(-\frac{c_*}{\sqrt{\mu_Y^{-1}\sigma_x^2}} < \inf_{0\le t\le 1} W(t) \le \sup_{0\le t\le 1} W(t) < \frac{c^*}{\sqrt{\mu_Y^{-1}\sigma_x^2}}\bigg) \\ &= \sum_{k=-\infty}^{\infty} (-1)^k P\bigg(-\frac{c_*}{\sqrt{\mu_Y^{-1}\sigma_x^2}} + \frac{k(c^*+c_*)}{\sqrt{\mu_Y^{-1}\sigma_x^2}} < W(1) < \frac{c^*}{\sqrt{\mu_Y^{-1}\sigma_x^2}} + \frac{k(c^*+c_*)}{\sqrt{\mu_Y^{-1}\sigma_x^2}}\bigg). \end{aligned}$$

The probability of making a decision before exhaustion of the budget thus equals 1 minus the above (since $P(\tau_D(L,U)=\tau_E(B))\to 0$ as $B\to\infty$).

The limit in the non-zero mean case can also be written down, but it creates difficulties because of the centering; recall formula (4.1).

5. THE BINARY-PLUS-PASSIVE DESIGN

As mentioned in the introduction, the first part and main focus of Pozdnyakov and Steele [6] is on the "binary-plus-passive design" in which case the sensor sends $+1$, -1 or nothing

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to the fusion center, depending on the value of the weight of the observation data. If the weight is "large" in some sense, the sensor transmits $+1$ to the fusion agent, if the weight is "small" it transmits -1 and nothing at all (which, *mathematically*, would correspond to transmitting a zero) for weights of intermediate size. A decision is made as soon as the random walk constituted by the sum of the weights hits either an upper level U or a lower level $-L$ ($L, U > 0$), after which a positive (negative) decision is made if the random walk hits the upper (lower) level.

In addition, one assumes that every transmission is subject to some cost $c(\cdot)$, and that the whole project is financed by some budget constraints.

For the key questions considered in Pozdnyakov and Steele [6], we refer back to the introduction.

Here is now the mathematical model: Let $X_1, X_2, \ldots$ be the i.i.d. weights, where

$$P(X = 1) = p, \quad P(X = -1) = q, \quad P(X = 0) = r \quad (0 \le p, q, r \le 1, \quad p + q + r = 1).$$

The cost function is defined as

$$y = c(x) = \begin{cases} 1, & \text{for} \quad x = \pm 1, \\ K^{-1}, & \text{for} \quad x = 0. \end{cases}$$

Clearly,

$$\mu_X = E\,X = p - q, \qquad \mu_Y = E\,Y = E\,c(X) = p + q + rK^{-1}.$$

If $r = 0$, the weights correspond to the steps in a random walk. If $r > 0$ the random walk stays put for a geometric number of time points after each step before continuing (where the probability of no stop equals $1 - r = p + q$). Pozdnyakov and Steele [6] call this a "lazy" random walk. We prefer, however to think of this as *a gambler's ruin problem with delays*, in view of the fact that we may interpret the probabilities of a positive (negative) decision as the respective probabilities of winning in a gambler's ruin problem in which the players, at times, pause for a geometric while.

As for limit theorems for large budgets we obtain the following result by inserting the parameters from this design into the various earlier strong laws; the first one is mentioned in Pozdnyakov and Steele [6], Section 5.1 (in a slightly weaker form).

COROLLARY 5.1: *For the binary-plus-passive design, such that costs have positive, finite mean, but otherwise are arbitrary, we have*

$$\frac{\tau_E(B)}{B} \xrightarrow{a.s} \frac{1}{p + q + rK^{-1}}, \quad \text{as} \quad B \to \infty,$$

$$\frac{S_{\tau_E(B)}}{B} \xrightarrow{a.s} \frac{p - q}{p + q + rK^{-1}}, \quad \text{as} \quad B \to \infty.$$

If, in addition, $p > q$, then

$$\frac{\tau_D(L, U)}{U} \xrightarrow{a.s} \frac{1}{p - q}, \quad \text{as} \quad U \to \infty,$$

$$\frac{T_{\tau_D(L,U)}}{U} \xrightarrow{a.s} \frac{p + q + rK^{-1}}{p - q}, \quad \text{as} \quad U \to \infty.$$

Central limit theorems may, of course, also be stated, by elementary, but tedious variance computations.

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Two problems that were not considered in Pozdnyakov and Steele [6] are

- the probability of a positive (negative) decision, and
- the duration of the process.

Omitting all financial considerations, the corresponding results from Gut [4] provide an immediate answer.

Corollary 5.2:

(i) *Let α be the probability of a positive decision. Then*

$$\alpha = P\left(\sum_{k=1}^{\tau_D(L,U)} Y_k = U\right) = \begin{cases} \dfrac{1-(p/q)^L}{(q/p)^U-(p/q)^L}, & \text{when} \quad p \neq q, \\ \dfrac{L}{U+L}, & \text{when} \quad p = q. \end{cases} \tag{5.1}$$

with $\beta = 1-\alpha$ being the probability of a negative decision.

(ii) *For the expected time until a decision, we have*

$$E\,\tau_D(L,U) = \begin{cases} \dfrac{1}{p^2-q^2}\Big(U-\big(U+L\big)\dfrac{1-(p/q)^L}{(q/p)^U-(p/q)^L}\Big), & \text{when} \quad p \neq q, \\ \dfrac{UL}{1-r}, & \text{when} \quad p = q. \end{cases} \tag{5.2}$$

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