Ergod. Th. & Dynam. Sys., page 1 of 19 © The Author(s), 2026. Published by Cambridge University Press.
doi:10.1017/etds.2026.10330

Unique Bernoulli Gibbs states and g-measures

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(*Received* 23 *April* 2025 *UTC and accepted in revised form* 12 *June* 2026 *UTC*)

Abstract. A sufficient condition for the Gibbs states of a shift-invariant specification on a one-dimensional lattice to be the g-chains for some continuous function g is obtained. This is then used to derive criteria under which there is a unique Gibbs state, which is also shift-invariant and Bernoulli.

Key words: Gibbs state, g-measure, Bernoulli
2020 Mathematics Subject Classification: 37A60 (Primary); 37A50, 60G10, 82B26 (Secondary)

1. *Introduction*

We consider Gibbs states on sequence spaces, that is, probability measures for which there is a specified set of functions that determine the probabilities of configurations on any finite subset conditioned on the configuration outside that subset. Let $X = S^I$ be a sequence space, where S is a finite set with the discrete topology, and I is a countable (infinite) set. The topology on X is the product topology, with respect to which X is compact, and $\mathcal{B}$ denotes the corresponding Borel σ-algebra. If $x \in S^{\Lambda'}$ ($\Lambda' \subseteq I$) and $\Lambda \subseteq \Lambda'$, then x_Λ denotes the natural projection of x onto S^Λ and $[x]_\Lambda$ is the cylinder set $\{y \in X : y_\Lambda = x_\Lambda\}$; similarly, if $s \in S$ and $i \in I$, $[s]_i$ denotes $\{y \in X : y_i = s\}$. The sub-σ-algebra of $\mathcal{B}$ generated by $\{[x]_\Lambda : x \in X\}$ is denoted by $\mathcal{B}_\Lambda$, and $\mathcal{C}$ denotes the set of finite, non-empty subsets of I.

A *specification* on X is a collection $\mathcal{V} = \{\phi_\Lambda\}_{\Lambda \in \mathcal{C}}$ of non-negative Borel functions on X such that, for all $\Lambda \in \mathcal{C}$ and $x \in X$,

$$\sum_{y \in [x]_{\Lambda^c}} \phi_\Lambda(y) = 1, \tag{1.1}$$

and for $K \subseteq \Lambda$,

$$\phi_\Lambda(x) = \phi_K(x) \sum_{y \in [x]_{K^c}} \phi_\Lambda(y). \tag{1.2}$$

https://doi.org/10.1017/etds.2026.10330 Published online by Cambridge University Press

The specification $\mathcal{V}$ is said to be continuous if ϕ_Λ is continuous for all $\Lambda \in \mathcal{C}$, and positive if $\phi_\Lambda > 0$ for all $\Lambda \in \mathcal{C}$. If $I = \mathbb{Z}^d$ $(d \geq 1)$, then $T = \{T_i\}_{i \in I}$ denotes the d-dimensional shift on X, that is, $(T_i x)_j = x_{i+j}$ $(x \in X,\ \ i, j \in \mathbb{Z}^d)$, and $\mathcal{V}$ is said to be T-invariant if $\phi_{i+\Lambda}(x) = \phi_\Lambda(T_i x)$ for all $i \in \mathbb{Z}^d$ and $\Lambda \in \mathcal{C}$. (As usual, when $d = 1$, we equate T with T_1.)

A probability measure μ on $(X, \mathcal{B})$ is a *Gibbs state* for a specification $\mathcal{V} = \{\phi_\Lambda\}_{\Lambda \in \mathcal{C}}$ if

$$\mu([x]_\Lambda \mid \mathcal{B}_{\Lambda^c})(x) = \phi_\Lambda(x) \quad \text{a.e. } (\mu)$$

for all $\Lambda \in \mathcal{C}$ (by a.e., we mean almost everywhere). The set of Gibbs states is denoted by $\mathcal{G}(\mathcal{V})$.

If $\mathcal{V}$ is continuous, there exists at least one Gibbs state, and natural questions concern the structure of $\mathcal{G}(\mathcal{V})$, in particular, uniqueness and any ergodic or mixing properties of Gibbs states. In general, $\mathcal{G}(\mathcal{V})$ is a compact, convex subset of $M(X)$ (the set of probability measures on $(X, \mathcal{B})$), and its extreme points are those states for which the tail σ-field $\bigcap_{\Lambda \in \mathcal{C}} \mathcal{B}_{\Lambda^c}$ is trivial. If, in addition, $I = \mathbb{Z}^d$ and $\mathcal{V}$ is T-invariant, then at least one of the Gibbs states is T-invariant; the T-invariant Gibbs states form a convex subset of $\mathcal{G}(V)$, the extreme points of which are the ergodic ones. Thus, a unique T-invariant Gibbs state μ is necessarily ergodic, but if it is also the only Gibbs state, then it follows from the triviality of the tail field that (T, μ) has some strong mixing properties. However, from the dynamical systems perspective, the system with the strongest mixing properties is one that is isomorphic to, and hence indistinguishable from, a Bernoulli shift (by which we mean the invertible dynamical system generated by a sequence of independent identically distributed random variables taking values in a finite set). While it is known that for those specifications that are attractive (or ferromagnetic), uniqueness of the Gibbs state implies the Bernoulli property (see [1, 12, 13]), in general, this is not the case.

Dobrushin [7] showed that if $\mathcal{V} = \{\phi_\Lambda\}_{\Lambda \in \mathcal{C}}$ is a continuous specification such that

$$\sup_{i \in I} \sum_{s \in S} \sum_{j \neq i} \sup_{\substack{x_{\{j\}^c} = y_{\{j\}^c} \\ x_i = y_i = s}} |\phi_{\{i\}}(x) - \phi_{\{i\}}(y)| < 2,$$

then $\mathcal{V}$ has a unique Gibbs state μ. This result holds for all countable I and does not depend on I having a group structure. If $I = \mathbb{Z}^d$ (in fact, if I is any amenable group) and $\mathcal{V}$ is T-invariant, then μ is T-invariant and the corresponding dynamical system (T, μ) is Bernoulli (see [13]).

Of particular interest in statistical mechanics are those specifications determined by an interaction potential $\Phi = \{\Phi_\Lambda\}_{\Lambda \in \mathcal{C}}$, that is, a set of functions on X such that Φ_Λ is $\mathcal{B}_\Lambda$-measurable. The corresponding specification $\mathcal{V}_\Phi$ is given by

$$\phi_\Lambda(x) = \exp(-H_\Lambda^\Phi(x)) \left\{ \sum_{y \in [x]_{\Lambda^c}} \exp(-H_\Lambda^\Phi(y)) \right\}^{-1}, \tag{1.3}$$

where

$$H_\Lambda^\Phi(x) = \sum_{\{K \in \mathcal{C} : K \cap \Lambda \neq \emptyset\}} \Phi_K(x).$$

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Usually, Φ is required to satisfy a summability condition that ensures H_Λ^Φ is continuous, and hence that $\mathcal{V}_\Phi$ is continuous and positive. When $I = \mathbb{Z}^d$, the interaction potential Φ is said to be T-invariant if $\Phi_{\Lambda+i}(x) = \Phi_\Lambda(T_i x)$ for all $\Lambda \in \mathcal{C}$ and $i \in \mathbb{Z}^d$, in which case, $\mathcal{V}_\Phi$ is also T-invariant. An example that has been studied extensively is the well-known Ising model on $\{-1, 1\}^{\mathbb{Z}^d}$, where $\Phi_\Lambda \equiv 0$ for $|\Lambda| \neq 2$. In particular, we note that in dimensions $d \geq 2$, the nearest-neighbour model (where $\Phi_{\{i,j\}} \equiv 0$ unless $|i - j| = 1$) gives examples of non-unique Gibbs states. When $d = 1$, the situation is quite different. Here, the nearest-neighbour model has a unique Gibbs state; more generally, if $\Phi_{\{i,j\}}(x) = -J(|i - j|)x_i x_j$, where $\{J(n)\}$ is a non-negative sequence, there is a unique Gibbs state provided $\sum_{i=1}^n i J(i) = o(\sqrt{\log n})$ [17]. In contrast, if $J(n) = \beta n^{-\alpha}$, there is a phase transition (that is, non-uniqueness) for large enough β when $1 < \alpha \leq 2$ [9]. (A more thorough survey of such results can be found in [11].)

For potentials in general, Ruelle [19] showed that when $I = \mathbb{Z}$ and $\Phi = \{\Phi_\Lambda\}_{\Lambda \in \mathcal{C}}$ is T-invariant, there is a unique T-invariant Gibbs state provided

$$\sum_{\Lambda \ni 0} \operatorname{diam}(\Lambda) \|\Phi_\Lambda\| < \infty. \tag{1.4}$$

Gallavotti [10] and Ledrappier [16] independently proved it has the Bernoulli property. Coelho and Quas [6] pointed out that by combining Ruelle's characterization [18] of the T-invariant Gibbs states of Φ as equilibrium states of the function

$$A_\Phi(x) = \sum_{\min \Lambda = 0} \Phi_\Lambda(x)$$

with the results of Walters [21], (1.4) could be replaced by the weaker hypothesis

$$\sum_{\min \Lambda = 0} \operatorname{diam}(\Lambda) \|\Phi_\Lambda\| < \infty. \tag{1.5}$$

Berbee [3] considered continuous specifications $\{\phi_\Lambda\}_{\Lambda \in \mathcal{C}}$ determined as follows. There is a sequence $\{\Lambda_n\}_{n=0}^\infty \subset \mathcal{C}$ such that $\Lambda_n \nearrow I$ and positive $\mathcal{B}_{\Lambda_{n-1}^c}$-measurable functions $f_n \in C(X)$ $(n \geq 0)$ (with $\Lambda_{-1} = \emptyset$) such that

$$\phi_{\Lambda_n}(x) = \Big(\prod_{j=0}^n f_j(x) \Big) \Big(\sum_{y \in [x]_{\Lambda_n^c}} \prod_{j=0}^n f_j(y) \Big)^{-1} \quad (n \geq 0,\ x \in X).$$

Any positive continuous specification can be described in this way (see §2). For a positive function f on X and $\Lambda \subseteq I$, let

$$r_\Lambda(f) = \sup\{f(x)/f(y) : x_\Lambda = y_\Lambda\}, \quad \bar{r}_\Lambda(f) = \inf\{f(x)/f(y) : x_\Lambda = y_\Lambda\}.$$

(Note that $\bar{r}_\Lambda(f) = r_\Lambda(f)^{-1}$.) Berbee showed that if

$$\sum_{n=0}^\infty \prod_{k=0}^n \inf_{i \geq 0} \bar{r}_{\Lambda_{i+k}}(f_i) = \infty, \tag{1.6}$$

then there is a unique Gibbs state. Like Dobrushin's result, this holds for any countable I and for non-invariant, as well as invariant, specifications. Taking $I = \mathbb{Z}$ and

$$\Lambda_{2k} = [-k, k], \quad \Lambda_{2k+1} = [-k, k+1] \quad (k \geq 0) \tag{1.7}$$

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(here and later, we use the usual notation for an interval $\mathcal{I} \subseteq \mathbb{R}$ to denote $\mathcal{I} \cap \mathbb{Z}$), and comparing hypotheses for the relevant f_n for an interaction potential (2.7), it can be seen that Berbee's hypothesis is weaker than Ruelle's (1.4); however, while in some cases it is weaker than that of Coelho and Quas (1.5), in others, it is not.

The hypotheses (1.4)–(1.5) can be viewed as measures of the continuity of $A(\Phi)$ in terms of its variation on sites outside finite subsets, or equivalently, in terms of the rate at which $\|\phi_\Lambda\|$ decreases as $\mathrm{diam}(\Lambda)$ increases. Similarly, (1.6) is a measure of the continuity of the functions f_n, which, in the case of an interaction potential Φ with the f_n defined by (2.7), is also related to the continuity of A_Φ. In this paper, we consider whether this type of continuity hypothesis can be relaxed. We restrict our attention to the case $I = \mathbb{Z}$ since (1.4) and (1.5) are only valid in this case, and while Berbee's result holds for any I, in the case of T-invariant specifications, it would seem that (1.6) rules out any genuinely multi-dimensional examples. Indeed, as has already been noted, the nearest-neighbour Ising model provides examples of non-uniqueness in dimensions greater than one. From hereon, X denotes $S^{\mathbb{Z}}$.

As Dobrushin [7] pointed out, any positive specification $\mathcal{V} = \{\phi_\Lambda\}_{\Lambda \in \mathcal{C}}$ is completely determined by $\{\phi_{\{n\}} : n \in \mathbb{Z}\}$; consequently, if $\mathcal{V}$ is also T-invariant, it is determined by $\phi_{\{0\}}$ alone. Thus, it seems natural to seek conditions for uniqueness and Bernoullicity in terms of $\phi_{\{0\}}$, or equivalently, the ratios $\{\phi_{\{0\}}(s_0 x)/\phi_{\{0\}}(t_0 x) : s, t \in S,\ x \in X\}$, where $s_n x \in X$ ($s \in S,\ x \in X,\ n \in \mathbb{Z}$) is defined by

$$(s_n x)_i = \begin{cases} s, & i = n, \\ x_i, & i \neq n. \end{cases}$$

We show the following theorem.

THEOREM 1.1. *Let $\mathcal{V} = \{\phi_\Lambda\}_{\Lambda \in \mathcal{C}}$ be a positive, continuous, T-invariant specification on X, and let*

$$f(x) = \phi_{\{0\}}(x)/\phi_{\{0\}}(s_0 x) \quad (x \in X)$$

for some $s \in S$. If, for some $0 < \alpha \leq 1$,

$$\prod_{i=0}^{n} r_{(-\infty,i]}(f) = O(n^{1-\alpha})$$

and

$$\sum_{i=2^n}^{2^{n+1}} (\log r_{[-i,\infty)}(f))^{2\alpha} = o(1),$$

then $\mathcal{V}$ has a unique Gibbs state μ, which is T-invariant, and (T, μ) is Bernoulli.

In particular, there is a unique, Bernoulli Gibbs state if

$$\max\{\log r_{(-\infty,n]}(f),\ \log r_{[-n,\infty)}(f)\} \leq \beta n^{-1} + a_n \quad (n \geq 1)$$

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for some $\beta < 1/2$ and non-negative series $\sum a_n < \infty$. In fact, we prove Theorem 1.1 under slightly weaker hypotheses (Theorem 5.1), which allows the following version to be deduced when the specification is given in terms of an interaction potential.

THEOREM 1.2. *Let $\Phi = \{\Phi_\Lambda\}_{\Lambda \in \mathcal{C}}$ be a T-invariant interaction potential on X. If*

$$\sum_{\substack{\Lambda \ni 0 \\ \Lambda \cap [n,\infty) \neq \emptyset}} \operatorname{var}(\Phi_\Lambda) \leq \beta n^{-1} + a_n \quad (n \geq 1) \tag{1.8}$$

for some $\beta < 1/2$ and $a_n \geq 0$ such that $\sum a_n < \infty$, then Φ has a unique Gibbs state μ, which is T-invariant, and (T, μ) is Bernoulli.

This includes all interaction potentials satisfying (1.4). In some cases, such as when $\sup\{|\Lambda| : \Phi_\Lambda \not\equiv 0\} < \infty$, (1.8) is also weaker than (1.5), but in others, it is not. Comparison with (1.6) is less straightforward, since (1.6) applies to non-invariant specifications and arbitrary increasing sequences $\{\Lambda_n\}_{n=0}^{\infty} \subset \mathcal{C}$. Consideration of the Ising model suggests Theorems 1.1 and 1.2 can give better results for uniqueness in some cases, but the main difference is that the Bernoulli property is obtained.

The idea of the proof of Theorem 5.1 is to characterize a Gibbs state μ in terms of its conditional probabilities

$$g(x) = \mu([x]_{\{0\}} \mid \mathcal{B}_{(-\infty,-1]})(x) \quad (x \in X),$$

in other words, as a g-chain; uniqueness, T-invariance and the Bernoulli property can then be determined from similar results for g-chains. Here, we use a result (Theorem 4.5) that is a formulation based on the techniques of Johansson, Öberg and Pollicott [**15**]. The estimate of the variation of $\log g$ needed for Theorem 4.5 is provided by Theorem 4.2. This result may be of some interest in its own right as it gives a sufficient condition for the set of Gibbs states of a T-invariant specification to be the set of g-chains for some continuous function g. (A characterization of g-chains which are also Gibbs states, as well as a discussion of the problem of determining whether a Gibbs state is a g-chain, can be found in [**5**].) Theorem 4.2 follows from Proposition 4.1, which gives a general criterion for the set of Gibbs states to be the same as the set of g-chains for some continuous function g, and Proposition 3.1, which is an adaptation of Berbee's method of ratio bounds in [**3**].

Lastly, we consider some alternative hypotheses which guarantee a unique T-invariant Gibbs state, but not the Bernoulli property.

2. *Specifications and Gibbs states*

We start with some preliminary remarks about specifications and Gibbs states; a more detailed account can be found in [**11**], for example. The following notation will be used throughout: if $x \in S^{\Lambda_1}$, $y \in S^{\Lambda_2}$ are sequences and $\Lambda_1 \cap \Lambda_2 = \emptyset$, then $xy \in S^{\Lambda_1 \cup \Lambda_2}$ denotes the sequence such that

$$(xy)_i = \begin{cases} x_i, & i \in \Lambda_1, \\ y_i, & i \in \Lambda_2. \end{cases}$$

The set of bounded Borel functions on X is denoted by $B(X)$.

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Let $X = S^{\mathbb{Z}}$ be a sequence space as described in the introduction, and let $\mathcal{V} = \{\phi_\Lambda\}_{\Lambda\in\mathcal{C}}$ be a specification on X. Then $\mathcal{V}$ determines a set of probability measures $\pi^{\mathcal{V}}_\Lambda(\,\cdot \mid x)$ $(\Lambda \in \mathcal{C},\ x \in X)$ as follows:

$$\pi^{\mathcal{V}}_\Lambda(f \mid x) = \sum_{y\in[x]_\Lambda} \phi_\Lambda(y) f(y) \quad (f \in C(X)),$$

and $\mu \in \mathcal{G}(\mathcal{V})$ if and only if

$$\mu([x]_\Lambda \mid \mathcal{B}_{\Lambda^c})(x) = \pi^{\mathcal{V}}_\Lambda([x]_\Lambda \mid x) \quad \text{a.e. } (\mu) \quad (\Lambda \in \mathcal{C}).$$

Thus, if $\mathcal{V}$ is continuous, $\mathcal{G}(\mathcal{V})$ is closed (in the weak-* topology), and moreover, any weak-* limit of measures of the form

$$\pi^{\mathcal{V}}_{\Lambda_n}(\,\cdot \mid x^{(n)}) \quad (x^{(n)} \in X,\ \Lambda_n \nearrow \mathbb{Z})$$

is a Gibbs state for $\mathcal{V}$. Since $M(X)$ is compact, it follows that there is at least one Gibbs state μ and the uniqueness of μ is equivalent to the uniform convergence (for each $f \in C(X)$) of $\pi^{\mathcal{V}}_\Lambda(f \mid x)$ to $\mu(f)$ as $\Lambda \nearrow \mathbb{Z}$.

Similarly, if $\mathcal{V}$ is T-invariant, any weak-* limit of measures of the form

$$\mu_n(f) = n^{-1} \sum_{i=0}^{n-1} \pi^{\mathcal{V}}_{[0,n-1]}(fT^i \mid x^{(n)}) \quad (f \in C(X),\ x^{(n)} \in X)$$

is a T-invariant Gibbs state for $\mathcal{V}$. By compactness, there is always one such measure μ and its uniqueness is equivalent to

$$\lim_{n\to\infty} n^{-1} \sum_{i=0}^{n-1} \pi^{\mathcal{V}}_{[0,n-1]}(fT^i \mid x) = \mu(f)$$

uniformly in x for all $f \in C(X)$.

Note that if ϕ_Λ is given for some $\Lambda \in \mathcal{C}$, then (1.2) determines ϕ_K for all $K \subseteq \Lambda$. Moreover, if (1.2) holds for the pairs K, Λ and K', K, where $K' \subseteq K \subseteq \Lambda$, then it holds for K', Λ. Thus, to define a specification, it is enough to consider a sequence $\{\Lambda_n\}_{n=0}^\infty \subset \mathcal{C}$ such that $\Lambda_n \nearrow \mathbb{Z}$, and specify the ϕ_{Λ_n} so that (1.1) and (1.2) hold for Λ_{n-1}, Λ_n $(n \geq 1)$.

Let $\mathcal{V}$ be positive; note that for positive specifications, (1.2) is equivalent to

$$\frac{\phi_K(x)}{\phi_K(x^*_K x_{K^c})} = \frac{\phi_\Lambda(x)}{\phi_\Lambda(x^*_K x_{K^c})} \quad (K \subseteq \Lambda,\ x, x^* \in X). \tag{2.1}$$

If $\Lambda \in \mathcal{C}$ and $\{i_0, \ldots, i_n\}$ is an enumeration of Λ, with $\Lambda_j = \{i_0, \ldots, i_j\}$ $(0 \leq j \leq n)$, then for any $x, x^* \in X$, we have

$$\begin{aligned}\frac{\phi_\Lambda(x)}{\phi_\Lambda(x^*_\Lambda x_{\Lambda^c})} &= \frac{\phi_\Lambda(x)}{\phi_\Lambda(x^*_{\Lambda_{n-1}} x_{\Lambda^c_{n-1}})} \cdot \frac{\phi_\Lambda(x^*_{\Lambda_{n-1}} x_{\Lambda^c_{n-1}})}{\phi_\Lambda(x^*_\Lambda x_{\Lambda^c})} \\ &= \frac{\phi_{\Lambda_{n-1}}(x)}{\phi_{\Lambda_{n-1}}(x^*_{\Lambda_{n-1}} x_{\Lambda^c_{n-1}})} \cdot \frac{\phi_{\{i_n\}}(x^*_{\Lambda_{n-1}} x_{\Lambda^c_{n-1}})}{\phi_{\{i_n\}}(x^*_\Lambda x_{\Lambda^c})} \qquad \text{(by (2.1))},\end{aligned}$$

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and so it follows that

$$\frac{\phi_\Lambda(x)}{\phi_\Lambda(x^*_\Lambda x_{\Lambda^c})} = \prod_{j=0}^{n} \frac{\phi_{\{i_j\}}(x^*_{\Lambda_{j-1}} x_{\Lambda^c_{j-1}})}{\phi_{\{i_j\}}(x^*_{\Lambda_j} x_{\Lambda^c_j})} \quad (x, x^* \in X,\ n \geq 0). \tag{2.2}$$

(Here and later, Λ_{-1} is taken to be $\emptyset$.) This, together with (1.1), determines ϕ_Λ. In particular, any positive specification $\mathcal{V} = \{\phi_\Lambda\}_{\Lambda \in \mathcal{C}}$ is completely determined by the $\phi_{\{n\}}$ ($n \in \mathbb{Z}$). Moreover, $\mathcal{V}$ is T-invariant provided $\phi_{\{n\}} = \phi_{\{0\}} \circ T^n$ for all $n \in \mathbb{Z}$, in which case, it is determined solely by $\phi_{\{0\}}$.

Conversely, given a sequence of positive functions $\{\phi_{\{n\}}\}_{n\in\mathbb{Z}} \subset B(X)$ such that $\sum_{y\in[x]_{\{n\}}} \phi_{\{n\}}(y) = 1$ for all $x \in X$, and $\{\Lambda_n\}_{n=0}^\infty \subset \mathcal{C}$ such that $\Lambda_n \nearrow \mathbb{Z}$ and $|\Lambda_n \setminus \Lambda_{n-1}| = 1$ ($n \geq 0$), define ϕ_{Λ_n} by (1.1) and

$$\frac{\phi_{\Lambda_n}(x)}{\phi_{\Lambda_n}(x^*_{\Lambda_n} x_{\Lambda^c_n})} = \prod_{j=0}^{n} \frac{\phi_{\{i_j\}}(x^*_{\Lambda_{j-1}} x_{\Lambda^c_{j-1}})}{\phi_{\{i_j\}}(x^*_{\Lambda_j} x_{\Lambda^c_j})} \quad (x, x^* \in X,\ n \geq 1),$$

where $\{i_n\} = \Lambda_n \setminus \Lambda_{n-1}$. Clearly, the ϕ_{Λ_n} satisfy (2.1) and so determine a specification, which is independent of the choice of $\{\Lambda_n\}$ since a specification is determined solely by the $\phi_{\{n\}}$. Thus, any sequence of positive functions $\{\phi_{\{n\}}\}_{n\in\mathbb{Z}} \subset B(X)$ such that $\sum_{y\in[x]_{\{n\}}} \phi_{\{n\}}(y) = 1$ for all $x \in X$ determines a unique positive specification. In particular, any positive $f \in B(X)$ determines a (unique) T-invariant specification $\mathcal{V} = \{\phi_\Lambda\}_{\Lambda\in\mathcal{C}}$ such that

$$\phi_{\{n\}}(x) = \frac{f(T^n x)}{\sum_{y\in[T^n x]_{\{0\}}} f(y)} \quad (x \in X,\ n \in \mathbb{Z}).$$

By a *specification sequence*, we mean a sequence $\{\Lambda_n, f_n\}_{n=0}^\infty \subset \mathcal{C} \times B(X)$ such that $\Lambda_{n-1} \subset \Lambda_n$, $|\Lambda_n \setminus \Lambda_{n-1}| = 1$, and f_n is positive and $\mathcal{B}_{\Lambda^c_{n-1}}$-measurable; it is a specification sequence for the specification $\mathcal{V} = \{\phi_\Lambda\}_{\Lambda\in\mathcal{C}}$ if

$$\phi_{\Lambda_n}(x) = \Big(\prod_{j=0}^{n} f_j(x)\Big)\Big(\sum_{y\in[x]_{\Lambda^c_n}} \prod_{j=0}^{n} f_j(y)\Big)^{-1} \quad (n \geq 0,\ x \in X). \tag{2.3}$$

If $\mathcal{V}$ is positive, such a sequence exists for any choice of $\{\Lambda_n\}$. Indeed, let $x^* \in X$ and define

$$f_n(x) = \frac{\phi_{\{i_n\}}(x^*_{\Lambda_{n-1}} x_{\Lambda^c_{n-1}})}{\phi_{\{i_n\}}(x^*_{\Lambda_n} x_{\Lambda^c_n})} \quad (x \in X,\ n \geq 0), \tag{2.4}$$

where $\Lambda_n \setminus \Lambda_{n-1} = \{i_n\}$. Then f_n is suitably measurable, and it follows from (2.2) that (2.3) holds. If $\mathcal{V}$ is continuous, so are the f_n, and if $\mathcal{V}$ is also T-invariant, then a consequence of (2.4) is that if

$$f(x) = \phi_{\{0\}}(x)/\phi_{\{0\}}(s_0 x) \quad (x \in X) \tag{2.5}$$

for some $s \in S$, then a specification sequence $\{[0, n],\ f_n\}_{n=0}^\infty$ can be chosen so that $f_0 = f$ and

$$r_{[-n,i+j]}(f_j) = r_{[-n-j,i]}(f_0) \leq r_{[-n,i]}(f) \quad (i, j, n \geq 0). \tag{2.6}$$

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Conversely, any specification sequence $\{\Lambda_n, f_n\}_{n=0}^{\infty}$ such that $\Lambda_n \nearrow \mathbb{Z}$ determines a (unique) specification $\mathcal{V} = \{\phi_\Lambda\}_{\Lambda\in\mathcal{C}}$ via (2.3), since (2.1) holds for the pairs Λ_{n-1}, Λ_n.

Any positive, continuous, T-invariant specification $\mathcal{V}$ can be determined by a T-invariant interaction potential $\Phi = \{\Phi_\Lambda\}_{\Lambda\in\mathcal{C}}$ such that for some $s \in S$,

$$\sup_{x\in X} \sum_{\Lambda\in\mathcal{C},\ \Lambda\ni 0} |\Phi_\Lambda(x) - \Phi_\Lambda(s_0 x)| < \infty$$

(see [20] for details). If $\sum_{\Lambda\ni 0} \|\Phi_\Lambda\| < \infty$, the functions

$$f_n(x) = \exp\left(- \sum_{\substack{\Lambda\ni n \\ \Lambda\cap[0,n-1]=\emptyset}} \Phi_\Lambda(x)\right) \quad (n \geq 0,\ x \in X) \tag{2.7}$$

determine a specification sequence $\{[0, n], f_n\}_{n=0}^{\infty}$ for $\mathcal{V}$, and since Φ is T-invariant,

$$\max\{\log r_{(-\infty,i+n]}(f_i),\ \log r_{[-n,\infty)}(f_i)\} \leq \sum_{\substack{\Lambda\cap[n+1,\infty)\neq\emptyset \\ 0\in\Lambda}} \mathrm{var}(\Phi_\Lambda) \quad (i, n \geq 0). \tag{2.8}$$

3. *A ratio bound*

Given a specification sequence $\{[0, n], f_n\}_{n=0}^{\infty}$, define $L_{m,n}\colon B(X) \to B(X)$ $(0 \leq m \leq n)$ by

$$L_{m,n} f(x) = \sum_{y\in[x]_{[m,n]^c}} \prod_{i=m}^{n} f_i(y) f(y) \quad (f \in B(X)),$$

and let

$$\rho_k^{(n)}(x, y) = \inf\left\{ \frac{L_{m,m+n}\mathbf{1}[\zeta](x)}{L_{m,m+n}\mathbf{1}[\zeta](y)} : \zeta \in S^{[m,m+k]},\ m \geq 0\right\},$$
$$\rho_{-1}^{(n)}(x, y) = \inf\left\{ \frac{L_{m,m+n}1(x)}{L_{m,m+n}1(y)} : m \geq 0\right\} \quad (k, n \geq 0,\ x, y \in X).$$

In addition to the usual notation for sets of integers, we denote the sets of negative and non-positive integers by $\mathbb{N}^-$ and $\mathbb{Z}^-$, respectively.

PROPOSITION 3.1. *Let $\{[0, n], f_n\}_{n=0}^{\infty}$ be a specification sequence, $A \subseteq \mathbb{N}^-$, and*

$$v_k = \inf_{i\geq 0} \bar{r}_{A\cup[0,i+k]}(f_i) \quad (k \geq 0).$$

Then

$$\liminf_{n\to\infty} \inf_{x_A=y_A} \rho_0^{(n)}(x, y)\rho_0^{(n)}(y, x) \geq \lim_{n\to\infty} \left(\sum_{k=0}^{n} \prod_{j=0}^{k} v_j\right)^2 \left(1 + \sum_{k=0}^{n} \prod_{j=0}^{k} v_j\right)^{-2}.$$

We split the proof of Proposition 3.1 into two lemmas. The proof of the first one is based on that of [3, Proposition 2.1].

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LEMMA 3.2. *Let* $\{f_n\}_{n=0}^{\infty} \subset B(X)$ *and* A *be as above, and let* $x, y \in X$ *be such that* $x_A = y_A$. *If* $\{v_k\}_{k=0}^{\infty}$ *is an increasing, positive sequence such that*

$$v_k \le \inf_{i \ge 0} \bar{r}_{A \cup [0, i+k]}(f_i) \quad (k \ge 0),$$

then

$$\rho_k^{(n)}(x, y) \ge v_k \rho_{k-1}^{(n-1)}(x, y) + \sum_{j=k}^{n-1} (v_{j+1} - v_j) \rho_j^{(n-1)}(x, y) \quad (0 \le k \le n).$$

Proof. We denote $\rho_k^{(n)}(x, y)$ by $\rho_k^{(n)}$, and $[\zeta]_{[i,j]}$ by $[\zeta]_i^j$ ($\zeta \in S^{\Lambda}$, $[i, j] \subseteq \Lambda \subseteq \mathbb{Z}$). Let

$$f_m^{(k)}(w) = \inf\{f_m(z) : z \in [x]_A \cap [w]_0^{m+k}\}$$

and

$$\phi_m^{(k+1)}(w) = f_m^{(k+1)}(w) - f_m^{(k)}(w) \quad (k, m \ge 0,\ w \in X).$$

Note that

$$f_m^{(k)}(w) \ge v_k f_m(w) \quad (w \in [x]_A)$$

and

$$f_m \ge f_m^{(k)} + \sum_{j=k+1}^{N} \phi_m^{(j)} \quad (N > k).$$

Let $m \ge 0$, $n \ge 1$ and $\zeta \in S^{[m,m+k]}$ for some $0 \le k \le n$. Since $f_m^{(j)}(w)$, $\phi_m^{(j)}(w)$ depend only on $w_{[m,m+j]}$, we can identify them with the functions induced by the natural projection onto $S^{[m,m+j]}$. Then

$$\begin{aligned} L_{m+1,m+n}(f_m^{(k)} \mathbf{1}[\zeta])(\zeta_{\{m\}} x_{\{m\}^c}) &= f_m^{(k)}(\zeta) L_{m+1,m+n}(\mathbf{1}[\zeta]_{m+1}^{m+k})(x) \\ &\ge \rho_{k-1}^{(n-1)} f_m^{(k)}(\zeta) L_{m+1,m+n}(\mathbf{1}[\zeta]_{m+1}^{m+k})(y) \\ &= \rho_{k-1}^{(n-1)} L_{m+1,m+n}(f_m^{(k)} \mathbf{1}[\zeta])(\zeta_{\{m\}} y_{\{m\}^c}) \\ &\qquad (0 \le k \le n,\ n \ge 1,\ m \ge 0), \end{aligned}$$

and

$$\begin{aligned} &L_{m+1,m+n}(\phi_m^{(j)} \mathbf{1}[\zeta])(\zeta_{\{m\}} x_{\{m\}^c}) \\ &\quad = \sum_{\eta \in S^{[m+k+1,m+j]}} \phi_m^{(j)}(\zeta\eta) L_{m+1,m+n}(\mathbf{1}[\zeta\eta]_{m+1}^{m+j})(x) \\ &\quad \ge \rho_{j-1}^{(n-1)} \sum_{\eta \in S^{[m+k+1,m+j]}} \phi_m^{(j)}(\zeta\eta) L_{m+1,m+n}(\mathbf{1}[\zeta\eta]_{m+1}^{m+j})(y) \\ &\quad = \rho_{j-1}^{(n-1)} L_{m+1,m+n}(\phi_m^{(j)} \mathbf{1}[\zeta])(\zeta_{\{m\}} y_{\{m\}^c}) \\ &\qquad\qquad (1 \le k+1 \le j \le n,\ m \ge 0). \end{aligned}$$

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Therefore,

$$\begin{aligned}
L_{m,m+n}\mathbf{1}[\zeta](x) &\geq L_{m+1,m+n}(f_m^{(k)}\mathbf{1}[\zeta])(\zeta_{\{m\}}x_{\{m\}^c}) \\
&\quad + \sum_{j=k+1}^{n} L_{m+1,m+n}(\phi_m^{(j)}\mathbf{1}[\zeta])(\zeta_{\{m\}}x_{\{m\}^c}) \\
&\geq \rho_{k-1}^{(n-1)} L_{m+1,m+n}(f_m^{(k)}\mathbf{1}[\zeta])(\zeta_{\{m\}}y_{\{m\}^c}) \\
&\quad + \sum_{j=k+1}^{n} \rho_{j-1}^{(n-1)} L_{m+1,m+n}(\phi_m^{(j)}\mathbf{1}[\zeta])(\zeta_{\{m\}}y_{\{m\}^c}) \\
&= \sum_{j=k}^{n-1} (\rho_{j-1}^{(n-1)} - \rho_j^{(n-1)}) L_{m+1,m+n}(f_m^{(j)}\mathbf{1}[\zeta])(\zeta_{\{m\}}y_{\{m\}^c}) \\
&\quad + \rho_{n-1}^{(n-1)} L_{m+1,m+n}(f_m^{(n)}\mathbf{1}[\zeta])(\zeta_{\{m\}}y_{\{m\}^c}) \\
&\geq \sum_{j=k}^{n-1} (\rho_{j-1}^{(n-1)} - \rho_j^{(n-1)}) v_j L_{m+1,m+n}(f_m\mathbf{1}[\zeta])(\zeta_{\{m\}}y_{\{m\}^c}) \\
&\quad + \rho_{n-1}^{(n-1)} v_n L_{m+1,m+n}(f_m\mathbf{1}[\zeta])(\zeta_{\{m\}}y_{\{m\}^c}) \\
&= \left(\rho_{n-1}^{(n-1)} v_n + \sum_{j=k}^{n-1} (\rho_{j-1}^{(n-1)} - \rho_j^{(n-1)}) v_j \right) L_{m,m+n}\mathbf{1}[\zeta](y)
\end{aligned}$$

for $0 \leq k \leq n-1,\ m \geq 0$, and

$$L_{m,m+n}\mathbf{1}[\zeta](x) = \rho_{n-1}^{(n-1)} v_n L_{m,m+n}\mathbf{1}[\zeta](y)$$

for $k = n \geq 1$. Hence,

$$\rho_k^{(n)} \geq \rho_{n-1}^{(n-1)} v_n + \sum_{j=k}^{n-1} (\rho_{j-1}^{(n-1)} - \rho_j^{(n-1)}) v_j \quad (0 \leq k \leq n,\ n \geq 1)$$

(where the empty sum is taken to be 0). Rearranging, we have

$$\rho_k^{(n)} \geq \rho_{k-1}^{(n-1)} v_k + \sum_{j=k}^{n-1} \rho_j^{(n-1)} (v_{j+1} - v_j) \quad (0 \leq k \leq n,\ n \geq 1),$$

as required. □

Now, given an increasing sequence $\{v_k\}_{k=0}^{\infty}$ such that $0 < v_k \leq 1$ for all k, consider $p_k^{(n)}$ $(-1 \leq k \leq n,\ n \geq 0)$, defined recursively by

$$p_{-1}^{(n)} = 1, \tag{3.1}$$

$$p_0^{(0)} = v_0, \tag{3.2}$$

$$p_k^{(n)} = v_k p_{k-1}^{(n-1)} + \sum_{j=k}^{n-1} (v_{j+1} - v_j) p_j^{(n-1)} \quad (0 \leq k \leq n,\ n \geq 1). \tag{3.3}$$

(As before, when $k = n$, the empty sum is taken to be 0.)

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LEMMA 3.3. *Let* $p_k^{(n)}$ $(k \geq -1,\ n \geq 0)$ *and* $\{v_k\}_{k=0}^{\infty}$ *be as above. Then*

$$\lim_{n\to\infty} p_0^{(n)} \geq \lim_{n\to\infty} \left(\sum_{k=0}^{n} \prod_{j=0}^{k} v_j \right) \left(1 + \sum_{k=0}^{n} \prod_{j=0}^{k} v_j \right)^{-1}.$$

Proof. It follows from (3.3) that for $0 \leq k \leq n$, $n \geq 1$,

$$p_k^{(n+1)} - p_k^{(n)} = v_k(p_{k-1}^{(n)} - p_{k-1}^{(n-1)}) + (v_{n+1} - v_n)p_n^{(n)} + \sum_{j=k}^{n-1}(v_{j+1} - v_j)(p_j^{(n)} - p_j^{(n-1)}),$$

and from (3.1)–(3.2) that

$$p_0^{(1)} - p_0^{(0)} = (v_1 - v_0)v_0 \geq 0.$$

Hence, if $n \geq 1$, then $p_k^{(n+1)} \geq p_k^{(n)}$ for $0 \leq k \leq n$ provided $p_k^{(n)} \geq p_k^{(n-1)}$ for $0 \leq k \leq n-1$. Since $p_0^{(1)} \geq p_0^{(0)}$, it follows inductively that $p_k^{(n+1)} \geq p_k^{(n)}$ for all $0 \leq k \leq n$. Thus,

$$p_k = \lim_{n\to\infty} p_k^{(n)} \quad (k \geq -1)$$

is well defined, and

$$p_k = v_k p_{k-1} + \sum_{j=k}^{\infty}(v_{j+1} - v_j)p_j \quad (k \geq 0).$$

Therefore,

$$\begin{aligned} p_k - p_{k+1} &= v_k p_{k-1} - v_{k+1}p_k + (v_{k+1} - v_k)p_k \\ &= v_k(p_{k-1} - p_k), \end{aligned}$$

and so

$$p_k - p_{k+1} = (1 - p_0) \prod_{j=0}^{k} v_j.$$

Summing gives

$$p_0 - p_{n+1} = (1 - p_0) \sum_{k=0}^{n} \prod_{j=0}^{k} v_j.$$

Since $p_{n+1} \geq 0$,

$$p_0 \geq \left(\sum_{k=0}^{n} \prod_{j=0}^{k} v_j \right) \left(1 + \sum_{k=0}^{n} \prod_{j=0}^{k} v_j \right)^{-1},$$

from which the result follows. □

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Proof of Proposition 3.1. It follows from Lemma 3.2 and the Cauchy–Schwarz inequality that

$$(\rho_k^{(n)}(x,y)\rho_k^{(n)}(y,x))^{\frac{1}{2}}$$
$$\geq v_k(\rho_{k-1}^{(n-1)}(x,y)\rho_{k-1}^{(n-1)}(y,x))^{\frac{1}{2}} + \sum_{j=k}^{n-1}(v_{j+1}-v_j)(\rho_j^{(n-1)}(x,y)\rho_j^{(n-1)}(y,x))^{\frac{1}{2}}$$
$$(0 \leq k \leq n,\ n \geq 1) \tag{3.4}$$

for any $x, y \in X$ such that $x_A = y_A$. Note also that $\rho_{-1}^{(n)}(x,y)\rho_{-1}^{(n)}(y,x) = 1$ for all $n \geq 0$ and $x, y \in X$. Define $p_k^{(n)}$ $(-1 \leq k \leq n,\ n \geq 0)$ inductively as in (3.1)–(3.3) with $\{v_k\}$ as in the statement of Proposition 3.1. Then it follows from (3.1)–(3.4) that

$$0 \leq p_k^{(n)} \leq \inf_{x_A = y_A}(\rho_k^{(n)}(x,y)\rho_k^{(n)}(y,x))^{\frac{1}{2}} \leq 1 \quad (-1 \leq k \leq n,\ n \geq 0).$$

The result now follows from Lemma 3.3. □

4. *Gibbs states and g-chains*

Let

$$\mathcal{G}_X = \left\{ g \in B(X) : g \text{ is } \mathcal{B}_{\mathbb{Z}^-}\text{-measurable},\ g \geq 0,\ \sum_{s\in S} g(s_0 x) = 1 \text{ for all } x \in X \right\},$$

and let $g \in \mathcal{G}_X$. A measure $\mu \in M(X)$ is a g-chain if

$$\mu([x]_{\{n\}} \mid \mathcal{B}_{(-\infty,n-1]})(x) = g(T^n x) \quad \text{a.e. } (\mu)$$

for all $n \in \mathbb{Z}$. A T-invariant g-chain is usually referred to as a g-measure. If $g \in \mathcal{G}_X$ is continuous, there is always at least one g-measure.

PROPOSITION 4.1. *Let $\mathcal{V} = \{\phi_\Lambda\}_{\Lambda\in\mathcal{C}}$ be a positive, continuous, T-invariant specification. If $\pi^{\mathcal{V}}_{[0,n]}([x]_{\{0\}} \mid x)$ converges to a continuous limit in $x \in X$ as $n \to \infty$, then*

$$g_{\mathcal{V}}(x) = \lim_{n\to\infty} \pi^{\mathcal{V}}_{[0,n]}([x]_{\{0\}} \mid x) \quad (x \in X)$$

is a positive, continuous function in $\mathcal{G}_X$, and $\mu \in \mathcal{G}(\mathcal{V})$ if and only if μ is a $g_{\mathcal{V}}$-chain.

Proof. Let

$$g_{\mathcal{V}}(x) = \lim_{n\to\infty} \pi^{\mathcal{V}}_{[0,n]}([x]_{\{0\}} \mid x) \quad (x \in X) \tag{4.1}$$

as above; by hypothesis, $g_{\mathcal{V}}$ is continuous. Note that $g_{\mathcal{V}}(x)$ is $\mathcal{B}_{\mathbb{Z}^-}$-measurable, and since

$$\inf \phi_{\{0\}} \leq \pi^{\mathcal{V}}_{[0,n]}([x]_{\{0\}} \mid x) \leq \sup \phi_{\{0\}} \quad (n \geq 0,\ x \in X),$$

it follows that $g_{\mathcal{V}}$ is positive. Clearly, $\sum_{s\in S} g_{\mathcal{V}}(s_0 x) = 1$ for all $x \in X$, and so $g_{\mathcal{V}} \in \mathcal{G}_X$. Moreover, it follows from the Martingale theorem that

$$\mu([x]_{\{n\}} \mid \mathcal{B}_{(-\infty,n-1]})(x) = g_{\mathcal{V}}(T^n x) \quad \text{a.e. } (\mu)$$

for any Gibbs state μ of $\mathcal{V}$ when $n = 0$, and hence, by the T-invariance of $\mathcal{V}$, for all $n \in \mathbb{Z}$; thus, any Gibbs state is a $g_{\mathcal{V}}$-chain.

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Conversely, if μ is a $g_{\mathcal{V}}$-chain, then

$$\begin{aligned}\mu([x]_{\{n\}} \mid \mathcal{B}_{\{n\}^c})(x) &= \lim_{m\to\infty} \mu([x]_{\{n\}} \mid \mathcal{B}_{(-\infty,m]\setminus\{n\}})(x)\\ &= \lim_{m\to\infty} \frac{\prod_{i=n}^{m} g_{\mathcal{V}}(T^i x)}{\sum_{s\in S}\prod_{i=n}^{m} g_{\mathcal{V}}(T^i(s_n x))} \quad \text{a.e. } (\mu) \quad (n\in\mathbb{Z}).\end{aligned}$$

Given $n\in\mathbb{Z}$, consider $k > m > n$. From (4.1), we have

$$\begin{aligned}\prod_{i=n}^{m} g_{\mathcal{V}}(T^i x) &= \lim_{k\to\infty} \prod_{i=n}^{m} \pi^{\mathcal{V}}_{[i,k]}([x]_{\{i\}} \mid x)\\ &= \lim_{k\to\infty} \pi^{\mathcal{V}}_{[n,k]}([x]_{[n,m]} \mid x)\\ &= \lim_{k\to\infty} \sum_{w\in[x]_{[m+1,k]^c}} \pi^{\mathcal{V}}_{\{n\}}([x]_{\{n\}} \mid w)\, \pi^{\mathcal{V}}_{[n+1,k]}([w]_{[n+1,k]} \mid x)\end{aligned}$$

for all $x\in X$. Therefore, for all $s, t\in S$, $x\in X$,

$$\begin{aligned}\frac{\mu([s]_n \mid \mathcal{B}_{\{n\}^c})(x)}{\mu([t]_n \mid \mathcal{B}_{\{n\}^c})(x)} &\le \lim_{m\to\infty}\lim_{k\to\infty} \sup_{w\in[x]_{[m+1,k]^c}} \frac{\pi^{\mathcal{V}}_{\{n\}}([s]_n \mid w)}{\pi^{\mathcal{V}}_{\{n\}}([t]_n \mid w)}\\ &= \frac{\phi_{\{n\}}(s_n x)}{\phi_{\{n\}}(t_n x)} \quad \text{a.e. } (\mu).\end{aligned}$$

It follows that $\mu([s]_n \mid \mathcal{B}_{\{n\}^c})(x) = \phi_{\{n\}}(s_n x)$ a.e. (μ) for all $s\in S$, $n\in\mathbb{Z}$, and so $\mu\in\mathcal{G}(\mathcal{V})$. □

THEOREM 4.2. *Let $\mathcal{V} = \{\phi_\Lambda\}_{\Lambda\in\mathcal{C}}$ be a positive, continuous, T-invariant specification with specification sequence $\{[0,n],\ f_n\}_{n=0}^{\infty}$. If*

$$\sum_{k=0}^{\infty}\prod_{j=0}^{k}\inf_{i\ge 0}\bar{r}_{(-\infty,i+j]}(f_i) = \infty,$$

then

$$g_{\mathcal{V}}(x) = \lim_{n\to\infty}\pi^{\mathcal{V}}_{[0,n]}([x]_{\{0\}} \mid x)$$

is well defined, the convergence is uniform in x, and it determines a positive function $g_{\mathcal{V}}\in\mathcal{G}_X\cap C(X)$ such that

$$r_{[-n,0]}(g_{\mathcal{V}}) \le (1+R_n^{-1})^2,$$

where

$$R_n = \sum_{k=0}^{\infty}\prod_{j=0}^{k}\inf_{i\ge 0}\bar{r}_{[-n,i+j]}(f_i) \quad (n\ge 0).$$

Moreover, $\mu\in\mathcal{G}(\mathcal{V})$ if and only if μ is a $g_{\mathcal{V}}$-chain.

Proof. Given $\{f_n\}$, let $\rho_k^{(n)}(\,\cdot\,,\,\cdot\,)$ be as in §3. Note that

$$\rho_0^{(n)}(x,y)\rho_0^{(n)}(y,x) \le \frac{\pi^{\mathcal{V}}_{[0,n]}([x]_{\{0\}} \mid x)}{\pi^{\mathcal{V}}_{[0,n]}([x]_{\{0\}} \mid y)}.$$

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Then since $\mathcal{V}$ is positive and continuous, Proposition 3.1 (with $A = \mathbb{N}^-$) implies that

$$\lim_{n\to\infty} \sup \left\{\pi^{\mathcal{V}}_{[0,n]}([x]_{\{0\}} \mid x) - \pi^{\mathcal{V}}_{[0,n]}([x]_{\{0\}} \mid y) : x, y \in X,\ x_{\mathbb{N}^-} = y_{\mathbb{N}^-}\right\} = 0. \quad (4.2)$$

Given $x \in X$, consider the sequences

$$\sup_{y\in[x]_{\mathbb{N}^-}} \pi^{\mathcal{V}}_{[0,n]}([x]_{\{0\}} \mid y), \qquad \inf_{y\in[x]_{\mathbb{N}^-}} \pi^{\mathcal{V}}_{[0,n]}([x]_{\{0\}} \mid y) \quad (n \geq 0),$$

which are respectively decreasing, increasing in n and, hence, convergent. It follows from (4.2) that they converge to the same limit, and so

$$g_{\mathcal{V}}(x) = \lim_{n\to\infty} \pi^{\mathcal{V}}_{[0,n]}([x]_{\{0\}} \mid x) \quad (x \in X)$$

is well defined for all $x \in X$. Furthermore,

$$\sup_{x\in X} |g_{\mathcal{V}}(x) - \pi^{\mathcal{V}}_{[0,n]}([x]_{\{0\}} \mid x)| \leq \sup_{x_{\mathbb{N}^-}=y_{\mathbb{N}^-}} \left\{\pi^{\mathcal{V}}_{[0,n]}([x]_{\{0\}} \mid x) - \pi^{\mathcal{V}}_{[0,n]}([x]_{\{0\}} \mid y)\right\},$$

and so the convergence is uniform. Thus, the limit is continuous, and it follows from Proposition 4.1 that $g_{\mathcal{V}} \in \mathcal{G}_X$, $g_{\mathcal{V}}$ is positive, and $\mu \in \mathcal{G}(\mathcal{V})$ if and only if μ is a $g_{\mathcal{V}}$-chain.

Applying Proposition 3.1 again with $A = [-n, -1]$ (and $\{f_n\}$ as before) gives

$$\begin{aligned}\bar{r}_{[-n,0]}(g_{\mathcal{V}}) &= \inf_{x_A=y_A} \lim_{m\to\infty} \frac{\pi^{\mathcal{V}}_{[0,m]}([x]_{\{0\}} \mid x)}{\pi^{\mathcal{V}}_{[0,m]}([x]_{\{0\}} \mid y)} \\ &\geq \left(\frac{R_n}{1+R_n}\right)^2,\end{aligned}$$

as required. □

Remark 4.3. Proposition 4.1 and Theorem 4.2 can easily be generalized to include non-invariant specifications $\mathcal{V}$, in which case, $g_{\mathcal{V}}$ is replaced by a sequence of continuous functions $\{g_n\}_{n=-\infty}^{\infty}$ such that

$$g_n(x) = \mu([x]_{\{n\}} \mid \mathcal{B}_{(-\infty,n-1]})(x) \quad (n \in \mathbb{Z}).$$

Note that, in general, $g_n(x) \neq g_0(T^n x)$.

The following result will be useful for estimating the variation of $g_{\mathcal{V}}$.

LEMMA 4.4. *Let $\mathcal{V} = \{\phi_\Lambda\}_{\Lambda\in\mathcal{C}}$ be a positive, continuous, T-invariant specification with specification sequence $\{[0, n],\ f_n\}_{n=0}^{\infty}$. If*

$$\limsup_{n\to\infty}\ n^{\alpha-1} \prod_{i=0}^{n} \sup_{j\geq 0} r_{(-\infty,i+j]}(f_j) \leq K \quad (4.3)$$

for some $0 < \alpha \leq 1,\ 0 < K < \infty$, then

$$g_{\mathcal{V}}(x) = \lim_{n\to\infty} \pi^{\mathcal{V}}_{[0,n]}([x]_{\{0\}} \mid x)$$

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is well defined, and

$$\limsup_{n\to\infty} \frac{\log r_{[-n,0]}(g_{\mathcal{V}})}{(\log \sup_{i\geq 0} r_{[-n,\infty)}(f_i))^{\alpha}} \leq 2\,\Gamma(\alpha)^{-1}K.$$

Proof. Note that (4.3) implies

$$\sum_{n=0}^{\infty}\prod_{j=0}^{n}\inf_{i\geq 0}\bar{r}_{(-\infty,i+j]}(f_i)=\infty,$$

and so it follows from Theorem 4.2 that $g_{\mathcal{V}}\in\mathcal{G}_X$ is well defined and

$$\log r_{[-n,0]}(g_{\mathcal{V}})\leq 2\log(1+R_n^{-1})\leq 2R_n^{-1}\quad (n\geq 0), \tag{4.4}$$

where

$$R_n=\sum_{k=1}^{\infty}\prod_{j=0}^{k-1}\inf_{i\geq 0}\bar{r}_{[-n,i+j]}(f_i)$$

as before. Note that

$$R_n\geq\sum_{k=1}^{\infty}a_k t_n^k,$$

where

$$a_k=\prod_{j=0}^{k-1}\inf_{i\geq 0}\bar{r}_{(-\infty,i+j]}(f_i),\quad t_n=\inf_{i\geq 0}\bar{r}_{[-n,\infty)}(f_i).$$

Note also that a_n is decreasing and the power series $\sum_{k=1}^{\infty}a_k t^k$ is convergent for $0\leq t<1$. For such series, we have the following Tauberian theorem (see [**8**, p. 447]): for $0<\alpha<\infty$,

$$\sum_{k=1}^{\infty}a_k t^k\sim(1-t)^{-\alpha}\quad\text{as } t\nearrow 1\iff a_n\sim\Gamma(\alpha)^{-1}n^{\alpha-1}\quad\text{as } n\to\infty.$$

Thus, if

$$\limsup_{n\to\infty} n^{\alpha-1}a_n^{-1}\leq K,$$

where $0<\alpha\leq 1$, $0<K<\infty$, then

$$\lim_{t\to 1^-}\left((1-t)^{\alpha}\sum_{k=0}^{\infty}a_k t^k\right)^{-1}\leq\Gamma(\alpha)^{-1}K,$$

and since $1-t\sim|\log t|$ as $t\nearrow 1$, and $t_n\nearrow 1$ as $n\to\infty$, it follows from (4.4) that

$$\begin{aligned}\limsup_{n\to\infty}\left(\log\sup_{i\geq 0}r_{[-n,\infty)}(f_i)\right)^{-\alpha}\log r_{[-n,0]}(g_{\mathcal{V}})&=2\limsup_{n\to\infty}|\log t_n|^{-\alpha}\log r_{[-n,0]}(g_{\mathcal{V}})\\&\leq 2\,\Gamma(\alpha)^{-1}K,\end{aligned}$$

as required. □

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The next result follows from the results and techniques of Johansson, Öberg and Pollicott in [15], although it is not formulated there; for an explicit proof, see [14].

THEOREM 4.5. *Let $g \in \mathcal{G}_X$ be positive and continuous. If*

$$\lim_{n\to\infty} \sum_{i=\lceil \lambda^{n-1} \rceil}^{\lceil \lambda^{n} \rceil} (\log r_{[-i,0]}(g))^2 = 0 \tag{4.5}$$

for some $\lambda > 1$, then there is a unique g-chain $\mu \in M(X)$, which is T-invariant and Bernoulli.

Note that if (4.5) holds for some $\lambda > 1$, then it holds for all such λ.

5. *Proof of the main results*

THEOREM 5.1. *Let $\mathcal{V} = \{\phi_\Lambda\}_{\Lambda\in\mathcal{C}}$ be a positive, continuous, T-invariant specification on X with specification sequence $\{[0, n], f_n\}_{n=0}^{\infty}$. If, for some $0 < \alpha \le 1$,*

$$\prod_{i=0}^{n} \sup_{j\ge 0} r_{(-\infty,i+j]}(f_j) = O(n^{1-\alpha}) \tag{5.1}$$

and

$$\sum_{i=2^n}^{2^{n+1}} \Big(\log \sup_{j\ge 0} r_{[-i,\infty)}(f_j)\Big)^{2\alpha} = o(1), \tag{5.2}$$

then $\mathcal{V}$ has a unique Gibbs state μ, which is T-invariant, and (T, μ) is Bernoulli.

Proof. It follows from (5.1), Theorem 4.2 and Lemma 4.4 that there is a positive, continuous $g_{\mathcal{V}} \in \mathcal{G}_X$ such that any Gibbs state for $\mathcal{V}$ is a $g_{\mathcal{V}}$-chain and

$$\log r_{[-n,0]}(g_{\mathcal{V}}) = O\left(\Big(\log \sup_{i\ge 0} r_{[-n,\infty)}(f_i)\Big)^{\alpha}\right).$$

The result now follows from (5.2) and Theorem 4.5. □

Theorem 1.1 follows immediately from Theorem 5.1 and (2.5)–(2.6). If the specification is described in terms of an interaction potential $\Phi = \{\Phi_\Lambda\}_{\Lambda\in\mathcal{C}}$, then it follows from (2.7) and (2.8) that (5.1) and (5.2) hold provided there are constants $K > 0$ and $0 < \alpha \le 1$ such that

$$\sum_{i=0}^{n} \sum_{\substack{\Lambda\ni 0 \\ \Lambda\cap[i+1,\infty)\neq\emptyset}} \operatorname{var}(\Phi_\Lambda) \le (1-\alpha)\log n + K \quad (n \ge 1)$$

and

$$\lim_{n\to\infty} \sum_{i=2^n}^{2^{n+1}} \Bigg(\sum_{\substack{\Lambda\ni 0 \\ \Lambda\cap[i+1,\infty)\neq\emptyset}} \operatorname{var}(\Phi_\Lambda)\Bigg)^{2\alpha} = 0.$$

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This will be the case if

$$\sum_{\substack{\Lambda \ni 0 \\ \Lambda \cap [n+1,\infty) \neq \emptyset}} \operatorname{var}(\Phi_\Lambda) \leq (1-\alpha)n^{-1} + a_n \quad (n \geq 1)$$

for some $1/2 < \alpha \leq 1$ and $\sum a_n < \infty$, that is, if (1.8) holds with $\beta = 1 - \alpha$; thus, Theorem 1.2 follows.

Since

$$\sum_{n=0}^{\infty} \sum_{\substack{\Lambda \ni 0 \\ \Lambda \cap [n+1,\infty) \neq \emptyset}} \operatorname{var}(\Phi_\Lambda) \leq \sum_{\Lambda \ni 0} \operatorname{diam}(\Lambda) \|\Phi_\Lambda\|,$$

Theorem 1.2 includes potentials satisfying (1.4). Comparison with (1.5) and (1.6) is less straightforward; in some cases, its hypotheses are weaker, but not always. Note however that Theorem 1.2 guarantees a unique Gibbs state (absence of symmetry breakdown), whereas (1.5) only guarantees uniqueness of T-invariant Gibbs states, and (1.6) does not include the Bernoulli property.

By way of comparison, consider the Ising model on $\{-1, 1\}^{\mathbb{Z}}$ with pair interactions

$$\Phi_{\{i,j\}}(x) = -\frac{\beta}{2} J(|i-j|) x_i x_j, \tag{5.3}$$

where $\beta > 0$ (the inverse temperature) is a constant and $|J(n)| \leq n^{-\alpha}$ for some $\alpha > 1$. Since Berbee's result applies to non-invariant specifications and does not depend on any group structure, there is no loss of generality in assuming that $I = \mathbb{Z}$ and

$$\Lambda_{2k} = [-k, k], \quad \Lambda_{2k+1} = [-k, k+1] \quad (k \geq 0)$$

(as in (1.7)), other than the assumption that $|\Lambda_n \setminus \Lambda_{n-1}| = 1$.

The corresponding specification sequence $\{\Lambda_n, f_n\}$ is given by

$$f_n(x) = \exp\left(\frac{\beta}{2} \sum_{j=1}^{\infty} |i_n - i_{n+j}|^{-\alpha} x_{i_n} x_{i_{n+j}}\right) \quad (n \geq 0),$$

where i_n is the unique element of $\Lambda_n \setminus \Lambda_{n-1}$. Note that $\inf_{i \geq 0} \bar{r}_{\Lambda_{i+k}}(f_i) = \bar{r}_{\Lambda_k}(f_0)$,

$$\bar{r}_{\Lambda_{2k}}(f_0) = \exp\left(-2\beta \sum_{j=k+1}^{\infty} j^{-\alpha}\right),$$

$$\bar{r}_{\Lambda_{2k+1}}(f_0) = \exp\left(-2\beta \sum_{j=k+1}^{\infty} j^{-\alpha} + \beta(k+1)^{-\alpha}\right)$$

and

$$r_{(-\infty,n]}(f_0) = r_{[-n,\infty)}(f_0) = \exp\left(\beta \sum_{j=n+1}^{\infty} j^{-\alpha}\right).$$

Thus, for $\alpha > 2$, Berbee's condition and Theorem 1.1 both give uniqueness and the Bernoulli property for all β (as do the results in [**6**, **19**]), whereas for $1 < \alpha < 2$, no conclusion can be drawn. For the borderline case $\alpha = 2$, no conclusion can be drawn from

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the results in [6, 19], but Berbee's condition gives uniqueness for $\beta \leq \frac{1}{4}$; from Theorem 1.1, we get uniqueness and the Bernoulli property for $\beta < \frac{1}{2}$.

Note that a better uniqueness result for the case $J(n) = n^{-2}$ is obtained in [2]. In this case, since $J(n) \geq 0$ for all n, the interaction potential is ferromagnetic, and as is well known, there is a critical inverse temperature $\beta_c > 0$ such that there is a unique Gibbs state when $0 \leq \beta < \beta_c$, but non-unique T-invariant Gibbs states for $\beta > \beta_c$. It is shown in [2] that $\beta_c > 1$ (with non-uniqueness at β_c). In particular, there is a unique Gibbs state when $\beta \leq 1$; moreover, it follows from results in [12] that this Gibbs state is T-invariant and Bernoulli.

6. *Uniqueness without the Bernoulli property*

A recent result of Berger *et al* [4] proves uniqueness of the g-measure for $g \in \mathcal{G}_X$ such that

$$\limsup_{n\to\infty} n^{1/2} \log r_{[-n,0]}(g) < 2. \tag{6.1}$$

Using this result instead of Theorem 4.5 gives uniqueness of the T-invariant Gibbs state under hypotheses that are slightly weaker in some cases than those of Theorem 5.1, at the expense of the Bernoulli property.

THEOREM 6.1. *Let $\mathcal{V} = \{\phi_\Lambda\}_{\Lambda\in\mathcal{C}}$ be a positive, continuous, T-invariant specification on X with specification sequence $\{[0, n], f_n\}_{n=0}^{\infty}$. If, for some $0 < \alpha \leq 1$ and $K > 0$,*

$$\limsup_{n\to\infty} n^{\alpha-1} \prod_{i=0}^{n} \sup_{j\geq 0} r_{(-\infty,i+j]}(f_j) \leq K \tag{6.2}$$

and

$$\limsup_{n\to\infty} n^{1/2} \Big(\log \sup_{j\geq 0} r_{[-n,\infty)}(f_j) \Big)^{\alpha} < K^{-1}\Gamma(\alpha), \tag{6.3}$$

then $\mathcal{V}$ has a unique T-invariant Gibbs state.

Proof. It follows from (6.2) and Theorem 4.2 that any Gibbs state for $\mathcal{V}$ is a $g_{\mathcal{V}}$-chain, where

$$g_{\mathcal{V}}(x) = \lim_{n\to\infty} \pi^{\mathcal{V}}_{[0,n]}([x]_{\{0\}} \mid x),$$

and from Lemma 4.4 that $g_{\mathcal{V}}$ satisfies (6.1) provided (6.3) holds; thus the result follows. □

Remark 6.2. Note that there is not necessarily a unique Gibbs state, unless $\mathcal{V}$ is also attractive, in which case, the unique T-invariant Gibbs state is also the unique Gibbs state, and is a Bernoulli measure for T (see [12]).

If the specification is described in terms of an interaction potential $\Phi = \{\Phi_\Lambda\}_{\Lambda\in\mathcal{C}}$, then it follows from (2.7) and (2.8) that Φ has a unique T-invariant Gibbs state provided there are constants $K > 0$ and $0 < \alpha \leq 1$ such that

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$$\sum_{i=0}^{n} \sum_{\substack{\Lambda \ni 0 \\ \Lambda \cap [i+1,\infty) \neq \emptyset}} \operatorname{var}(\Phi_\Lambda) \leq (1-\alpha) \log n + \log K \quad (n \geq 1)$$

and

$$\limsup_{n\to\infty} n^{1/2} \left(\sum_{\substack{\Lambda \ni 0 \\ \Lambda \cap [n+1,\infty) \neq \emptyset}} \operatorname{var}(\Phi_\Lambda) \right)^{\alpha} < K^{-1}\Gamma(\alpha).$$

Since $\Gamma(1/2) = \sqrt{\pi}$, this will be the case if, for example,

$$\sum_{\substack{\Lambda \ni 0 \\ \Lambda \cap [n,\infty) \neq \emptyset}} \operatorname{var}(\Phi_\Lambda) \leq \frac{1}{2n} \quad (n \geq 1).$$

(Note that this does not imply (5.2) holds.)

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