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On the time periodic thermistor problem

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In this paper we study a nonlocal parabolic/elliptic system which models thermistor behaviour in cases where heat losses to the surrounding gas play a significant role. The existence of time periodic solutions for the system is established through Faedo-Galerkin approximations and the Leray–Schauder degree theory. We show that for the small gas pressure case, the temperature of the time periodic solutions is positive. Moreover we consider the long time behaviour of the system and prove the existence of a uniform attractor. Finally, the finite dimensionality of the attractor is discussed.

1 Introduction

The classical system of parabolic/elliptic equations modelling thermistor behaviour is given as follows:

$$u_t - \Delta u = \nabla[\sigma(u)\phi\nabla\phi], \tag{1.1}$$

$$-\nabla[\sigma(u)\nabla\phi] = 0. \tag{1.2}$$

Here u and ϕ are the temperature and the electric potential in the thermistor, respectively, and $\sigma(u)$ is the electric conductivity. The above system has a long history [16], and has been the subject of a variety of mathematical investigations in the past decade. We refer in particular to the work of Cimatti [10]–[13], and the results by various other authors [4, 5, 7, 17, 24, 31, 32, 37] and the references therein. The system under consideration often has two features which make analysis challenging: a degeneracy in the equations and mixed boundary conditions. An important step with regard to the first difficulty was taken by Xu [33, 34], who introduced the concept of a 'capacity solution' in the time dependent case.

Recent interest in the problem of the modelling of micromachined microsensors has led to the addition of a nonlocal term to one of the equations, to account for the physically important effect of heat losses to the surrounding gas [6, 26]. Specifically, the classical system (1.1)–(1.2) is replaced by the following nonlocal equations

$$u_t - \Delta u + \eta \int_\Omega G(x, y)u(y)\, dy = \nabla[\sigma(u)\phi\nabla\phi], \tag{1.3}$$

$$-\nabla[\sigma(u)\nabla\phi] = 0. \tag{1.4}$$

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To avoid spurious negative temperature points, this system has also been formulated as an 'obstacle' problem [3]. Indeed, we recall that the presence of the nonlocal term renders invalid maximum principle/order arguments. For the steady state (i.e. purely elliptic) case, results in Allegretto & Barbanova [2] and Freitas & Sweers [18] showed that some form of the maximum principle was indeed valid if the surrounding gas pressure was sufficiently small.

It is the purpose of this paper to consider the long time behaviour of the solutions to the nonlocal thermistor system (1.3)–(1.4) in the physically important case of periodic input. We show the existence of a uniform attractor and estimate its Hausdorff dimension. We also prove that the temperature of periodic solutions is positive if the gas pressure (i.e. η) is small. To the best of our knowledge, no previous long time results are known even for the classical system, and there are no positivity results for nonlocal parabolic equations.

Let Ω denote the three-dimensional domain occupied by the microsensor. Its boundary is denoted by $\partial\Omega$ which is decomposed into two parts Γ_D and Γ_N. We assume that Ω is smooth enough and Γ_D is nonempty. Specific descriptions about the domain can be found in Xie [31, 32]. The unknowns are associated with the following boundary conditions

$$u|_{\partial\Omega} = 0,\ \phi|_{\Gamma_D} = \phi_0(x,t),\ \left.\frac{\partial\phi}{\partial\vec{n}}\right|_{\Gamma_N} = 0. \tag{1.5}$$

Here the potential source ϕ_0 is time periodic with period T. Moreover, u satisfies either a periodic condition

$$u(x,\cdot) = u(x,\cdot + T), \tag{1.6}$$

or an initial condition

$$u(x,0) = u_0(x). \tag{1.7}$$

We first consider the periodic case (1.6). The existence of a time periodic solution is established through the Faedo–Galerkin method and the Leray–Schauder degree theory. Next we are interested in the positivity of the time periodic solutions. As we mentioned before, the temperature u could be somewhere negative if η is big. We will show that, for all those potential sources ϕ_0 which satisfy a certain growth property, there exists an η_0 independent of the specific choice of ϕ_0 such that the temperature of all time periodic solutions is positive for all $0 \leqslant \eta < \eta_0$.

Finally, we study the initial value case (1.7) and consider the long time behaviour of the non-autonomous system (1.3)–(1.5) and (1.7). We show that there exists a uniform attractor in $L^2(\Omega)$ and that its Hausdorff dimension is finite. The main difficulty related to this problem is the lack of uniqueness of the weak solution for a given $u_0(x) \in L^2(\Omega)$. Specifically, to obtain uniqueness, a certain regularity of the gradient of ϕ, say,

$$|\nabla\phi| \in L^p(\Omega) \text{ for some } p > 2, \tag{1.8}$$

is needed. Since in our case ϕ satisfies a mixed boundary condition, such a regularity (1.8) is not generally available. From a result in Xie [32], we know that if $u(x,t)$ is Hölder

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continuous for each t, (1.8) will be true, but this will require $u_0(x)$ to be Hölder continuous as well. To overcome this difficulty, our idea is to show that for all positive time t, any weak solution $u(x,t)$ will be Hölder continuous. Moreover for a given Hölder continuous initial value, the corresponding solution will also be Hölder continuous for all time t. Thus the long time behaviour of (1.3)–(1.5) in $L^2(\Omega)$ (i.e. for all $u_0(x) \in L^2(\Omega)$) is the same as in $C^\alpha(\bar{\Omega})$ (i.e. for all $u_0(x) \in C^\alpha(\bar{\Omega})$).

We denote by $L^p(\Omega)$ the standard Lebesgue spaces with norm $\|w\|_{L^p(\Omega)} = (\int_\Omega |w|^p)^{1/p}$. The standard inner product and norm in $L^2(\Omega)$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. Let $H^1(\Omega)$ be the standard Sobolev space with norm $\|w\|_{H^1(\Omega)} = (\int_\Omega (|w|^2 + |\nabla w|^2)^{1/2}$. Its dual space is denoted by $H^{-1}(\Omega)$. The closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$ is denoted by $H_0^1(\Omega)$. For simplicity, we write $H^1_{\Gamma_D}(\Omega) = H_0^1(\Omega \cup \Gamma_N)$. Let X be a general normed space, the space $L^p(0,T;X)$ consists of functions from $(0,T)$ into X with $\|w\|_{L^p(0,T;X)} = (\int_0^T \|w\|_X dt)^{1/p} < \infty$. For $0 < \alpha < 1$, we denote by $C^{\alpha,\alpha/2}(Q_T)$ the collection of all Hölder continuous functions on $Q_T = [0,T] \times \bar{\Omega}$. Details of these spaces and norms can be found in Adams [1] and Ladyzenskaja *et al.* [25]. Other notation will be given below.

For simplicity, we write either

$$W(T,\Omega) = \{u | u(x,\cdot) = u(x,\cdot + T),\ \ u \in L^2(0,T;H_0^1(\Omega)),\ u_t \in L^2(0,T;H^{-1}(\Omega))\},$$

or

$$W(T,\Omega) = \{u | u \in L^2(0,T;H_0^1(\Omega)),\ u_t \in L^2(0,T;H^{-1}(\Omega))\},$$

corresponding to cases (1.6) or (1.7), respectively.

A tuple $(u,\ \phi)$ is called a weak solution of (1.3)–(1.4) if $u \in W(T,\Omega)$ and $\phi - \phi_0 \in L^\infty(0,T;H^1_{\Gamma_D}(\Omega))$ and satisfies that, for almost every t,

$$\int_\Omega \left[u_t v + \nabla u \nabla v + \eta \int_\Omega G(x,y)u(y)\,dy v\right] = -\int_\Omega \sigma(u)\phi\nabla\phi\nabla v,\ \forall v \in H_0^1(\Omega), \tag{1.9}$$

$$\int_\Omega \sigma(u)\nabla\phi\nabla\psi = 0,\ \forall \psi \in H^1_{\Gamma_D}(\Omega). \tag{1.10}$$

Before proceeding, we give some general assumptions on the given data and the coefficients.

A1. There are two positive finite numbers σ_0 and σ_1 such that $\sigma_0 \leqslant \sigma(s) \leqslant \sigma_1$. Moreover, $\sigma(s)$ is Lipschitz continuous with Lipschitz constant l.

A2. $\sup_{x,y} |G(x,y)| \leqslant G_0$ and $\int_\Omega \int_\Omega G(x,y)w(x)w(y)dxdy \geqslant 0$ for all $w(x) \in L^2(\Omega)$.

A3. $\sup_{x,t} |\phi_0| < \infty$. Moreover, ϕ_0 can be extended to be a Lipschitz function over Q_T which satisfies the same boundary condition as ϕ. This extension will also be denoted by ϕ_0.

Observe that we have assumed σ is nondegenerate, to obtain the existence of a uniform attractor and its finite dimensionality. Thus the long time behaviour for the 'capacity solution' case remains open.

The paper is structured as follows. The existence of time periodic solutions is presented in § 2. The positivity of time periodic solutions is obtained in § 3. In the last section, we first

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show the Hölder continuity of the weak solutions for positive time t. As a consequence, we obtain the existence of a uniform attractor and its finite dimensionality in $L^2(\Omega)$.

In the sequel, c, c_i always denotes some generic positive constants, which may depend upon the boundary conditions and the various bounds of the coefficients but are independent of the initial value and the time t except an explicit specification. Moreover their values may vary from one step to the next.

2 The existence of time periodic solutions

We apply the Leray–Schauder degree theory and the Galerkin method to show the existence of time periodic solutions. Let w_n, $n = 1, 2, \ldots$ be a countable basis of $H_0^1(\Omega)$. Without loss of generality, we may assume $(w_n, w_m) = 1$, if $n = m$ and 0, otherwise. The n-dimensional subspace spanned by $w_1, \ldots, w_n$ is denoted by H_n. We also write $X = \{d(t) | d(t) \in C([0, T]), d(\cdot) = d(\cdot + T)\}$ and X^n the n-power cartesian product of X. A function $u^n(x, t) = \sum_{j=1}^n d_j^n(t) w_j(x)$ is called a Galerkin approximation solution of $u(x, t)$ if it satisfies

$$\int_\Omega \left[u_t^n w_j + \nabla u^n \nabla w_j + \eta \int_\Omega G(x, y) u^n(y) dy w_j \right] = - \int_\Omega \sigma(u^n) \phi^n \nabla \phi^n \nabla w_j, \quad j = 1, \ldots, n, \tag{2.1}$$

$$\int_\Omega \sigma(u^n) \nabla \phi^n \nabla \psi = 0, \ \forall \psi \in H_{\Gamma_D}^1(\Omega). \tag{2.2}$$

Here $d_j^n(t) \in X$. For simplicity, we write

$$\begin{aligned} F_j(z) &= \eta \int_\Omega \int_\Omega G(x, y) z(y) dy w_j, \\ E_j(z) &= - \int_\Omega \sigma(z) \varphi \nabla \varphi \nabla w_j, \\ a_{ij} &= (\nabla w_i, \nabla w_j), \end{aligned}$$

where φ denotes the unique solution of (1.10) with u replaced by z. By the assumption on w_n, we obtain that

$$d_t^n + A^n d^n = E^n(u^n) - F^n(u^n), \tag{2.3}$$

with

$$\begin{aligned} d^n &= \left(d_1^n, \ldots, d_n^n\right)^*, \\ E^n(z) &= (E_1(z), \ldots, E_n(z))^*, \\ F^n(z) &= (F_1(z), \ldots, F_n(z))^*, \\ A^n &= (a_{ij})_{n \times n}. \end{aligned}$$

Here $*$ denotes the transpose of a row vector. Let $b(t) = (b_1(t), \ldots, b_2(t))^* \in X^n$ and write $v = \sum_{j=1}^n b_j w_j$. For $0 \leqslant \lambda \leqslant 1$, we define a family of mappings $T_\lambda : b(t) \to d(t)$from X^n

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into itself with $d(t) = (d_1(t), \dots, d_n(t))^* \in X^n$ satisfying the following linear system:

$$d_t + A^n d = \lambda[E^n(v) - F^n(v)]. \tag{2.4}$$

Since $v \in C(0, T; H_0^1(\Omega))$, there exists a unique solution φ of (1.10) with u replaced by v. By the weak maximum principle and the general estimates of linear elliptic equations we have

$$\sup_{x,t} |\varphi| \leqslant \sup_{x,t} |\phi_0|, \quad \sup_t \|\nabla\varphi\| \leqslant c \sup_t \|\nabla\phi_0\|. \tag{2.5}$$

Thus, $\sup_t |E_j(v)| < \infty$. By the assumption of $G(x, y)$, we also have $\sup_t |F_j(v)| < \infty$. Moreover, $E_j(v)$ and $F_j(v)$ are time periodic. Therefore, according to the general theory of linear ordinary differential equations, there exists a unique periodic solution of (2.4). Actually, the solution is given by

$$d(t) = \lambda e^{-A^n t}\left[\int_0^t e^{A^n s}[E^n(v) - F^n(v)]ds + (e^{A^n T} - I_{n\times n})^{-1}\int_0^T e^{A^n s}[E^n(v) - F^n(v)]ds\right]. \tag{2.6}$$

Here $I_{n\times n}$ is the $n \times n$ identity matrix. Thus for each λ, T_λ is well defined. We observe from (2.6) that $\{T_\lambda b(t) |\ \|b(t)\|_{X^n} \leqslant 1\}$ is equi-continuous and equi-bounded. Consequently T_λ is compact for each λ.

Lemma 2.1 *T_λ is continuous for each λ.*

Proof Let $b^{(m)}(t)$ be a convergent sequence of X^n with limit $b(t)$. Then it follows from (2.6) that

$$\begin{aligned}\left\|T_\lambda b^{(m)} - T_\lambda b\right\|_{X^n} &\leqslant \sup_t \left\|e^{A^n t}\right\| \left(\lambda + \left\|e^{A^n T} - I_{n\times n}\right\|^{-1}\right) \\ &\times \int_0^T \left[\left|E^n\left(v^{(m)}\right) - E^n(v)\right| + \left|F^n\left(v^{(m)}\right) - F^n(v)\right|\right] ds.\end{aligned} \tag{2.7}$$

Here $v^{(m)} = \sum_{j=1}^n b_j^{(m)} w_j$, similarly for v. By the definitions and the estimates (2.5) we have that

$$\left|E_j\left(v^{(m)}\right) - E_j(v)\right| \leqslant c\int_\Omega \left[\left|\sigma\left(v^{(m)}\right) - \sigma(v)\right||\nabla\varphi| + \left|\varphi^{(m)} - \varphi\right||\nabla\varphi| + \left|\nabla\left(\varphi^{(m)} - \varphi\right)\right|\right] |\nabla w_j| \tag{2.8}$$

$$\left|F_j\left(v^{(m)}\right) - F_j(v)\right| \leqslant \sum_{i=1}^n \left|d_i^{(m)} - d_i\right| \left|\int_\Omega\int_\Omega G(x, y) w_i(y) w_j(x) dxdy\right|. \tag{2.9}$$

Thus

$$\sup_t \left|F_j\left(v^{(m)}\right) - F_j(v)\right| \to 0. \tag{2.10}$$

Since $v^{(m)} \to v$ in $\Omega \times [0, T]$ and $\sigma(s)$ is continuous, the Lebesgue convergence theorem gives that

$$\int_{Q_T} \left|\sigma\left(v^{(m)}\right) - \sigma(v)\right||\nabla\varphi|\ |\nabla w_j| \to 0. \tag{2.11}$$

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It follows from (1.10) that

$$\int_\Omega \left|\nabla\left(\varphi^{(m)}-\varphi\right)\right|^2 \leqslant c\int_\Omega \left|\sigma\left(v^{(m)}\right)-\sigma(v)\right|^2|\nabla\varphi|^2, \tag{2.12}$$

which implies that

$$\int_{Q_T} \left|\nabla\left(\varphi^{(m)}-\varphi\right)\right|^2 \to 0. \tag{2.13}$$

Therefore

$$\int_{Q_T} \left|\nabla\left(\varphi^{(m)}-\varphi\right)\right| |\nabla w_j| \to 0. \tag{2.14}$$

The Poincare Inequality and (2.13) give that (if necessary, after passing to a subsequence) $\varphi^{(m)} \to \varphi$ *a.e.* in $\Omega\times(0,T)$. We apply the Lebesgue convergence theorem again, and obtain that

$$\int_{Q_T} \left|\varphi^{(m)}-\varphi\right||\nabla\varphi|\nabla w_j| \to 0. \tag{2.15}$$

Finally by combining (2.10), (2.11), (2.14) and (2.15) we obtain that

$$\int_0^T \left[\left|E^n\left(v^{(m)}\right)-E^n(v)\right|+\left|F^n\left(v^{(m)}\right)-F^n(v)\right|\right] \to 0, \tag{2.16}$$

which implies that $\| T_\lambda b_j^{(m)} - T_\lambda b_j\|_{X^n} \to 0$. Thus T_λ is continuous for each λ. □

Lemma 2.2 *There exists a constant β independent of λ such that, for all $d(t)\in X^n$ satisfying $T_\lambda d = d$, $\|d\|_{X^n} \leqslant \beta$.*

Proof We multiply both sides of (2.4) by d^* to obtain

$$\frac{1}{2}\frac{d}{dt}|d|^2 + d^*A^n d = \lambda d^*[E^n(w)-F^n(w)]. \tag{2.17}$$

Here $w=\sum_{j=1}^n d_j w_j$. By the definition of A^n and the Poincare Inequality, there exists a positive constant ν such that

$$d^*A^n d = \|\nabla w\|^2 \geqslant \nu|d|^2.$$

Furthermore,

$$d^*F^n(w) = \int_\Omega\int_\Omega G(x,y)w(y)w(x)dydx \geqslant 0,$$
$$|d^*E^n(w)| = \left|\int_\Omega \sigma(w)\varphi\nabla\varphi\nabla w\right| \leqslant c+\frac{1}{2}\|\nabla w\|^2,$$

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where the assumptions on $G(x, y)$, the Schwartz Inequality and the estimates (2.5) have been used. Thus, we obtain

$$\frac{d}{dt}|d|^2 + v|d|^2 \leqslant c. \tag{2.18}$$

Integrating (2.18) from 0 to T gives $\int_0^T |d(s)|^2 ds \leqslant \frac{c}{v}T$. Thus, there exists $t_0 \in [0, T]$ such that $|d(t_0)|^2 \leqslant \frac{c}{v}$. Now for any $t \in [t_0, t_0 + T]$, we integrate (2.18) from t_0 to t, and obtain that

$$|d(t)|^2 \leqslant cT + \frac{c}{v}. \tag{2.19}$$

Thus Lemma (2.2) follows immediately from (2.19). □

Lemma 2.3 *There exists at least one time periodic solution to the approximation system* (2.1)–(2.2) *for each n.*

Proof From the previous discussion we conclude that the family of mappings T_λ satisfies all the conditions of the Leray–Schauder degree theory. Therefore the topological degrees of T_0 and T_1 are the same. Since 0 is the unique time periodic fixed point of T_0, the degree of T_0 is $+1$. Thus T_1 is of degree $+1$ and has at least one fixed point. By the definition of T_1, this fixed point is a time periodic solution of the system (2.1)–(2.2). □

Theorem 2.1 *There exists at least one time periodic solution to the system* (1.3)–(1.4).

Proof Similar to (2.17)–(2.19), we obtain

$$\int_0^T \|\nabla u^n\|^2 ds, \quad \sup_t \|u^n\|^2 \leqslant c(T). \tag{2.20}$$

Thus, there exists a convergent subsequence of u^n (denoted by u^n itself for the sake of simplicity) and a function $u \in W[T, \Omega]$ such that

$$u^n \to \text{ } u \text{ weakly star in } L^\infty(0, T; L^2(\Omega)), \text{ and weakly in } L^2\big(0, T; H_0^1(\Omega)\big).$$

By a similar discussion as in Cimatti [12], up to another extracted subsequence, we conclude that

$$u^n \to u \text{ strongly in } L^2(Q_T).$$

Furthermore, ϕ^n satisfies a similar bound as in (2.5), thus there exists a function ϕ such that

$$\phi^n \to \phi \text{ weakly star in } L^\infty\big(0, T; H_{\Gamma_D}^1\big) \text{ and in } L^\infty(Q_T).$$

Now we fix an m, and for any w_j in H_m we pass to the limit in (2.1)–(2.2) with respect to n to obtain (1.9)–(1.10) for all $v \in H_m$. Since $\cup_{m=1}^\infty H_m$ is dense in $H_0^1(\Omega)$, we conclude that (1.9) is satisfied for all $v \in H_0^1(\Omega)$. Thus, (u, ϕ) is a time periodic solution. □

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3 The positivity of the time periodic solutions

The purpose of this section is to show that if η is small, the temperature u of periodic solutions of (1.9)–(1.10) is positive. We assume here that $G(x,y) \equiv 1$ (this may be weakened), and require a specific nature of the input $\phi_0(x,t)$ as given below. In particular, we ask that ϕ_0 satisfy the M-property stated below. We begin with some preliminary considerations.

Let $0 < \sigma_0 < a(x) < \sigma_1$, and assume z solves

$$-\nabla[a(x)\nabla z] = 0, \tag{3.1}$$

$$z|_{\Gamma_0} = 0, \quad z|_{\Gamma_1} = 1, \tag{3.2}$$

$$\left.\frac{\partial z}{\partial \vec{n}}\right|_{\Gamma_N} = 0, \tag{3.3}$$

where $\Gamma_D = \Gamma_0 \cup \Gamma_1$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$ is empty.

Lemma 3.1 *Let $w(x) > 0$ in Ω be a smooth function. Then there exists a positive constant m independent of the specific a, z such that $\int_\Omega a(x)|\nabla z|^2 w dx > m$.*

Proof If not, there exist sequences $\{a_i(x)\}$, $\{z_i(x)\}$ with $\alpha < a_i(x) < \beta$ and $z_i(x)$ the corresponding solutions of (3.1)–(3.3) such that $\int_\Omega a(x)|\nabla z|^2 w dx \to 0$ as $i \to \infty$. Now $\{z_i(x)\}$ are bounded in $C^\alpha(\bar{\Omega})$. Thus, without loss of generality, $z_i \to z$ in C^{α_1} for some $\alpha_1 < \alpha$. Clearly, $z = 1$ on Γ_1, $z = 0$ on Γ_0. Now $\int_\Omega |\nabla z_i|^2$ is bounded. Without loss of generality, $z_i \to z$ also in $L^2(\Omega)$. Let K be any compact subdomain of Ω, then $\int_K |\nabla z_i|^2 \to 0$. But $\int_K \varphi \nabla z_i = -\int_K \nabla\varphi z_i \to -\int_K \nabla\varphi z$ for all $\varphi \in C_0^\infty(K)$. Thus $\nabla z = 0$ in K, which implies $z = constant$. Since K is arbitrary, $z = constant$ in $\bar{\Omega}$, which contradicts $z = 1$ on Γ_1, $z = 0$ on Γ_0. □

Next we define the following:

Definition 3.1 For a given $M > 0$, a function $f \in L^2(0,T)$ has the M-property if and only if

$$\|f\|_{L^2(0,T)} \leqslant M\|f\|_{L^1(0,T)}.$$

Examples of collections of functions which satisfy this property are:

(a) There exist c_1, $c_2 > 0$ such that $c_1 \leqslant f \leqslant c_2$.

(b) There exist c_1, $c_2 > 0$ such that $\sup|f| \leqslant c_1$ and $\sup|f'| \leqslant c_2$.

(c) If t_0 is a point such that $\sup|f| = f(t_0) = c_1 > 0$, then there exists an interval I with $t_0 \in I$ and for any $t \in I$, $|f'(t)| \leqslant c_2 > 0$.

In each of these cases, $M = M(c_1, c_2)$ and does not depend upon the specific f. We observe that physical limitations will force most possible practical inputs to satisfy the M-property for some M.

Now let u and ϕ solve the system (1.3)–(1.4) with $G(x,y) \equiv 1$, $\phi_0 = \phi_0(t)$, $\Gamma_D = \Gamma_0 \cup \Gamma_1$.

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Theorem 3.1 *Suppose u, ϕ are smooth, periodic and $\phi_0^2(t)$ satisfies the M-property. There exists $\eta_0 > 0$, dependent on the data, but independent of specific ϕ_0 except through the M-property, such that if $0 \leqslant \eta < \eta_0$ all periodic solutions (u, ϕ) satisfy $u > 0$.*

Proof Through the transformation of $z = \frac{\phi}{\phi_0}$, we obtain that u and z satisfy

$$u_t - \Delta u + \eta \int_\Omega u = \phi_0^2(t)\nabla[\sigma(u)z\nabla z], \tag{3.4}$$

$$-\nabla[\sigma(u)\nabla z] = 0. \tag{3.5}$$

Here u satisfies the same initial and boundary conditions as before, but z satisfies

$$z|_{\Gamma_0} = 1,\ z|_{\Gamma_1} = 0,\ \left.\frac{\partial z}{\partial \vec{n}}\right|_{\Gamma_N} = 0. \tag{3.6}$$

Let η be small enough so that

$$-\Delta w_1 + \eta \int_\Omega w_1 = \lambda_1 w_1 \tag{3.7}$$

has a positive eigenvalue/eigenvector $\lambda_1 = \lambda_1(\eta)$, $w_1 = w_1(\eta)$ corresponding to Dirichlet conditions. Note that $\lambda_1(\eta)$ is bounded away from zero as $\eta \to 0$. Then

$$\left(\int_\Omega w_1 u\right)_t + \lambda_1 \left(\int_\Omega w_1 u\right) = \phi_0^2(t) \int_\Omega \sigma(u)|\nabla z|^2 w_1.$$

Observe that $\int_\Omega \sigma(u)|\nabla z|^2 w_1$ is bounded above and below by Lemma 3.1. Write the value of this integral as $\tau(t)$ and obtain

$$e^{\lambda_1 t} \int_\Omega w_1 u\Big|_t = \int_\Omega w_1 u\Big|_0 + \int_0^t e^{\lambda_1 s}\phi_0^2(s)\tau(s)ds.$$

Since u is periodic, we have

$$\int_\Omega w_1 u\Big|_t = \frac{e^{-\lambda_1 t}}{e^{\lambda_1 T} - 1} \int_0^T e^{\lambda_1 s}\phi_0^2(s)\tau(s)\,ds + e^{-\lambda_1 t} \int_0^t e^{\lambda_1 s}\phi_0^2(s)\tau(s)ds.$$

Thus, replace u by $u/\int_0^T \phi_0^2(s)ds$ in (3.4) (then $\int_0^T \phi_0^2(s)ds = 1$ in (3.4) and $\int_0^T \phi_0^4(s)ds \leqslant M^2$). We conclude that

$$\int_\Omega w_1 u\Big|_t > C, \tag{3.8}$$

for some positive constant C independent of u.

From equations (3.4), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 + \eta\left(\int_\Omega u\right)^2 = \phi_0^2(t)\int_\Omega \sigma(u)z\nabla z\nabla u \leqslant \frac{1}{2}\|\nabla u\|^2 + c\phi_0^4(t)\int_\Omega \sigma^2(u)z^2|\nabla z|^2,$$

for some constant c, while (3.5)–(3.6) and the Poincare Inequality then yield for some

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c_1, c_2:

$$\frac{d}{dt}\|u\|^2 + c_1\|u\|^2 \leqslant c_2\phi_0^4(t).$$

We obtain by periodicity:

$$\|u\|^2(t) \leqslant c_2\left[1 + \frac{1}{e^{c_1 T}-1}\right]\int_0^T e^{c_1 s}\phi_0^4(s)ds.$$

Thus, $\|u\|^2(t)$ is bounded by the M-property, and we conclude that

$$\left|\int_\Omega u\right| < D,$$

for some constant D independent of the specific $\phi_0(t)$, but dependent on the M-property of ϕ_0, the domain and the bounds of the coefficients.

Finally, we have

$$\int_\Omega uw_1 \geqslant C \text{ and } \left|\int_\Omega u\right| \leqslant D.$$

Now $u_t - \Delta u + \eta\int_\Omega u \geqslant 0$. Put $Lw = w_t - \Delta w$ with $w(x,0) = w(x,T)$. We obtain that for any $t \in [0,T]$:

$$\int_\Omega \left\{u + \eta L^{-1}\left[\int_\Omega u\right]\right\} w_1 \geqslant \int_\Omega \left\{u - \eta \sup_t \left|\int_\Omega u\right| L^{-1}(1)\right\} w_1 \geqslant \frac{C}{2},$$

if η is small enough as $|\int_\Omega u|$ and $L^{-1}(1)$ are bounded. On the other hand, by the same calculation,

$$\int_K \left\{u + \eta L^{-1}\left[\int_\Omega u\right]\right\} w_1 \geqslant \frac{C}{4},$$

where K is a compact subset of Ω. Since $w_1 > 0$ in K and $u + \eta L^{-1}[\int_\Omega u] \geqslant 0$,

$$\int_K \left\{u + \eta L^{-1}\left[\int_\Omega u\right]\right\} \geqslant c > 0.$$

However, since

$$L\left[u + \eta L^{-1}\left(\int_\Omega u\right)\right] \geqslant 0,$$

we can apply Harnack's Inequality [15, 25], and obtain for $t_1 > t_0$

$$\sup_K \left[u + \eta L^{-1}\left(\int_\Omega u\right)\right]\Big|_{t=t_0} \leqslant C \inf_K \left[u + \eta L^{-1}\left(\int_\Omega u\right)\right]\Big|_{t=t_1}.$$

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Since the problem is periodic, $u + \eta L^{-1}(\int_\Omega u)$ at t_0 and $t_0 + T$ is the same, i.e.

$$\inf_K \left[u + \eta L^{-1} \left(\int_\Omega u \right) \right] \geqslant c > 0,$$

for any $t \in [0, T]$.

Let v be the solution of the following problem:

$$-\Delta v = 0 \text{ in } \Omega - K, \quad v = 1 \text{ on } \partial K, \quad v = 0 \text{ on } \partial\Omega.$$

Then $u + \eta L^{-1}(\int_\Omega u) \geqslant cv$ in $\Omega - K$. We also have $u + \eta L^{-1}(\int_\Omega u) \geqslant c$ in K.

Thus, if we choose η small enough, (independent of ϕ_0, except through the M-property), we obtain $u > 0$ in $\Omega \times [0, T]$. □

4 The existence of a uniform attractor and its dimension

4.1 The Hölder continuity of the weak solutions

In this section we consider the Hölder continuity of the weak solutions for positive time t. For the case of a Hölder continuous initial value, we show that the solution belongs to $C^{\alpha,\alpha/2}(Q_T)$ and thus is unique. The results are obtained through Campanato space type arguments together with a cut-off method and are summarized in Theorem 4.1 and Theorem 4.2 below. As we shall see, Theorem 4.1 will play a significant role in the study of the long time behaviour of the weak solutions. Moreover, instead of the methods proposed here, one may apply the known results in Xu [35] and Yuan [36], and classical regularity theorems for parabolic equations, to obtain Theorem 4.2 as well. In our proof below we also give the explicit dependence of Hölder bound on the time (exponential decay) and the initial data.

Before proceeding, we recall some notation and results related to Campanato Spaces. For $0 \leqslant t_0 < t_1$, we denote $\Omega \times (t_0, t_1]$ by Q_{t_0,t_1}. For simplicity, if $t_0 = 0$, we write it as Q_{t_1}. A point $(x, t) \in Q_{t_0,t_1}$ is denoted by z. Let $B_r(x_0)$ be the ball centred at x_0 with radius r and $Q_r(z_0)$ be the cylinder $B_r(x_0) \times (t_0 - r^2, t_0]$. Then we define

$$\Omega[x_0, r] = B_r(x_0) \cap \Omega, \quad Q[z_0, r] = Q_r(z_0) \cap Q_{t_0,t_1}.$$

Moreover, for $\mu \geqslant 0$, $\mathscr{L}^{2,\mu}(\Omega)$ and $\mathscr{L}^{2,\mu}(Q_{t_0,t_1})$ denote the Campanato spaces on Ω and Q_{t_0,t_1} associated with the standard norms $\|.\|_{2,\mu,\Omega}$ and $\|.\|_{2,\mu,Q_{t_0,t_1}}$, respectively. We refer interested readers to Kufner *et al.* [23], Troianiello [28] and Yin [37] for details on these spaces and norms.

Let δ_0 denote the Hölder exponent as stated in the De Giorgi–Nash theorem (see Troianiello [28] and Yin [37]. In the what follows, all α, α_i are in $(0, \delta_0)$ and μ_0, μ_1 are nonnegative numbers such that $\mu_0 < N - 2 + 2\delta_0$ and $\mu_1 < N + 2\delta_0$ where $N = 3$ is the dimension of Ω . They may differ from one step to the next. Furthermore, $(\mu - 2)^+ = \max\{0, \mu - 2\}$.

Theorem 4.1 *Let $t_0 > 0$ and $h > 0$. There are generic constants $\rho_1 > 0$ and $\rho_2 > 0$ which only depend upon h, the bounds of the coefficients, the boundary conditions and $|Q_{t_0,t_0+2h}|$*

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and are independent of t_0 *and the initial value* u_0 *such that the weak solution* u *of* (1.3)–(1.4) *satisfies*

$$\|u\|_{C^{\alpha_0,\alpha_0/2}(Q_{t_0+h,t_0+2h})} \leqslant \rho_1 + \rho_2 e^{-\nu t_0}\|u_0\|, \tag{4.1}$$

for all $0 < \alpha_0 < \delta_0$. *Here* ν *is a positive constant dependent on the domain* Ω *only.*

Proof Let $0 \leqslant \xi(t) \leqslant 1$ be a smooth function such that $\xi(t) = 0$ for $t \leqslant t_0$ and $\xi(t) = 1$ for $t \geqslant t_0 + h$. Furthermore, assume $|\xi_t| \leqslant \beta$ for some constant $\beta > 0$. Let (u, ϕ) be a weak solution of equations (1.3)–(1.4) and consider

$$(\xi u)_t - \Delta(\xi u) + \xi\eta \int_\Omega G(x,y)u(y,t)\,dy = \xi\nabla[\sigma(u)\phi\nabla\phi] + \xi_t u, \quad \xi u(x,t_0) = 0, \quad \xi u|_{\partial\Omega} = 0. \tag{4.2}$$

It follows from Theorem 3.5.1 of [32] that for all $0 \leqslant \mu_0 < N - 2 + 2\delta_0$

$$\|\nabla\phi_n\|_{2,\mu_0,\Omega} \leqslant c. \tag{4.3}$$

By Theorem 1.17 of Troianiello [28], we have

$$r^{-\mu_0}\int_{t_0-r^2}^{t_0}\int_{B_r(x_0)} |\nabla\phi_n|^2 dxdt < cr^2. \tag{4.4}$$

Thus,

$$\sup_{z_0\in Q_{t_0,t_0+2h},r>0} r^{-(\mu_0+2)}\int_{Q_r(z_0)} |\nabla\phi_n|^2 dz < c. \tag{4.5}$$

By using a result of Campanato [8], we obtain that for all $0 \leqslant \mu_1 < N + 2\delta_0$

$$\|\nabla\phi_n\|_{2,\mu_1,Q_{t_0,t_0+2h}} \leqslant c. \tag{4.6}$$

Since $\sigma(u_n)\phi_n \in L^\infty(Q_{t_0,t_0+2h})$, we have $\sigma(u_n)\phi_n\nabla\phi_n \in \mathscr{L}^{2,\mu_1}(Q_{t_0,t_0+2h})$. It follows from Theorem 1 in Yin [37] and (4.2) that

$$\|\xi\nabla u\|_{2,\mu_1,Q_{t_0,t_0+2h}} \leqslant c\big[\|\xi\sigma(u)\phi\nabla\phi\|_{2,\mu_1,Q_{t_0,t_0+2h}} + \|\xi_t u\|_{2,(\mu_1-2)^+,Q_{t_0,t_0+2h}} + \|\xi u\|_{L^2(t_0,t_0+2h;H^1(\Omega))}\big]. \tag{4.7}$$

First, we apply the Energy Inequality and Gronwall Lemma to (1.9) and obtain

$$\|u\|(t) \leqslant c + e^{-\nu t}\|u_0\|, \tag{4.8}$$

and

$$\int_{t_0}^{t_0+2h} \|\nabla u\|^2 \leqslant c(1 + e^{-\nu t_0}\|u_0\|)^2. \tag{4.9}$$

Thus,

$$\|\xi u\|_{L^2(t_0,t_0+2h;H^1(\Omega))} \leqslant c(1 + e^{-\nu t_0}\|u_0\|). \tag{4.10}$$

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Applying (4.6) and (4.10) to (4.7) yields

$$\|\xi\nabla u\|_{2,\mu_1,Q_{t_0,t_0+2h}} \leqslant c\big(1+e^{-\nu t_0}\|u_0\|^2+\|\xi_t u\|_{2,(\mu_1-2)^+,Q_{t_0,t_0+2h}}\big), \tag{4.11}$$

Now for $0 \leqslant \mu_2 \leqslant 2$, $(\mu_2-2)^+=0$. Since $\mathscr{L}^{2,0}(Q_{t_0,t_0+2h})$ is isomorphic to $L^2(Q_{t_0,t_0+2h})$, we obtain

$$\|\xi\nabla u\|_{2,\mu_2,Q_{t_0,t_0+2h}} \leqslant c(1+e^{-\nu t_0}\|u_0\|), \tag{4.12}$$

where (4.8) has been used. Equation (4.12) implies that

$$\|\xi u\|_{2,\mu_2+2,Q_{t_0,t_0+2h}} \leqslant c(1+e^{-\nu t_0}\|u_0\|), \tag{4.13}$$

Now for any $0 \leqslant \mu \leqslant N+2\delta_0$, $(\mu-2)^+ \leqslant 3$, since in our case $N=3$. Thus we conclude from (4.11) and (4.13) that

$$\|\xi\nabla u\|_{2,\mu,Q_{t_0,t_0+2h}} \leqslant c(1+e^{-\nu t_0}\|u_0\|), \tag{4.14}$$

for all $0 \leqslant \mu \leqslant N+2\delta_0$. Consequently,

$$\|\xi u\|_{2,\mu+2,Q_{t_0,t_0+2h}} \leqslant c(1+e^{-\nu t_0}\|u_0\|), \tag{4.15}$$

In particular, for $\mu = N+2+2\alpha_0$ with $\alpha_0 < \delta_0$,

$$\|u\|_{2,N+2+2\alpha_0,Q_{t_0+h,t_0+2h}} \leqslant c(1+e^{-\nu t_0}\|u_0\|), \tag{4.16}$$

where the definition of ξ and (4.15) have been used. Thus, (4.1) follows from (4.16) immediately. □

Theorem 4.2 *If $u_0 \in C^\alpha(\bar{\Omega})$ and $u_0|_{\partial\Omega}=0$ then the weak solution is Hölder continuous.*

Proof We rewrite $u=w+z$. Here z is the unique solution of the simple equation

$$z_t-\Delta z=0,\quad z(x,0)=u_0,\quad z|_{\partial\Omega}=0, \tag{4.17}$$

and w satisfies

$$w_t-\Delta w=-\eta\int_\Omega G(x,y)u(y)dy+\nabla[\sigma(u)\phi\nabla\phi], \tag{4.18}$$

$$w(x,0)=0,\quad w|_{\partial\Omega}=0. \tag{4.19}$$

By a classic result in Ladyzenskaja *et al.* [25], $z \in C^{\alpha,\alpha/2}(Q_T)$. By a similar discussion as in Theorem 4.1, we also have $w \in C^{\alpha,\alpha/2}(Q_T)$. This completes the proof of the theorem. □

From Theorem 4.2, for a given initial value $u_0 \in C^\alpha(\bar{\Omega})$ with $u_0|_{\partial\Omega}=0$, the function u is Hölder continuous. Thus the potential ϕ satisfies the regularity (1.8) with some $p>3$ due to Lemma 5.3.2 in Xie [32], which implies that the weak solution is unique. Moreover, the following proposition holds (see Allegretto *et al.* [3]).

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Theorem 4.3 *Let* (u_i, ϕ_i), $i = 1, 2$, *be two* $C^{\alpha,\alpha/2}(\bar{Q}_T)$ *solutions to* (1.3)–(1.5) *corresponding to the initial data* u_0^i, $i = 1, 2$ *and the same* ϕ_0. *Write* $w = u_1 - u_2$, $\varphi = \phi_1 - \phi_2$, $w_0 = u_0^1 - u_0^2$. *Under the previous assumptions, there exists a constant* $c(t) > 0$ *such that*

$$\|w\|^2 + \int_0^t \|\nabla w(s)\|^2 ds + \int_0^t \|\nabla \varphi(s)\|^2 ds \leqslant c(t)\|w_0\|^2. \tag{4.20}$$

4.2 The uniform attractor and its dimension

Since the number h in Theorem 4.1 is arbitrary, we conclude that if the weak solution starts from an initial value in $L^2(\Omega)$ it will enter into the space $C^\alpha(\bar{\Omega})$ immediately. Since $C^\alpha(\bar{\Omega})$ is a subspace of $L^2(\Omega)$, we conclude that the uniform attractor in $C^\alpha(\bar{\Omega})$, if any, will be the same as in $L^2(\Omega)$. Thus the long time behaviour of the system in $L^2(\Omega)$ is identical to that in $C^\alpha(\bar{\Omega})$. From Theorem 4.2, the weak solution corresponding to an initial value $u_0(x) \in C^\alpha(\bar{\Omega})$ is unique. Thus the solution operator of (1.3)–(1.4) is well defined. Therefore in the rest of the paper we will focus on the case $u_0(x) \in C^\alpha(\bar{\Omega})$ only.

Following the notation of Haraux [21] and Vishik & Chepyzhov [29], we first briefly recall some definitions for the reader's convenience. Let E be a Banach space subject to the action of a two-parameter family of mappings $\{U(t,\tau)\} = \{U(t,\tau), t \geqslant \tau\}$, $U(t,\tau) : E \to E$, $t \geqslant \tau$.

Definition 4.1 A family of operators $\{U(t,\tau)\}$ is said to be a process in E if

1. $U(t,\tau) = U(t,s)U(s,\tau)$ $\forall t \geqslant s \geqslant \tau$.
2. $U(\tau,\tau) = I$ is the identity operator $\forall \tau$.

Moreover, a process $\{U(t,\tau)\}$ is said to be periodic with period T if

3. $U(t+T, \tau+T) = U(t,\tau)$ $\forall t \geqslant \tau$.

Definition 4.2 A set $\mathscr{B} \in E$ is said to be uniformly attracting with respect to $\tau \in R$ for a process $\{U(t,\tau)\}$ if for all τ and for any set A that is bounded in E

$$\sup_{\tau} d(U(t+\tau,\tau)A, \mathscr{B}) \to 0 \quad \text{as } t \to \infty.$$

Here $d(A_1, A_2)$ denotes the usual semi-distance of A_1 and A_2. Furthermore, a process is said to be uniformly asymptotically compact if $\mathscr{B}$ is also compact.

Definition 4.3 A closed subset of E is said to be a uniform attractor for a process $\{U(t,\tau)\}$ if it is the minimal closed uniformly attracting set for this process.

Definition 4.4 A curve $u(s) \in E$, $s \in R$, is called a complete trajectory of a process $\{U(t,\tau)\}$ if $U(t,\tau)u(\tau) = u(t)$ $\forall t \geqslant \tau$; $t, \tau \in R$. The kernel $\mathscr{K}$ of a process $\{U(t,\tau)\}$ consists of all of its bounded complete trajectories. The set $\mathscr{K}(s) = \{u(s) : u(\cdot) \in \mathscr{K}\}$ of values of complete trajectories $u(s)$ with $t = s$ is called the kernel section of this process at the time $t = s$.

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We refer the readers to Haraux [21] and Vishik & Chepyzhov [29] and the references therein for more specific descriptions of complete trajectories, kernels, kernel sections and so on.

We first show the existence of a uniform attractor for the initial boundary value problem (1.3)–(1.5) and (1.7). As in the previous sections, the driving source $\phi_0(x,t)$ is time-periodic with period T. We define the family of two-parameter operators $\{U(t,\tau)\}$: $C^\alpha(\bar{\Omega}) \to C^\alpha(\bar{\Omega})$, $U(t,\tau)u_\tau = u(t)$. Here $u(t)$ is the unique solution of the problem but with the initial value condition replaced by

$$u(x,\tau) = u_\tau(x), \quad u_\tau(x) \in C^\alpha(\bar{\Omega}), \quad u_\tau|_{\partial\Omega} = 0. \tag{4.21}$$

We observe that all conditions in Definition 4.1 are satisfied. Thus, $\{U(t,\tau)\}$ defines a periodic process. Moreover, it follows from Theorem 4.3 that $U(t,\tau)$ is jointly continuous with respect to the potential $\phi_0(x,t)$ and the initial value $u_0(x)$. Let

$$\mathscr{B} = \{w | w \in C^\alpha(\bar{\Omega}), w|_{\partial\Omega} = 0, \text{ and } \|w\|_{C^\alpha(\bar{\Omega})} \leqslant \rho_1 + 1\}, \tag{4.22}$$

where ρ_1 is the positive constant in Theorem 4.1. According to Theorem 4.1, $\mathscr{B}$ is a uniform attracting set of $U(t,\tau)$. Now write

$$\mathscr{B}_1 = \cup_{t\geqslant 0} \cup_{\tau\in[0,T)} U(t+\tau+2h,\tau)\mathscr{B}. \tag{4.23}$$

Here h is a positive constant as specified in Theorem 4.1. We observe that $\mathscr{B}_1$ is also a uniform attracting set. Since inequality (4.1) is satisfied for all $0 < \alpha_0 < \delta_0$, we obtain that $\mathscr{B}_1$ is a bounded subset of $C^{\alpha_0}(\bar{\Omega})$ for all $\alpha < \alpha_0 < \delta_0$. Thus, by the compact imbedding theorem $\mathscr{B}_1$ is precompact in $C^\alpha(\bar{\Omega})$. This implies that the periodic process $U(t,\tau)$ is uniformly asymptotically compact. By a result in Vishik & Chepyzhov [29] (see Theorem 2.1), we conclude the following:

Theorem 4.4 *Under the previous assumptions, the system* (1.3)–(1.5) *has a uniform attractor* $\mathscr{A}$ *in* $C^\alpha(\bar{\Omega})$ *which is nonempty and compact. Furthermore,*

$$\mathscr{A} = \bigcup_{t\in[0,T)} \mathscr{K}(t). \tag{4.24}$$

Here $\mathscr{K}(t)$ *are the kernel sections of* $\{U(t,\tau)\}$*, which have the following properties:*

$$\mathscr{K}(t+T) = \mathscr{K}(t) \quad \forall t \in R, \tag{4.25}$$

$$U(t,\tau)\mathscr{K}(\tau) = \mathscr{K}(t), \quad t \geqslant \tau, \ t, \ \tau \in R. \tag{4.26}$$

In the rest of the paper, we deal with the finite dimensionality of the attractor $\mathscr{A}$ by considerations based on arguments related to these [9, 20, 29]. We first find an upper bound of the dimensions of the kernel sections $\mathscr{K}(t)$. Then by an extension of a result in Vishik & Chepyzhov [29], we conclude that the dimension of the uniform attractor $\mathscr{A}$ is also bounded.

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In this section, besides the assumptions (A1)–(A3), we further assume

A4. $\sigma(s)$ is continuously differentiable and there exist two positive constants σ_2 and θ_0 such that $|\sigma'(s)| \leqslant \sigma_2 < \infty$ and $|\sigma'(s_1) - \sigma'(s_2)| \leqslant \sigma_2 |s_1 - s_2|^{\theta_0}$ for all $s,\ s_1,\ s_2 \geqslant 0$.

We linearize equations (1.3)–(1.4) about (u, ϕ) and obtain

$$\frac{d\bar{u}}{dt} = L(u, \bar{u}), \tag{4.27}$$

$$\nabla[\sigma(u)\nabla\bar{\phi}] = -\nabla[\sigma'(u)\bar{u}\nabla\phi]. \tag{4.28}$$

Here $u(t) = U(t,\tau)u_\tau$ with $u_\tau \in \mathscr{K}(\tau)$. The unknowns satisfy the following initial and boundary conditions:

$$\bar{u}|_{\partial\Omega} = 0, \quad \bar{u}(x,\tau) = \bar{u}_\tau, \tag{4.29}$$

$$\bar{\phi}|_{\Gamma_D} = 0, \quad \frac{\partial\bar{\phi}}{\partial\vec{n}}|_{\Gamma_N} = 0. \tag{4.30}$$

Moreover, the operator $L(u, w)$ is given by

$$L(u, w) = \Delta w + L_1(w) + L_2(u, w), \tag{4.31}$$

with

$$L_1(w) = -\eta \int_\Omega G(x, y)w(y)dy, \tag{4.32}$$

$$L_2(u, w) = \nabla[\sigma(u)\bar{\phi}_w\nabla\phi + \sigma(u)\phi\nabla\bar{\phi}_w + \sigma'(u)\phi w\nabla\phi]. \tag{4.33}$$

Here $\bar{\phi}_w$ is the solution of (4.28) and (4.30) with $\bar{u}$ replaced by w. For the case of $w = \bar{u}$, it is simply denoted by $\bar{\phi}$ as before.

Theorem 4.5 *For any $w,\ v \in H_0^1(\Omega)$, the operator L satisfies*

$$|(L(u, w), v)| \leqslant c\|\nabla w\|\ \|\nabla v\|, \tag{4.34}$$

$$(-L(u, w), w) \geqslant \frac{1}{2}\|\nabla w\|^2 - c\|w\|^2. \tag{4.35}$$

Consequently, the system (4.27)–(4.30) *possesses a unique weak solution.*

Proof We recall that there exists a generic constant $c > 0$ such that

$$\|\phi\|_{L^\infty(\Omega)}(t),\ \|\nabla\phi\|_{L^p(\Omega)}(t) \leqslant c \quad \text{for all } t, \tag{4.36}$$

where $p > 3$ is some positive constant, see Lemma 5.3.2 of Xie [32]. Now we estimate the terms of $L(u, w)$. We first have

$$|(L_1(w), v)| \leqslant c\|w\|\ \|v\|. \tag{4.37}$$

Next we replace $\bar{u}$ by w in (4.28), then multiply its both sides by $\bar{\phi}_w$ and integrate it over

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Ω to obtain

$$\int_\Omega \sigma(u)|\nabla\bar{\phi}_w|^2 = -\int_\Omega \sigma'(u)w\nabla\phi\nabla\bar{\phi}_w \leqslant \sigma_2\|\nabla\phi\|_{L^p(\Omega)}\ \|w\|_{L^{2p/p-2}(\Omega)}\ \|\nabla\bar{\phi}_w\|, \tag{4.38}$$

which gives that

$$\|\nabla\bar{\phi}_w\| \leqslant c\|w\|_{L^{2p/p-2}(\Omega)}. \tag{4.39}$$

Now we are ready to estimate $|(L_2(u,w),v)|$. In fact, it follows from (4.33) that

$$\begin{aligned}|(L_2(u,w),v)| &= |(\sigma(u)\bar{\phi}_w\nabla\phi + \sigma(u)\phi\nabla\bar{\phi}_w + \sigma'(u)\phi w\nabla\phi, \nabla v))\\ &\leqslant [\sigma_1\|\nabla\phi\|_{L^p(\Omega)}\ \|\bar{\phi}_w\|_{L^{2p/p-2}(\Omega)} + \sigma_1\|\phi\|_{L^\infty(\Omega)}\ \|\nabla\bar{\phi}_w\|\\ &\quad + \sigma_2\|\phi\|_{L^\infty(\Omega)}\ \|\nabla\phi\|_{L^p(\Omega)}\ \|w\|_{L^{2p/p-2}(\Omega)}]\ \|\nabla v\|.\end{aligned} \tag{4.40}$$

By the Sobolev imbedding theorem, $\|\bar{\phi}_w\|_{L^{2p/p-2}(\Omega)} \leqslant c\|\nabla\bar{\phi}_w\|$, we conclude from (4.40) that

$$|(L_2(u,w),v)| \leqslant c\|w\|_{L^{2p/p-2}(\Omega)}\ \|\nabla v\|. \tag{4.41}$$

Finally, thanks to (4.37) and (4.41), we have

$$\begin{aligned}|(L(u,w),v)| &\leqslant \|\nabla w\|\ \|\nabla v\| + |(L_1(w),v)| + |(L_2(u,w),v)|\\ &\leqslant \|\nabla w\|\ \|\nabla v\| + c\big(\|w\|\ \|v\| + \|w\|_{L^{2p/p-2}(\Omega)}\ \|\nabla v\|\big).\end{aligned} \tag{4.42}$$

Then (4.34) follows directly from (4.42), the Sobolev imbedding theorem and the Poincare Inequality.

On the other hand

$$\begin{aligned}(-L(u,w),w) &= \|\nabla w\|^2 + (-L_1(w),w) + (-L_2(u,w),w)\\ &\geqslant \|\nabla w\|^2 - c\|w\|_{L^{2p/p-2}(\Omega)}\ \|\nabla w\|,\end{aligned} \tag{4.43}$$

where the property $(-L_1(w),w) \geqslant 0$ and (4.41) are used. From the Sobolev interpolation inequality we have

$$\|w\|^2_{2p/(p-2)} \leqslant c\|w\|^{2-2N/p}\|\nabla w\|^{2N/p}. \tag{4.44}$$

which together with Young inequality and (4.43) give (4.35).

From the properties (4.34) and (4.35), we conclude that the operator $-L$ is continuous and coercive. Thus the existence of a unique solution to the problem (4.27)–(4.30) is just a direct application of the results for abstract Cauchy problems presented in Wloka [30]. □

Denote the process generated by the problem (4.27)–(4.30) by $\{U'(t,\tau)\}$, i.e., $\bar{u}(t) = U'(t,\tau)\bar{u}_\tau$. The following theorem holds

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Theorem 4.6 *The process* $\{U(t,\tau)\}$ *is uniformly quasi-differentiable on the kernel sections* $\{\mathscr{K}(\tau)\}_{\tau\in[0,T)}$ *in* $L^2(\Omega)$ *and* $\{U'(t,\tau)\}$ *is one of its differentials, i.e.*

$$\lim_{\delta\to 0}\sup_{0<\|u_\tau^1-u_\tau^2\|\leqslant\delta}\frac{\left\|U(t,\tau)u_\tau^1-U(t,\tau)u_\tau^2-U'(t,\tau)\left(u_\tau^1-u_\tau^2\right)\right\|}{\left\|u_\tau^1-u_\tau^2\right\|}=0, \tag{4.45}$$

for all $u_\tau^1, u_\tau^2\in\mathscr{K}(\tau)$, $\tau\in[0,T)$ *and* $t\geqslant\tau$.

The proof of this theorem is lengthy and similar to that of Theorem 4.3, thus we leave it to interested readers.

Next we estimate the dimension of the kernel sections $\mathscr{K}(\tau)$, $\tau\in[0,T)$. Let us introduce the following quantities, as in Chepyzhov & Vishik [9], Constantin *et al.* [14] and Temam [27]

$$q_m=\liminf_{t\to 0}\sup_{\tau\in[0,T)}\sup_{u_\tau\in\mathscr{K}(\tau)}\left(\frac{1}{t}\int_\tau^{\tau+t}Tr_mLds\right). \tag{4.46}$$

Here L is the operator defined in (4.31), and Tr_mL is the m-dimensional trace of L defined by $Tr_mL=\sup_Q TrLQ$ with the supremum taken over all the orthogonal projectors Q in $L^2(\Omega)$ on the space QL^2 of dimension m belonging to the domain of L (see [9]).

Before we estimate the bounds of the uniform attractors $\mathscr{A}$, we give an extension to the Proposition 3.2. of Chepyzhov & Vishik [9]. First, let us recall some basic definitions. Let E be a metric space and $Y\subset E$ be a subset of E. Given two positive numbers d and ε, we write

$$\mu_H(Y,d,\varepsilon)=\inf\sum_i r_i^d, \tag{4.47}$$

$$\mu_H(Y,d)=\lim_{\varepsilon\to 0}\mu_H(Y,d,\varepsilon)=\sup_{\varepsilon>0}\mu_H(Y,d,\varepsilon). \tag{4.48}$$

Here the infimum in (4.47) is for all coverings of Y by balls B_{r_i} of E with radius $r_i\leqslant\varepsilon$. Then the Hausdorff dimension of Y in E is defined by

$$\dim_H(Y)=\inf\{d:\ \mu_H(Y,d)=0\}. \tag{4.49}$$

Similarly, let $J(\varepsilon,Y)$ be the minimum number of balls of E of radius ε which is necessary to cover Y, then the fractal dimension of Y in E is defined by

$$\dim_F(Y)=\inf\{d:\ \mu_F(Y,d)=0\}, \tag{4.50}$$

with

$$\mu_F(Y,d)=\limsup_{\varepsilon\to 0}\varepsilon^d J(\varepsilon,Y). \tag{4.51}$$

Let K_0 be a compact subset in E and S be a mapping from $K_0\times[0,T]$ to E such that $S(y,0)=y$ for all $y\in K_0$. We assume that the mapping S is Hölder continuous with respect to both y and t with Hölder exponents ϵ_1 and ϵ_2, respectively, i.e. for all y_1,

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$y_2 \in K_0$ and $t_1, t_2 \in [0, T]$,

$$\|S(y_1, t_1) - S(y_2, t_2)\|_E \leqslant v\left(\|y_1 - y_2\|_E^{\epsilon_1} + |t_1 - t_2|^{\epsilon_2}\right). \tag{4.52}$$

The following theorem is an extension of Proposition 3.2 of Chepyzhov & Vishik [9], where the case $\epsilon_1 = \epsilon_2 = 1$ has been discussed.

Theorem 4.7 *Let K_0 and S be the compact set and the mapping described above, and also let $K_t = S(K_0, t)$, $t \in [0, T]$ and $Y = \cup_{t\in[0,T]} K_t$, then*

$$\dim_F(Y) \leqslant \frac{1}{\epsilon_1} \dim_F(K_0) + \frac{1}{\epsilon_2}, \tag{4.53}$$

$$\dim_H(Y) \leqslant \frac{1}{\epsilon_1} \dim_H(K_0) + \frac{1}{\epsilon_2}. \tag{4.54}$$

Proof We fist consider the fractal dimension of Y. Given $\varepsilon > 0$, We cover K_0 by a family of balls $\{B_{\varepsilon'}(y_j)\}_{j=1}^J$ with $\varepsilon' = (\frac{\varepsilon}{2v})^{\frac{1}{\epsilon_1}}$. Next we give a partition $\{t_i\}_{i=0}^{M+1}$ of $[0, T]$ with $t_0 = 0$ and $t_{M+1} = T$ such that

$$0 < t_{i+1} - t_i \leqslant \left(\frac{\varepsilon}{2v}\right)^{\frac{1}{\epsilon_2}}, \quad M \leqslant T\left(\frac{2v}{\varepsilon}\right)^{\frac{1}{\epsilon_2}}. \tag{4.55}$$

Now for any point (y, t) in $Y \times [0, T]$, we can choose some y_j and t_i such that $\|y - y_j\|_E \leqslant \varepsilon'$ and $|t - t_i| \leqslant (\frac{\varepsilon}{2v})^{\frac{1}{\epsilon_2}}$. Consequently,

$$\|S(y, t) - S(y_j, t_i)\|_E \leqslant v\left(\frac{\varepsilon}{2v} + \frac{\varepsilon}{2v}\right) = \varepsilon \tag{4.56}$$

This implies that $B_\varepsilon(S(y_j, t_i))$, $j = 1, \ldots, J$, $i = 1, \ldots, M$ is a covering of Y. Thus, by definition

$$\begin{aligned}
\mu_F\left(Y, \frac{1}{\epsilon_1}d + \frac{1}{\epsilon_2}\right) &= \limsup_{\varepsilon\to 0} \varepsilon^{\frac{1}{\epsilon_1}d + \frac{1}{\epsilon_2}} JM \\
&\leqslant T\,(2v)^{\frac{1}{\epsilon_1}d + \frac{1}{\epsilon_2}} \limsup_{\varepsilon\to 0} \left(\varepsilon'\right)^d J \\
&= T\,(2v)^{\frac{1}{\epsilon_1}d + \frac{1}{\epsilon_2}}\, \mu_F(K_0, d),
\end{aligned} \tag{4.57}$$

where the last inequality in (4.55) is used. Thus, (4.53) follows immediately.

Similarly, to prove (4.54), we cover K_0 with balls $\{B_{\varepsilon_j}(y_j)\}_{j=1}^J$, where $\varepsilon_j \leqslant (\frac{\varepsilon}{2v})^{\frac{1}{\epsilon_1}}$. For each j, we give a partition $\{t_i^j\}_{i=0}^{M_j+1}$ such that

$$0 < t_{i+1}^j - t_i^j \leqslant (\varepsilon_j)^{\frac{\epsilon_1}{\epsilon_2}}, \quad M \leqslant T(\varepsilon_j)^{-\frac{\epsilon_1}{\epsilon_2}}. \tag{4.58}$$

Then for each point (y, t), we can choose y_j and t_i^j such that $\|S(y, t) - S(y_j, t_i^j)\|_E \leqslant 2v(\varepsilon_j)^{\epsilon_1} \leqslant \varepsilon$. Therefore, the family of balls

$$\left\{B_{2v(\varepsilon_j)^{\epsilon_1}}\left(S\left(y_j, t_i^j\right)\right) : j = 1, \ldots, J,\ i = 1, \ldots, M_j\right\}$$

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is a covering of Y with radii less than ε. Hence

$$\mu_H\left(Y, \frac{1}{\epsilon_1}d + \frac{1}{\epsilon_2}\right) \leqslant \sum_{j=1}^{J}\sum_{i=1}^{M_j}[2v(\varepsilon_j)^{\epsilon_1}]^{\frac{1}{\epsilon_1}d+\frac{1}{\epsilon_2}}$$
$$= \sum_{j=1}^{J} M_j[2v(\varepsilon_j)^{\epsilon_1}]^{\frac{1}{\epsilon_1}d+\frac{1}{\epsilon_2}}$$
$$\leqslant T(2v)^{\frac{1}{\epsilon_1}d+\frac{1}{\epsilon_2}}\sum_{j=1}^{J}(\varepsilon_j)^d, \tag{4.59}$$

which implies (4.54). □

Theorem 4.8 *Under assumptions* (A1)–(A4)*, the Hausdorff dimensions in* $L^2(\Omega)$ *of the kernel sections of* $U(t,\tau)$ *are bounded and*

$$\dim_H(\mathscr{K}(\tau)) \leqslant m_0, \qquad \forall \tau \in [0,T), \tag{4.60}$$

where m_0 *depends upon the boundary conditions, the various bounds of the coefficients and the domain.*

Proof Let Q_m be an m-dimensional orthogonal projector in $L^2(\Omega)$, and $\{w_j\}_{j=1}^m$ be an orthonormal basis in $Q_m L^2(\Omega)$. We recall that

$$TrLQ_m = \sum_{j=1}^{m}(L(u,w_j),w_j). \tag{4.61}$$

Thus, it follows from (4.35) that

$$TrLQ_m \leqslant -\frac{1}{2}\sum_{j=1}^{m}\|\nabla w_j\|^2 + c\sum_{j=1}^{m}\|w_j\|^2. \tag{4.62}$$

However, by Lemma 2.1 of Temam [27], we have

$$\sum_{j=1}^{m}\|\nabla w_j\|^2 \geqslant cm^{1+2/N}. \tag{4.63}$$

Substituting (4.63) into (4.62) yields

$$TrLQ_m \leqslant -\frac{1}{2}c_1 m^{1+2/N} + c_2 m \leqslant -c_3 m^{1+2/N} + c_4, \tag{4.64}$$

where Young's inequality is used. Then by the definition (4.46) of q_m we obtain

$$q_m \leqslant -c_3 m^{1+2/N} + c_4. \tag{4.65}$$

Thus, if $m > (\frac{c_4}{c_3})^{\frac{N}{N+2}}$, $q_m < 0$. By Theorem 4.1 of Chepyzhov & Vishik [9], we conclude

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that the Hausdorff dimensions of the kernel sections $\mathscr{K}(\tau)$ are bounded and (4.60) is satisfied with m_0 the minimal integer such that $m > (\frac{c_4}{c_3})^{\frac{N}{N+2}}$. □

Finally, the estimate of the Hausdorff dimension of the uniform attractor $\mathscr{A}$ is summarized in the following theorem

Theorem 4.9 *Under assumptions* (A1)–(A4)*, the Hausdorff dimension in* $L^2(\Omega)$ *of the uniform attractor* $\mathscr{A}$ *is bounded and satisfies*

$$\dim_H(\mathscr{A}) \leqslant m_0 + \frac{2}{\alpha}. \tag{4.66}$$

Proof We only have to show that $U(t,0)$ satisfies a similar property to (4.52). In fact, let $u_1, u_2 \in \mathscr{K}(0)$ and $t_1, t_2 \geqslant 0$, then

$$\|U(t_1,0)u_1 - U(t_2,0)u_2\| \leqslant \|U(t_1,0)u_1 - U(t_1,0)u_2\| + \|U(t_1,0)u_2 - U(t_2,0)u_2\|. \tag{4.67}$$

In view of Theorem 4.3,

$$\|U(t_1,0)u_1 - U(t_1,0)u_2\| \leqslant c\|u_1 - u_2\|. \tag{4.68}$$

Since $U(t,0)u_2 \in C^{\alpha,\frac{\alpha}{2}}(Q_T)$, we obtain

$$\|U(t_1,0)u_2 - U(t_2,0)u_2\| \leqslant c|t_1 - t_2|^{\frac{\alpha}{2}}. \tag{4.69}$$

Thus $U(t,0)$ satisfies the property (4.52) with $\epsilon_1 = 1$ and $\epsilon_2 = \frac{\alpha}{2}$. According to Theorems 4.7–4.8, (4.66) holds. □

5 Conclusions

In this paper, we have determined the positivity of periodic solutions to a nonlocal thermistor system if the surrounding gas pressure is small. We have also considered the long time behaviour of initial value problem solutions and showed the existence of a uniform attractor. Finally, the Hausdorff dimension of the attractor was estimated. We believe these results to be new even for the classical thermistor system (i.e. (1.1)–(1.2)). The degenerate case involving 'capacity solutions' remains open.

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