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DIVISIBILITY AMONG POWER GCD MATRICES AND POWER LCM MATRICES

SHAOFANG HONG

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Abstract

For any integers x and y, let (x, y) and $[x, y]$ stand for the greatest common divisor and the least common multiple of x and y, respectively. Let a, b and n be positive integers, and let $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. We denote by (S^a) and $[S^a]$ the $n \times n$ matrices having the ath power of (x_i, x_j) and $[x_i, x_j]$, respectively, as the (i, j)-entry. Bourque and Ligh ['On GCD and LCM matrices', *Linear Algebra Appl.* **174** (1992), 65–74] showed that if S is factor closed (that is, S contains all positive divisors of any element of S), then the GCD matrix (S) divides the LCM matrix $[S]$ (written as $(S) \mid [S]$) in the ring $M_n(\mathbb{Z})$ of $n \times n$ matrices over the integers. Hong ['Divisibility properties of power GCD matrices and power LCM matrices', *Linear Algebra Appl.* **428** (2008), 1001–1008] proved that $(S^a) \mid (S^b)$, $(S^a) \mid [S^b]$ and $[S^a] \mid [S^b]$ in the ring $M_n(\mathbb{Z})$ when $a \mid b$ and S is a divisor chain (namely, there is a permutation σ of order n such that $x_{\sigma(1)} \mid \cdots \mid x_{\sigma(n)}$). In this paper, we show that if $a \mid b$ and S is factor closed, then $(S^a) \mid (S^b)$, $(S^a) \mid [S^b]$ and $[S^a] \mid [S^b]$ in the ring $M_n(\mathbb{Z})$. The proof is algebraic and p-adic. Our result extends the Bourque–Ligh theorem. Finally, several interesting conjectures are proposed.

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Keywords and phrases: power GCD matrix, power LCM matrix, factor-closed set, divisibility, p-adic valuation.

1. Introduction

For arbitrary positive integers x and y, we denote by (x, y) the greatest common divisor of x and y and $[x, y]$ their least common multiple. Throughout, let a, b and n be positive integers, and let f be an arithmetic function. Let $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. Let $(f(x_i, x_j))$ (abbreviated by $(f(S))$) and $(f[x_i, x_j])$ (abbreviated by $(f[S])$) stand for the $n \times n$ matrices whose (i, j)-entry is $f((x_i, x_j))$ and $f([x_i, x_j])$, respectively. Evidently, the matrices $(f(S))$ and $(f[S])$ are symmetric. The set S is called *factor closed* (FC) if all positive divisors of any element of S are in S, and is called *gcd closed* if $(x_i, x_j) \in S$ for all integers i and j with $1 \le i, j \le n$. An FC set is clearly gcd closed but the converse is not true. For example, if $a > 1$ is an integer, then the set $S = \{a, 2a, \ldots, na\}$ is gcd closed but not factor closed.

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In 1875, Smith [26] published his renowned result stating that if S is an FC set, then $\det(f(S)) = \prod_{k=1}^{n}(f*\mu)(x_k)$, where μ is the Möbius function, and $f*\mu$ is the Dirichlet convolution of f and μ. There have been many generalisations of Smith's determinant (see [2–7, 12, 13, 16, 28, 30]). The nonsingularity of Smith's matrices was first considered by Bourque and Ligh in [4] and subsequently studied in [10, 14, 15, 19, 20, 24]. The asymptotic behaviour of the eigenvalues of Smith's matrices has also been investigated (see [1, 11, 22, 23]).

Let $\mathbb{Z}$ denote the ring of integers and let $|T|$ stand for the cardinality of a finite set T. When $f(x) = x$, the matrices $(S) := (f(S))$ and $[S] := (f[S])$ are called the *GCD matrix* and the *LCM matrix*, respectively. In 1992, Bourque and Ligh [4] showed that $(S) \mid [S]$ in the ring $M_n(\mathbb{Z})$ of $n \times n$ matrices over the integers if S is FC. That is, there is $A \in M_n(\mathbb{Z})$ such that $[S] = (S)A$ or, equivalently, $[S] = A^T(S)$. Let $x, y \in S$ with $y < x$. If $y \mid x$ and the conditions $y \mid z \mid x$ and $z \in S$ imply that $z \in \{y, x\}$, then y is called a *greatest-type divisor* of x in S. We denote by $G_S(x)$ the set of all the greatest-type divisors of x in S. Obviously, $G_S(x) = G_{S(x)}(x)$ if $S(x)$ is the set of all positive divisors of x in S. The concept of greatest-type divisor was introduced by Hong in [14] and played a key role in his solution [14, 15] of the Bourque–Ligh conjecture [4]. In 2002, Hong [17] proved that there exist gcd-closed sets S with $\max_{x\in S}\{|G_S(x)|\} = 2$ such that $(S) \nmid [S]$ in $M_n(\mathbb{Z})$. Feng *et al.* [9] used the concept of greatest-type divisor to study the divisibility of a power LCM matrix by a power GCD matrix defined on the gcd-closed set S in $M_n(\mathbb{Z})$. Let $C_S := \{f : (f*\mu)(d) \in \mathbb{Z} \text{ whenever } d \mid \operatorname{lcm}(S)\}$, where $\operatorname{lcm}(S)$ means the least common multiple of all the elements of S. From [18, Lemma 4.2(i)], the conditions $d \mid \operatorname{lcm}(S)$ and $f \in C_S$ yield $f(d) \in \mathbb{Z}$. Li and Tan [25] proved that $(f(S)) \mid (f[S])$ if S is gcd closed such that $\max_{x\in S}\{|G_S(x)|\} = 1$ and f is multiplicative with $f \in D_S := \{f \in C_S : f(u) \mid f(v) \text{ whenever } u \mid v \text{ and } u, v \in S\}$ and $(f(S))$ is nonsingular.

When $f = \xi_a$ with $\xi_a(x) := x^a$, the matrices $(S^a) := (f(S))$ and $[S^b] := (f[S])$ are called the ath *power GCD matrix* and ath *power LCM matrix*, respectively. Hong [21] initiated the study of divisibility among power GCD matrices and among power LCM matrices and proved that $(S^a) \mid (S^b)$, $(S^a) \mid [S^b]$ and $[S^a] \mid [S^b]$ in the ring $M_n(\mathbb{Z})$ when $a \mid b$ and S is a divisor chain (that is, there is a permutation σ of order n on $\{1, 2, \ldots, n\}$ such that $x_{\sigma(1)} \mid \cdots \mid x_{\sigma(n)}$). Hong's results were extended by Tan and Li [27] to the case of finitely many coprime divisor chains. In 2022, Zhu [29] and, in 2023, Zhu and Li [31] showed that if $a \mid b$ and S is a gcd-closed set with $\max_{x\in S}\{|G_S(x)|\} = 1$, then $(S^a) \mid (S^b)$, $(S^a) \mid [S^b]$ and $(S^a) \mid [S^b]$ in the ring $M_n(\mathbb{Z})$.

In this paper, we examine the divisibility among the power GCD matrices and the power LCM matrices defined on factor-closed sets. The main result of this paper can be stated as the following theorem.

THEOREM 1.1. *Let S be a factor-closed set and let a and b be positive integers with $a \mid b$. In the ring $M_{|S|}(\mathbb{Z})$, the ath power GCD matrix (S^a) divides both the bth power GCD matrix (S^b) and the bth power LCM matrix $[S^b]$, and the ath power LCM matrix $[S^a]$ divides the bth power LCM matrix $[S^b]$.*

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The proof of Theorem 1.1 is algebraic and p-adic. Setting $a = b$, Theorem 1.1 becomes the Bourque–Ligh theorem presented in [7].

COROLLARY 1.2 [7]. *Let S be a factor-closed set and let a be a positive integer. Then, the ath power LCM matrix $[S^a]$ is divisible by the ath power GCD matrix (S^a) in the ring $M_{|S|}(\mathbb{Z})$.*

This paper is organised as follows. In Section 2, we establish the results on the divisibility among power GCD and among power LCM matrices on factor-closed sets. As a corollary, Theorem 1.1 follows immediately. Section 3 is devoted to some interesting remarks and conjectures.

2. Divisibility in the factor-closed case

In this section, we investigate the divisibility among power GCD and among power LCM matrices on factor-closed sets. As usual, for any given prime number p and any integer x, we denote by $v_p(x)$ the *p-adic valuation of x*, that is, the largest nonnegative integer r such that p^r divides x. We write $p^l \| x$ if $l = v_p(x)$. We begin with a well-known result of Bourque and Ligh.

LEMMA 2.1 [6]. *Let $S = \{x_1, \ldots, x_n\}$ be an FC set. Then, $(S^a)^{-1} = ((x_i, x_j)^a)^{-1} = (\alpha_{ij})$, where*

$$\alpha_{ij} = \sum_{\substack{x_i \mid x_k \\ x_j \mid x_k}} \frac{\mu(x_k/x_i)\mu(x_k/x_j)}{\phi_a(x_k)}$$

and ϕ_a is the Jordan totient function, that is, $\phi_a(x_k) = (\xi_a * \mu)(x_k) = \sum_{d \mid x_k} d^a \mu(x_k/d)$.

LEMMA 2.2. *Let a and b be positive integers with $a \mid b$. Then, for any positive integer m,*

$$\frac{\phi_b(m)}{\phi_a(m)} = \prod_{p \mid m} p^{(b-a)(v_p(m)-1)} \cdot \frac{p^b - 1}{p^a - 1} \in \mathbb{Z}.$$

PROOF. Let $m \geq 1$ be an integer. If $m = 1$, then $\phi_a(m) = (\xi_a * \mu)(1) = 1$ and $\phi_b(m) = 1$. So Lemma 2.2 is true when $m = 1$. In what follows, we let $m \geq 2$. Write $m = p_1^{e_1} \cdots p_r^{e_r}$, where $r \geq 1$, $p_1, \ldots, p_r$ are r distinct primes and $e_1, \ldots, e_r$ are positive integers. Since $\phi_a = \xi_a * \mu$ and $\phi_b = \xi_b * \mu$, and ξ_a, ξ_b and μ are multiplicative functions, ϕ_a and ϕ_b are multiplicative functions. It then follows that

$$\begin{aligned}\frac{\phi_b(m)}{\phi_a(m)} &= \prod_{i=1}^{r} \frac{\phi_b(p_i^{e_i})}{\phi_a(p_i^{e_i})} = \prod_{i=1}^{r} \frac{(\xi_b * \mu)(p_i^{e_i})}{(\xi_a * \mu)(p_i^{e_i})} = \prod_{i=1}^{r} \frac{\xi_b(p_i^{e_i})\mu(1) + \xi_b(p_i^{e_i-1})\mu(p_i)}{\xi_a(p_i^{e_i})\mu(1) + \xi_a(p_i^{e_i-1})\mu(p_i)} \\ &= \prod_{i=1}^{r} \frac{(p_i^{e_i})^b - (p_i^{e_i-1})^b}{(p_i^{e_i})^a - (p_i^{e_i-1})^a} = \prod_{i=1}^{r} (p_i^{e_i-1})^{b-a} \cdot \frac{p_i^b - 1}{p_i^a - 1} = \prod_{p^l \| m} p^{(b-a)(l-1)} \cdot \frac{p^b - 1}{p^a - 1}.\end{aligned}$$

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Since $a \mid b$, we have $(p^b - 1)/(p^a - 1) \in \mathbb{Z}$. Thus, $\phi_b(m)/\phi_a(m) \in \mathbb{Z}$ and Lemma 2.2 is proved. □

The first main result of this section is the following theorem.

THEOREM 2.3. *Let S be a factor-closed set, and let a and b be positive integers with $a \mid b$. Then, $(S^b) = (S^a)A$, where $A = (a_{ij})$ is an integer matrix of order $|S|$ with*

$$a_{ij} := \begin{cases} 0 & \text{if } x_i \nmid x_j, \\ \displaystyle\sum_{x_i \mid x_k \mid x_j} \mu\Big(\frac{x_k}{x_i}\Big) \prod_{p \mid x_k} p^{(b-a)(\nu_p(x_k)-1)} \sum_{t=0}^{b/a-1} p^{at} & \text{otherwise.} \end{cases}$$

PROOF. It is clear that $\phi_b(p^l) = p^{bl} - p^{b(l-1)} = p^{b(l-1)}(p^b - 1)$ for the prime power p^l $(l \geq 1)$. Set $f_{x_j,b}(x) := (x, x_j)^b$ for all $x \in \mathbb{Z}$. Since S is factor closed, by Lemma 2.1,

$$\begin{aligned} ((S^a)^{-1}(S^b))_{ij} &= \sum_{t=1}^{n} (S^a)^{-1}_{it} \cdot (S^b)_{tj} = \sum_{t=1}^{n} (x_t, x_j)^b \alpha_{it} = \sum_{t=1}^{n} (x_t, x_j)^b \sum_{\substack{x_i \mid x_k \\ x_t \mid x_k}} \frac{\mu(x_k/x_i)\mu(x_k/x_t)}{\phi_a(x_k)} \\ &= \sum_{x_i \mid x_k} \phi_a(x_k)^{-1} \mu\Big(\frac{x_k}{x_i}\Big) \sum_{\substack{t=1 \\ x_t \mid x_k}}^{n} (x_t, x_j)^b \mu\Big(\frac{x_k}{x_t}\Big) = \sum_{x_i \mid x_k} \frac{\mu(x_k/x_i)}{\phi_a(x_k)} \sum_{d \mid x_k} (d, x_j)^b \mu\Big(\frac{x_k}{d}\Big) \\ &= \sum_{x_i \mid x_k} \frac{\mu(x_k/x_i)}{\phi_a(x_k)} \sum_{d \mid x_k} f_{x_j,b}(d) \mu\Big(\frac{x_k}{d}\Big) = \sum_{x_i \mid x_k} \frac{\mu(x_k/x_i)}{\phi_a(x_k)} \cdot (f_{x_j,b} * \mu)(x_k). \end{aligned} \tag{2.1}$$

Since $\gcd(x, d_1 d_2) = \gcd(x, d_1) \cdot \gcd(x, d_2)$ whenever $(d_1, d_2) = 1$, the function $f_{x_j,b}$ is a multiplication function. Computing $(f_{x_j,b} * \mu)(p^l)$ for the prime power p^l $(l \geq 1)$,

$$\begin{aligned} (f_{x_j,b} * \mu)(p^l) &= \sum_{d \mid p^l} (d, x_j)^b \mu\Big(\frac{p^l}{d}\Big) = \sum_{e=0}^{l} (p^e, x_j)^b \mu(p^{l-e}) \\ &= (p^l, x_j)^b \mu(1) + (p^{l-1}, x_j)^b \mu(p) = (p^l, x_j)^b - (p^{l-1}, x_j)^b \\ &= \begin{cases} 0 & \text{if } p^l \nmid x_j \Leftrightarrow \nu_p(x_j) \leq l-1, \\ \phi_b(p^l) & \text{if } p^l \mid x_j \Leftrightarrow \nu_p(x_j) \geq l. \end{cases} \end{aligned}$$

Thus,

$$(f_{x_j,b} * \mu)(x_k) = \begin{cases} \phi_b(x_k) & \text{if } x_k \mid x_j, \\ 0 & \text{if } x_k \nmid x_j. \end{cases} \tag{2.2}$$

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It then follows from (2.1) and (2.2) that

$$((S^a)^{-1}(S^b))_{ij} = \sum_{\substack{x_i \mid x_k \\ x_k \mid x_j}} \frac{\mu(x_k/x_i)}{\phi_a(x_k)} \cdot (f_{x_j,b} * \mu)(x_k) + \sum_{\substack{x_i \mid x_k \\ x_k \nmid x_j}} \frac{\mu(x_k/x_i)}{\phi_a(x_k)} \cdot (f_{x_j,b} * \mu)(x_k)$$

$$= \sum_{\substack{x_i \mid x_k \\ x_k \mid x_j}} \frac{\mu(x_k/x_i)}{\phi_a(x_k)} \phi_b(x_k) = \begin{cases} 0 & \text{if } x_i \nmid x_j, \\ \displaystyle\sum_{x_i \mid x_k \mid x_j} \frac{\phi_b(x_k)}{\phi_a(x_k)} \mu\Big(\frac{x_k}{x_i}\Big) & \text{if } x_i \mid x_j. \end{cases} \tag{2.3}$$

By (2.3) and Lemma 2.2,

$$((S^a)^{-1}(S^b))_{ij} = \begin{cases} 0 & \text{if } x_i \nmid x_j, \\ \displaystyle\sum_{x_i \mid x_k \mid x_j} \mu\Big(\frac{x_k}{x_i}\Big) \prod_{p^l \| x_k} p^{(b-a)(l-1)} \sum_{t=0}^{b/a-1} (p^a)^t & \text{if } x_i \mid x_j, \end{cases}$$

$$= a_{ij}.$$

This concludes the proof of Theorem 2.3. □

We can now give the second main result of this section.

THEOREM 2.4. *Let S be a factor-closed set, and let a and b be positive integers with $a \mid b$. Then, $[S^b] = (S^a)B$, where $B = (b_{ij})$ is an integer matrix of order $|S|$ with*

$$b_{ij} := \sum_{x_i \mid x_k} \Big(\frac{x_j}{(x_k, x_j)}\Big)^b \mu\Big(\frac{x_k}{x_i}\Big) \frac{\phi_b(x_k)}{\phi_a(x_k)} \delta(x_k, x_j)$$

and

$$\delta(x_k, x_j) := \begin{cases} 0 & \text{if } p^l \mid x_j \text{ for some } p^l \| x_k, \\ 1 & \text{otherwise.} \end{cases}$$

PROOF. Set $g_{x_j,b}(x) := x^b/(x, x_j)^b$ for all $x \in \mathbb{Z}$. Since S is factor closed, Lemma 2.1 yields

$$((S^a)^{-1}[S^b])_{ij} = \sum_{t=1}^{n} ((S^a)^{-1})_{it} [S^b]_{tj} = \sum_{t=1}^{n} \alpha_{it} [x_t, x_j]^b = \sum_{t=1}^{n} \frac{x_t^b x_j^b}{(x_t, x_j)^b} \cdot \alpha_{it}$$

$$= x_j^b \sum_{t=1}^{n} \frac{x_t^b}{(x_t, x_j)^b} \sum_{\substack{x_i \mid x_k \\ x_t \mid x_k}} \frac{1}{\phi_a(x_k)} \mu\Big(\frac{x_k}{x_i}\Big) \mu\Big(\frac{x_k}{x_t}\Big)$$

$$= x_j^b \sum_{x_i \mid x_k} \frac{\mu(x_k/x_i)}{\phi_a(x_k)} \sum_{\substack{t=1 \\ x_t \mid x_k}}^{n} \frac{x_t^b}{(x_t, x_j)^b} \mu\Big(\frac{x_k}{x_t}\Big)$$

$$= x_j^b \sum_{x_i \mid x_k} \frac{\mu(x_k/x_i)}{\phi_a(x_k)} \sum_{d \mid x_k} \frac{d^b}{(d, x_j)^b} \mu\Big(\frac{x_k}{d}\Big)$$

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$$= x_j^b \sum_{x_i \mid x_k} \frac{\mu(x_k/x_i)}{\phi_a(x_k)} \sum_{d \mid x_k} g_{x_j,b}(d)\mu\left(\frac{x_k}{d}\right)$$
$$= x_j^b \sum_{x_i \mid x_k} \frac{\mu(x_k/x_i)}{\phi_a(x_k)} (g_{x_j,b} * \mu)(x_k). \tag{2.4}$$

For any coprime positive integers d_1 and d_2,

$$g_{x_j,b}(d_1 d_2) = \frac{d_1^b d_2^b}{(d_1 d_2, x_j)^b} = \frac{d_1^b}{(d_1, x_j)^b} \cdot \frac{d_2^b}{(d_2, x_j)^b} = g_{x_j,b}(d_1) g_{x_j,b}(d_2),$$

so $g_{x_j,b}$ is a multiplicative function. Now let $l = \nu_p(x_k)$. Calculating $(g_{x_j,b} * \mu)(p^l)$,

$$(g_{x_j,b} * \mu)(p^l) = \sum_{e=0}^{l} g_{x_j,b}(p^e)\mu(p^{l-e}) = g_{x_j,b}(p^l) - g_{x_j,b}(p^{l-1}) = \frac{p^{lb}}{(p^l, x_j)^b} - \frac{p^{(l-1)b}}{(p^{l-1}, x_j)^b}.$$

If $p^l \mid x_j$, then $\nu_p(x_j) \geq l$ and

$$\frac{p^{lb}}{(p^l, x_j)^b} - \frac{p^{(l-1)b}}{(p^{l-1}, x_j)^b} = \frac{p^{lb}}{p^{lb}} - \frac{p^{(l-1)b}}{p^{(l-1)b}} = 1 - 1 = 0.$$

If $p^l \nmid x_j$, then $\nu_p(x_j) \leq l - 1$ and so

$$\frac{p^{lb}}{(p^l, x_j)^b} - \frac{p^{(l-1)b}}{(p^{l-1}, x_j)^b} = \frac{p^{lb}}{p^{b \cdot \nu_p(x_j)}} - \frac{p^{(l-1)b}}{p^{b \cdot \nu_p(x_j)}} = \frac{p^{lb}}{p^{b \cdot \nu_p(x_j)}}\left(1 - \frac{1}{p^b}\right) = \frac{\phi_b(p^l)}{(p^l, x_j)^b}.$$

Hence,

$$(g_{x_j,b} * \mu)(p^l) = \begin{cases} 0 & \text{if } p^l \mid x_j \Leftrightarrow \nu_p(x_j) \geq l = \nu_p(x_k), \\ \dfrac{\phi_b(p^l)}{(p^l, x_j)^b} & \text{if } p^l \nmid x_j \Leftrightarrow \nu_p(x_j) \leq l - 1 = \nu_p(x_k) - 1. \end{cases}$$

It then follows from

$$(g_{x_j,b} * \mu)(x_k) = \prod_{p^l \| x_k} (g_{x_j,b} * \mu)(p^l)$$

that $(g_{x_j,b} * \mu)(x_k) = 0$ if there exists a prime power p^l with $p^l \| x_k$ such that $p^l \mid x_j$. Furthermore, if $p^l \nmid x_j$ for any prime power p^l with $p^l \| x_k$, then

$$(g_{x_j,b} * \mu)(x_k) = \prod_{\substack{p^l \| x_k \\ p^l \nmid x_j}} \frac{p^{lb}}{p^{b\nu_p(x_j)}}\left(1 - \frac{1}{p^b}\right) = \frac{1}{\prod\limits_{\substack{p^l \| x_k \\ p^l \nmid x_j}} p^{b\nu_p(x_j)}} \prod_{\substack{p^l \| x_k \\ p^l \nmid x_j}} p^{lb}\left(1 - \frac{1}{p^b}\right) = \frac{\phi_b(x_k)}{\gcd(x_k, x_j)^b}.$$

Therefore,

$$(g_{x_j,b} * \mu)(x_k) = \begin{cases} 0 & \text{if } p^l \mid x_j \text{ for some prime power } p^l \| x_k, \\ \dfrac{\phi_b(x_k)}{\gcd(x_k, x_j)^b} & \text{otherwise.} \end{cases} \tag{2.5}$$

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Substituting (2.5) into (2.4) yields

$$((S^a)^{-1}[S^b])_{ij} = \sum_{x_i|x_k} \left(\frac{x_j}{\gcd(x_k, x_j)}\right)^b \mu\left(\frac{x_k}{x_i}\right) \frac{\phi_b(x_k)}{\phi_a(x_k)} \delta(x_k, x_j) = b_{ij},$$

where

$$\delta(x_k, x_j) := \begin{cases} 0 & \text{if } p^l \mid x_j \text{ for some prime power } p^l \| x_k, \\ 1 & \text{otherwise.} \end{cases}$$

This finishes the proof of Theorem 2.4. □

The following result is also due to Bourque and Ligh.

LEMMA 2.5 [7]. *Let S be an FC set. Then, $[S^a]^{-1} = (\beta_{ij})$, where*

$$\beta_{ij} = \frac{1}{(x_i x_j)^a} \sum_{\substack{x_i|x_k \\ x_j|x_k}} \frac{x_k^{2a} v(x_k)}{r(x_k)^a \phi_a(x_k)} \mu\left(\frac{x_k}{x_i}\right) \mu\left(\frac{x_k}{x_j}\right),$$

with v and r being multiplicative functions defined for a prime power p^e by $v(p^e) = -1$ and $r(p^e) = p$.

The third main result of this section is the following theorem.

THEOREM 2.6. *Let S be a factor-closed set and let a and b be positive integers with $a \mid b$. Then, $[S^b] = [S^a]C$, where $C = (c_{ij})$ is an integer matrix of order $|S|$ with*

$$c_{ij} := \sum_{x_i|x_k} (-1)^\alpha \left(\frac{x_k}{x_i}\right)^a \mu\left(\frac{x_k}{x_i}\right) \prod_{\substack{p|x_k \\ \nu_p(x_j) \geq \nu_p(x_k)}} p^{\nu_p(x_j)b - \nu_p(x_k)a}$$
$$\times \prod_{\substack{p|x_k \\ \nu_p(x_j) < \nu_p(x_k)}} \frac{p^{(b-a)(\nu_p(x_k)-1)}(p^{b-a}-1)}{p^a - 1} \prod_{\substack{p|x_j \\ p \nmid x_k}} p^{\nu_p(x_j)b}$$

and $\alpha = \sum_{p|x_k,\, \nu_p(x_j) < \nu_p(x_k)} 1$.

PROOF. By Lemma 2.5,

$$c_{ij} = ([S^a]^{-1}[S^b])_{ij} = \sum_{t=1}^n ([S^a]^{-1})_{it} \cdot [S^b]_{tj}$$
$$= \sum_{t=1}^n \beta_{it} [x_t, x_j]^b = \sum_{t=1}^n \frac{x_t^b x_j^b}{(x_t, x_j)^b} \beta_{it}$$
$$= x_j^b \sum_{t=1}^n \frac{x_t^b}{(x_t, x_j)^b} \frac{1}{x_i^a x_t^a} \sum_{\substack{x_i|x_k \\ x_t|x_k}} \frac{x_k^{2a} v(x_k)}{r(x_k)^a \phi_a(x_k)} \mu\left(\frac{x_k}{x_i}\right) \mu\left(\frac{x_k}{x_t}\right)$$
$$= \frac{x_j^b}{x_i^a} \sum_{x_i|x_k} \frac{x_k^{2a} v(x_k) \mu(x_k/x_i)}{r(x_k)^a \phi_a(x_k)} \sum_{\substack{t=1 \\ x_t|x_k}}^n \frac{x_t^{b-a}}{(x_t, x_j)^b} \mu\left(\frac{x_k}{x_t}\right).$$

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Define the arithmetic function $h_{x_j,a,b}$ by $h_{x_j,a,b}(x) := x^{b-a}/(x, x_j)^b$ for all positive integers x. Since S is factor closed,

$$\begin{aligned} c_{ij} &= \frac{x_j^b}{x_i^a} \sum_{x_i \mid x_k} \frac{x_k^{2a} \nu(x_k) \mu(x_k/x_i)}{r(x_k)^a \phi_a(x_k)} \sum_{d \mid x_k} \frac{d^{b-a}}{(d, x_j)^b} \mu\left(\frac{x_k}{d}\right) \\ &= \frac{x_j^b}{x_i^a} \sum_{x_i \mid x_k} \frac{x_k^{2a} \nu(x_k) \mu(x_k/x_i)}{r(x_k)^a \phi_a(x_k)} \sum_{d \mid x_k} h_{x_j,a,b}(d) \mu\left(\frac{x_k}{d}\right) \\ &= \frac{x_j^b}{x_i^a} \sum_{x_i \mid x_k} \frac{x_k^{2a} \nu(x_k) \mu(x_k/x_i)}{r(x_k)^a \phi_a(x_k)} (h_{x_j,a,b} * \mu)(x_k). \end{aligned}$$

For all positive integers d_1 and d_2 with $(d_1, d_2) = 1$,

$$h_{x_j,a,b}(d_1 d_2) = \frac{(d_1 d_2)^{b-a}}{(d_1 d_2, x_j)^b} = \frac{d_1^{b-a}}{(d_1, x_j)^b} \cdot \frac{d_2^{b-a}}{(d_2, x_j)^b} = h_{x_j,a,b}(d_1) h_{x_j,a,b}(d_2),$$

so $h_{x_j,a,b}$ is a multiplicative function. Computing $(h_{x_j,a,b} * \mu)(p^l)$ for $l = \nu_p(x_k)$,

$$(h_{x_j,a,b} * \mu)(p^l) = h_{x_j,a,b}(p^l) - h_{x_j,a,b}(p^{l-1}) = \frac{p^{l(b-a)}}{(p^l, x_j)^b} - \frac{p^{(l-1)(b-a)}}{(p^{l-1}, x_j)^b}.$$

If $p^l \mid x_j$, then $\nu_p(x_j) \geq l = \nu_p(x_k)$, and so

$$\frac{p^{l(b-a)}}{(p^l, x_j)^b} - \frac{p^{(l-1)(b-a)}}{(p^{l-1}, x_j)^b} = \frac{p^{l(b-a)}}{p^{lb}} - \frac{p^{(l-1)(b-a)}}{p^{(l-1)b}} = \frac{1}{p^{la}} - \frac{1}{p^{(l-1)a}} = \frac{1 - p^a}{p^{la}}.$$

If $p^l \nmid x_j$, then $\nu_p(x_j) \leq l - 1 = \nu_p(x_k) - 1$ and so

$$\frac{p^{l(b-a)}}{(p^l, x_j)^b} - \frac{p^{(l-1)(b-a)}}{(p^{l-1}, x_j)^b} = \frac{p^{l(b-a)} - p^{(l-1)(b-a)}}{p^{b\nu_p(x_j)}} = \frac{p^{l(b-a)}(1 - 1/p^{b-a})}{p^{b\nu_p(x_j)}}.$$

So,

$$(h_{x_j,a,b} * \mu)(p^l) = \begin{cases} \dfrac{1 - p^a}{p^{la}} & \text{if } p^l \mid x_j \Leftrightarrow \nu_p(x_j) \geq l = \nu_p(x_k), \\ \dfrac{p^{l(b-a)}(1 - 1/p^{b-a})}{p^{b \cdot \nu_p(x_j)}} & \text{if } p^l \nmid x_j \Leftrightarrow \nu_p(x_j) \leq l - 1 = \nu_p(x_k) - 1. \end{cases}$$

Thus,

$$\begin{aligned} (h_{x_j,a,b} * \mu)(x_k) &= \prod_{p^l \| x_k} (h_{x_j,a,b} * \mu)(p^l) = \prod_{\substack{p^l \| x_k \\ \nu_p(x_j) \geq l}} (h_{x_j,a,b} * \mu)(p^l) \cdot \prod_{\substack{p^l \| x_k \\ \nu_p(x_j) < l}} (h_{x_j,a,b} * \mu)(p^l) \\ &= \prod_{\substack{p^l \| x_k \\ \nu_p(x_j) \geq l}} \frac{1 - p^a}{(p^l)^a} \cdot \prod_{\substack{p^l \| x_k \\ \nu_p(x_j) < l}} \frac{p^{(l-1)(b-a)}(p^{b-a} - 1)}{(p^{\nu_p(x_j)})^b}. \end{aligned}$$

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Moreover,

$$\phi_a(x_k) = x_k^a \prod_{p|x_k} \left(1 - \frac{1}{p^a}\right) = x_k^a \prod_{p^l \| x_k} \left(1 - \frac{1}{p^a}\right).$$

Hence,

$$\begin{aligned} c_{ij} &= \frac{x_j^b}{x_i^a} \sum_{x_i|x_k} \frac{x_k^{2a} v(x_k)\mu(x_k/x_i)}{r(x_k)^a x_k^a \cdot \prod_{p^l \| x_k}(1 - 1/p^a)} \prod_{\substack{p^l \| x_k \\ \nu_p(x_j) \geq l}} \frac{1 - p^a}{(p^l)^a} \prod_{\substack{p^l \| x_k \\ \nu_p(x_j) < l}} \frac{p^{(l-1)(b-a)}(p^{b-a} - 1)}{(p^{\nu_p(x_j)})^b} \\ &= x_j^b \sum_{x_i|x_k} \left(\frac{x_k}{x_i}\right)^a \mu\left(\frac{x_k}{x_i}\right) \frac{v(x_k)}{\prod_{p|x_k}(p^a - 1)} \prod_{\substack{p^l \| x_k \\ \nu_p(x_j) \geq l}} \frac{1 - p^a}{(p^l)^a} \prod_{\substack{p^l \| x_k \\ \nu_p(x_j) < l}} \frac{p^{(l-1)(b-a)}(p^{b-a} - 1)}{(p^{\nu_p(x_j)})^b}. \end{aligned}$$

It then follows that

$$\begin{aligned} c_{ij} &= \sum_{x_i|x_k} \left(\frac{x_k}{x_i}\right)^a \mu\left(\frac{x_k}{x_i}\right) v(x_k) \prod_{\substack{p|x_k \\ \nu_p(x_j) \geq \nu_p(x_k)}} \frac{p^{\nu_p(x_j)b}}{p^{\nu_p(x_k)a}} \prod_{\substack{p|x_k \\ \nu_p(x_j) < \nu_p(x_k)}} \frac{p^{\nu_p(x_j)b}}{p^{\nu_p(x_j)b}} \\ &\quad \times \prod_{\substack{p|x_k \\ \nu_p(x_j) \geq \nu_p(x_k)}} \frac{1 - p^a}{p^a - 1} \cdot \prod_{\substack{p|x_k \\ \nu_p(x_j) < \nu_p(x_k)}} \frac{p^{(b-a)(\nu_p(x_k)-1)}(p^{b-a} - 1)}{p^a - 1} \prod_{\substack{p|x_j \\ p \nmid x_k}} p^{\nu_p(x_j)b} \\ &= \sum_{x_i|x_k} \left(\frac{x_k}{x_i}\right)^a \mu\left(\frac{x_k}{x_i}\right) (-1)^{\sum_{p|x_k, \nu_p(x_j) < \nu_p(x_k)} 1} \prod_{\substack{p|x_k \\ \nu_p(x_j) \geq \nu_p(x_k)}} p^{\nu_p(x_j)b - \nu_p(x_k)a} \\ &\quad \times \prod_{\substack{p|x_k \\ \nu_p(x_j) < \nu_p(x_k)}} \frac{p^{(b-a)(\nu_p(x_k)-1)}(p^{b-a} - 1)}{p^a - 1} \prod_{\substack{p|x_j \\ p \nmid x_k}} p^{\nu_p(x_j)b} \end{aligned}$$

as required. This completes the proof of Theorem 2.6. □

Finally, it is easy to see that the conclusion of Theorem 1.1 follows immediately from Theorems 2.3, 2.4 and 2.6.

3. Remarks and conjectures in the gcd-closed case

For any set R of positive integers and for any $x \in R$, we define $g_R(x) := |G_R(x)|$ to be the number of greatest-type divisors of x in R. For brevity, we write $g(x)$ for $g_R(x)$.

DEFINITION 3.1. Let S be a set of positive integers. Let $x \in S$ with $g(x) \geq 2$.

(i) We say that the two distinct greatest-type divisors y_1 and y_2 of x in S *satisfy the condition* $\mathcal{G}$ if $[y_1, y_2] = x$ and $(y_1, y_2) \in G_S(y_1) \cap G_S(y_2)$.

(ii) We say that x *satisfies the condition* $\mathcal{G}$ if any two distinct greatest-type divisors of x in S satisfy the condition $\mathcal{G}$.

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It should be pointed out that if S is gcd closed, then the definition of condition $\mathcal{G}$ given above coincides with that of condition C given in [9].

DEFINITION 3.2. Let S be a set of positive integers. We say that the set S *satisfies the condition* $\mathcal{G}$ if, for any element x in S, either $g(x) \leq 1$, or $g(x) \geq 2$ and x satisfies the condition $\mathcal{G}$.

For instance, let $S = \{2, 3, 4, 5, 9, 11, 12, 15, 33, 36, 55, 165\}$. Then, S is not gcd closed and S does not satisfy the condition $\mathcal{G}$. In fact, $g(2) = g(3) = g(5) = g(11) = 0$, $g(4) = g(9) = 1$, $g(12) = g(15) = g(33) = g(36) = g(55) = 2$ and $g(165) = 3$. Since $G_S(12) = \{3, 4\}$ and $(3, 4) = 1 \notin G_S(3) \cap G_S(4)$, it follows that 12 does not satisfy the condition $\mathcal{G}$. Likewise, each of 15, 33 and 55 does not satisfy the condition $\mathcal{G}$. Since $G_S(36) = \{9, 12\}$ and $[9, 12] = 36$ as well as $(9, 12) = 3 \in G_S(9) \cap G_S(12)$, 9 and 12 satisfy the condition $\mathcal{G}$ and so 36 satisfies the condition $\mathcal{G}$. Since $G_S(165) = \{15, 33, 55\}$, $[15, 33] = 165$ and $(15, 33) = 3 \in G_S(15) \cap G_S(33)$, 15 and 33 satisfy the condition $\mathcal{G}$. Similarly, one can check that 15 and 55 satisfy the condition $\mathcal{G}$, and 33 and 55 satisfy the condition $\mathcal{G}$. Therefore, 165 satisfies the condition $\mathcal{G}$. In a similar way, one can easily check that the set $T_1 = \{1, 3, 4, 5, 9, 11, 12, 15, 33, 36, 55, 165\}$ is a gcd-closed set satisfying the condition $\mathcal{G}$, the set $T_2 = \{1, 3, 5, 6, 10, 11, 15, 33, 55, 165\}$ is not gcd closed but satisfies the condition $\mathcal{G}$ since $\gcd(6, 10) = 2 \notin S$, and the set $T_3 = \{1, 2, 3, 5, 11, 12, 15, 33, 55, 165\}$ is gcd closed but does not satisfy the condition $\mathcal{G}$.

THEOREM 3.3. *Every factor-closed set is a gcd-closed set satisfying the condition* $\mathcal{G}$.

PROOF. Let S be a factor-closed set. First of all, we show that S is gcd closed. For any $x, y \in S$, since $(x, y) \mid x$ and $x \in S$ and S is factor closed, we must have $(x, y) \in S$. Hence, S is gcd closed as desired.

Next, we show that S satisfies the condition $\mathcal{G}$. Clearly, $1 \in S$ and $g(1) = 0$. In the following, we let $x \in S$ and $x > 1$. If x is a prime power, say, p^i with $i \geq 1$, then $\{p, p^2, \ldots, p^{i-1}\} \subseteq S$ and $G_S(x) = \{p^{i-1}\}$, which implies that $g(x) = 1$. In what follows, we assume that x contains at least two prime divisors, say, $x = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where the e_i are positive integers and $p_1, \ldots, p_k$ are $k \geq 2$ distinct primes. Since S is factor closed, it follows that for any k-tuple $(r_1, r_2, \ldots, r_k) \in \mathbb{Z}^k$ with $0 \leq r_i \leq e_i$ for $1 \leq i \leq k$, one has $p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \in S$. It then follows that

$$G_S(x) = \{p_1^{e_1-1} p_2^{e_2} \cdots p_k^{e_k}, p_1^{e_1} p_2^{e_2-1} \cdots p_k^{e_k}, \ldots, p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k-1}\} = \left\{\frac{x}{p_1}, \frac{x}{p_2}, \ldots, \frac{x}{p_k}\right\}.$$

Then, $g(x) = k \geq 2$, and for any two distinct integers i and j with $1 \leq i < j \leq k$, we have

$$\left(\frac{x}{p_i}, \frac{x}{p_j}\right) = \frac{x}{p_i p_j} \in G_S\left(\frac{x}{p_i}\right) \cap G_S\left(\frac{x}{p_j}\right)$$

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and

$$\left[\frac{x}{p_i}, \frac{x}{p_j}\right] = [p_1^{e_1} \cdots p_i^{e_i-1} \cdots p_j^{e_j} \cdots p_k^{e_k}, p_1^{e_1} \cdots p_i^{e_i} \cdots p_j^{e_j-1} \cdots p_k^{e_k}]$$
$$= p_1^{e_1} \cdots p_i^{e_i} \cdots p_j^{e_j} \cdots p_k^{e_k} = x.$$

Thus, x/p_i and x/p_j satisfy the condition $\mathcal{G}$, which implies that the element $x \in S$ satisfies the condition $\mathcal{G}$. Therefore, Theorem 3.3 is proved. □

For any factor-closed set S and for any integer $a > 1$, the set $aS := \{ax : x \in S\}$ is not factor closed since $a \in aS$, $1 \mid a$ and $1 \notin aS$. One can easily check that if S is a gcd-closed set satisfying the condition $\mathcal{G}$, then so is the set aS. Bourque and Ligh [4] showed that for any factor-closed set S, one has $(S) \mid [S]$ in the ring $M_{|S|}(\mathbb{Z})$ of $|S| \times |S|$ matrices over the integers. If a and b are positive integers with $a \mid b$, then $(S^a) \mid (S^b)$, $(S^a) \mid [S^b]$ and $[S^a] \mid [S^b]$ when the set S is a divisor chain (see [21]), or is a gcd-closed set with $\max_{x \in S}\{|G_S(x)|\} = 1$ (see [29, 31]), or is a factor-closed set (from this paper). We believe that such factorisations are still true when S is a gcd-closed set satisfying the condition $\mathcal{G}$.

CONJECTURE 3.4. Let a and b be positive integers with $a \mid b$ and let S be a gcd-closed set satisfying the condition $\mathcal{G}$. Then, $(S^a) \mid (S^b)$, $(S^a) \mid [S^b]$ and $[S^a] \mid [S^b]$ in $M_{|S|}(\mathbb{Z})$.

CONJECTURE 3.5. Let a and b be positive integers with $a \mid b$. Let $S = S_1 \cup S_2$ with S_1 and S_2 being gcd-closed sets satisfying the condition $\mathcal{G}$ such that $\gcd(\operatorname{lcm}(S_1), \operatorname{lcm}(S_2)) = 1$ and $\min(S) > 1$. Then, $(S^a) \mid (S^b)$, $(S^a) \mid [S^b]$ and $[S^a] \mid [S^b]$ in $M_{|S|}(\mathbb{Z})$.

CONJECTURE 3.6. Let $a \geq 1$ be an integer. Let S_1 and S_2 be gcd-closed sets such that $\gcd(\operatorname{lcm}(S_1), \operatorname{lcm}(S_2)) = 1$. Then, (i) $(S_1^a) \mid [S_1^a]$ if and only if S_1 satisfies the condition $\mathcal{G}$. (ii) If $S = S_1 \cup S_2$ and $\min(S) > 1$, then $(S^a) \mid [S^a]$ if and only if S_1 and S_2 satisfy the condition $\mathcal{G}$.

A general question is to characterise the couples (S, f) with S a set of distinct positive integers, and f integer-valued and multiplicative for which the matrix $(f[S])$ is divisible by the matrix $(f(S))$ in the ring $M_{|S|}(\mathbb{Z})$. If S is factor closed or multiple closed, the answer to this question is known (see, for instance, [7, 18]). If S is gcd closed, we give a conjectural answer to this question as the following conjecture.

CONJECTURE 3.7. Let S be a gcd-closed set satisfying the condition $\mathcal{G}$ and let f be a multiplicative function such that $f(d) \in \mathbb{Z}$ whenever $d \mid \operatorname{lcm}(S)$ and $f(u) \mid f(v)$ whenever $u \mid v$ and $u, v \in S$. Then, $(f(S))$ divides $(f[S])$ in $M_n(\mathbb{Z})$.

CONJECTURE 3.8. Let $S = S_1 \cup S_2$ with S_1 and S_2 being gcd-closed sets satisfying the condition $\mathcal{G}$ such that $\gcd(\operatorname{lcm}(S_1), \operatorname{lcm}(S_2)) = 1$ and $\min(S) > 1$. Let f be a multiplicative function such that $f(d) \in \mathbb{Z}$ whenever $d \mid \operatorname{lcm}(S)$, and $f(u) \mid f(v)$ whenever $u \mid v$ and $u, v \in S$. Then, $(f(S))$ divides $(f[S])$ in $M_n(\mathbb{Z})$.

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Some evidence to support Conjecture 3.8 is given in [8]. An interesting problem is to give explicit formulas for the determinants of the matrices $(f(S))$ and $(f[S])$ when S is a gcd-closed set satisfying the condition $\mathcal{G}$. Another general question is to characterise the couples (S, f) with S gcd closed and f multiplicative such that the matrix $(f[S])$ is nonsingular. We will explore this interesting question in the near future. Regarding the counterexamples to the Bourque–Ligh conjecture [4] that the LCM matrix $[S]$ defined on any gcd-closed set S is nonsingular, we surprisingly found that all the known gcd-closed sets S with the LCM matrix $[S]$ being singular, given in [10, 14, 15, 19, 20, 24], do not satisfy the condition $\mathcal{G}$. We believe that this phenomenon is true in general. In other words, we propose the following conjectures that are closely relating to the conjecture raised in [16].

CONJECTURE 3.9. Let $a \geq 1$ be an integer. Then, the ath power LCM matrix $[S^a]$ defined on any gcd-closed set S satisfying the condition $\mathcal{G}$ is nonsingular.

CONJECTURE 3.10. Let $a \geq 1$ be an integer. Let $S = S_1 \cup S_2$ with S_1 and S_2 being gcd-closed sets satisfying the condition $\mathcal{G}$ such that $\gcd(\operatorname{lcm}(S_1), \operatorname{lcm}(S_2)) = 1$ and $\min(S) > 1$. Then, the ath power LCM matrix $[S^a]$ defined by S is nonsingular.

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SHAOFANG HONG, Mathematical College,
Sichuan University, Chengdu 610064, PR China
e-mail: sfhong@scu.edu.cn

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