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Number Theory

Sums of three cubes over a function field

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Abstract

We use a function field version of the circle method to prove that a positive proportion of elements in $\mathbb{F}_q[t]$ are representable as a sum of three cubes of minimal degree from $\mathbb{F}_q[t]$, assuming a suitable form of the Ratios Conjecture and that $\mathrm{char}(\mathbb{F}_q) > 3$. The analogue of this conjecture for quadratic Dirichlet L-functions is known for large fixed q, via recent developments in homological stability.

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1. Introduction

A folklore conjecture in number theory states that any $k \in \mathbb{Z}$, not congruent to ± 4 modulo 9, admits a representation as a sum of three cubes of integers. The state of the art around this difficult problem is summarised in recent work by Wang [57], where a modern form of the circle method is developed to prove the conjecture for a positive proportion of integers k. Wang's work is highly conditional, depending on an automorphy conjecture and GRH for certain Hasse–Weil L-functions, as well as the Ratios Conjecture from random matrix theory and a Square-Free Sieve Conjecture [57, Conjecture 1.6].

In this paper we investigate the analogous problem over the function field $K = \mathbb{F}_q(t)$, with the aim of removing as many of the hypotheses as possible from [57]. Let $\mathcal{O} = \mathbb{F}_q[t]$ denote the ring of integers of K and let $k \in \mathcal{O}$. In stark contrast to the case of $\mathbb{Z}$, when $\mathrm{char}(\mathbb{F}_q) \neq 3$ and $q \notin \{2, 4, 7, 13, 16\}$, Serre and Vaserstein [51, Lemma 1] have explicitly constructed *linear functions* $a_i k + b_i$, with $a_i \in \mathbb{F}_q^\times$ and $b_i \in \mathbb{F}_q$, such that

$$\sum_{1 \leqslant i \leqslant 3} (a_i k + b_i)^3 = k.$$

Vaserstein also gives more complicated polynomial constructions if $q \in \{7, 13\}$. In each case, the degrees of $x, y, z \in \mathcal{O}$ solving $x^3 + y^3 + z^3 = k$ are $\geqslant \deg k$, whereas one might hope for solutions of degree $\sim \frac{1}{3} \deg k$. Also, [51] only constructs a bounded finite number of solutions for each k, and thus it remains open to produce an infinitude of solutions, or even a number of solutions tending to infinity with $\deg k$.

According to [24], when $q = 16$ there is no known polynomial construction, but a direct adaptation of [15, Lemme 4.5] shows that there are no local obstructions to the representation of k as a sum of three cubes from $\mathcal{O}$ and so it is natural to conjecture that most, or perhaps all, $k \in \mathcal{O}$ admit such a representation. When $q \in \{2, 4\}$, there are local obstructions at places of residue cardinality 4, because all cubes in $\mathbb{F}_4$ lie in the proper subfield $\mathbb{F}_2$; but most, or perhaps all, $k \in \mathcal{O}$ may still satisfy the Hasse principle, with a positive proportion then being representable. If $\mathrm{char}(\mathbb{F}_q) = 3$, then $\{x^3 + y^3 + z^3 : x, y, z \in \mathcal{O}\}$ has density 0 in $\mathcal{O}$, however, because $x^3 + y^3 + z^3 = (x + y + z)^3$.

To each smooth hyperplane section $c_1 x_1 + \cdots + c_6 x_6 = 0$ of the hypersurface

$$x_1^3 + \cdots + x_6^3 = 0$$

in $\mathbb{P}^5_K$, for $\boldsymbol{c} = (c_1, \ldots, c_6) \in \mathcal{O}^6$, we shall see in § 3 that we can associate a Hasse–Weil L-function $L(s, \boldsymbol{c})$ over K. Thanks to Grothendieck and Deligne [20,27], we know that these L-functions are actually rational functions of q^{-s} that satisfy the Grand Riemann Hypothesis (GRH); that is, their zeros lie on the line $\Re(s) = \frac{1}{2}$. Our main result relies on mean-value statistics of the ratio $1/L(s, \boldsymbol{c})$ over boxes of vectors $\boldsymbol{c} \in \mathcal{O}^6$; we need an asymptotic formula for

$$\sum_{\substack{\boldsymbol{c} \in \mathcal{O}^6 \\ \deg(c_1), \ldots, \deg(c_6) \leqslant Z}} \frac{1}{L(s_1, \boldsymbol{c}) L(s_2, \boldsymbol{c})},$$

as $Z \to \infty$, where the distance from $\Re(s_1), \Re(s_2)$ to $\frac{1}{2}$ is sufficiently small compared to the quality of the error term. This is given by Conjecture 3.6 (R2), a special case of the Ratios Conjecture. The following is our main result.

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Theorem 1.1. *Suppose* $\mathrm{char}(\mathbb{F}_q) > 3$ *and assume the Ratios Conjecture* 3.6 *for* $L(s, \boldsymbol{c})$, *as* $\boldsymbol{c} \in \mathcal{O}^6$ *varies. Then each of the following sets has positive lower density in* $\mathcal{O}$.

1. $\{x^3 + y^3 + z^3 : x, y, z \in \mathcal{O}$ *monic*$\}$.
2. $\{k : x^3 + y^3 + z^3 = k$ *is soluble in* $\mathcal{O}$ *with max* $\{\deg x, \deg y, \deg z\} \leqslant \lceil \frac{\deg k}{3} \rceil\}$.

We remark that $\deg(x^3 + y^3 + z^3) = 3 \max\{\deg x, \deg y, \deg z\}$ for all monic polynomials $x, y, z \in \mathcal{O}$, when $\mathrm{char}(\mathbb{F}_q) > 3$, whence the set in (1) is a subset of the set in (2). In parts (6) and (7) of Theorem 11.1, we will strengthen Theorem 1.1, showing that we can take $x, y, z \in \mathcal{O}$ to be monic polynomials lying in any fixed subset of $\mathcal{O}$ of positive lower density. Throughout this paper, we think of q as being fixed. Thus all implied constants, and unspecified positive densities, are allowed to depend on q.

The set in (2) is known in Waring's problem as the set of *strict sums of three cubes*, in the sense of Carlitz [23, p. 2964]. We have already seen that the conclusions of the theorem are false if $\mathrm{char}(\mathbb{F}_q) = 3$, where the sets in (1) and (2) have density 0. If $\mathrm{char}(\mathbb{F}_q) = 2$, then most of the proof of Theorem 1.1 still goes through, but some of the ingredients would need to be modified. We shall comment further on this in § 11.

The Ratios Conjectures were originally stated for automorphic L-functions over $\mathbb{Q}$ by Conrey, Farmer, et al. [16,17]. Their natural extension to $\mathbb{F}_q(t)$ has been explained by Andrade and Keating [2]. The precise form of the Ratios Conjecture we need will be presented in § 3. It is plausible that the homological stability framework of Bergström, Diaconu, Petersen and Westerland [5], together with Miller, Patzt, Petersen and Randal-Williams [43], could eventually resolve the particular form of the Ratios Conjecture needed in the present paper, for sufficiently large values of q. We defer the details of this to § 11.2, where we shall define a *q-restricted* form of the Ratios Conjecture, together with a strengthening of Theorem 1.1, in the shape of Theorem 11.1. Still working under the Ratios Conjecture, one can show that the set

$$\{k : x^3 + y^3 + z^3 = k \text{ is soluble in } \mathcal{O} \text{ with } \max\{\deg x, \deg y, \deg z\} \leqslant \tfrac{\deg k}{3} + A\}$$

has lower density approaching 1 as $A \to \infty$, by adapting the proof of [54, Theorem 1.6]. Moreover, one can show that this set has upper density < 1 for any fixed $A \in \mathbb{R}$, by adapting local density arguments of Diaconu from [21, §1]. Both tasks are deferred to the sequel [8], in order to keep the present paper as clean as possible.

The Ratios Conjecture has already found application in the theory of *rational points* on elliptic curves over K. Thus, following the work of Katz–Sarnak [36], and Conrey et al. [17,18], it has been shown that Goldfeld's Conjecture follows from the Ratios Conjecture for a suitable family of elliptic curve L-functions over K. This conjecture states that there is a density $\frac{1}{2}$ of curves with prescribed analytic rank $r \in \{0, 1\}$, and therefore implies the Birch and Swinnerton-Dyer Conjecture (BSD) for a density 1 of curves. Whereas L-function ratios are directly related to analytic ranks and BSD, their connection to *integral points* and sums of three cubes in Theorem 1.1 is much less direct.

Our proof of Theorem 1.1 uses the second moment method and requires a detailed analysis of the counting function

$$N(P) = \#\big\{\boldsymbol{x} \in \mathcal{O}^6 : x_1^3 + \cdots + x_6^3 = 0,\ |x_1| = \cdots = |x_6| = |P|\big\}, \tag{1.1}$$

for given $P \in \mathcal{O}$, where $|\cdot|$ denotes the usual absolute value on K. We shall be interested in the size of $N(P)$ as $|P| \to \infty$. The Batyrev–Manin Conjecture suggests that an asymptotic formula of the shape $N(P) \sim c|P|^3$ should hold, for a suitable constant $c > 0$. The best progress towards this is to be found in recent work of Glas and Hochfilzer [26], who use a function field version of the circle method to prove that

$$N(P) = O_\varepsilon(|P|^{3+\varepsilon}), \tag{1.2}$$

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for any $\varepsilon > 0$, assuming only that $\mathrm{char}(\mathbb{F}_q) \neq 3$. It is worth emphasising that this bound is completely unconditional, unlike the parallel picture over $\mathbb{Z}$, where the pioneering work of Hooley [29,33] and Heath-Brown [28] is conditional on GRH. Assuming that $\mathrm{char}(\mathbb{F}_q) > 3$, our proof of Theorem 1.1 entirely rests on our ability to remove the ε from the upper bound (1.2). Ultimately we shall only manage to do so at the cost of assuming the Ratios Conjecture for the Hasse–Weil L-functions $L(s, \boldsymbol{c})$, as $\boldsymbol{c} \in \mathcal{O}^6$ varies. In fact, $L(s, \boldsymbol{c})$ is only defined for generic vectors $\boldsymbol{c}$, whereas we shall be able to handle the contribution from nongeneric vectors unconditionally, finding that they encode special subvarieties that contribute to $N(P)$.

Our argument will broadly follow the path laid down by Wang [57] over $\mathbb{Z}$, but with a variety of differences and refinements, some of which we have chosen to highlight here.

1. The L-functions $L(s, \boldsymbol{c})$ carry less information over $\mathbb{F}_q(t)$ than over $\mathbb{Q}$, since q-adic sums of local coefficients are collapsed into single global coefficients. Nonetheless, these are still the only L-functions that we need in our analysis, provided we choose our weight functions carefully, as suggested in the next item of this list.
2. In order to avoid introducing character twists as in [12], which would enlarge and complicate the form of the Ratios Conjecture needed, we restrict the class of weight functions that we use. This allows us to cleanly factor out the relevant oscillatory integral using symmetry ideas of Glas–Hochfilzer [26].
3. The precise automorphy hypotheses and Square-Free Sieve Conjecture required in [57] become unconditional over $\mathbb{F}_q(t)$. For the former, we use Poincaré duality to carefully produce poles of exterior square L-functions. For the latter, we use an argument of Poonen [45] on square-free values of multivariate polynomials.
4. In Theorem 7.2 we prove a simple version of the Ekedahl sieve in positive characteristic for square-free moduli, which is sufficient for the applications in this paper. (A general prime moduli version for arbitrary global fields has been worked out by Bhargava–Shankar–Wang [7].)
5. The bias in exponential sums that was discovered in [56, Lemma 7.7] was only worked out for diagonal cubic forms in six variables; in Proposition 10.6 we extend the argument to handle arbitrary nonsingular senary cubic forms. This requires new geometric insight on quadric bundles, via work of Beauville [4].

Working over function fields, it is also natural to ask what happens in the limit as $q \to \infty$. In this limit, methods of Coskun–Starr [19] on rational curves in cubic hypersurfaces come into play. It follows from recent work of Kitagawa [38] that

$$\lim_{\substack{q \to \infty \\ \mathrm{char}(\mathbb{F}_q) \neq 2,3}} \frac{N(P)}{q^5 |P|^3} = 1.$$

In addition, recent work of the second author [25] implies that $N(P) \ll_{\deg P} q^5 |P|^3$ as long as $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$, where the implied constant may depend on $\deg P$ but not on q. Combining such estimates for $N(P)$ with the second moment method, as in the proof of part (6) of Theorem 11.1, might then produce an unconditional analogue of Theorem 1.1 in the large q limit. We leave the details to the interested reader.

2. Background

In this section we collect some basic facts about the function field $K = \mathbb{F}_q(t)$, which is equipped with the ring of integers $\mathcal{O} = \mathbb{F}_q[t]$. Let $K_\infty = \mathbb{F}_q((t^{-1}))$ be the field of Laurent series in t^{-1}. Let $\mathcal{O}^+$ be the set of monic polynomials $r \in \mathcal{O}$; this is analogous to the set of positive integers in $\mathbb{Z}$. For $M \in \mathbb{R}$, we shall write $\widehat{M} := q^M$. Any $\alpha \in K_\infty \setminus \{0\}$ can be written uniquely as

$$\alpha = \sum_{i \leqslant M} a_i t^i, \quad a_M \neq 0, \tag{2.1}$$

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for some $M \in \mathbb{Z}$ and $a_i \in \mathbb{F}_q$. Define $|\alpha| := \widehat{M}$; then $|\cdot|$ naturally extends to K_∞ the absolute value on K induced by t^{-1}. Moreover, K_∞ is the completion of K with respect to this absolute value. The analogue of the unit interval in K_∞ is given by

$$\mathbb{T} := \{\alpha \in K_\infty : |\alpha| < 1\}.$$

Since K_∞ is a local field, it can be endowed with a unique Haar measure $\mathrm{d}\alpha$ such that $\int_{\mathbb{T}} \mathrm{d}\alpha = 1$. We shall extend the absolute value to K_∞^n by $|\boldsymbol{\alpha}| := \max_{1 \leqslant i \leqslant n} |\alpha_i|$ and the Haar measure by $\mathrm{d}\boldsymbol{\alpha} := \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n$, for $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in K_\infty^n$. Finally, for each prime $\varpi \in \mathcal{O}^+$, we have an associated absolute value $|\cdot|_\varpi$ on K, given by $|a|_\varpi = q^{-\nu_\varpi(a) \deg \varpi}$. We denote by K_ϖ the completion of K with respect to this absolute value and we write $\mathcal{O}_\varpi := \{\alpha \in K_\varpi : |\alpha|_\varpi \leqslant 1\}$.

Farey dissection

Dirichlet's approximation theorem holds over K. That is, for any $\alpha \in \mathbb{T}$ and $Q \in \mathbb{N} := \mathbb{Z}_{\geqslant 1}$, there exist $a \in \mathcal{O}$ and $r \in \mathcal{O}^+$ with $\gcd(a, r) = 1$ and $|a| < |r| \leqslant \widehat{Q}$ such that $|r\alpha - a| < \widehat{Q}^{-1}$. By the ultrametric property, this is enough to obtain an analogue of a *Farey dissection* of the unit interval

$$\mathbb{T} = \bigsqcup_{\substack{r \in \mathcal{O}^+ \\ |r| \leqslant \widehat{Q}}} \bigsqcup_{\substack{|a| < |r| \\ \gcd(a,r)=1}} \{\alpha \in \mathbb{T} : |r\alpha - a| < \widehat{Q}^{-1}\}, \tag{2.2}$$

for any $Q \geqslant 1$. A full proof of this fact can be found in [12, Lemma 4.2], for example.

Characters

For $\alpha \in K_\infty$ given by (2.1), we define

$$\psi : K_\infty \to \mathbb{C}^\times, \quad \psi(\alpha) = e\left(\frac{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a_{-1})}{p}\right) \quad (\text{with } p = \mathrm{char}(\mathbb{F}_q))$$

and set $\psi(0) = 1$, where as usual $e(x) := \exp(2\pi i x)$ for $x \in \mathbb{R}$. It is easy to see that ψ is a nontrivial additive character of K_∞, and that for $x \in K_\infty$ and $N \in \mathbb{Z}_{\geqslant 0}$, we have

$$\int_{|\alpha| < \widehat{N}^{-1}} \psi(\alpha x) \mathrm{d}\alpha = \begin{cases} \widehat{N}^{-1} & \text{if } |x| < \widehat{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Note that if $x \in \mathcal{O}$, then (2.3) implies

$$\int_{\mathbb{T}} \psi(\alpha x) \mathrm{d}\alpha = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In addition, we will make frequent use of the following formula for exponential sums. If $r, a \in \mathcal{O}$ are such that $r \neq 0$, then

$$\frac{1}{|r|} \sum_{\substack{x \in \mathcal{O} \\ |x| < |r|}} \psi\left(\frac{ax}{r}\right) = \begin{cases} 1 & \text{if } r \mid a, \\ 0 & \text{otherwise.} \end{cases}$$

Poisson summation

We call a function $w : K_\infty^n \to \mathbb{C}$ *smooth* if it is locally constant. Denote by $S(K_\infty^n)$ the space of all smooth functions $w : K_\infty^n \to \mathbb{C}$ with compact support. For such functions the Poisson summation formula [12,

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Lemma 2.1] holds in the following form.

Lemma 2.1. *Let $f \in K_\infty[x_1, \dots, x_n]$ and let $w \in S(K_\infty^n)$. Then we have*

$$\sum_{\boldsymbol{z} \in \mathcal{O}^n} w(\boldsymbol{z})\psi(f(\boldsymbol{z})) = \sum_{\boldsymbol{c} \in \mathcal{O}^n} \int_{K_\infty^n} w(\boldsymbol{u})\psi(f(\boldsymbol{u}) + \boldsymbol{c} \cdot \boldsymbol{u}) d\boldsymbol{u}.$$

Delta method

Given a homogeneous cubic polynomial $F \in \mathcal{O}[x_1, \dots, x_n]$ and a weight function $w \in S(K_\infty^n)$, we are interested in the counting function

$$N_F(w, P) = \sum_{\substack{\boldsymbol{x} \in \mathcal{O}^n \\ F(\boldsymbol{x})=0}} w\Big(\frac{\boldsymbol{x}}{P}\Big). \tag{2.4}$$

For a parameter $Q \geqslant 1$ to be specified later, we deduce from (2.2) and (2.3) that

$$N_F(w, P) = \sum_{\substack{r \in \mathcal{O}^+ \\ |r| \leqslant \widehat{Q}}} {\sum_{|a|<|r|}}' \int_{|\theta|<|r|^{-1}\widehat{Q}^{-1}} S(a/r + \theta) \mathrm{d}\theta,$$

where $\sum'_{|a|<|r|}$ means that we sum over $a \in \mathcal{O}$ with $\gcd(a, r) = 1$ only, and where

$$S(\alpha) = \sum_{\boldsymbol{x} \in \mathcal{O}^n} \psi(\alpha F(\boldsymbol{x}))w(\boldsymbol{x}/P)$$

for $\alpha \in \mathbb{T}$. As explained in [12, §4], we can use Poisson summation in Lemma 2.1 to evaluate $S(\theta + a/r)$, giving

$$N_F(w, P) = |P|^n \sum_{\substack{r \in \mathcal{O}^+ \\ |r| \leqslant \widehat{Q}}} |r|^{-n} \int_{|\theta|<|r|^{-1}\widehat{Q}^{-1}} \sum_{\boldsymbol{c} \in \mathcal{O}^n} S_r(\boldsymbol{c}) I_r(\theta, \boldsymbol{c}) \mathrm{d}\theta, \tag{2.5}$$

where

$$S_r(\boldsymbol{c}) := {\sum_{|a|<|r|}}' \sum_{|\boldsymbol{x}|<|r|} \psi\left(\frac{aF(\boldsymbol{x}) - \boldsymbol{c} \cdot \boldsymbol{x}}{r}\right) \tag{2.6}$$

and

$$I_r(\theta, \boldsymbol{c}) := \int_{K_\infty^n} w(\boldsymbol{x})\psi\left(\theta P^3 F(\boldsymbol{x}) + \frac{P\boldsymbol{c} \cdot \boldsymbol{x}}{r}\right) \mathrm{d}\boldsymbol{x}. \tag{2.7}$$

Moreover, we will also consider averages of $I_r(\theta, \boldsymbol{c})$ of the form

$$I_r(\boldsymbol{c}) := \int_{|\theta|<|r|^{-1}\widehat{Q}^{-1}} I_r(\theta, \boldsymbol{c}) \mathrm{d}\theta, \tag{2.8}$$

in which case we may write

$$N_F(w, P) = |P|^n \sum_{\substack{r \in \mathcal{O}^+ \\ |r| \leqslant \widehat{Q}}} |r|^{-n} \sum_{\boldsymbol{c} \in \mathcal{O}^n} S_r(\boldsymbol{c}) I_r(\boldsymbol{c}). \tag{2.9}$$

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Ultimately Q will be chosen so that $|P|^{3/2} \asymp \widehat{Q}$.

The expression (2.5) is the starting point for our work and from now on we will mostly be concerned with estimating the integrals $I_r(\theta, \boldsymbol{c})$ and the sums $S_r(\boldsymbol{c})$. To keep notation simple, it is convenient to introduce the normalised sum

$$S_r^\natural(\boldsymbol{c}) := |r|^{-(n+1)/2} S_r(\boldsymbol{c}).$$

If $r = r_1 r_2$, where both $r_1, r_2 \in \mathcal{O}$ are monic with $\gcd(r_1, r_2) = 1$, the Chinese remainder theorem readily implies that

$$S_r(\boldsymbol{c}) = S_{r_1}(\boldsymbol{c}) S_{r_2}(\boldsymbol{c}). \tag{2.10}$$

Therefore the prime factorisation of r will play an important role in our analysis. The letter ϖ will generally denote a prime in $\mathcal{O}^+$.

Notation

Whenever the letter F appears from now on, we take

$$F = x_1^3 + \cdots + x_n^3,$$

with $n = 6$. However, we will often work with a general choice of n to clarify the nature of arguments. Let

$$F^*(\boldsymbol{c}) = \prod (c_1^{3/2} \pm c_2^{3/2} \pm \cdots \pm c_n^{3/2}) \in \mathcal{O}[\boldsymbol{c}] \tag{2.11}$$

be the (primitive) dual form associated to F; it has degree $3 \cdot 2^{n-2}$. Let $V \subset \mathbb{P}^{n-1}$ be the hypersurface defined by $F = 0$. The dual form F^* defines the dual variety $V^* \subset \mathbb{P}^{n-1}$ of V, which parameterises hyperplanes that intersect V tangentially. Alternatively, V^* may be described as the closure of the image under the Gauss map $V \to \mathbb{P}^{n-1}$ given by $\boldsymbol{x} \mapsto \nabla F(\boldsymbol{x})$. From this description it is clear that

$$F(\boldsymbol{x}) \mid F^*(\nabla F(\boldsymbol{x})), \tag{2.12}$$

as an identity in $K[x_1, \ldots, x_n]$. Let

$$\mathcal{S}_0 = \{\boldsymbol{c} \in \mathcal{O}^n : F^*(\boldsymbol{c}) = 0\}, \quad \mathcal{S}_1 = \{\boldsymbol{c} \in \mathcal{O}^n : F^*(\boldsymbol{c}) \neq 0\}. \tag{2.13}$$

For each $\boldsymbol{c} \in \mathcal{S}_1$, let

$$\mathcal{R}_{\boldsymbol{c}}^{\mathrm{G}} = \{r \in \mathcal{O}^+ : \gcd(r, F^*(\boldsymbol{c})) = 1\}, \quad \mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}} = \{r \in \mathcal{O}^+ : \varpi \mid r \Rightarrow \varpi \mid F^*(\boldsymbol{c})\}. \tag{2.14}$$

Given $r \in \mathcal{O}^+$ and nonzero $B \in \mathcal{O}$, we will sometimes write $r \mid B^\infty$ to mean that $\varpi \mid r \Rightarrow \varpi \mid B$. This is equivalent to $\mathrm{rad}(r) \mid B$, where $\mathrm{rad}(r)$ is the radical of r. In particular, we note that $\mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}} = \{r \in \mathcal{O}^+ : \mathrm{rad}(r) \mid F^*(\boldsymbol{c})\}$.

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To connect $S_r(\boldsymbol{c})$ to Hasse–Weil L-functions, we define further quantities. Given a finite field k, let $\mathcal{V}(k)$ be the set of k-points on the variety $F(\boldsymbol{x}) = 0$ in $\mathbb{P}^5_k$, let $\mathcal{V}_{\boldsymbol{c}}(k)$ be the set of k-points on the variety $F(\boldsymbol{x}) = \boldsymbol{c} \cdot \boldsymbol{x} = 0$ in $\mathbb{P}^5_k$, and let

$$E_F(k) := \#\mathcal{V}(k) - \#\mathbb{P}^4(k), \quad E_{\boldsymbol{c}}(k) := \#\mathcal{V}_{\boldsymbol{c}}(k) - \#\mathbb{P}^3(k),$$

where $\#\mathbb{P}^d(k) = ((\#k)^{d+1} - 1)/((\#k) - 1)$. Next, let

$$E_F^\natural(k) := (\#k)^{-2} E_F(k), \quad E_{\boldsymbol{c}}^\natural(k) := (\#k)^{-3/2} E_{\boldsymbol{c}}(k). \tag{2.15}$$

For primes $\varpi \nmid \boldsymbol{c}$, it is known that

$$S_{\varpi}^\natural(\boldsymbol{c}) = E_{\boldsymbol{c}}^\natural(\mathcal{O}/\varpi\mathcal{O}) - |\varpi|^{-1/2} E_F^\natural(\mathcal{O}/\varpi\mathcal{O}); \tag{2.16}$$

see [34, Lemma 7] for a general reference over prime fields, which carries over directly to prime-power fields. Moreover, if $\varpi \nmid F^*(\boldsymbol{c})$ then [12, Lemma 5.2] gives

$$S_{\varpi^l}(\boldsymbol{c}) = 0 \quad (l \geqslant 2). \tag{2.17}$$

Basic strategy

In order to analyse the quantity $N_F(w, P)$, we will decompose the right-hand side of (2.9) into three main pieces:

1. the contribution from vectors $\boldsymbol{c} \in \mathcal{S}_1$,
2. the contribution from the zero vector $\boldsymbol{c} = \boldsymbol{0}$, and
3. the contribution from nonzero vectors $\boldsymbol{c} \in \mathcal{S}_0$,

where the sets $\mathcal{S}_0$ and $\mathcal{S}_1$ are as in (2.13). The explicit definitions of these three pieces can be found in (8.1), (9.1) and (10.1), respectively.

In the following proof sketch, we refer to each section either by its number, or by a representative definition or result contained within it. Thus the paper can be read either linearly or by focusing on particular sections of interest. The three pieces will be tied together in § 11.

The first piece will be handled by approximating $S_r(\boldsymbol{c})$ in terms of coefficients of Hasse–Weil L-functions, which are analysed in § 3 using the Ratios Conjecture. In order to measure the quality of the approximation, we construct various Euler products in Definition 4.1, whose coefficients we then investigate using L-function techniques and other methods. This includes exponential sum estimates such as Lemma 6.1, and moment estimates such as Theorem 7.1. The latter is based on the Ekedahl sieve. Finally, we will also need to show that the integral $I_r(\boldsymbol{c})$ typically decays unless $\widehat{Q}/|r|$ is very small. This is achieved in § 5.

The second piece is standard and we will find that moduli r of small absolute value $|r|$ dominate it. This relies on the tail estimate Lemma 9.2, for suitable averages of $S_r(\boldsymbol{0})$.

The third piece is based on a conductor-dropping phenomenon on the set $\mathcal{S}_0$. In general, $S_r(\boldsymbol{c})$ would be expected to vary wildly with $\boldsymbol{c}$ and r. However, when $\boldsymbol{c}$ is restricted to the set $\mathcal{S}_0$, we will show that $S_r(\boldsymbol{c})$ typically resembles a coefficient of an Euler product that is independent of $\boldsymbol{c}$. A key input for this is Proposition 10.6, which establishes a certain bias in $S_r(\boldsymbol{c})$ that we mentioned earlier in the introduction. Using this bias, which is independent of $\boldsymbol{c}$, we are roughly then left with summing the integrals $I_r(\boldsymbol{c})$ over $\boldsymbol{c}$, which is achieved through the Poisson summation formula.

Technical results

We proceed to record some useful results that will often be appealed to during the course of our argument.

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Lemma 2.2. *Let $R \geqslant 0$ and $B \in \mathcal{O} \setminus \{0\}$. Then*

$$\#\{r \in \mathcal{O}^+ : r \mid B^\infty,\ |r| = \widehat{R}\} \ll_\varepsilon (\widehat{R}|B|)^\varepsilon.$$

In particular, if $\boldsymbol{c} \in \mathcal{S}_1$, then $\#\{r \in \mathcal{R}_{\boldsymbol{c}}^B : |r| = \widehat{R}\} \ll_\varepsilon |\boldsymbol{c}|^\varepsilon \widehat{R}^\varepsilon$.

Proof. $\sum_{r|B^\infty} \mathbf{1}_{|r|=\widehat{R}} \leqslant \sum_{r|B^\infty} (\widehat{R}/|r|)^\varepsilon = \widehat{R}^\varepsilon \prod_{\varpi|B} (1 - |\varpi|^{-\varepsilon})^{-1} \ll_\varepsilon \widehat{R}^\varepsilon |B|^\varepsilon$. □

Lemma 2.3. *If $Z, R \in \mathbb{R}$ and $A, \varepsilon > 0$, then we have*

$$\sum_{\boldsymbol{c} \in \mathcal{S}_1 :\, |\boldsymbol{c}| \leqslant \widehat{Z}} \#\{r \in \mathcal{R}_{\boldsymbol{c}}^B : |r| \leqslant \widehat{R}\}^A \ll_{A,\varepsilon} \widehat{Z}^n \widehat{R}^\varepsilon. \tag{2.18}$$

Proof. By replacing A with $\lceil A \rceil$, we may assume $A \in \mathbb{N}$. We now proceed based on the size of $\widehat{Z}$. If $\widehat{Z} < \widehat{R}^A$, we use Lemma 2.2. Now suppose $\widehat{Z} \geqslant \widehat{R}^A$. Let Σ denote the left-hand side of (2.18). Since $A \in \mathbb{N}$, we have

$$\Sigma = \sum_{\boldsymbol{c} \in \mathcal{S}_1 :\, |\boldsymbol{c}| \leqslant \widehat{Z}} \sum_{\substack{u_1, \dots, u_A \in \mathcal{R}_{\boldsymbol{c}}^B \\ |u_i| \leqslant \widehat{R}}} 1 = \sum_{\substack{u_1, \dots, u_A \in \mathcal{O}^+ \\ |u_i| \leqslant \widehat{R}}} \sum_{\boldsymbol{c} \in \mathcal{S}_1 :\, |\boldsymbol{c}| \leqslant \widehat{Z}} \mathbf{1}_{\mathrm{rad}(u_1 \cdots u_A) | F^*(\boldsymbol{c})}.$$

Let $k = \mathrm{rad}(u_1 \cdots u_A)$, noting that $|k| \leqslant |u_1 \cdots u_A| \leqslant \widehat{R}^A \leqslant \widehat{Z}$. Breaking into residue classes modulo k, we see that the inner sum over $\boldsymbol{c}$ is at most

$$\sum_{\substack{\boldsymbol{u} \bmod k \\ F^*(\boldsymbol{u}) \equiv 0 \bmod k}} \#\Big\{\boldsymbol{c} \in \mathcal{S}_1 :\ |\boldsymbol{c}| \leqslant \widehat{Z},\ \boldsymbol{c} \equiv \boldsymbol{u} \bmod k\Big\} \ll \frac{\widehat{Z}^n}{|k|^n} \sum_{\substack{\boldsymbol{u} \bmod k \\ F^*(\boldsymbol{u}) \equiv 0 \bmod k}} 1.$$

Since k is square-free, it follows from the Chinese remainder theorem and the Schwartz–Zippel lemma that the sum over $\boldsymbol{u}$ is $O(|k|^{n-1+\varepsilon})$, whence

$$\begin{aligned} \Sigma &\ll_\varepsilon \sum_{u_1, \dots, u_A \in \mathcal{O}^+ :\, |u_i| \leqslant \widehat{R}} \frac{\widehat{Z}^n}{|\mathrm{rad}(u_1 \cdots u_A)|^{1-\varepsilon}} \\ &\leqslant \sum_{r \in \mathcal{O}^+ :\, |r| \leqslant \widehat{R}^A} \frac{\widehat{Z}^n}{|r|^{1-\varepsilon}} \sum_{u_1, \dots, u_A \in \mathcal{O}^+ :\, |u_i| \leqslant \widehat{R},\ u_i | r^\infty} 1. \end{aligned}$$

Lemma 2.2 now yields

$$\Sigma \ll_{A,\varepsilon} \sum_{r \in \mathcal{O}^+ :\, |r| \leqslant \widehat{R}^A} \widehat{Z}^n |r|^{\varepsilon - 1} (\widehat{R}|r|)^\varepsilon \ll_\varepsilon \widehat{Z}^n \widehat{R}^{(2A+1)\varepsilon},$$

from which the statement follows on redefining ε. □

At several points later, we shall require nontrivial bounds on certain level sets. Let $\overline{K}$ denote the algebraic closure of K. For each place v of K, fix an embedding of $\overline{K}$ into the algebraic closure $\overline{K}_v$ of K_v. The absolute value $|\cdot|_v$ on K_v extends uniquely to $\overline{K}_v$. Restricting from $\overline{K}_v$ to $\overline{K}$, we thus obtain an absolute value $|\cdot|_v$ on $\overline{K}$.

Lemma 2.4. *Let $B, \lambda \in \mathbb{R}$ and $z \in \overline{K}$. Uniformly over places v of K, we have*

$$\#\{x \in \mathcal{O} : |x| \leqslant \widehat{B},\ |x - z|_v \leqslant \widehat{\lambda}\} \ll_K 1 + \widehat{B}^{\mathbf{1}_{v \neq \infty}} \widehat{\lambda}. \tag{2.19}$$

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Proof. This is trivial if $v = \infty$. If $v = \varpi \neq \infty$, we write Ω_z for the set appearing on the left hand side, which we may assume is nonempty. Fix an element $x_0 \in \Omega_z$ and note that $|x - z|_v \leqslant \widehat{\lambda} \Leftrightarrow |x - x_0|_v \leqslant \widehat{\lambda} \Leftrightarrow \varpi^l \mid x - x_0$, where $l := -\lfloor \lambda / \deg \varpi \rfloor \in \mathbb{Z}$. Hence we can write $x = x_0 + y\varpi^l$ with $y \in \mathcal{O}$, to get

$$\#\Omega_z \leqslant \#\{y \in \mathcal{O} : |\varpi|^l \, |y - z_0| \leqslant \widehat{B}\},$$

where $z_0 := -x_0/\varpi^l \in K$. By the '$v = \infty$ case' of (2.19), the desired result follows, since $\widehat{B}/|\varpi|^l = \widehat{B}/q^{l \deg \varpi} = \widehat{B} q^{(\deg \varpi)\lfloor \lambda / \deg \varpi \rfloor} \leqslant \widehat{B}\widehat{\lambda}$. □

Lemma 2.5. *Suppose $f \in K[x]$ has leading term ax^d with $a \neq 0$ and $d \geqslant 1$. Let $B, \lambda \in \mathbb{R}$. Then, uniformly over places v of K, we have*

$$\#\{x \in \mathcal{O} : |x| \leqslant \widehat{B}, \ |f(x)|_v \leqslant \widehat{\lambda}\} \ll_K d\,(1 + \widehat{B}^{\mathbf{1}_{v \neq \infty}} (\widehat{\lambda}/|a|_v)^{1/d}). \tag{2.20}$$

Proof. We use the idea behind Lemma 1 of [10]. By replacing f with f/a, we may assume f is monic. Let $z_1, \dots, z_d \in \overline{K}$ be the roots of f. Then

$$\{x \in \overline{K} : |f(x)|_v \leqslant \widehat{\lambda}\} \subseteq \bigcup_{1 \leqslant i \leqslant d} \{x \in \overline{K} : |x - z_i|_v \leqslant \widehat{\lambda}^{1/d}\}.$$

Summing (2.19) over $z = z_i$ gives the desired result. □

Given $f \in \mathcal{O}[x]$ and $r \in \mathcal{O}^+$, let $N(f; r)$ be the number of solutions $x \in \mathcal{O}/r\mathcal{O}$ to $f(x) \equiv 0 \bmod r$. Similarly, for $g \in \mathcal{O}[y_1, \dots, y_n]$, define

$$N(g; r) := \#\{(y_1, \dots, y_n) \in (\mathcal{O}/r\mathcal{O})^n : g(y_1, \dots, y_n) \equiv 0 \bmod r\}. \tag{2.21}$$

Lemma 2.6. *Suppose $f \in \mathcal{O}[x]$ has leading term ax^d with $a \neq 0$ and $d \geqslant 1$. Then there exists $A_d > 0$ such that for any $r \in \mathcal{O}^+$ we have*

$$N(f; r) \leqslant A_d^{\omega(r)} |\gcd(r, a)|^{1/d} |r|^{1 - d^{-1}}. \tag{2.22}$$

Moreover, if r is square-free then

$$N(f; r) \leqslant d^{\omega(r)} |\gcd(r, a)| \ll_{a,d,\varepsilon} |r|^{\varepsilon}. \tag{2.23}$$

Proof. By the Chinese remainder theorem it suffices to assume $r = \varpi^l$ is a prime power. Next, observe that $N(f; r) = N(f + rx^d; r)$ and $\min(v_\varpi(a), v_\varpi(a + r)) \leqslant v_\varpi(r)$. Therefore, after possibly replacing f with $f + rx^d$, we may assume that $a \mid r$. Then to get (2.22), we apply (2.20) with $(\widehat{B}, \widehat{\lambda}, v) = (|r|, |r|^{-1}, \varpi)$, noting that $|a|_\varpi^{-1} = q^{v_\varpi(a)} = |a| = |\gcd(r, a)|$. Moreover, if $l = 1$, then (2.23) holds because $N(f; \varpi) \leqslant d\,|\gcd(\varpi, a)|$. □

Before stating a useful consequence of this result, we first prove a technical result which allows us to make a convenient change of variables.

Lemma 2.7. *Let $n \geqslant 1$. Let S be a finite subset of $\mathcal{O}[y_1, \dots, y_n] \setminus \{0\}$. Then there exists $\boldsymbol{a} = (a_2, \dots, a_n) \in \mathcal{O}^{n-1}$ such that for all $g \in S$, we have*

$$\deg_{y_1}(\boldsymbol{a}^* g) = \deg(\boldsymbol{a}^* g) = \deg(g),$$

where $\boldsymbol{a}^* g := g(y_1, y_2 + a_2 y_1, \dots, y_n + a_n y_1) \in \mathcal{O}[y_1, \dots, y_n]$.

Proof. For each $g \in S$, let $h_g \in \mathcal{O}[y_1, \dots, y_n] \setminus \{0\}$ be the leading homogeneous part of g, so that $\deg h_g = \deg g$ and $\deg(g - h_g) < \deg g$. Then for any $g \in S$ and $\boldsymbol{a} \in R^{n-1}$ (for any K-algebra R), the

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desired property is equivalent to

$$h_g(1, a_2, \dots, a_n) \neq 0.$$

But h_g is homogeneous, so this defines a *nonempty* open subscheme of $\mathbb{A}_K^{n-1}$. Since S is finite and $\mathcal{O}^{n-1}$ is Zariski dense in $\mathbb{A}_K^{n-1}$, the lemma follows. □

Corollary 2.8. *Fix a nonconstant polynomial $g \in \mathcal{O}[y_1, \dots, y_n]$, where $n \geqslant 1$. Then there exists $A_g > 0$ such that for any $r \in \mathcal{O}^+$ we have*

$$N(g; r) \leqslant A_g^{\omega(r)} |r|^{n-(\deg g)^{-1}}.$$

Proof. This is analogous to [44, Lemma 4.10] (in which the homogeneity assumed is unimportant). By Lemma 2.7, we can make an $\mathcal{O}$-linear change of variables in order to assume that $\deg_{y_1}(g) = \deg(g)$. Then (2.22) immediately implies the desired bound. □

Beyond Corollary 2.8, we need a strong estimate for square-free polynomials.

Lemma 2.9. *Fix a nonzero polynomial $g \in \mathcal{O}[y_1, \dots, y_n]$, where $n \geqslant 1$. Assume g is square-free as an element of $K[y_1, \dots, y_n]$. Then there exists $A_g > 0$ such that for all cube-free $r \in \mathcal{O}^+$ we have*

$$N(g; r) \leqslant A_g^{\omega(r)} |r|^{n-1}.$$

Proof. Assume $r = \varpi^l$. When $l = 1$, the Lang–Weil estimate for the quasi-projective variety $g = 0$ implies $N(g; \varpi) \ll_g |\varpi|^{n-1} = |r|^{n-1}$. When $l = 2$, care is needed because K is not perfect. Nonetheless, in [45, final paragraph of §7 (in "Proof of Theorem 3.4")], it is shown that $N(g; \varpi^2) \ll_g |\varpi|^{2n-2} = |r|^{n-1}$. This completes the proof. □

Finally, we need an affine dimension growth bound available from [12].

Lemma 2.10 [12, Lemma 2.8]. *Let $B \geqslant 0$. Any affine K-variety W (equipped with an integral model over $\mathcal{O}$) satisfies $\#\{\boldsymbol{y} \in W(\mathcal{O}) : |\boldsymbol{y}| \leqslant \widehat{B}\} \ll_W \widehat{B}^{\dim W}$.*

3. Ratios Conjecture and applications

We start generally and then specialise. Let K^{sep} be the separable closure of K. Fix a prime $\ell \neq \mathrm{char}(\mathbb{F}_q)$. For a smooth proper K-variety X, define the ℓ-adic cohomology

$$H_\ell^i(X) := H^i(X \times_K K^{\mathrm{sep}}, \mathbb{Q}_\ell).$$

This is an ℓ-adic representation of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$, pure of weight i by Deligne's resolution of the Weil Conjectures [20]. The exterior square representation $\bigwedge^2 H_\ell^i(X)$, which will soon play an important role, is pure of weight $2i$.

In general, for an ℓ-adic representation[1] M of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$, pure of weight $w \in \mathbb{Z}$, we make the following definitions.

1. Let M^{I_v} be the inertia invariants of M at v. Let $\alpha_{M,j}^0(v)$ (for $1 \leqslant j \leqslant \dim M^{I_v}$) be the eigenvalues of geometric Frobenius on M^{I_v}.
2. Let $\alpha_{M,j}(v) := \alpha_{M,j}^0(v)/(\#k_v)^{w/2}$, where k_v is the residue field of $\mathcal{O}_v$.
3. Let

$$L_v(s, M) := \prod_j (1 - \alpha_{M,j}(v)|k_v|^{-s})^{-1}$$

[1]More precisely, M is a finite-dimensional $\mathbb{Q}_\ell$-vector space equipped with a continuous group homomorphism $\rho\colon \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{GL}(M)$.

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be the *analytically normalised* local factor at v. Let

$$L(s, M) := \prod_{\varpi} L_{\varpi}(s, M)$$

(with $\varpi \neq \infty$) and

$$\Lambda(s, M) := \prod_{v} L_v(s, M)$$

(including $v = \infty$), so that $\Lambda(*) = L(*)L_\infty(*)$.

Let V and $V_{\boldsymbol{c}}$ be the K-varieties in $\mathbb{P}^5_K$ defined by $F(\boldsymbol{x}) = 0$ and $F(\boldsymbol{x}) = \boldsymbol{c} \cdot \boldsymbol{x} = 0$, respectively. Let $L(s, V) = L(s, H^4_\ell(V)/H^4_\ell(\mathbb{P}^5))$, and for $\boldsymbol{c} \in \mathcal{S}_1$ let

$$L(s, \boldsymbol{c}) = L(s, H^3_\ell(V_{\boldsymbol{c}})), \quad L(s, \boldsymbol{c}, \textstyle\bigwedge^2) = L(s, \bigwedge^2 H^3_\ell(V_{\boldsymbol{c}})).$$

Let $\alpha_{V,j}(v)$, $\alpha_{\boldsymbol{c},j}(v)$, $\alpha_{\boldsymbol{c},\bigwedge^2,j}(v)$ be the corresponding normalised eigenvalues. The local factors are independent of ℓ, by [35, Theorem 5.46], and the number of normalised eigenvalues in each case is $\leqslant \binom{10}{2} = 45$, by classical Betti number calculations for smooth projective hypersurfaces over $\mathbb{C}$, which follow for instance from [3, Corollary 17.3.8].

For $r \in \mathcal{O}^+$, define $\lambda_V(r)$, $\lambda_{\boldsymbol{c}}(r)$, $\lambda_{\boldsymbol{c},\bigwedge^2}(r)$ to be the *rth coefficients* of the Euler products $L(s, V)$, $L(s, \boldsymbol{c})$, $L(s, \boldsymbol{c}, \bigwedge^2)$, respectively. More precisely, for prime $\varpi \in \mathcal{O}^+$ let $\lambda_*(\varpi^k)$ be the coefficient of $|\varpi|^{-ks}$ in $L_\varpi(s, *)$, and extend multiplicatively to define $\lambda_*(r)$ for $r \in \mathcal{O}^+$. (We have to carefully define $\lambda_*(r)$, because for any ϖ, ϖ' the sizes $|\varpi|$, $|\varpi'|$ are multiplicatively dependent.)

Let $\mathsf{T} = \frac{2\pi}{\log q}$. Note that q^s is invariant under translation by $i\mathsf{T}$, so

$$L(s, *) = L(s + i\mathsf{T}\mathbb{Z}, *). \tag{3.1}$$

Let $\zeta_K(s) := \prod_\varpi (1 - |\varpi|^{-s})^{-1} = \sum_{r \in \mathcal{O}^+} |r|^{-s} = (1 - q^{1-s})^{-1}$. For later convenience we define

$$L(s, \boldsymbol{c}, 2) := L(s, \boldsymbol{c}, \textstyle\bigwedge^2)/\zeta_K(s), \tag{3.2}$$

and $L_v(s, \boldsymbol{c}, 2) := L_v(s, \boldsymbol{c}, \bigwedge^2)/\zeta_{K,v}(s)$.

Proposition 3.1. *Let $\boldsymbol{c} \in \mathcal{S}_1$. Let $L(s)$ be one of $\zeta_K(s)$, $L(s, V)$, $L(s, \boldsymbol{c})$, $L(s, \boldsymbol{c}, \bigwedge^2)$, $L(s, \boldsymbol{c}, 2)$. Let $\Lambda(s) = L(s)L_\infty(s)$ be the corresponding completed L-function.*

1. *There exist unique polynomials $P_0, P_1, P_2 \in 1 + z\,\mathbb{R}[z]$, where the roots of P_i are complex numbers of size $q^{-i/2}$, such that*

$$\Lambda(s) = \frac{P_1(q^{-s})}{P_0(q^{-s})P_2(q^{-s})}.$$

2. $\deg P_0, \deg P_2 \ll 1$ *and* $\deg P_1 \ll 1 + \log|F^*(\boldsymbol{c})|$.
3. *At each place v, the local factor $L_v(s) = \prod_\alpha (1 - \alpha q^{-s})^{-1}$ has real coefficients, and the inverse roots α satisfy the Ramanujan bound $|\alpha| \leqslant 1$.*
4. *We have $1/L(s) \ll_\varepsilon |\boldsymbol{c}|^\varepsilon$ for $\Re(s) \geqslant \frac{1}{2} + \varepsilon$.*
5. *The $\widehat{R}^{-s}$ coefficient of $1/L(s) \in \mathbb{R}[[q^{-s}]]$ is $\ll_\varepsilon |\boldsymbol{c}|^\varepsilon \widehat{R}^{1/2+\varepsilon}$ for all $R \in \mathbb{Z}_{\geqslant 0}$.*

Proof. (1): For $\zeta_K(s)$, $L(s, V)$, $L(s, \boldsymbol{c})$, $L(s, \boldsymbol{c}, \bigwedge^2)$, this follows directly from [35, proof of Theorem 5.58], since the corresponding ℓ-adic sheaves $\mathbb{Q}_\ell$, $H^4_\ell(V)/H^4_\ell(\mathbb{P}^5)$, $H^3_\ell(V_{\boldsymbol{c}})$, $\bigwedge^2 H^3_\ell(V_{\boldsymbol{c}})$ are 'weakly polarisable' (by Poincaré duality on V and $V_{\boldsymbol{c}}$, which gives a symmetric pairing on $H^4_\ell(V)$ and a skew-symmetric pairing on $H^3_\ell(V_{\boldsymbol{c}})$, and thus a symmetric pairing on the tensor squares thereof). By Proposition 3.2 and the formula $\zeta_K(s) = 1/(1 - q^{1-s})$, the result then follows for $L(s, \boldsymbol{c}, 2)$ by (3.2).

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(2): This is explained in [12, §3.4] using Swan conductors.
(3): This follows from [35, Theorem 5.46].
(4): This follows from (1)–(3) and the Hadamard three circle theorem as in [12, proof of Lemma 8.4].
(5): Let $s = \frac{1}{2} + \varepsilon + i\tau$ and integrate $\widehat{R}^s/L(s) \ll_\varepsilon |\boldsymbol{c}|^\varepsilon \widehat{R}^{1/2+\varepsilon}$ over $\tau \in \mathbb{R}/\mathsf{T}\mathbb{Z}$. □

Proposition 3.2. *Let $\boldsymbol{c} \in \mathcal{S}_1$. Then $L(s, \boldsymbol{c}, \bigwedge^2)$ has a pole at $s = 1$.*

Proof. Let $M = H^3_\ell(V_{\boldsymbol{c}})$. Poincaré duality gives a perfect *skew-symmetric* pairing

$$\psi \colon M \wedge M \to \mathbb{Q}_\ell(-3),$$

where $\mathbb{Q}_\ell(-3)$ denotes the Tate motive of weight 6. Since ψ is surjective, its dual

$$\psi^\vee \colon \mathbb{Q}_\ell(3) \to M^\vee \wedge M^\vee$$

is injective. But ψ is perfect, so that it induces an isomorphism

$$\psi' \colon M \to \mathrm{Hom}(M, \mathbb{Q}_\ell(-3)) = M^\vee(-3).$$

After twisting $\psi^\vee$ by $\mathbb{Q}_\ell(-6)$, we thus obtain an injection $\mathbb{Q}_\ell(-3) \to M \wedge M$. Thus $(M \wedge M)(3)$ (which is pure of weight 0) has a nonzero space of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-invariants. So by [42, correction to Theorem 2.1] (which builds on [40,41]), it follows that the L-function $L(s, (M \wedge M)(3)) = L(s, \boldsymbol{c}, \bigwedge^2)$ has a pole at $s = 1$. □

Proposition 3.3 (Kisin). *Fix a prime $\varpi \in \mathcal{O}^+$ and a tuple $\boldsymbol{b} \in \mathcal{O}^6_\varpi$ with $F^*(\boldsymbol{b}) \neq 0$. Then there exists an integer $l \geqslant 0$, depending only on ϖ and $\boldsymbol{b}$, such that for all tuples $\boldsymbol{a} \in \mathcal{O}^6_\varpi$ with $\boldsymbol{a} \equiv \boldsymbol{b} \bmod \varpi^{1+l}$, we have $F^*(\boldsymbol{a}) \neq 0$ and $L_\varpi(s, \boldsymbol{a}) = L_\varpi(s, \boldsymbol{b})$.*

Proof. This follows directly from [37, case (2) of Theorem 5.1]. □

Lemma 3.4. *For each $\boldsymbol{b} \in \mathcal{O}^6_\varpi$ with $F^*(\boldsymbol{b}) \neq 0$, let $l(\varpi, \boldsymbol{b})$ be the smallest integer $l \geqslant 0$ verifying Proposition 3.3. Then for any integer $l \geqslant 0$, the set*

$$\{\boldsymbol{b} \in \mathcal{O}^6_\varpi : F^*(\boldsymbol{b}) \neq 0,\ l(\varpi, \boldsymbol{b}) \leqslant l\} \tag{3.3}$$

is closed under translation by $\varpi^{l+1}\mathcal{O}^6_\varpi$, and its measure tends to 1 as $l \to \infty$.

Proof. We first show that $l(\varpi, \boldsymbol{b})$ is locally constant. Let $\boldsymbol{b}, \boldsymbol{c} \in \mathcal{O}^6_\varpi$ with $F^*(\boldsymbol{b}) \neq 0$ and $\boldsymbol{c} \equiv \boldsymbol{b} \bmod \varpi^{1+l(\varpi,\boldsymbol{b})}$. Then $F^*(\boldsymbol{c}) \neq 0$ and $L_\varpi(s, \boldsymbol{c}) = L_\varpi(s, \boldsymbol{b})$, and thus $l(\varpi, \boldsymbol{c}) \leqslant l(\varpi, \boldsymbol{b})$ (since $\boldsymbol{a} \equiv \boldsymbol{c} \bmod \varpi^{1+l(\varpi,\boldsymbol{b})} \Rightarrow \boldsymbol{a} \equiv \boldsymbol{b} \bmod \varpi^{1+l(\varpi,\boldsymbol{b})}$). But then $\boldsymbol{b} \equiv \boldsymbol{c} \bmod \varpi^{1+l(\varpi,\boldsymbol{c})}$, so $l(\varpi, \boldsymbol{b}) \leqslant l(\varpi, \boldsymbol{c})$, whence $l(\varpi, \boldsymbol{b}) = l(\varpi, \boldsymbol{c})$. So if $\boldsymbol{b}$ lies in (3.3) for some $l \geqslant 0$, then (3.3) is indeed closed under translation by $\varpi^{l+1}\mathcal{O}^6_\varpi$.

We now turn to measures. Let $A \in \mathbb{Z}_{\geqslant 0}$. The set $S_A = \{\boldsymbol{c} \in \mathcal{O}^6_\varpi : v_\varpi(F^*(\boldsymbol{c})) \leqslant A\}$ is compact. Yet the function $\boldsymbol{b} \mapsto l(\varpi, \boldsymbol{b})$ on S_A is continuous. So (3.3) contains S_A for all $l \gg_A 1$, say. But by Corollary 2.8, the measure of S_A tends to 1 as $A \to \infty$. □

Let $\mu_{\boldsymbol{c}}(r)$ be the rth coefficient of the Euler product $L(s, \boldsymbol{c})^{-1} = \prod_\varpi L_\varpi(s, \boldsymbol{c})^{-1}$. The following result collects together some facts about averages of $\mu_{\boldsymbol{c}}(r)$ over vectors $\boldsymbol{c}$.

Proposition 3.5. *Let $r_1, r_2 \in \mathcal{O}^+$. Let $\mathbb{E}_{\boldsymbol{c} \in S}[f]$ be the average of f over S. The limit*

$$\bar{\mu}_{F,2}(r_1, r_2) := \lim_{Z \to \infty} \mathbb{E}_{\boldsymbol{c} \in \mathcal{S}_1 : |\boldsymbol{c}| \leqslant \widehat{Z}} [\mu_{\boldsymbol{c}}(r_1)\mu_{\boldsymbol{c}}(r_2)]$$

exists. Moreover, $\bar{\mu}_{F,2}(r_1, r_2)\bar{\mu}_{F,2}(r_1', r_2') = \bar{\mu}_{F,2}(r_1 r_1', r_2 r_2')$ if $\gcd(r_1 r_2, r_1' r_2') = 1$.

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Now let $\varpi \in \mathcal{O}^+$ be a prime, and let $l, l_1, l_2 \geqslant 0$ be integers. Then

$$\bar{\mu}_{F,2}(\varpi^{l_1}, \varpi^{l_2}) \ll_\varepsilon |\varpi|^{(l_1+l_2)\varepsilon}. \tag{3.4}$$

Furthermore,

$$\bar{\mu}_{F,2}(\varpi, 1) = \lambda_V(\varpi)|\varpi|^{-1/2} + O(|\varpi|^{-1}), \quad \bar{\mu}_{F,2}(\varpi^2, 1) = 1 + O(|\varpi|^{-1}), \tag{3.5}$$

$$\bar{\mu}_{F,2}(\varpi, \varpi) = 1 + O(|\varpi|^{-1}). \tag{3.6}$$

Proof. This result is directly analogous to [57, Proposition 6.1], and is purely local. For the reader's convenience, we highlight the main points. Existence and multiplicativity of the limit follow from Lemma 3.4, the bound $|\mu_{\boldsymbol{c}}(r)| \ll_\varepsilon |r|^\varepsilon$ (which follows from Proposition 3.1(3)) and the Chinese remainder theorem. The bound $|\mu_{\boldsymbol{c}}(r)| \ll_\varepsilon |r|^\varepsilon$ implies (3.4). The estimates in (3.5) and (3.6) are subtler; they relate to the *rank* and *homogeneity type* [46, §1] of our geometric family of L-functions $L(s, \boldsymbol{c})$.

Recall the definition of $E^\natural_{\boldsymbol{c}}(k)$ from (2.15). The following hold uniformly over primes $\varpi \in \mathcal{O}^+$ (where $k_\varpi = \mathcal{O}/\varpi\mathcal{O}$ and $k_{\varpi,2}$ is the unique quadratic field extension of k_ϖ):

$$\mathbb{E}_{\boldsymbol{c} \in k_\varpi^6}[E^\natural_{\boldsymbol{c}}(k_\varpi)\mathbf{1}_{\varpi \nmid F^*(\boldsymbol{c})}] = \lambda_V(\varpi)|\varpi|^{-1/2} + O(|\varpi|^{-1}), \tag{3.7}$$

$$\mathbb{E}_{\boldsymbol{c} \in k_\varpi^6}[E^\natural_{\boldsymbol{c}}(k_{\varpi,2})\mathbf{1}_{\varpi \nmid F^*(\boldsymbol{c})}] = 1 + O(|\varpi|^{-1}), \tag{3.8}$$

$$\mathbb{E}_{\boldsymbol{c} \in k_\varpi^6}[E^\natural_{\boldsymbol{c}}(k_\varpi)^2\mathbf{1}_{\varpi \nmid F^*(\boldsymbol{c})}] = 1 + O(|\varpi|^{-1}). \tag{3.9}$$

This is proven for prime fields in [53, Corollary 1.7]; the proof there carries over directly to arbitrary finite fields of characteristic > 3.

Finally, observe that if $\varpi \nmid F^*(\boldsymbol{c})$, then $L_\varpi(s, \boldsymbol{c})^{-1} = \prod_{1 \leqslant j \leqslant 10}(1 - \alpha_{\boldsymbol{c},j}(\varpi)|\varpi|^{-s})$, so

$$\mu_{\boldsymbol{c}}(\varpi) = -\sum_{1 \leqslant j \leqslant 10} \alpha_{\boldsymbol{c},j}(\varpi) = E^\natural_{\boldsymbol{c}}(k_\varpi),$$
$$\mu_{\boldsymbol{c}}(\varpi^2) = \sum_{1 \leqslant i < j \leqslant 10} \alpha_{\boldsymbol{c},i}(\varpi)\alpha_{\boldsymbol{c},j}(\varpi) = \tfrac{1}{2}(E^\natural_{\boldsymbol{c}}(k_\varpi)^2 + E^\natural_{\boldsymbol{c}}(k_{\varpi,2})),$$

because $E^\natural_{\boldsymbol{c}}(k_\varpi) = -\sum_{1 \leqslant j \leqslant 10} \alpha_{\boldsymbol{c},j}(\varpi)$ and $E^\natural_{\boldsymbol{c}}(k_{\varpi,2}) = -\sum_{1 \leqslant j \leqslant 10} \alpha_{\boldsymbol{c},j}(\varpi)^2$ by the Grothendieck–Lefschetz trace formula. Thus (3.7)–(3.9) imply (3.5) and (3.6). □

Informally, Proposition 3.5 tells us

$$\sum_{r_1, r_2 \in \mathcal{O}^+} \frac{\bar{\mu}_{F,2}(r_1, r_2)}{|r_1|^{s_1}|r_2|^{s_2}} \approx \prod_{\varpi} \Big[1 + |\varpi|^{-s_1-s_2} + \sum_{1 \leqslant j \leqslant 2} (\lambda_V(\varpi)|\varpi|^{-s_j-1/2} + |\varpi|^{-2s_j})\Big]$$
$$\approx \zeta_K(s_1 + s_2) \prod_{1 \leqslant j \leqslant 2} (L(s_j + \tfrac{1}{2}, V)\zeta_K(2s_j)),$$

where the terms indexed by $1 \leqslant j \leqslant 2$ arise from (3.5) and the trivial symmetry identity $\bar{\mu}_{F,2}(\varpi^l, 1) = \bar{\mu}_{F,2}(1, \varpi^l)$. Motivated by this, let

$$A_{F,2}(s_1, s_2) = \frac{\zeta_K(s_1 + s_2)^{-1}}{\prod_{1 \leqslant j \leqslant 2}(\zeta_K(2s_j)L(s_j + \frac{1}{2}, V))} \sum_{r_1, r_2 \in \mathcal{O}^+} \frac{\bar{\mu}_{F,2}(r_1, r_2)}{|r_1|^{s_1}|r_2|^{s_2}},$$

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which by Proposition 3.5 converges absolutely for $\Re(s_1), \Re(s_2) > \frac{1}{3}$. The expression $A_{F,2}$ appears as the 'leading constant' in the Ratios Conjecture for

$$\frac{1}{L(s_1, \boldsymbol{c})L(s_2, \boldsymbol{c})}.$$

Let $\sigma(Z) = \frac{1}{2} + \frac{1}{Z}$. For $\boldsymbol{c} \in \mathcal{S}_1$, let

$$\Phi^{\boldsymbol{c},1}(s) = \zeta_K(2s)^{-1} L(s + \tfrac{1}{2}, V)^{-1} L(s, \boldsymbol{c})^{-1}.$$

For convenience, let $a_{\boldsymbol{c},1}(r)$ be the rth coefficient of the Euler product $\Phi^{\boldsymbol{c},1}(s)$.

The Ratios Recipe [17, §5.1], directly adapted to function fields as in [2], produces the following Ratios Conjecture (R2), even with a power-saving error term $O(\widehat{Z}^{-\delta})$ *independent of* β for $\beta \leqslant \delta$, say. The derivation over K is exactly the same as the derivation over $\mathbb{Q}$ in [57, §6.3]. We emphasise that the summand on the right-hand side of (3.10) does not depend on $\boldsymbol{c}$; a similar point is made after [57, Conjecture 1.5]. For additional context on random matrix predictions for L-functions, we note that the L-functions $L(s, \boldsymbol{c})$ over $\boldsymbol{c} \in \mathcal{S}_1$ form a *geometric family* in the sense of [46, pp. 534–535].

Conjecture 3.6 (R2). *There exists a constant* $\beta \in [0, 1]$ *such that if*

$$s_j = \beta + \sigma(Z) + i\tau_j,$$

then uniformly over $Z \in \mathbb{N}$ *and* $\tau_1, \tau_2 \in \mathbb{R}$, *we have*

$$\sum_{\boldsymbol{c} \in \mathcal{S}_1 : |\boldsymbol{c}| \leqslant \widehat{Z}} \Phi^{\boldsymbol{c},1}(s_1)\Phi^{\boldsymbol{c},1}(s_2) = \sum_{\boldsymbol{c} \in \mathcal{S}_1 : |\boldsymbol{c}| \leqslant \widehat{Z}} (\zeta_K(s_1 + s_2) + O(\widehat{Z}^{-6\beta}))\, A_{F,2}(s_1, s_2). \tag{3.10}$$

(In fact one would expect (R2) to hold with $\beta = 0$, but the approach of [5,43] suggests that a small positive value of β might be more tractable, at least for large fixed q. Also, the 'slope' 6 in the exponent 6β is what we need for our application to Conjecture 3.7 below, but arbitrarily large slope should be permissible.)

For a vertically T-periodic function $f(s) = f(\sigma + i\tau)$, define the *vertical average*

$$\mathbb{E}_{\Re(s)=\sigma}[f(s)] := \frac{1}{\mathsf{T}} \int_{\mathbb{R}/\mathsf{T}\mathbb{Z}} f(\sigma + i\tau)\, \mathrm{d}\tau.$$

Note that if f is holomorphic on a vertical strip $\Re(s) \in I$, then $\mathbb{E}_{\Re(s)=\sigma}[f(s)]$ is independent of $\sigma \in I$, by Cauchy's integral theorem over rectangles of height T.

Conjecture 3.7. *Let* $Z, R \in \mathbb{Z}$ *with* $R \leqslant 3Z$. *If* $\sigma_0 > 1/2$, *then*

$$\sum_{\boldsymbol{c} \in \mathcal{S}_1 : |\boldsymbol{c}| \leqslant \widehat{Z}} \left| \mathbb{E}_{\Re(s)=\sigma_0}[\Phi^{\boldsymbol{c},1}(s) \cdot \widehat{R}^s] \right|^2 \ll \widehat{Z}^6 \widehat{R}. \tag{3.11}$$

Proof assuming (R2). By (3.1) and GRH, the left-hand side of (3.11) is independent of $\sigma_0 > \frac{1}{2}$. If $\widehat{Z} \ll 1$ then (3.11) is trivial, so suppose $Z \geqslant 2$. We find, after shifting contours to $\Re(s) = \beta + \sigma(Z) \in (\frac{1}{2}, 2]$ and expanding squares using self-duality of $\Phi^{\boldsymbol{c},1}$, the left-hand side of (3.11) equals

$$\Sigma_0 := \sum_{\boldsymbol{c} \in \mathcal{S}_1 : |\boldsymbol{c}| \leqslant \widehat{Z}} \mathbb{E}_{\Re(s_1), \Re(s_2) = \beta + \sigma(Z)} [\Phi^{\boldsymbol{c},1}(s_1)\Phi^{\boldsymbol{c},1}(s_2) \cdot \widehat{R}^{s_1+s_2}].$$

After switching the order of $\boldsymbol{c}$ and $\boldsymbol{s}$ in Σ_0, and plugging in (3.10) for each $\boldsymbol{s}$, we get

$$\Sigma_0 = \Sigma_2 + O(\widehat{Z}^{6-6\beta}\Sigma_1^2), \tag{3.12}$$

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where $\Sigma_1 := \mathbb{E}_{\Re(s)=\beta+\sigma(Z)}[|\widehat{R}^s|] = \widehat{R}^{\beta+\sigma(Z)}$ and

$$\Sigma_2 := \sum_{\boldsymbol{c}\in\mathcal{S}_1:\,|\boldsymbol{c}|\leqslant\widehat{Z}} \mathbb{E}_{\Re(s_1),\Re(s_2)=\beta+\sigma(Z)}[\zeta_K(s_1+s_2)A_{F,2}(s_1,s_2)\cdot\widehat{R}^{s_1+s_2}].$$

Since $R \leqslant 3Z$ and $\widehat{Z}^{1/Z} = q \ll 1$, we certainly have $\Sigma_1 \ll \widehat{Z}^{3\beta}\widehat{R}^{1/2}$.

Let $\delta = \frac{1}{20}$; then $\frac{1}{2} - \delta \geqslant \frac{1}{3} + \delta$. In Σ_2 we shift $\Re(s_1)$ from $\beta+\sigma(Z)$ to $\frac{1}{2}$, then shift $\Re(s_2)$ from $\beta+\sigma(Z)$ to $\frac{1}{2}-\delta$. This yields $\Sigma_2 = \Sigma_3 + O(\Sigma_4)$, where

$$\Sigma_3 := \sum_{\boldsymbol{c}\in\mathcal{S}_1:\,|\boldsymbol{c}|\leqslant\widehat{Z}} \mathbb{E}_{\Re(s_1)=\frac{1}{2}}[\tfrac{1}{\mathsf{T}}\tfrac{2\pi}{\log q}A_{F,2}(s_1,1-s_1)\cdot\widehat{R}]$$

comes from the residue $(\log q)^{-1}$ of $\zeta_K(s_1+s_2) = (1-q^{1-s_1-s_2})^{-1}$ at $s_2 = 1-s_1$, and where we bound the contribution from $\Re(s_1) = \frac{1}{2}$ and $\Re(s_2) = \frac{1}{2}-\delta$ by

$$\Sigma_4 := \sum_{\boldsymbol{c}\in\mathcal{S}_1:\,|\boldsymbol{c}|\leqslant\widehat{Z}} \widehat{R}^{1-\delta} \ll \widehat{Z}^6\widehat{R}^{1-\delta},$$

on using the bounds $A_{F,2}(s_1,s_2) \ll 1$ and $\zeta_K(s_1+s_2) \ll_\delta 1$. Next we note that

$$\Sigma_3 \ll \widehat{Z}^6\widehat{R},$$

since $A_{F,2}(s_1,1-s_1) \ll 1$ in Σ_3. But

$$\Sigma_0 \leqslant O(\widehat{Z}^{6-6\beta}\Sigma_1^2) + \Sigma_3 + O(\Sigma_4),$$

by (3.12). Plugging in our estimates for Σ_1, Σ_3 and Σ_4, we finally get (3.11). □

Conjecture 3.8. *Let $Z, R \in \mathbb{Z}$ with $R \leqslant 3Z$. Then*

$$\sum_{\boldsymbol{c}\in\mathcal{S}_1:\,|\boldsymbol{c}|\leqslant\widehat{Z}} \Big|\sum_{r\in\mathcal{O}^+} \mathbf{1}_{|r|=\widehat{R}}\cdot a_{\boldsymbol{c},1}(r)\Big|^2 \ll \widehat{Z}^6\widehat{R}. \tag{3.13}$$

Proof assuming Conjecture 3.7. Since $R \in \mathbb{Z}$, we may write

$$\mathbf{1}_{|r|=\widehat{R}} = \mathbb{E}_{\Re(s)=\sigma_0}[(\widehat{R}/|r|)^s]$$

for each $r \in \mathcal{O}^+$. Now apply (3.11). □

4. Euler product factorisations

Recall the definition (2.14) of $\mathcal{R}_{\boldsymbol{c}}^{\mathrm{G}}$ and $\mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}}$. Given $\boldsymbol{c} \in \mathcal{S}_1$, consider the factorisation $\Phi = \Phi^{\mathrm{G}}\Phi^{\mathrm{B}}$, where

$$\begin{aligned}
\Phi_\varpi(\boldsymbol{c},s) &:= \sum_{l\geqslant 0}|\varpi|^{-ls}S^\natural_{\varpi^l}(\boldsymbol{c}),\\
\Phi(\boldsymbol{c},s) &:= \prod_{\varpi}\Phi_\varpi(\boldsymbol{c},s) = \sum_{r\in\mathcal{O}^+}|r|^{-s}S^\natural_r(\boldsymbol{c}),\\
\Phi^{\mathrm{G}}(\boldsymbol{c},s) &:= \prod_{\varpi\nmid F^*(\boldsymbol{c})}\Phi_\varpi(\boldsymbol{c},s) = \sum_{r\in\mathcal{R}_{\boldsymbol{c}}^{\mathrm{G}}}S^\natural_r(\boldsymbol{c})|r|^{-s},\\
\Phi^{\mathrm{B}}(\boldsymbol{c},s) &:= \prod_{\varpi\mid F^*(\boldsymbol{c})}\Phi_\varpi(\boldsymbol{c},s) = \sum_{r\in\mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}}}S^\natural_r(\boldsymbol{c})|r|^{-s}.
\end{aligned} \tag{4.1}$$

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For convenience, let $L^{\mathrm{G}}(*) := \prod_{\varpi \nmid F^*(\boldsymbol{c})} L_\varpi(*)$ and $L^{\mathrm{B}}(*) := \prod_{\varpi \mid F^*(\boldsymbol{c})} L_\varpi(*)$.

Definition 4.1. Let $\Phi^{\boldsymbol{c},1}(s) := L(s,\boldsymbol{c})^{-1} L(\frac{1}{2}+s, V)^{-1} \zeta_K(2s)^{-1}$ as before, and let

$$\Phi^{\boldsymbol{c},2}(s) := L^{\mathrm{G}}(2s, \boldsymbol{c}, 2)^{-1},$$
$$\Phi^{\boldsymbol{c},3}(s) := \Phi^{\mathrm{G}}(\boldsymbol{c}, s) L(s,\boldsymbol{c}) L(\tfrac{1}{2}+s, V) \zeta_K(2s) L^{\mathrm{G}}(2s, \boldsymbol{c}, 2).$$

For each $j \in \{1,2,3\}$, let $a_{\boldsymbol{c},j}(r)$ be the rth coefficient of the Euler product $\Phi^{\boldsymbol{c},j}(s)$.

Multiplying out the definitions of $\Phi^{\boldsymbol{c},j}(s)$ above, we find that

$$\prod_{1 \leqslant j \leqslant 3} \Phi^{\boldsymbol{c},j}(s) = \Phi^{\mathrm{G}}(\boldsymbol{c}, s).$$

Thus, in order to use (3.13) (which concerns $\Phi^{\boldsymbol{c},1}$) to bound $\Phi^{\mathrm{G}}(\boldsymbol{c}, s)$, we will first need to control the factors $\Phi^{\boldsymbol{c},2}$ and $\Phi^{\boldsymbol{c},3}$ on average over $\boldsymbol{c}$.

Proposition 4.2. *Let $Z, R \in \mathbb{Z}$ and $A, \varepsilon \in \mathbb{R}_{>0}$. Then*

$$\sum_{\boldsymbol{c} \in \mathcal{S}_1 : |\boldsymbol{c}| \leqslant \widehat{Z}} \Big| \sum_{r \in \mathcal{O}^+ : |r| = \widehat{R}} a_{\boldsymbol{c},2}(r) \Big|^A \ll_{A,\varepsilon} \widehat{Z}^6 \widehat{R}^{A/4+\varepsilon}. \tag{4.2}$$

Proof. If $\widehat{R} \leqslant 1$ then (4.2) is trivial, so suppose $\widehat{R} > 1$. By Hölder's inequality, we may assume A is an even integer. Let $\Sigma(Z,R)$ denote the left-hand side of (4.2) and put

$$\widetilde{\Sigma}(Z,R) := \Sigma(Z,R)/\widehat{Z}^6.$$

We claim that for $Z \geqslant AR$ we have

$$\widetilde{\Sigma}(Z,R) = \widetilde{\Sigma}(AR,R). \tag{4.3}$$

To prove this, we write $|\sum_{|r|=\widehat{R}} a_{\boldsymbol{c},2}(r)|^A = \sum_{|r_1|=\cdots=|r_A|=\widehat{R}} a_{\boldsymbol{c},2}(r_1)\cdots a_{\boldsymbol{c},2}(r_A)$, on noting that $a_{\boldsymbol{c},2}(r) \in \mathbb{R}$. If $r_1 \cdots r_A = M$, then (by smooth proper base change) the function $\boldsymbol{c} \mapsto a_{\boldsymbol{c},2}(r_1)\cdots a_{\boldsymbol{c},2}(r_A)$ is constant on each fibre of the map

$$\{\boldsymbol{c} \in \mathcal{S}_1 : |\boldsymbol{c}| \leqslant \widehat{Z},\ (M, F^*(\boldsymbol{c})) = 1\} \to \{\boldsymbol{c} \in (\mathcal{O}/M\mathcal{O})^6 : \gcd(M, F^*(\boldsymbol{c})) = 1\}.$$

Each fibre has cardinality $(\widehat{Z}/|M|)^6$, since $1 < |M| \leqslant \widehat{R}^A \leqslant \widehat{Z}$. The claim follows.

By GRH (Proposition 3.1(5)), we have $\Sigma(Z,R) \ll_{A,\varepsilon} \widehat{Z}^{6+\varepsilon}\widehat{R}^{A/4+\varepsilon}$ for all $\widehat{Z} > 0$. (The exponent is $A/4$ rather than $A/2$, because the Euler product of $\Phi^{\boldsymbol{c},2}(s)$, as given in Definition 4.1, is supported on *square* moduli due to the appearance of $2s$.) Thus $\widetilde{\Sigma}(Z,R) \ll_{A,\varepsilon} \widehat{R}^{A/4+\varepsilon}$ for $Z \leqslant AR$, and so for all Z, by (4.3). □

Proposition 4.3. *For $\boldsymbol{c} \in \mathcal{S}_1$, primes $\varpi \in \mathcal{O}^+$, and integers $l \geqslant 1$, we have*

$$a_{\boldsymbol{c},3}(\varpi) \cdot \mathbf{1}_{\varpi \nmid F^*(\boldsymbol{c})} = 0, \quad a_{\boldsymbol{c},3}(\varpi^2) \cdot \mathbf{1}_{\varpi \nmid F^*(\boldsymbol{c})} \ll |\varpi|^{-1/2}, \quad a_{\boldsymbol{c},3}(\varpi^l) \ll_\varepsilon |\varpi|^{l\varepsilon}.$$

Proof. If $\varpi \mid F^*(\boldsymbol{c})$, then $\Phi^{\mathrm{G}}_\varpi(\boldsymbol{c}, s) = 1$. If $\varpi \nmid F^*(\boldsymbol{c})$, then (2.17) implies

$$\Phi^{\mathrm{G}}_\varpi(\boldsymbol{c}, s) = 1 + S^{\natural}_\varpi(\boldsymbol{c})|\varpi|^{-s} = 1 - \lambda_{\boldsymbol{c}}(\varpi)|\varpi|^{-s} - \lambda_V(\varpi)|\varpi|^{-1/2-s},$$

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because $S^\natural_\varpi(\boldsymbol{c}) = E^\natural_{\boldsymbol{c}}(\mathcal{O}/\varpi\mathcal{O}) - |\varpi|^{-1/2}E^\natural_F(\mathcal{O}/\varpi\mathcal{O})$ by (2.16) and

$$\lambda_{\boldsymbol{c}}(\varpi) = \sum_{1\leqslant j\leqslant 10}\alpha_{\boldsymbol{c},j}(\varpi) = -E^\natural_{\boldsymbol{c}}(\mathcal{O}/\varpi\mathcal{O}), \quad \lambda_V(\varpi) = \sum_{1\leqslant j\leqslant 22}\alpha_{V,j}(\varpi) = E^\natural_F(\mathcal{O}/\varpi\mathcal{O}),$$

by the Grothendieck–Lefschetz trace formula. In either case, $a_{\boldsymbol{c},3}(\varpi^l) \ll_\varepsilon |\varpi|^{l\varepsilon}$ follows from the Ramanujan bound $|\alpha| \leqslant 1$ in Proposition 3.1(3).

Now suppose $\varpi \nmid F^*(\boldsymbol{c})$. Multiplying $\Phi^{\mathrm{G}}_\varpi(\boldsymbol{c}, s)$ by

$$\begin{aligned} L_\varpi(s,\boldsymbol{c}) &= 1 + \lambda_{\boldsymbol{c}}(\varpi)|\varpi|^{-s} + \lambda_{\boldsymbol{c}}(\varpi^2)|\varpi|^{-2s} + O(|\varpi|^{-3s}),\\ L_\varpi(\tfrac{1}{2}+s,V) &= 1 + \lambda_V(\varpi)|\varpi|^{-1/2-s} + O(|\varpi|^{-1-2s}),\\ \zeta_K(2s)L_\varpi(2s,\boldsymbol{c},2) = L_\varpi(2s,\boldsymbol{c},\textstyle\bigwedge^2) &= 1 + \lambda_{\boldsymbol{c},\bigwedge^2}(\varpi)|\varpi|^{-2s} + O(|\varpi|^{-4s}),\end{aligned}$$

and discarding all terms of the form $O(|\varpi|^{-s}\times|\varpi|^{-2s})$ or $O(|\varpi|^{-s}\times|\varpi|^{-1/2-s})$, we get

$$\Phi^{\boldsymbol{c},3}_\varpi(s) = 1 + O(|\varpi|^{-1/2-2s}) + O(|\varpi|^{-3s}),$$

since $\lambda_{\boldsymbol{c}}(\varpi)^2 = \lambda_{\boldsymbol{c}}(\varpi^2) + \lambda_{\boldsymbol{c},\bigwedge^2}(\varpi)$. This completes the proof of the proposition. □

Corollary 4.4. *Let $Z, R \in \mathbb{Z}$ and $A, \varepsilon \in \mathbb{R}_{>0}$. Then*

$$\sum_{\boldsymbol{c}\in\mathcal{S}_1:\,|\boldsymbol{c}|\leqslant\widehat{Z}}\Bigg(\sum_{r\in\mathcal{O}^+:\,|r|\leqslant\widehat{R}}|a_{\boldsymbol{c},3}(r)|\Bigg)^A \ll_{A,\varepsilon} \widehat{Z}^6\widehat{R}^{(1/3+\varepsilon)A}.$$

Proof. By multiplicativity (of $|a_{\boldsymbol{c},3}|$) and positivity, we have

$$\sum_{r\in\mathcal{O}^+:\,|r|\leqslant\widehat{R}}|a_{\boldsymbol{c},3}(r)| \leqslant \sum_{d\in\mathcal{R}^{\mathrm{B}}_{\boldsymbol{c}}:\,|d|\leqslant\widehat{R}}|a_{\boldsymbol{c},3}(d)| \sum_{e\in\mathcal{R}^{\mathrm{G}}_{\boldsymbol{c}}:\,|e|\leqslant\widehat{R}}|a_{\boldsymbol{c},3}(e)|. \tag{4.4}$$

But by Proposition 4.3, we have $a_{\boldsymbol{c},3}(d) \ll_\varepsilon |d|^\varepsilon$ and

$$\sum_{e\in\mathcal{R}^{\mathrm{G}}_{\boldsymbol{c}}:\,|e|\leqslant\widehat{R}}\frac{|a_{\boldsymbol{c},3}(e)|}{\widehat{R}^{1/3+\varepsilon}} \leqslant \sum_{e\in\mathcal{R}^{\mathrm{G}}_{\boldsymbol{c}}}\frac{|a_{\boldsymbol{c},3}(e)|}{|e|^{1/3+\varepsilon}} \leqslant \prod_{\varpi}\left(1 + \frac{O(|\varpi|^{-1/2})}{|\varpi|^{2/3+2\varepsilon}} + \frac{O_\varepsilon(|\varpi|^\varepsilon)}{|\varpi|^{1+3\varepsilon}}\right) \ll_\varepsilon 1,$$

So by (4.4), the quantity to be estimated is

$$\ll_{A,\varepsilon} \widehat{R}^{(1/3+2\varepsilon)A}\sum_{\boldsymbol{c}\in\mathcal{S}_1:\,|\boldsymbol{c}|\leqslant\widehat{Z}}\#\{d\in\mathcal{R}^{\mathrm{B}}_{\boldsymbol{c}} : |d|\leqslant\widehat{R}\}^A.$$

Now the claimed bound follows from Lemma 2.3. □

Conjecture 4.5. *Let $\varepsilon \in (0,1]$. Let $Z, R \in \mathbb{Z}$ with $R \leqslant 3Z$. Then*

$$\sum_{\boldsymbol{c}\in\mathcal{S}_1:\,|\boldsymbol{c}|\leqslant\widehat{Z}}\Bigg|\sum_{r\in\mathcal{R}^G_{\boldsymbol{c}}}\mathbf{1}_{|r|=\widehat{R}}\cdot S^\natural_r(\boldsymbol{c})\Bigg|^{2-\varepsilon} \ll_\varepsilon \widehat{Z}^6\widehat{R}^{(2-\varepsilon)/2}. \tag{4.5}$$

Proof assuming Conjecture 3.8. Let $\boldsymbol{c} \in \mathcal{S}_1$ and $R_1, R_2, R_3 \in \mathbb{Z}_{\geqslant 0}$. Writing

$$\Sigma^{\boldsymbol{c},j}(R_j) := \sum_{r_j\in\mathcal{O}^+:\,|r_j|=\widehat{R}_j} a_{\boldsymbol{c},j}(r_j)$$

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for brevity, we have

$$\sum_{r\in\mathcal{R}_{\boldsymbol{c}}^{\mathrm{G}}:\,|r|=\widehat{R}} S_r^{\natural}(\boldsymbol{c}) = \sum_{R_1+R_2+R_3=R}\ \prod_{1\leqslant j\leqslant 3}\Sigma^{\boldsymbol{c},j}(R_j),$$

since $\Phi^{\mathrm{G}} = \Phi^{\boldsymbol{c},1}\Phi^{\boldsymbol{c},2}\Phi^{\boldsymbol{c},3}$. Let $\beta := 2-\varepsilon$. Let

$$W_1(R_1) := \widehat{R}_1^{\eta}, \quad W_2(R_1) := \widehat{R}_1^{-(\beta-1)\eta},$$

for some $\eta > 0$ to be chosen. Let

$$\mathcal{I}_1 := \sum_{R_1+R_2+R_3=R} W_1(R_1).$$

Using Hölder's inequality in the form $1 = \frac{\beta-1}{\beta} + \frac{1}{\beta}$ over the set $\{R_1+R_2+R_3 = R\}$ (and then exponentiating by β), we obtain

$$\Big|\sum_{r\in\mathcal{R}_{\boldsymbol{c}}^{\mathrm{G}}:\,|r|=\widehat{R}} S_r^{\natural}(\boldsymbol{c})\Big|^{\beta} \leqslant \mathcal{I}_1^{\beta-1}\sum_{R_1+R_2+R_3=R} W_2(R_1)\prod_{1\leqslant j\leqslant 3}|\Sigma^{\boldsymbol{c},j}(R_j)|^{\beta}, \tag{4.6}$$

since $\beta \geqslant 1$ and $W_1^{\beta-1}W_2 = 1$. Here

$$\mathcal{I}_1 \leqslant \sum_{R_2,R_3\geqslant 0}\frac{\widehat{R}^{\eta}}{(\widehat{R}_2\widehat{R}_3)^{\eta}} \ll_{\eta} \widehat{R}^{\eta}.$$

Upon summing (4.6) over $|\boldsymbol{c}| \leqslant \widehat{Z}$, we thus find that the left-hand side of (4.5) is

$$\ll_{\eta} \sum_{R_1+R_2+R_3=R}(\widehat{R}/\widehat{R}_1)^{(\beta-1)\eta}\sum_{\boldsymbol{c}\in\mathcal{S}_1:\,|\boldsymbol{c}|\leqslant\widehat{Z}}\ \prod_{1\leqslant j\leqslant 3}|\Sigma^{\boldsymbol{c},j}(R_j)|^{\beta}. \tag{4.7}$$

Let $(\gamma_1,\gamma_2,\gamma_3) := (2, 4\beta/\varepsilon, 4\beta/\varepsilon)$. Then $\sum_{1\leqslant j\leqslant 3}\beta/\gamma_j = 1$, so by Hölder over $\boldsymbol{c}$, we obtain

$$\sum_{\boldsymbol{c}\in\mathcal{S}_1:\,|\boldsymbol{c}|\leqslant\widehat{Z}}\ \prod_{1\leqslant j\leqslant 3}|\Sigma^{\boldsymbol{c},j}(R_j)|^{\beta} \leqslant \prod_{1\leqslant j\leqslant 3}\mathscr{M}_j^{\beta/\gamma_j}, \tag{4.8}$$

where $\mathscr{M}_j = \sum_{\boldsymbol{c}\in\mathcal{S}_1:\,|\boldsymbol{c}|\leqslant\widehat{Z}}|\Sigma^{\boldsymbol{c},j}(R_j)|^{\gamma_j}$. By Conjecture 3.8 with R_1 in place of R, we have

$$\mathscr{M}_1 \ll \widehat{Z}^6\widehat{R}_1,$$

since $R_1 \leqslant R \leqslant 3Z$ by assumption. Also, by (4.2) and Corollary 4.4, we have

$$\mathscr{M}_2 \ll_{\gamma_2} \widehat{Z}^6\widehat{R}_2^{11\gamma_2/40} \quad \text{and} \quad \mathscr{M}_3 \ll_{\gamma_3} \widehat{Z}^6\widehat{R}_3^{11\gamma_3/30}.$$

Therefore, if $R_1+R_2+R_3 = R$, then the right-hand side of (4.8) is

$$\ll_{\beta} \widehat{Z}^{\sum_j 6\beta/\gamma_j}\widehat{R}_1^{\beta/2}\widehat{R}_2^{11\beta/40}\widehat{R}_3^{11\beta/30} = \widehat{Z}^6(\widehat{R}^{1/2}\widehat{R}_2^{-9/40}\widehat{R}_3^{-2/15})^{\beta}.$$

Plugging (4.8) into (4.7) now bounds the left-hand side of (4.5) by

$$\ll_{\eta,\beta}\sum_{R_1+R_2+R_3=R}\frac{(\widehat{R}/\widehat{R}_1)^{(\beta-1)\eta}\widehat{Z}^6\widehat{R}^{\beta/2}}{(\widehat{R}_2^{9/40}\widehat{R}_3^{2/15})^{\beta}}. \tag{4.9}$$

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Let $\eta = \min(9/40, 2/15)$. Then $R_1 + R_2 + R_3 = R \Rightarrow (\widehat{R}/\widehat{R}_1)^\eta \leqslant \widehat{R}_2^{9/40}\widehat{R}_3^{2/15}$. Since $\beta - 1 \geqslant 0$, it follows that the quantity (4.9) is

$$\ll_\beta \sum_{R_1+R_2+R_3=R} \frac{\widehat{Z}^6\widehat{R}^{\beta/2}}{\widehat{R}_2^{9/40}\widehat{R}_3^{2/15}} \leqslant \sum_{R_2,R_3\geqslant 0} \frac{\widehat{Z}^6\widehat{R}^{\beta/2}}{\widehat{R}_2^{9/40}\widehat{R}_3^{2/15}} \ll \widehat{Z}^6\widehat{R}^{\beta/2}.$$

This implies the desired inequality (4.5). □

5. Integral estimates

Let $\gamma \in K_\infty$, $\boldsymbol{w} \in K_\infty^n$ and $G \in K_\infty[x_1, \dots, x_n]$. For a fixed smooth weight function $w\colon K_\infty^n \to \mathbb{R}$ with compact support, we are interested in integrals of the form

$$J_{G,w}(\gamma, \boldsymbol{w}) := \int_{K_\infty^n} w(\boldsymbol{x})\psi(\gamma G(\boldsymbol{x}) + \boldsymbol{w}\cdot\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$

together with their averages

$$J_{G,w}^\Gamma(\boldsymbol{w}) = \int_{|\gamma|<\widehat{\Gamma}} J_{G,w}(\gamma, \boldsymbol{w})\mathrm{d}\gamma,$$

for $\Gamma \in \mathbb{Z}$. (In fact, the dependence of the latter integral on Γ is mild, as we will soon see.) With this notation we have $I_r(\theta, \boldsymbol{c}) = J_{F,w}(\theta P^3, P\boldsymbol{c}/r)$ in (2.7) and $I_r(\boldsymbol{c}) = |P|^{-3}J_{F,w}^\Gamma(P\boldsymbol{c}/r)$ in (2.8), where $\Gamma = -\deg(r) - Q + 3\deg(P)$.

When w is the indicator function of $\mathbb{T}^n$, we shall write $J_G(\gamma, \boldsymbol{w})$ and $J_G^\Gamma(\boldsymbol{w})$ for the sake of brevity. Moreover, we let H_G be the maximum of the absolute values of the coefficients of G and refer to it as the height of G.

This level of generality, where G is allowed to vary, will be useful when changing variables to analyse $J_{F,w}^\Gamma(\boldsymbol{w})$. Let us recall Lemmas 2.4 and 2.7 of [12].

Lemma 5.1. *Let $G \in K_\infty[x_1, \dots, x_n]$ and $\boldsymbol{w} \in K_\infty^n$. Suppose $|\boldsymbol{w}| \geqslant 1$ and $|\boldsymbol{w}| > H_G$. Then*

$$\int_{\mathbb{T}^n} \psi(G(\boldsymbol{x}) + \boldsymbol{w}\cdot\boldsymbol{x})\, d\boldsymbol{x} = 0.$$

Lemma 5.2. *Let*

$$\Omega = \{\boldsymbol{x} \in \mathbb{T}^n\colon |\gamma\nabla G(\boldsymbol{x}) + \boldsymbol{w}| \leqslant H_G \max\{1, |\gamma|^{1/2}\}\}.$$

Then we have

$$J_G(\gamma, \boldsymbol{w}) = \int_\Omega \psi(\gamma G(\boldsymbol{x}) + \boldsymbol{w}\cdot\boldsymbol{x})d\boldsymbol{x}.$$

We now fix a nonsingular cubic form $F \in \mathcal{O}[x_1, \dots, x_n]$, which we will eventually take to be our initial cubic form $x_1^3 + \cdots + x_6^3$, but is allowed to be arbitrary in this section. Before we start our investigation, we impose the following conditions on the weight function w.

Hypothesis 5.3. *Let $H(\boldsymbol{x}) = (\frac{\partial^2 F(\boldsymbol{x})}{\partial x_i \partial x_j})$ be the Hessian associated to F.*

(i) *If $\boldsymbol{x} \in \mathrm{supp}(w)$, then $\det H(\boldsymbol{x}) \neq 0$.*
(ii) *There exist $\boldsymbol{x_0} \in K_\infty^n$ with $|\boldsymbol{x}_0| \leqslant 1$, and an integer $L \geqslant 0$, such that*

$$w(\boldsymbol{x}) = \begin{cases} 1 & \text{if } |\boldsymbol{x} - \boldsymbol{x}_0| < \widehat{L}^{-1}, \\ 0 & \text{else.} \end{cases}$$

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If $\mathrm{char}(\mathbb{F}_q) \in \{2,3\}$, then $\det H(\boldsymbol{x})$ vanishes identically and so (i) is impossible in this case. Therefore, all of our results in this section have the implicit assumption that $\mathrm{char}(\mathbb{F}_q) > 3$. Moreover, condition (ii) is nonrestrictive, since any compactly supported smooth function $w' \colon K_\infty^n \to \mathbb{R}$ can be written as $w' = s_1 w_1 + \cdots + s_m w_m$, where $s_i \in \mathbb{R}$ and w_i are indicator functions of some compact ball in K_∞^n. Part (ii) of Hypothesis 5.3 implies the crucial symmetry property

$$w(\boldsymbol{x}) = w((1+t^{-L}\gamma)\boldsymbol{x}) \quad \text{for all } \gamma \in \mathbb{T}. \tag{5.1}$$

In other words, w is invariant under scaling by $1 + t^{-L}\mathbb{T}$. Throughout this section, all implied constants are allowed to depend on the choice of w and in particular implicitly also on the parameter L and the vector $\boldsymbol{x}_0$ in the definition of w.

For the purposes of Theorem 1.1, we will be focusing on diagonal cubic forms. In this setting we shall work with the following explicit weight function.

Definition 5.4. Let $F(\boldsymbol{x}) = a_1 x_1^3 + \cdots + a_n x_n^3$, for $a_1, \dots, a_n \in \mathbb{F}_q^\times$ and let $\boldsymbol{x}_0 \in (\mathbb{F}_q^\times)^n$ be such that $F(\boldsymbol{x}_0) = 0$. Then we define a weight function $w = w_{\boldsymbol{x}_0}$ by the formula

$$w(\boldsymbol{x}) = \begin{cases} 1 & \text{if } |\boldsymbol{x} - \boldsymbol{x}_0| < 1, \\ 0 & \text{else.} \end{cases}$$

We check that such weight functions satisfy Hypothesis 5.3 when F is a diagonal cubic form with coefficients in $\mathbb{F}_q^\times$ and $\mathrm{char}(\mathbb{F}_q) > 3$. Condition (i) can then be rephrased as $x_1 \cdots x_n \neq 0$ for all $\boldsymbol{x} \in \mathrm{supp}(w)$, which is obvious. Condition (ii) clearly holds with $L = 0$.

Returning to the setting of general weight functions w satisfying Hypothesis 5.3, we begin with the following estimate.

Lemma 5.5. *For $\Gamma \in \mathbb{Z}$ and $\boldsymbol{w} \in K_\infty^n$, we have*

$$J_{F,w}^{\Gamma}(\boldsymbol{w}) \ll_{w,F} (1+|\boldsymbol{w}|)^{1-n/2}.$$

Moreover, if $|\boldsymbol{w}| \gg 1$, then $J_{F,w}(\gamma, \boldsymbol{w}) = 0$ unless $|\gamma| \asymp |\boldsymbol{w}|$.

Proof. In this proof all implied constants are allowed to depend on F. Upon making the change of variables $\boldsymbol{y} = t^L(\boldsymbol{x} - \boldsymbol{x}_0)$, it follows that

$$J_{F,w}(\gamma, \boldsymbol{w}) = \widehat{L}^{-n} \psi(\boldsymbol{w} \cdot \boldsymbol{x_0}) J_G(\gamma, t^{-L}\boldsymbol{w}),$$

where $G(\boldsymbol{y}) = F(t^{-L}\boldsymbol{y} + \boldsymbol{x}_0)$. It is clear that $H_G \ll H_F \ll 1$. At this point we can use Lemma 5.1 to conclude that $J_{F,w}(\gamma, \boldsymbol{w}) = 0$ unless $|\boldsymbol{w}| \ll \max\{1, |\gamma|\}$. Furthermore, by Lemma 5.2

$$\begin{aligned} |J_{F,w}(\gamma, \boldsymbol{w})| &\leqslant \widehat{L}^{-n} \mathrm{meas}\{\boldsymbol{y} \in \mathbb{T}^n \colon |\gamma \nabla G(\boldsymbol{y}) + t^{-L}\boldsymbol{w}| \leqslant H_G \max\{1, |\gamma|^{1/2}\}\} \\ &\leqslant \mathrm{meas}\{\boldsymbol{x} \in \mathrm{supp}(w) \colon |\gamma \nabla F(\boldsymbol{x}) + \boldsymbol{w}| \ll \max\{1, |\gamma|^{1/2}\}\}, \end{aligned}$$

since $\nabla G(\boldsymbol{y}) = t^{-L} \nabla F(\boldsymbol{x})$. Let us denote the set whose measure we want to estimate by Ω. Since $\boldsymbol{0} \notin \mathrm{supp}(w)$, we must have $|\nabla F(\boldsymbol{x})| \asymp 1$ for all $\boldsymbol{x} \in \mathrm{supp}(w)$. If $|\boldsymbol{w}| \ll 1$, then Ω is empty unless $|\gamma| \ll 1$, so that we can use the trivial estimate $|J_{F,w}(\gamma, \boldsymbol{w})| \leqslant 1$ to conclude that $J_{F,w}^{\Gamma}(\boldsymbol{w}) \ll 1$ if $|\boldsymbol{w}| \ll 1$. In particular, we may assume $|\boldsymbol{w}| \gg 1$ for the rest of the proof. In this case the contribution from $|\gamma| \ll 1$ vanishes, since then Ω is empty, while for $|\gamma| \gg 1$ the defining inequality for Ω becomes

$$|\gamma \nabla F(\boldsymbol{x}) + \boldsymbol{w}| \ll |\gamma|^{1/2}.$$

This can only hold if $|\gamma| \asymp |\boldsymbol{w}|$. From this we deduce the statement that $J_{F,w}(\gamma, \boldsymbol{w}) = 0$ unless $|\gamma| \asymp |\boldsymbol{w}|$.

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It remains to give an upper bound. This will be similar to [12, proof of Lemma 7.3]. Let us now assume that $\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{x}'$ are both in Ω. Note that $\nabla F(\boldsymbol{x}+\boldsymbol{x}') - \nabla F(\boldsymbol{x}) = H(\boldsymbol{x}+\boldsymbol{x}'/2)\boldsymbol{x}'$, so that we must then have

$$|H(\boldsymbol{x}+\boldsymbol{x}'/2)\boldsymbol{x}'| \ll |\gamma|^{-1/2}. \tag{5.2}$$

As $\det H(\boldsymbol{y}) \neq 0$ for all $\boldsymbol{y} \in \operatorname{supp}(w)$, compactness of $\operatorname{supp}(w)$ implies that we have $|\det H(\boldsymbol{y})| \gg 1$ for all $\boldsymbol{y} \in \operatorname{supp}(w)$. In addition, a straightforward computation shows that $\boldsymbol{x}+\boldsymbol{x}'/2 \in \operatorname{supp}(w)$, so that we can multiply (5.2) from the left by the inverse of $H(\boldsymbol{x}+\boldsymbol{x}'/2)$, whose entries have absolute value $O(1)$, to conclude that $|\boldsymbol{x}'| \ll |\gamma|^{-1/2} \ll |\boldsymbol{w}|^{-1/2}$. Finally, from this we obtain

$$J^{\Gamma}_{F,w}(\boldsymbol{w}) \ll \int_{|\gamma| \asymp |\boldsymbol{w}|} \operatorname{meas}(\Omega) \mathrm{d}\gamma \ll |\boldsymbol{w}|^{1-n/2}$$

when $|\boldsymbol{w}| \gg 1$, as desired. □

The following symmetry property of $J^{\Gamma}_{F,w}(\boldsymbol{w})$ generalises [26, Lemma 3.6].

Lemma 5.6. *Suppose $\lambda_1, \lambda_2 \in K_\infty^\times$ are such that $\lambda_1/\lambda_2 \in 1 + t^{-L}\mathbb{T}$. Then*

$$J^{\Gamma}_{F,w}(\lambda_1 \boldsymbol{w}) = J^{\Gamma}_{F,w}(\lambda_2 \boldsymbol{w}) \quad \textit{for any } \boldsymbol{w} \in K_\infty^n.$$

Proof. Let us write $\lambda = \lambda_1$ and show that for $\boldsymbol{w}$ fixed, the value of $J^{\Gamma}_{F,w}(\lambda \boldsymbol{w})$ only depends on the image of λ in the quotient group $K_\infty^\times/(1+t^{-L}\mathbb{T})$. We have

$$\begin{aligned} J^{\Gamma}_{F,w}(\lambda \boldsymbol{w}) &= \int_{|\gamma|<\widehat{\Gamma}} \int_{K_\infty^n} w(\boldsymbol{x}) \psi(\gamma F(\boldsymbol{x}) + \lambda \boldsymbol{w} \cdot \boldsymbol{x}) \mathrm{d}\boldsymbol{x} \mathrm{d}\gamma \\ &= |\lambda|^n \int_{K_\infty^n} w(\lambda^{-1}\boldsymbol{y}) \psi(\boldsymbol{w} \cdot \boldsymbol{y}) \int_{|\gamma|<\widehat{\Gamma}} \psi(\gamma \lambda^{-3} F(\boldsymbol{y})) \mathrm{d}\gamma \mathrm{d}\boldsymbol{y}, \end{aligned}$$

where we applied the change of variables $\boldsymbol{y} = \lambda \boldsymbol{x}$. It follows from (2.3) that

$$\int_{|\gamma|<\widehat{\Gamma}} \psi(\gamma \lambda^{-3} F(\boldsymbol{y})) \mathrm{d}\gamma = \begin{cases} \widehat{\Gamma} & \text{if } |F(\boldsymbol{y})| < \widehat{\Gamma}^{-1}|\lambda|^3, \\ 0 & \text{else.} \end{cases}$$

This formula clearly only depends on $|\lambda|$. Moreover, $w(\lambda^{-1}\boldsymbol{y})$ depends only on $\lambda \bmod 1 + t^{-L}\mathbb{T}$ by (5.1). Since we assume $L \geqslant 0$, any two elements of $K_\infty^\times$ whose image in $K_\infty^\times/(1+t^{-L}\mathbb{T})$ coincides must have the same absolute value, from which we deduce that $J^{\Gamma}_{F,w}(\lambda \boldsymbol{w})$ only depends on $\lambda \bmod 1 + t^{-L}\mathbb{T}$. □

The above establishes most of the analogue over K of [57, Proposition 8.1]. It still remains to show that $J^{\Gamma}_{F,w}(\boldsymbol{w})$ vanishes unless $|F^*(\boldsymbol{w})|$ is small. Working over function fields significantly simplifies the execution of the key ideas, because fewer dyadic ranges intervene. We start with an auxiliary lemma.

Lemma 5.7. *Let R be a compact set such that $R \subset \{\boldsymbol{x} \in K_\infty^n : \det H(\boldsymbol{x}) \neq 0\}$. Then there exists a constant $C = C(R) > 0$ such that if $\boldsymbol{w} \in K_\infty^n$ and $\boldsymbol{x} \in R$ satisfy*

$$|\nabla F(\boldsymbol{x}) - \boldsymbol{w}| < \widehat{C}^{-1},$$

it follows that there exists $\boldsymbol{s} \in K_\infty^n$ with $\nabla F(\boldsymbol{s}) = \boldsymbol{w}$ and

$$|\boldsymbol{s} - \boldsymbol{x}| < \widehat{C} |\nabla F(\boldsymbol{x}) - \boldsymbol{w}|.$$

Proof. Let $\boldsymbol{x} \in R$. Since $\det H(\boldsymbol{x}) \neq 0$, it follows from the inverse function theorem for local fields [48, Part II, §III.9, Theorem 2] that there exist open neighbourhoods $U, U' \subset K_\infty^n$, with $\det H(\boldsymbol{y}) \neq 0$ for any

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$\boldsymbol{y}$ in the closure of U, such that $\boldsymbol{x} \in U$, $\nabla F(\boldsymbol{x}) \in U'$ and $\nabla F\colon U \to U'$ is an isomorphism. In addition, after shrinking U' if necessary, we may assume that it is an open ball of radius $\widehat{C}_0^{-1}$ around $\nabla F(\boldsymbol{x})$, for some real $C_0 = C_0(\boldsymbol{x}) \geqslant 0$. It follows that if $\boldsymbol{w} \in U'$, there exists $\boldsymbol{s} \in U$ such that $\nabla F(\boldsymbol{s}) = \boldsymbol{w}$. Therefore, we have

$$\boldsymbol{w} = \nabla F(\boldsymbol{s}) = \nabla F(\boldsymbol{s} - \boldsymbol{x}) + H(\boldsymbol{x})(\boldsymbol{s} - \boldsymbol{x}) + \nabla F(\boldsymbol{x}).$$

As $\boldsymbol{x} \in R$, we have $\det H(\boldsymbol{x}) \neq 0$, and hence

$$\boldsymbol{s} - \boldsymbol{x} = H(\boldsymbol{x})^{-1}(\boldsymbol{w} - \nabla F(\boldsymbol{x}) - \nabla F(\boldsymbol{s} - \boldsymbol{x})).$$

Moreover, as R is compact, the entries of $H(\boldsymbol{x})^{-1}$ are $O_R(1)$. It follows that there exist constants $C_1, C_2 \geqslant 0$ depending only on R and F such that

$$|\boldsymbol{s} - \boldsymbol{x}| \leqslant \widehat{C}_1|\boldsymbol{w} - \nabla F(\boldsymbol{x})| + \widehat{C}_2|\boldsymbol{s} - \boldsymbol{x}|^2.$$

Upon shrinking U if necessary, we may assume that $|\boldsymbol{s} - \boldsymbol{x}| < (2\widehat{C}_2)^{-1}$, so that we obtain

$$|\boldsymbol{s} - \boldsymbol{x}| \leqslant 2\widehat{C}_1|\boldsymbol{w} - \nabla F(\boldsymbol{x})|.$$

Since R is compact, we can cover it with finitely many open sets U and then take C to be the maximum of the constants that arise in this process. □

Lemma 5.8. *Assume $L = 0$ in condition (ii) on the weight function w. Let $\Gamma \in \mathbb{Z}$ and $\boldsymbol{w} \in K_\infty^n$. Then $J_{F,w}^\Gamma(\boldsymbol{w}) = 0$ unless $|F^*(\boldsymbol{w})| \ll 1 + |\boldsymbol{w}|^{\deg F^* - 1}$.*

Proof. Note that if $|\boldsymbol{w}| \ll 1$, then $|F^*(\boldsymbol{w})| \ll 1 + |\boldsymbol{w}|^{\deg F^* - 1}$ holds trivially, so that we may assume $|\boldsymbol{w}| \gg 1$ from now on. Similarly we can assume $1 \ll |\boldsymbol{w}| \ll \widehat{\Gamma}$, since otherwise $J_{F,w}^\Gamma(\boldsymbol{w}) = 0$ by Lemma 5.5. Using Lemma 5.5 again, it follows that

$$J_{F,w}^\Gamma(\boldsymbol{w}) = \int_{|\gamma| \asymp |\boldsymbol{w}|} J_{F,w}(\gamma, \boldsymbol{w}) \mathrm{d}\gamma.$$

Let $Z \in \mathbb{Z}_{>0}$ be such that $\widehat{Z} \asymp |\boldsymbol{w}|$. We will now consider the contribution to $J_{F,w}^\Gamma(\boldsymbol{w})$ from those γ with $|\gamma| = \widehat{Z}$. To do so, write $\gamma = at^Z(1 + \gamma')$, where $a \in \mathbb{F}_q^\times$ and $\gamma' \in \mathbb{T}$. Let $\boldsymbol{y} := \boldsymbol{x} - \boldsymbol{x}_0$ and $G(\boldsymbol{y}) := F(\boldsymbol{y} + \boldsymbol{x}_0)$, as in the proof of Lemma 5.5. Then let

$$\boldsymbol{x}' := (1 + \gamma')^{-1/2}\boldsymbol{x} = (1 + \gamma')^{-1/2}(\boldsymbol{y} + \boldsymbol{x}_0).$$

Note that since $(1 + \gamma')^{-1/2} - 1 \in \mathbb{T}$ and $L = 0$, property (5.1) of the weight function w implies $w(\boldsymbol{x}) = w(\boldsymbol{x}')$. In particular, after the change of variables $\boldsymbol{x} \mapsto (1 + \gamma')^{-1/2}\boldsymbol{x}$ by condition (ii) on w, we get

$$\begin{aligned}\int_{\mathbb{T}^n} w(\boldsymbol{x}')\psi(\gamma F(\boldsymbol{x}') + \boldsymbol{w} \cdot \boldsymbol{x}')\mathrm{d}\boldsymbol{x} &= \psi((1 + \gamma')^{-1/2}\boldsymbol{x}_0 \cdot \boldsymbol{w}) \int_{\mathbb{T}^n} \psi((1 + \gamma')^{-1/2}(at^Z G(\boldsymbol{y}) + \boldsymbol{y} \cdot \boldsymbol{w}))\mathrm{d}\boldsymbol{y} \\ &= \psi((1 + \gamma')^{-1/2}\boldsymbol{x}_0 \cdot \boldsymbol{w}) \int_{\Omega} \psi((1 + \gamma')^{-1/2}(at^Z G(\boldsymbol{y}) + \boldsymbol{y} \cdot \boldsymbol{w}))\mathrm{d}\boldsymbol{y},\end{aligned}$$

by Lemma 5.2, where

$$\Omega := \{\boldsymbol{y} \in \mathbb{T}^n \colon |at^Z \nabla G(\boldsymbol{y}) + \boldsymbol{w}| \ll \widehat{Z}^{1/2}\}$$

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and we used that $\gamma F(\boldsymbol{x}') = at^Z(1+\gamma')^{-1/2}F(\boldsymbol{y}+\boldsymbol{x}_0)$ since F is homogeneous of degree 3. In summary, we have proven the identity

$$J_{F,w}(\gamma, \boldsymbol{w}) = J_{F,w}((1+\gamma')^{-3/2}\gamma, (1+\gamma')^{-1/2}\boldsymbol{w}),$$

and applied stationary phase to the latter integral to get an integral over $\boldsymbol{y} \in \Omega$.

Upon expressing the integral over $\boldsymbol{y} \in \Omega$ in terms of $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{x}_0$, we find that

$$J_{F,w}(\gamma, \boldsymbol{w}) = \int_{\mathbb{T}^n} w(\boldsymbol{x}')\psi(\gamma F(\boldsymbol{x}') + \boldsymbol{w}\cdot\boldsymbol{x}')\mathrm{d}\boldsymbol{x}' = \int_{\Omega'} \psi((1+\gamma')^{-1/2}\Phi(\boldsymbol{x}))\mathrm{d}\boldsymbol{x},$$

where $\Phi(\boldsymbol{x}) := at^Z F(\boldsymbol{x}) + \boldsymbol{w}\cdot\boldsymbol{x}$ and

$$\Omega' := \{\boldsymbol{x} \in \mathrm{supp}(w) \colon |a\nabla F(\boldsymbol{x}) + t^{-Z}\boldsymbol{w}| \ll \widehat{Z}^{-1/2}\}.$$

The crucial observation here is that Ω' is independent of γ'. In particular, when considering the total contribution to $J^{\Gamma}_{F,w}(\boldsymbol{w})$ from the set $\{\gamma = at^Z(1+\gamma') : \gamma' \in \mathbb{T}\}$, we may exchange the order of integration and see that it is given by

$$\int_{\Omega'}\int_{\mathbb{T}} \psi((1+\gamma')^{-1/2}\Phi(\boldsymbol{x}))\mathrm{d}\gamma'\mathrm{d}\boldsymbol{x}.$$

Here Ω', Φ depend on Z and a, but all estimates below will be uniform over Z and a. Let us now take the Taylor expansion

$$(1+\gamma')^{-1/2} = \sum_{k\geqslant 0} a_k \gamma'^k,$$

where $a_k = \binom{-1/2}{k} \in \mathbb{F}_q$. This is well defined, because $|\gamma'| < 1$ and $\mathrm{char}(\mathbb{F}_q) \neq 2$. Since $|\gamma'| < 1$, the value of $\psi((1+\gamma')^{-1/2}\Phi(\boldsymbol{x}))$ only depends on $f(\gamma') = \sum_{k=0}^{k_0} a_k\gamma'^k$ if we choose k_0 sufficiently large. We may now apply Lemma 5.2 again with $n = 1$ to infer that the inner integral is given by

$$\int_{\Theta} \psi(f(\gamma')\Phi(\boldsymbol{x}))\mathrm{d}\gamma',$$

where

$$\Theta = \{\gamma' \in \mathbb{T} \colon |f'(\gamma')\Phi(\boldsymbol{x})| \leqslant \max\{1, |\Phi(\boldsymbol{x})|^{1/2}\}\}.$$

It is easy to see that $|f'(\gamma')| = 1$, so that Θ is nonempty only if $|\Phi(\boldsymbol{x})| \leqslant 1$, which we shall assume for the rest of the proof.

Since $\boldsymbol{x} \in \Omega'$, we have $\boldsymbol{x} \in \mathrm{supp}(w)$ and $|a\nabla F(\boldsymbol{x}) + t^{-Z}\boldsymbol{w}| \ll \widehat{Z}^{-1/2} \ll |\boldsymbol{w}|^{-1/2}$. Since we assume $|\boldsymbol{w}| \gg 1$ for a sufficiently large implied constant, we may invoke Lemma 5.7 to deduce the existence of $\boldsymbol{s} \in K^n_\infty$ such that $|\boldsymbol{s}-\boldsymbol{x}| \ll |a\nabla F(\boldsymbol{x}) + t^{-Z}\boldsymbol{w}| \ll |\boldsymbol{w}|^{-1/2}$ and $\nabla F(\boldsymbol{s}) = -a^{-1}t^{-Z}\boldsymbol{w}$. Using Taylor expansion at $\boldsymbol{s}$ it follows that

$$\begin{aligned} |aF(\boldsymbol{x}) + t^{-Z}\boldsymbol{w}\cdot\boldsymbol{x}| &= |aF(\boldsymbol{s}) + t^{-Z}\boldsymbol{w}\cdot\boldsymbol{s}| + O(|\boldsymbol{x}-\boldsymbol{s}|^2) \\ &= |aF(\boldsymbol{s}) + t^{-Z}\boldsymbol{w}\cdot\boldsymbol{s}| + O(|\boldsymbol{w}|^{-1}). \end{aligned}$$

Moreover, as $a\nabla F(\boldsymbol{s}) = -t^{-Z}\boldsymbol{w}$, we have $3aF(\boldsymbol{s}) = -t^{-Z}\boldsymbol{w}\cdot\boldsymbol{s}$ and hence

$$|aF(\boldsymbol{s}) + t^{-Z}\boldsymbol{w}\cdot\boldsymbol{s}| = |F(\boldsymbol{s})|.$$

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By (2.12), the ratio $F^*(\nabla F(\boldsymbol{x}))/F(\boldsymbol{x})$ is a homogeneous polynomial in $\mathcal{O}[x_1,\dots,x_n]$ of degree $2\deg F^*-3$, which immediately implies

$$|F^*(\nabla F(\boldsymbol{s}))| \ll |F(\boldsymbol{s})||\boldsymbol{s}|^{2\deg F^*-3}.$$

In addition, as $|\boldsymbol{s}-\boldsymbol{x}| \ll |\boldsymbol{w}|^{-1/2}$, we have $|\boldsymbol{s}| \asymp 1$. Together with $-a^{-1}t^{-Z}\boldsymbol{w} = \nabla F(\boldsymbol{s})$, this implies $|F^*(t^{-Z}\boldsymbol{w})| = |F^*(\nabla F(\boldsymbol{s}))| \ll |F(\boldsymbol{s})|$ and hence

$$|F^*(\boldsymbol{w})| \ll \widehat{Z}^{\deg F^*}(\widehat{Z}^{-1}|\Phi(\boldsymbol{x})| + |\boldsymbol{w}|^{-1}) \ll |\boldsymbol{w}|^{\deg F^*-1},$$

as desired. □

6. Estimates for exponential sums

As usual we assume that $K = \mathbb{F}_q(t)$ with $\mathcal{O} = \mathbb{F}_q[t]$ and $\mathrm{char}(\mathbb{F}_q) > 3$. Throughout this section we assume that

$$F(\boldsymbol{x}) = a_1x_1^3 + \cdots + a_nx_n^3,$$

where $a_1,\dots,a_n \in \mathbb{F}_q^\times$. Let $r \in \mathcal{O}^+$ and let $\boldsymbol{c} \in \mathcal{O}^n$. In this section we collect estimates for the exponential sums $S_r(\boldsymbol{c})$ (defined in (2.6)) and the normalisation $S_r^\natural(\boldsymbol{c}) = |r|^{-(n+1)/2}S_r(\boldsymbol{c})$. As we will see, the quality of our estimates is governed by the behaviour of the dual form $F^* \in \mathbb{F}_q[c_1,\dots,c_n]$, whose formula is recorded in (2.11) in the case $F = x_1^3+\cdots+x_n^3$.

By (2.10), it suffices to treat the case of prime power moduli when estimating $S_r(\boldsymbol{c})$. Define

$$\mathrm{sq}(c) = \prod_{\varpi^2|c}\varpi^{\nu_\varpi(c)} \quad \text{and} \quad \mathrm{cub}(c) = \prod_{\varpi^3|c}\varpi^{\nu_\varpi(c)} \tag{6.1}$$

for the square-full and cube-full parts of any $c \in \mathcal{O}$; if $c = 0$ we take $\mathrm{sq}(c) = \mathrm{cub}(c) = 0$. With this in mind we summarise the relevant facts in the following result.

Lemma 6.1. *Let $\varpi^e \in \mathcal{O}$ be a prime power.*

1. *If $e = 1$ then $S_\varpi^\natural(\boldsymbol{c}) \ll |\gcd(\varpi, \nabla F^*(\boldsymbol{c}))|^{1/2} \leqslant |\varpi|^{1/2}$.*
2. *If $e = 2$ then $S_{\varpi^2}^\natural(\boldsymbol{c}) \ll |\varpi|$.*
3. *If $e \geqslant 3$ then*

$$S_{\varpi^e}^\natural(\boldsymbol{c}) \ll |\varpi^e|^{1/2}\prod_{1\leqslant i\leqslant n}|\gcd(\varpi^e, \mathrm{sq}(c_i))|^{1/4}.$$

Proof. Part (1) is due to Hooley [32, Lemma 60]. Part (2) is straightforward and is recorded in the proof of [26, Eq. (4.7)], although we take the opportunity to correct the definition of $\{\varpi^e, c_i\}$ to be equal to 1 when $e = 2$. To see part (3) we invoke [26, Eq. (4.7)] to deduce that

$$S_{\varpi^e}^\natural(\boldsymbol{c}) \ll \frac{1}{|\varpi^e|^{(n+1)/2}}\cdot|\varpi^e|\prod_{1\leqslant i\leqslant n}|\varpi^e|^{1/2}\{\varpi^e, c_i\}^{1/4},$$

where $\{\varpi^e, c_i\} = |\varpi|^{-1}$ if $\varpi\|c_i$, with $\{\varpi^e, c_i\} = |\gcd(\varpi^e, c_i)|$ otherwise. The statement of the lemma follows on noting that $\{\varpi^e, c_i\} \leqslant |\gcd(\varpi^e, \mathrm{sq}(c_i))|$. □

The following result follows from combining Lemma 6.1 with multiplicativity.

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Lemma 6.2. *There exists a constant $A > 0$ such that*

$$|r|^{-1/2}|S_r^\natural(\boldsymbol{c})| \leqslant A^{\omega(r)} \prod_{1\leqslant i\leqslant n} |\gcd(\mathrm{cub}(r), \mathrm{sq}(c_i))|^{1/4}.$$

It will also be useful to record a crude upper bound, which will be helpful when $\boldsymbol{c} = \mathbf{0}$.

Lemma 6.3. *There exists a constant $A > 0$ such that*

$$|r|^{-1/2}|S_r^\natural(\boldsymbol{c})| \leqslant A^{\omega(r)} |\mathrm{cub}(r)|^{n/6}.$$

Proof. By multiplicativity, it suffices to show that $S_{\varpi^e}^\natural(\boldsymbol{c}) \ll |\varpi^e|^{1/2+\theta_e}$, for any prime power $\varpi^e \in \mathcal{O}$, where

$$\theta_e = \begin{cases} 0 & \text{if } e \leqslant 2, \\ n/6 & \text{if } e \geqslant 3. \end{cases}$$

For $e \leqslant 2$ this follows from parts (1) and (2) of Lemma 6.1. For $e \geqslant 3$, we apply the 1-dimensional Hua type estimate recorded in [26, §4.2]. □

The following classical recursive structure will soon prove useful.

Lemma 6.4. *If $\varpi \mid \boldsymbol{c}$ and $e \geqslant 4$, then*

$$S_{\varpi^e}(\boldsymbol{c}) = \begin{cases} 0 & \text{if } \varpi^2 \nmid \boldsymbol{c}, \\ |\varpi|^{3+2n} S_{\varpi^{e-3}}(\boldsymbol{c}/\varpi^2) & \text{if } \varpi^2 \mid \boldsymbol{c}. \end{cases}$$

Proof. After expressing $S_{\varpi^e}(\boldsymbol{c})$ in terms of one-dimensional sums, this is immediate from the function field analogues of [29, Eq. (44)–(45)], which hold since $\mathrm{char}(\mathbb{F}_q) \neq 3$. □

Since a priori (depending on one's convention[2]) the dual form is only defined up to scaling, and we will soon need to understand its relation to discriminants from work of Busé and Jouanolou [14, Definition 4.6], we record the following standard fact.

Proposition 6.5. *Let $R = \mathbb{F}_q[c_1, \dots, c_{n-1}]$. Then $F^*(c_1, \dots, c_{n-1}, 1) \in R$ is divisible by the discriminant $\Delta \in R$ of the cubic form*

$$F(y_1, \dots, y_{n-1}, -c_1y_1 - \cdots - c_{n-1}y_{n-1}) \in R[y_1, \dots, y_{n-1}].$$

Proof. Here Δ is defined by a specialisation process explained in [14, paragraph after Eq. (4.2.1)]. Write $\boldsymbol{c} = (c_1, \dots, c_{n-1}, 1)$ and let $\mathrm{disc}(F, \boldsymbol{c})$ be the discriminant associated in [50, §1.1] to the pair $(F(\boldsymbol{x}), \boldsymbol{c} \cdot \boldsymbol{x})$.

As explained in [53, Proposition 4.4], for example, we have $\mathrm{disc}(F, \boldsymbol{c})/F^*(\boldsymbol{c}) \in \mathbb{F}_q^\times$. (This is the only step where we use $\mathrm{char}(\mathbb{F}_q) > 3$.) Also

$$\mathrm{disc}(F, \boldsymbol{c}) = \pm \mathrm{disc}(F(y_1, \dots, y_{n-1}, -c_1y_1 - \cdots - c_{n-1}y_{n-1})),$$

by [53, Proposition 3.2]. It therefore follows that $\Delta/F^* \in \mathbb{F}_q^\times$, which suffices. □

The following result is the function field analogue of [57, Lemma 9.1 (2),(3)].

Lemma 6.6. *Let $\varpi^e \in \mathcal{O}$ be a prime power.*

1. *If $e = 2$ and $v_\varpi(F^*(\boldsymbol{c})) \leqslant 1$ then $S_{\varpi^2}^\natural(\boldsymbol{c}) \ll 1$.*
2. *If $e \geqslant 2 + v_\varpi(F^*(\boldsymbol{c}))$ then $S_{\varpi^e}(\boldsymbol{c}) = 0$.*

[2]Our definition of F^* differs from the classical 'eliminant' of [30, p. 37, footnote 2].

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Proof. Let $e > 1$ and assume that $\varpi \nmid \boldsymbol{c}$. We may appeal to an argument of Hooley [34, §6], which carries over to the function field setting to give

$$S_{\varpi^e}(\boldsymbol{c}) = \frac{|\varpi|}{|\varpi|-1}\big\{|\varpi|^e \nu_0(\varpi^e, \boldsymbol{c}) - |\varpi|^{e-1}\nu_1(\varpi^e, \boldsymbol{c})\big\}.$$

Here, for $i \in \{0,1\}$, $\nu_i(\varpi^e, \boldsymbol{c})$ is the number of $\boldsymbol{y}$ mod ϖ^e such that $F(\boldsymbol{y}) \equiv 0 \bmod \varpi^e$, with $\varpi \nmid \boldsymbol{y}$ and $\boldsymbol{c}.\boldsymbol{y} \equiv 0 \bmod \varpi^{e-i}$. Since F is a nonsingular cubic form with coefficients in $\mathbb{F}_q$, it follows that $\varpi \nmid \nabla F(\boldsymbol{y})$ for any $\boldsymbol{y}$ such that $\varpi \mid F(\boldsymbol{y})$ but $\varpi \nmid \boldsymbol{y}$. Hence $\nu_1(\varpi^e, \boldsymbol{c}) = |\varpi|^{n-1}\nu_0(\varpi^{e-1}, \boldsymbol{c})$, so that

$$S_{\varpi^e}(\boldsymbol{c}) = \frac{|\varpi|^e}{1-|\varpi|^{-1}}\big\{\nu_0(\varpi^e, \boldsymbol{c}) - |\varpi|^{n-2}\nu_0(\varpi^{e-1}, \boldsymbol{c})\big\}.$$

Since $\varpi \nmid \boldsymbol{c}$, we can assume without loss of generality that $\varpi \nmid c_n$. Eliminating y_n, we are left with the expression

$$S_{\varpi^e}(\boldsymbol{c}) = \frac{|\varpi|^e}{1-|\varpi|^{-1}}\big\{\nu(\varpi^e, \boldsymbol{c}) - |\varpi|^{n-2}\nu(\varpi^{e-1}, \boldsymbol{c})\big\},$$

where $\nu(\varpi^e, \boldsymbol{c})$ is the number of $\boldsymbol{y}' = (y_1, \dots, y_{n-1})$ mod ϖ^e such that $G(\boldsymbol{y}') \equiv 0 \bmod \varpi^e$ and $\varpi \nmid \boldsymbol{y}'$, where $G(\boldsymbol{y}') = F(c_n y_1, \dots, c_n y_{n-1}, -c_1 y_1 - \cdots - c_{n-1} y_{n-1})$. Part (2) for $\varpi \nmid \boldsymbol{c}$ now follows from taking $m = e-1$, $\delta \in [0, \frac{m-1}{2}]$ and $\ell = 0$ in the multivariate form of the Hensel-type lemma [11, Lemma 3.3], carried over to $\mathcal{O}$. Indeed, for any $\boldsymbol{u}'$ counted by $\nu(\varpi^{e-1}, \boldsymbol{c})$, we must have

$$v_\varpi(\nabla G(\boldsymbol{u}')) \leqslant (e-2)/2 = (m-1)/2,$$

or else $v_\varpi(F^*(\boldsymbol{c})) \geqslant e-1$ by the $\beta = \lceil \frac{e-1}{2} \rceil$ case of Lemma 6.7. Using Lemma 6.4 and arguing by induction on $v_\varpi(\boldsymbol{c})$, we can also conclude that part (2) holds if $\varpi \mid \boldsymbol{c}$.

Turning to part (1), we take $e = 2$ and note that $\varpi \nmid \boldsymbol{c}$ if $v_\varpi(F^*(\boldsymbol{c})) \leqslant 1$, since $\deg F^* > 1$. But then it follows that

$$S_{\varpi^2}(\boldsymbol{c}) = \frac{|\varpi|^2}{1-|\varpi|^{-1}}\big\{\nu(\varpi^2, \boldsymbol{c}) - |\varpi|^{n-2}\nu(\varpi, \boldsymbol{c})\big\}.$$

Pick $\boldsymbol{u}'$ counted by $\nu(\varpi, \boldsymbol{c})$. If $\varpi \nmid \nabla G(\boldsymbol{u}')$ then the contribution to the right hand side vanishes, by Hensel lifting. On the other hand, if $\varpi \mid \nabla G(\boldsymbol{u}')$ then Lemma 6.7 implies that the contribution to $\nu(\varpi^2, \boldsymbol{c})$ vanishes. Hence

$$S^\natural_{\varpi^2}(\boldsymbol{c}) \ll \frac{|\varpi|^2}{|\varpi^2|^{(n+1)/2}} \cdot |\varpi|^{n-2} \#\{\boldsymbol{u}' \bmod \varpi : \varpi \mid \nabla G(\boldsymbol{u}'),\ \varpi \nmid \boldsymbol{u}'\}.$$

It follows from a result of Zak [31, Katz's Appendix, Theorem 2] that the hypersurface $G = 0$ in $\mathbb{P}^{n-2}$ has only $O(1)$ singular points, whence the number of $\boldsymbol{u}'$ is $O(|\varpi|)$. It follows that

$$S^\natural_{\varpi^2}(\boldsymbol{c}) \ll \frac{|\varpi|^2}{|\varpi^2|^{(n+1)/2}} \cdot |\varpi|^{n-1} \ll 1,$$

which is what we wanted. □

Lemma 6.7. *Suppose $\varpi \nmid \boldsymbol{u}'$ and $\varpi^\beta \mid \nabla G(\boldsymbol{u}')$. Then $v_\varpi(F^*(\boldsymbol{c})) \geqslant \min(v_\varpi(G(\boldsymbol{u}')), 2\beta)$.*

Proof. Since $\varpi \nmid \boldsymbol{u}'$, we have $\varpi \nmid u_a$, say. Then by Proposition 6.5 and [14, Corollary 4.30; see the beginning of §4 for the definition of $\tilde{f}$ there], $F^*(\boldsymbol{c})$ is an $\mathcal{O}[\boldsymbol{u}'][1/u_a]$-linear combination of G and the pairwise products $(\partial G/\partial u_i)(\partial G/\partial u_j)$. Since $\varpi^{2\beta}$ divides each pairwise product, the desired divisibility of $F^*(\boldsymbol{c})$ follows. □

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In addition, we have the following estimate for averages over moduli belonging to $\mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}}$, which is a consequence of Lemma 4.4 in [26] and (6.8) in [26], together with the bound $\#\mathcal{R}(\mathbf{C}) \ll \widehat{C}^n$.

Proposition 6.8. *Let $Y, C \geqslant 1$. Then*

$$\sum_{\substack{|\boldsymbol{c}|=\widehat{C}\\ c_1\cdots c_n\neq 0\\ F^*(\boldsymbol{c})\neq 0}} \sum_{\substack{r\in\mathcal{R}_{\boldsymbol{c}}^{B}\\ |r|=\widehat{Y}}} |S_r(\boldsymbol{c})| \ll_{\varepsilon} \widehat{C}^{n+\varepsilon}\widehat{Y}^{1+n/2+\varepsilon}.$$

We end by recalling [26, Lemma 4.2] (a special case of [12, Lemma 8.5]), which concerns square-root cancellation of $S_r(\boldsymbol{c})$ over 'good' moduli r.

Proposition 6.9. *Let $Y \geqslant 1$ and $\boldsymbol{c} \in \mathcal{S}_1$. Then*

$$\sum_{\substack{r\in\mathcal{R}_{\boldsymbol{c}}^{G}\\ |r|=\widehat{Y}}} S_r(\boldsymbol{c}) \ll_{\varepsilon} |\boldsymbol{c}|^{\varepsilon}\widehat{Y}^{1+n/2+\varepsilon}.$$

7. Moments of bad exponential sums

Our primary goal in this section is to prove the following result.

Theorem 7.1. *Let $0 \leqslant R \leqslant 3Z$. There exist $\eta, \delta > 0$ such that if $2 \leqslant A \leqslant 2+\delta$, then*

$$\sum_{\substack{\boldsymbol{c}\in\mathcal{S}_1\\ |\boldsymbol{c}|\leqslant\widehat{Z}\\ c_1\cdots c_n\neq 0}} \Bigg| \sum_{\substack{r\in\mathcal{R}_{\boldsymbol{c}}^{B}\\ |r|=\widehat{R}}} |r|^{-1/2}|S_r^{\natural}(\boldsymbol{c})|\Bigg|^{A} \ll \widehat{Z}^n\widehat{R}^{-\eta}.$$

(As we will see in § 8, the complementary locus $\{\boldsymbol{c}\in\mathcal{S}_1 : c_1\cdots c_n = 0\}$ has essentially already been satisfactorily treated in [26].)

Let us denote by $\Sigma_A = \Sigma_A(Z,R)$ the sum on the left-hand side in the statement of this result. The trivial bounds $|S_r(\boldsymbol{c})| \leqslant |r|^{1+n}$ and $\#\{r\in\mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}} : |r| = \widehat{R}\} \leqslant \widehat{R}$ imply

$$\Sigma_A \leqslant \widehat{R}^{(1+n/2)\,\delta}\Sigma_2 \quad \text{for } 2 \leqslant A \leqslant 2+\delta. \tag{7.1}$$

To analyse Σ_2, it will be convenient to define

$$\mathcal{S}_2 = \{\boldsymbol{c}\in\mathcal{S}_1 : c_1\cdots c_n \neq 0\}$$

and

$$T_r(\boldsymbol{c}) = \prod_{\varpi^e\| r} \max\left\{1, \frac{|S_{\varpi^e}^{\natural}(\boldsymbol{c})|}{|\varpi^e|^{1/2}}\right\}.$$

Following the approach in [57, §9], we shall use different arguments according to the size of $T_r(\boldsymbol{c})$. Let $B \geqslant 0$ be a parameter to be chosen in due course and note that

$$\Sigma_2 \leqslant 2\Big(\Sigma^{(1)} + \Sigma^{(2)}\Big), \tag{7.2}$$

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where

$$\Sigma^{(1)} = \sum_{\substack{\boldsymbol{c}\in\mathcal{S}_2\\ |\boldsymbol{c}|\leqslant\widehat{Z}}} \Bigg| \sum_{\substack{r\in\mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}},\ |r|=\widehat{R}\\ T_r(\boldsymbol{c})<\widehat{B}}} |r|^{-1/2}|S_r^{\natural}(\boldsymbol{c})|\Bigg|^2, \tag{7.3}$$

$$\Sigma^{(2)} = \sum_{\substack{\boldsymbol{c}\in\mathcal{S}_2\\ |\boldsymbol{c}|\leqslant\widehat{Z}}} \Bigg| \sum_{\substack{r\in\mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}},\ |r|=\widehat{R}\\ T_r(\boldsymbol{c})\geqslant\widehat{B}}} |r|^{-1/2}|S_r^{\natural}(\boldsymbol{c})|\Bigg|^2. \tag{7.4}$$

7.1. Ekedahl sieve

A version of the geometric sieve for arbitrary global fields has been worked out by Bhargava, Shankar and Wang [7]. Their result sieves out prime moduli, but we need at least a partial version of the result for square-free moduli. This is folklore but we prove it for the reader's convenience. A similar idea is used in [6, §5, Case III].

Theorem 7.2. *Let $T, Y > 0$. Fix two polynomials $H_0, H_1 \in \mathcal{O}[c_1,\dots,c_n]$ that are relatively prime in $K[c_1,\dots,c_n]$. Then*

$$\#\left\{\boldsymbol{c}\in\mathcal{O}^n : \begin{array}{l}|\boldsymbol{c}|\leqslant\widehat{T},\ \exists\ \text{square-free } r\in\mathcal{O}^+ \text{such that}\\ r\mid \gcd(H_0(\boldsymbol{c}),H_1(\boldsymbol{c})) \text{ and } |r|=\widehat{Y}\end{array}\right\} \ll_\varepsilon \widehat{Y}^{\varepsilon}\left(\frac{\widehat{T}^n}{\widehat{Y}}+\widehat{T}^{n-1}\right).$$

Proof. Case 1: $Y \leqslant T$. Applying the Lang–Weil estimate for the quasi-projective variety $H_0 = H_1 = 0$, the count on the left is seen to be

$$\ll \sum_{\substack{r\in\mathcal{O}^+\\ |r|=\widehat{Y}}} \mu_K^2(r)|r|^{n-2+\varepsilon}\left(\frac{\widehat{T}}{|r|}\right)^n \ll_\varepsilon \frac{\widehat{T}^n}{\widehat{Y}^{1-\varepsilon}},$$

as desired.

Case 2: $Y > T$. By Lemma 2.7, we can make an $\mathcal{O}$-linear change of variables in order to assume that $\deg_{c_1}(H_0) = \deg(H_0)$ and $\deg_{c_1}(H_1) = \deg(H_1)$.

Let $H_2 \in \mathcal{O}[c_2,\dots,c_n]$ be the resultant of H_0 and H_1 with respect to c_1. Then any square-free $r\in\mathcal{O}^+$ such that $r \mid \gcd(H_0(\boldsymbol{c}),H_1(\boldsymbol{c}))$ must also satisfy[3]

$$r \mid \gcd(H_0(\boldsymbol{c}),H_2(c_2,\dots,c_n)).$$

If $c_2,\dots,c_n\in\mathcal{O}$ are specified, with $|c_2|,\dots,|c_n| \ll \widehat{T}$, then either $H_2(c_2,\dots,c_n)=0$ or else there are only $O_\varepsilon(\widehat{T}^{\varepsilon})$ choices for $r \mid H_2(c_2,\dots,c_n)$, by the divisor function bound. Note that if r is also specified, then the number of choices for $c_1\in\mathcal{O}$ with $|c_1|\ll\widehat{T}$ and $r\mid H_0(\boldsymbol{c})$ is

$$\ll N(g;r)\left(1+\frac{\widehat{T}}{|r|}\right),$$

in the notation of (2.21), where $g(u) = H_0(u,c_2,\dots,c_n)$. It follows from (2.23) that $N(g;r)\ll_\varepsilon |r|^{\varepsilon}$, uniformly over $c_2,\dots,c_n\in\mathcal{O}$.

Putting everything together, and recalling that $Y > T$, we obtain the overall contribution

$$\ll_\varepsilon \#\{\boldsymbol{c}\in\mathcal{O}^n : |\boldsymbol{c}|\leqslant\widehat{T},\ H_2=0\} + \widehat{T}^{n-1}\widehat{Y}^{\varepsilon}.$$

[3] In fact this implication is valid even without assuming r square-free; the key is that H_2 lies in the ideal (H_0,H_1) of $\mathcal{O}[\boldsymbol{c}]$. However, square-free is important when using Lang–Weil.

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Since H_0 and H_1 are coprime over K, it follows that H_2 is a nonzero polynomial, whence the first term is $O(\widehat{T}^{n-1})$ by Lemma 2.10. □

7.2. *Treatment of* $\Sigma^{(1)}$

The main task of this section is to prove the following result.

Proposition 7.3. *Let* $0 \leqslant R \leqslant 3Z$ *and let* $B \geqslant 0$. *Then*

$$\Sigma^{(1)} \ll_\varepsilon \widehat{B}^2 \widehat{Z}^{n+\varepsilon} \left(\frac{1}{\widehat{Z}^{\eta_0}} + \frac{1}{\widehat{R}^{1/(27\cdot 2^n)}} \right),$$

where $\eta_0 = \min\{1/p, 1/(3 \cdot 2^{n+1})\}$ *and* $p = \mathrm{char}(\mathbb{F}_q)$.

Recall the definition (7.3) of $\Sigma^{(1)}$. Any $r \in \mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}}$ admits a factorisation $r = r_1 s_1 s_2$, where r_1, s_1, s_2 are relatively coprime such that r_1 is square-free and s_1, s_2 are both square-full, and satisfy

$$\varpi^e \| s_1 \Rightarrow \max\{2, e-1\} > v_\varpi(F^*(\boldsymbol{c})),$$
$$\varpi^e \| s_2 \Rightarrow \max\{2, e-1\} \leqslant v_\varpi(F^*(\boldsymbol{c})).$$

It follows from parts (1)–(2) of Lemma 6.6 that $S_{s_1}^\natural(\boldsymbol{c}) \ll_\varepsilon |s_1|^\varepsilon$ for any $\varepsilon > 0$. On the other hand, from the definition of s_2 we have

$$\prod_{\varpi^e \| s_2} \varpi^{\max\{2,e-1\}} \mid F^*(\boldsymbol{c}).$$

Moreover, it is clear that the left hand side is a square-full element of $\mathcal{O}$ with absolute value $\prod_{\varpi^e \| s_2} |\varpi|^{\max\{2,e-1\}} \geqslant |s_2|^{2/3}$. We note that $\max\{|r_1|, |s_1|, |s_2|\} \geqslant |r|^{1/3} = \widehat{R}^{1/3}$. Let us temporarily write

$$U(\boldsymbol{c}) = \sum_{\substack{r = r_1 s_1 s_2 \in \mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}}, \ |r| = \widehat{R} \\ T_r(\boldsymbol{c}) < \widehat{B}}} |r|^{-1/2} |S_r^\natural(\boldsymbol{c})|,$$

so that

$$\Sigma^{(1)} = \sum_{\substack{\boldsymbol{c} \in \mathcal{S}_2 \\ |\boldsymbol{c}| \leqslant \widehat{Z}}} |U(\boldsymbol{c})|^2.$$

We will argue differently according to which of $|r_1|, |s_1|, |s_2|$ is the maximum in the factorisation $r = r_1 s_1 s_2$, exploiting the multiplicativity of the sum $S_r^\natural(\boldsymbol{c})$.

The contribution to $U(\boldsymbol{c})$ from the case $\max\{|r_1|, |s_1|, |s_2|\} = |r_1|$ is

$$\leqslant \widehat{B} \sum_{\substack{s_1, s_2 \in \mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}} \\ |s_1 s_2| \leqslant \widehat{R}}} \sum_{\substack{r_1 \in \mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}} \\ \widehat{R}^{1/3} \leqslant |r_1| \leqslant \widehat{R}}} |r_1|^{-1/2} |S_{r_1}^\natural(\boldsymbol{c})|,$$

since $|s_1 s_2|^{-1/2} |S_{s_1 s_2}^\natural(\boldsymbol{c})| \leqslant T_{s_1 s_2}(\boldsymbol{c}) \leqslant T_r(\boldsymbol{c}) < \widehat{B}$, by assumption. It follows from Lemma 2.2 that the number of available s_1, s_2 is $O_\varepsilon(\widehat{Z}^\varepsilon)$, on recalling that $F^*(\boldsymbol{c}) \neq 0$ and $R \leqslant 3Z$. Hence the overall contribution to $\Sigma^{(1)}$ from this case is

$$\ll_\varepsilon \widehat{B}^2 \widehat{Z}^\varepsilon \Sigma_0^{(1)},$$

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where

$$\Sigma_0^{(1)} = \sum_{\substack{\boldsymbol{c}\in\mathcal{S}_2\\ |\boldsymbol{c}|\leqslant\widehat{Z}}} \Bigg| \sum_{\substack{r\in\mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}}\\ \widehat{R}^{1/3}\leqslant|r|\leqslant\widehat{R}}} \mu^2(r)|r|^{-1/2}|S_r^{\natural}(\boldsymbol{c})|\Bigg|^2.$$

Next, the contribution to $U(\boldsymbol{c})$ from the case $\max\{|r_1|,|s_1|,|s_2|\} = |s_1|$ is

$$\leqslant \widehat{B} \sum_{\substack{r_1,s_2\in\mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}}\\ |r_1s_2|\leqslant\widehat{R}}} \sum_{\substack{s_1\in\mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}}\\ \widehat{R}^{1/3}\leqslant|s_1|\leqslant\widehat{R}}} |s_1|^{-1/2}|S_{s_1}^{\natural}(\boldsymbol{c})| \ll_{\varepsilon} \frac{\widehat{B}\widehat{Z}^{\varepsilon}}{\widehat{R}^{1/6}}.$$

Finally, the contribution to $U(\boldsymbol{c})$ from the case $\max\{|r_1|,|s_1|,|s_2|\} = |s_2|$ is

$$\leqslant \widehat{B} \sum_{\substack{r_1,s_1\in\mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}}\\ |r_1s_1|\leqslant\widehat{R}}} \sum_{\substack{s_2\in\mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}}\\ \widehat{R}^{1/3}\leqslant|s_2|\leqslant\widehat{R}\\ S_{s_2}^{\natural}(\boldsymbol{c})\neq 0}} 1 \ll_{\varepsilon} \widehat{B}\widehat{Z}^{\varepsilon}\mathbf{1}_{\boldsymbol{c}},$$

where

$$\mathbf{1}_{\boldsymbol{c}} = \begin{cases} 1 & \text{if there exists square-full } d\in\mathcal{O} \text{ s.t. } d \mid F^*(\boldsymbol{c}) \text{ and } |d| \geqslant \widehat{R}^{2/9},.\\ 0 & \text{otherwise.}\end{cases}$$

Putting everything together, it now follows that

$$\Sigma^{(1)} \ll_{\varepsilon} \widehat{B}^2\widehat{Z}^{\varepsilon}\left(\Sigma_0^{(1)} + \frac{\widehat{Z}^n}{\widehat{R}^{1/3}} + \max_{M\geqslant\frac{2}{9}R} N(M,Z)\right),$$

where

$$N(M,Z) = \#\left\{\boldsymbol{c}\in\mathcal{O}^n : \begin{array}{l}|\boldsymbol{c}|\leqslant\widehat{Z},\ c_1\dots c_nF^*(\boldsymbol{c})\neq 0\\ \exists \text{ square-full } d\in\mathcal{O} \text{ s.t. } d\mid F^*(\boldsymbol{c}) \text{ and } |d| = \widehat{M}\end{array}\right\}$$

The statement of Proposition 7.3 now follows from the conjunction of the following pair of results, both of which rely on the version of the geometric sieve for function fields from § 7.1.

Lemma 7.4. *We have*

$$\Sigma_0^{(1)} \ll_{\varepsilon} \frac{\widehat{Z}^{n+\varepsilon}}{\min\{\widehat{R}^{1/6},\widehat{Z}\}}.$$

Lemma 7.5. *We have*

$$N(M,Z) \ll_{\varepsilon} \widehat{Z}^{n+\varepsilon}\left(\frac{1}{\widehat{Z}^{1/p}} + \frac{1}{\min\{\widehat{Z},\widehat{M}\}^{1/(3\cdot 2^{n+1})}}\right),$$

where p is the characteristic of $\mathbb{F}_q$.

Proof of Lemma 7.4. It follows from multiplicativity and Lemma 6.1(1) that

$$S_r^{\natural}(\boldsymbol{c}) \ll_{\varepsilon} \widehat{Z}^{\varepsilon}|\gcd(r,\nabla F^*(\boldsymbol{c}))|^{1/2}.$$

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We pick a parameter $0 \leqslant Y \leqslant R$ and break the set $\mathcal{S}_2$ into two sets

$$\mathcal{S}_{2,1} = \Big\{ \boldsymbol{c} \in \mathcal{S}_2 : \exists \text{ square-free } d \in \mathcal{O} \text{ s.t. } d \mid \nabla F^*(\boldsymbol{c}) \text{ and } |d| > \widehat{Y} \Big\},$$
$$\mathcal{S}_{2,2} = \Big\{ \boldsymbol{c} \in \mathcal{S}_2 : \text{ any square-free } d \in \mathcal{O} \text{ s.t. } d \mid \nabla F^*(\boldsymbol{c}) \text{ satisfies } |d| \leqslant \widehat{Y} \Big\}.$$

To begin with, if $\boldsymbol{c} \in \mathcal{S}_{2,1}$ then we take $S_r^\natural(\boldsymbol{c}) \ll_\varepsilon \widehat{Z}^\varepsilon |r|^{1/2}$ and we note that there are $O_\varepsilon(\widehat{Z}^\varepsilon)$ choices of r, by Lemma 2.2. On the other hand, if $\boldsymbol{c} \in \mathcal{S}_{2,2}$ then we may take $S_r^\natural(\boldsymbol{c}) \ll_\varepsilon \widehat{Z}^\varepsilon \widehat{Y}^{1/2}$ and we also have $O_\varepsilon(\widehat{Z}^\varepsilon)$ choices of r overall. It follows that

$$\Sigma_0^{(1)} \ll_\varepsilon \widehat{Z}^\varepsilon \left(\#\mathcal{S}_{2,1} + \frac{\widehat{Y}\widehat{Z}^n}{\widehat{R}^{1/3}} \right).$$

Since $3 \cdot 2^{n-2} F^* = \boldsymbol{c} \cdot \nabla F^*$, we can apply the geometric sieve, in the form of Theorem 7.2 (with $H_0 = F^*$ and $H_1 = \partial F^*/\partial c_1$, noting that F^* is absolutely irreducible and $\deg H_1 < \deg H_0$), in order to deduce the bound

$$\#\mathcal{S}_{2,1} \ll_\varepsilon \widehat{Z}^\varepsilon \left(\frac{\widehat{Z}^n}{\widehat{Y}} + \widehat{Z}^{n-1} \right).$$

The statement of the lemma follows on combining these and taking $Y = \frac{1}{6}R$. □

Proof of Lemma 7.5. We shall reduce the problem to estimating the quantities

$$N_1(M_1, Z) = \#\left\{ \boldsymbol{c} \in \mathcal{O}^n : \begin{array}{l} |\boldsymbol{c}| \leqslant \widehat{Z},\ c_1 \dots c_n F^*(\boldsymbol{c}) \neq 0 \\ \exists \text{ square-full } d \in \mathcal{O} \text{ s.t. } d \mid F^*(\boldsymbol{c}) \text{ and } \widehat{M}_1 \leqslant |d| \leqslant \widehat{Z} \end{array} \right\}$$

and

$$N_2(M_2, Z) = \#\left\{ \boldsymbol{c} \in \mathcal{O}^n : \begin{array}{l} |\boldsymbol{c}| \leqslant \widehat{Z},\ c_1 \dots c_n F^*(\boldsymbol{c}) \neq 0 \\ \exists \text{ prime } \varpi \in \mathcal{O} \text{ s.t. } \varpi^2 \mid F^*(\boldsymbol{c}) \text{ and } |\varpi| > \widehat{M}_2 \end{array} \right\},$$

for suitable $M_1, M_2 \geqslant 0$.

Let $\boldsymbol{c} \in \mathcal{O}^n$ be counted by $N(M, R)$, so that $|\boldsymbol{c}| \leqslant \widehat{Z}$ and $c_1 \dots c_n F^*(\boldsymbol{c}) \neq 0$, and there exists a square-full $d \in \mathcal{O}$ such that $d \mid F^*(\boldsymbol{c})$ and $|d| = \widehat{M}$. Any such d admits a factorisation

$$d = \varpi_1^{e_1} \dots \varpi_m^{e_m},$$

where $e_1, \dots, e_m \geqslant 2$. Let $\eta \in (0, 1]$ be a parameter, to be chosen in due course. Suppose first that $|\varpi_i^{e_i}| \leqslant \widehat{Z}^\eta$ for all $1 \leqslant i \leqslant m$. Then we claim that $\boldsymbol{c}$ is counted by

$$N_1((1-\eta)\min\{M, Z\}, Z).$$

To see this, if $M \leqslant Z$ then in fact $\boldsymbol{c}$ is counted by $N_1(M, Z)$, which is satisfactory since

$$M \geqslant (1-\eta)M = (1-\eta)\min\{M, Z\}.$$

If $M > Z$ then we consider the square-full integer $d_1 = d/\varpi_1^{e_1}$, with absolute value $|d_1| = \widehat{M}/|\varpi_1^{e_1}| \geqslant \widehat{M}/\widehat{Z}^\eta$. If $\widehat{M}/|\varpi_1^{e_1}| \leqslant \widehat{Z}$ then $\boldsymbol{c}$ is counted by $N_1(M - \eta Z, Z)$, where

$$M - \eta Z \geqslant (1-\eta)Z = (1-\eta)\min\{M, Z\}.$$

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On the other hand, if $\widehat{M}/|\varpi_1^{e_1}| > \widehat{Z}$ then we repeat the argument for $d_2 = d_1/\varpi_2^{e_2}$. The claim follows on continuing in this way, since the process clearly terminates with a square-full divisor in the desired interval.

We now suppose, by symmetry, that $|\varpi_1^{e_1}| > \widehat{Z}^{\eta}$. If $|\varpi_1| \leqslant \widehat{Z}^{\eta/2}$ then we let $e_1' \leqslant e_1$ be the largest integer such that $|\varpi_1^{e_1'}| \leqslant \widehat{Z}^{\eta}$. It then follows that $|\varpi_1^{e_1'+1}| > \widehat{Z}^{\eta}$, whence

$$|\varpi_1^{e_1'}| > \frac{\widehat{Z}^{\eta}}{|\varpi_1|} > \widehat{Z}^{\eta/2}.$$

Moreover, we must have $e_1' \geqslant 2$ since $|\varpi_1| \leqslant \widehat{Z}^{\eta/2}$. Thus, since $\eta \leqslant 1$, we see that $\varpi_1^{e_1'}$ is a square-full divisor of $F^*(\boldsymbol{c})$ whose norm lies in the interval $(\widehat{Z}^{\eta/2}, \widehat{Z}]$. In this case, therefore, we deduce that $\boldsymbol{c}$ is counted by $N_1(\frac{\eta}{2}Z, Z)$. Finally, we must attend to the case that $|\varpi_1^{e_1}| > \widehat{Z}^{\eta}$ and $|\varpi_1| > \widehat{Z}^{\eta/2}$. But then it is immediate that $\boldsymbol{c}$ is counted by $N_2(\frac{\eta}{2}Z, Z)$.

In summary, we may conclude that

$$N(M, Z) \leqslant N_1((1-\eta)\min\{M, Z\}, Z) + N_1(\tfrac{\eta}{2}Z, Z) + N_2(\tfrac{\eta}{2}Z, Z), \tag{7.5}$$

for any $\eta \in (0, 1]$. We claim that

$$N_1(M_1, Z) \ll_{\varepsilon} \frac{\widehat{Z}^{n+\varepsilon}}{\widehat{M}_1^{1/2^{n+1}}} \tag{7.6}$$

and

$$N_2(M_2, Z) \ll_{\varepsilon} \widehat{Z}^{\varepsilon}\left(\frac{\widehat{Z}^{n}}{\widehat{M}_2} + \widehat{Z}^{n-1/p}\right), \tag{7.7}$$

for any $M_1, M_2 > 0$, where p is the characteristic of $\mathbb{F}_q$. Applying these in (7.5), we deduce that

$$N(M, Z) \ll_{\varepsilon} \widehat{Z}^{n+\varepsilon}\left(\frac{1}{\widehat{Z}^{1/p}} + \frac{1}{\widehat{Z}^{\eta/2^{n+2}}} + \frac{\widehat{Z}^{\eta/2^{n+1}}}{\min\{\widehat{M}, \widehat{Z}\}^{1/2^{n+1}}}\right),$$

since $\eta \min\{M, Z\} \leqslant \eta Z$. The statement of Lemma 7.5 easily follows on taking

$$\eta = \frac{2}{3}\frac{\min\{M, Z\}}{Z}.$$

Treatment of $N_1(M_1, Z)$

We will only make $N_1(M_1, Z)$ bigger by summing over all possible square-full d, and then breaking into residue classes modulo d. This gives

$$N_1(M_1, Z) \leqslant \sum_{\substack{d \in \mathcal{O} \text{ square-full} \\ \widehat{M}_1 \leqslant |d| \leqslant \widehat{Z}}} N(F^*; d)\left(\frac{\widehat{Z}}{|d|}\right)^n,$$

in the notation of (2.21). Let $\delta = 2^{-n}$. Since $N(F^*; d)$ is multiplicative in d, it follows from Rankin's trick that

$$N_1(M_1, Z) \leqslant \frac{\widehat{Z}^n}{\widehat{M}_1^{\delta/2-\varepsilon}} \prod_{\varpi}\left(1 + \sum_{e \geqslant 2} \frac{N(F^*; \varpi^e)}{|\varpi^e|^{n-\delta/2+\varepsilon}}\right)$$

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When $e = 2$, we have $N(F^*; \varpi^2) = O(|\varpi|^{2n-2})$ by Lemma 2.9. It follows that

$$N(F^*; \varpi^e) \leqslant |\varpi|^{(e-2)n} N(F^*; \varpi^2) = O(|\varpi|^{en-2}).$$

Once combined with Corollary 2.8, we conclude that

$$N_1(M_1, Z) \leqslant \frac{\widehat{Z}^n}{\widehat{M}_1^{\delta/2-\varepsilon}} \prod_{\varpi} \left(1 + O\left(\sum_{e \geqslant 2} \frac{|\varpi|^{en-\max\{2,\delta e\}}}{|\varpi^e|^{n-\delta/2+\varepsilon}}\right)\right),$$

since F^* is an absolutely irreducible form of degree $3 \times 2^{n-2} \leqslant 2^n$. But clearly

$$\sum_{e \geqslant 2} \frac{|\varpi|^{en-\max\{2,\delta e\}}}{|\varpi^e|^{n-\delta/2+\varepsilon}} \leqslant \sum_{2 \leqslant e \leqslant 2^{n+1}} \frac{|\varpi|^{e(1/2^{n+1}-\varepsilon)}}{|\varpi|^2} + \sum_{e \geqslant 2^{n+1}} \frac{1}{|\varpi|^{e(1/2^{n+1}+\varepsilon)}} \ll \frac{1}{|\varpi|^{1+2^{n+1}\varepsilon}},$$

since $\delta = 2^{-n}$. Thus the product over ϖ converges absolutely and the bound (7.6) follows on noting that $\widehat{M}_1^{\varepsilon} \leqslant \widehat{Z}^{\varepsilon}$.

Treatment of $N_2(M_2, Z)$

We follow the argument in Poonen [45, §7], noting that F^* is a square-free polynomial defined over $\mathbb{F}_q$. Poonen defines the polynomial $G \in \mathcal{O}[(y_{i,j})_{1 \leqslant i \leqslant n, 0 \leqslant j \leqslant p-1}]$ via

$$G = F^*\left(\sum_{0 \leqslant j \leqslant p-1} t^j y_{1,j}^p, \dots, \sum_{0 \leqslant j \leqslant p-1} t^j y_{n,j}^p\right),$$

where p is the characteristic of $\mathbb{F}_q$. It follows from n applications of [45, Lemma 7.2] that G is square-free as an element of $K[y_{i,j}]$. Let $Z_j = (Z - j)/p$. Then, as $y_{i,j}$ runs over elements of $\mathcal{O}$ with $|y_{i,j}| \leqslant \widehat{Z}_j$, so the n-tuples

$$\left(\sum_{0 \leqslant j \leqslant p-1} t^j y_{1,j}^p, \dots, \sum_{0 \leqslant j \leqslant p-1} t^j y_{n,j}^p\right)$$

run over elements $\mathbf{z} \in \mathcal{O}^n$ with $|\mathbf{z}| \leqslant \widehat{Z}$, with each element appearing exactly once. Hence it follows that

$$N_2(M_2, Z) = \#\left\{\boldsymbol{y} \in \mathcal{O}^{np} : \begin{array}{l} |y_{i,j}| \leqslant \widehat{Z}_j \text{ for } 1 \leqslant i \leqslant n \text{ and } 0 \leqslant j \leqslant p-1 \\ \exists \varpi \in \mathcal{O} \text{ s.t. } \varpi^2 \mid G(\boldsymbol{y}) \text{ and } |\varpi| > \widehat{M}_2 \end{array}\right\}.$$

Poonen's crucial observation is that we have $\varpi \mid G(\boldsymbol{y})$ and $\varpi \mid G'(\boldsymbol{y})$ whenever $\varpi^2 \mid G(\boldsymbol{y})$, where $G' = \frac{\partial G}{\partial t}$. Since G and G' are coprime as elements of $K[y_{i,j}]$, by [45, Lemma 7.3], we can therefore apply Theorem 7.2 to deduce that

$$N_2(M_2, Z) \ll_{\varepsilon} \widehat{Z}_0^{\varepsilon}\left(\frac{\widehat{Z}_0^{pn}}{\widehat{M}_2} + \widehat{Z}_0^{pn-1}\right) \ll_{\varepsilon} \widehat{Z}^{\varepsilon}\left(\frac{\widehat{Z}^n}{\widehat{M}_2} + \widehat{Z}^{n-1/p}\right),$$

since $Z_j \leqslant Z_0 \leqslant Z/p$ for all $0 \leqslant j \leqslant p-1$. This therefore establishes (7.7). □

7.3. *Treatment of* $\Sigma^{(2)}$

We now turn to the estimation of $\Sigma^{(2)}$, as defined in (7.4), with the main goal being to prove the following bound.

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Proposition 7.6. *Let $0 \leqslant R \leqslant 3Z$ and let $B \geqslant 0$. Then*

$$\Sigma^{(2)} \ll_\varepsilon \widehat{Z}^{n+\varepsilon}\left(\frac{1}{\widehat{Z}^{1/2}} + \frac{1}{\widehat{B}^{2/n}}\right).$$

Proof. Let r and $\boldsymbol{c}$ be counted by $\Sigma^{(2)}$. Then it follows from Lemma 6.2 that

$$\begin{aligned}\widehat{B} \leqslant T_r(\boldsymbol{c}) \leqslant \max_{s|r} \frac{|S_s^\natural(\boldsymbol{c})|}{|s|^{1/2}} &\ll_\varepsilon \max_{s|r}|s|^\varepsilon \prod_{1\leqslant i\leqslant n} |\gcd(\mathrm{cub}(s), \mathrm{sq}(c_i))|^{1/4}\\ &\ll_\varepsilon \widehat{R}^\varepsilon \max_{1\leqslant i\leqslant n}|\gcd(\mathrm{cub}(r), \mathrm{sq}(c_i))|^{n/4},\end{aligned}$$

since $\mathrm{cub}(s) \mid \mathrm{cub}(r)$. On relabelling $c_1, \dots, c_n$, we may assume that the maximum occurs at $i = n$ on the right hand side. According to part (2) of Lemma 6.6, moreover, we will have $S_r^\natural(\boldsymbol{c}) = 0$ unless $e \leqslant 1 + v_\varpi(F^*(\boldsymbol{c}))$ for any prime power $\varpi^e \| r$. If we let $d = \gcd(\mathrm{cub}(r), \mathrm{sq}(c_n))$, then d is a square-full element of $\mathcal{O}$ with

$$|d| \gg_\varepsilon \frac{\widehat{B}^{4/n}}{\widehat{R}^\varepsilon}.$$

Moreover, $|d| \leqslant \widehat{Z}$ since $d \mid c_n$ and $c_n \neq 0$. Finally, we note that $d \mid F^*(\boldsymbol{c})^2$ since $d \mid \mathrm{cub}(r)$ and $e \leqslant 2(e-1)$ for $e \geqslant 2$.

Let $D \in \mathbb{R}$ be such that $\widehat{D} = \widehat{B}^{4/n}/\widehat{R}^\varepsilon$. Then we have proved that

$$\Sigma^{(2)} \ll_\varepsilon \widehat{Z}^\varepsilon \sum_{\substack{d\in\mathcal{O}\text{ square-full}\\ \widehat{D}\ll_\varepsilon |d|\leqslant\widehat{Z}}} |d|^{1/2} \sum_{\substack{\boldsymbol{c}\in\mathcal{S}_2\\ |\boldsymbol{c}|\leqslant\widehat{Z}\\ d\mid\gcd(c_n, F^*(\boldsymbol{c})^2)}} |\mathrm{sq}(c_1)\cdots\mathrm{sq}(c_{n-1})|^{1/2},$$

since Lemma 2.2 implies that $\#\{r \in \mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}} : |r| = \widehat{R}\} \ll_\varepsilon \widehat{Z}^\varepsilon$. In dealing with this sum we shall make frequent use of the bound

$$\sum_{\substack{c\in\mathcal{O}\\ |c|\leqslant\widehat{Z}}} |\mathrm{sq}(c)|^{1/2} \ll \widehat{Z}\log\widehat{Z},$$

which readily follows on making the factorisation $c = c_0\,\mathrm{sq}(c)$, where c_0 is the square-free part of c, and noting that there are $O(\widehat{C}^{1/2})$ square-full elements of $\mathcal{O}$ with absolute value $\widehat{C}$.

We proceed by sorting the vectors $\boldsymbol{c}$ into two contributions. Let $\boldsymbol{c}' = (c_1, \dots, c_{n-1})$ and let $G(\boldsymbol{c}') = F^*(c_1, \dots, c_{n-1}, 0)^2$. We first deal with the contribution from those $\boldsymbol{c}$ for which $G(\boldsymbol{c}') \neq 0$. In this case we invert the order of summation and observe that there are $O_\varepsilon(\widehat{Z}^\varepsilon)$ choices of d for which $d \mid G(\boldsymbol{c}')$, by the standard estimate for the divisor function. Hence the contribution to $\Sigma^{(2)}$ from this case is

$$\begin{aligned}&\ll_\varepsilon \widehat{Z}^\varepsilon \sum_{\substack{|c_1|,\dots,|c_{n-1}|\leqslant\widehat{Z}\\ G(\boldsymbol{c}')\neq 0}} |\mathrm{sq}(c_1)\cdots\mathrm{sq}(c_{n-1})|^{1/2} \sum_{\substack{d\mid G(\boldsymbol{c}')\\ \widehat{D}\ll_\varepsilon |d|\leqslant\widehat{Z}}} |d|^{1/2}\#\Big\{c_n \in \mathcal{O} : |c_n| \leqslant \widehat{Z}, d \mid c_n\Big\}\\ &\ll_\varepsilon \frac{\widehat{Z}^{1+2\varepsilon}}{\widehat{D}^{1/2}} \sum_{|c_1|,\dots,|c_{n-1}|\leqslant\widehat{Z}} |\mathrm{sq}(c_1)\cdots\mathrm{sq}(c_{n-1})|^{1/2}\\ &\ll_\varepsilon \frac{\widehat{Z}^{n+3\varepsilon}}{\widehat{D}^{1/2}}.\end{aligned}$$

This is satisfactory for the proposition on observing that $\widehat{D} \gg \widehat{B}^{4/n}/\widehat{Z}^\varepsilon$.

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In order to deal with the remaining contribution from $\boldsymbol{c}$ for which $G(\boldsymbol{c}') = 0$, we note that for fixed $c_1, \dots, c_{n-2}$, there are only finitely many choices of c_{n-1} for which $G(\boldsymbol{c}') = 0$. This is due to the fact that the coefficient of $c_{n-1}^{\deg G}$ appears with a nonzero coefficient, as observed by Heath-Brown [28, Eq. (4.2)]. Hence we obtain the overall contribution

$$\begin{aligned}
&\ll_\varepsilon \widehat{Z}^{\varepsilon} \sum_{\substack{d \in \mathcal{O} \text{ square-full} \\ \widehat{D} \ll_\varepsilon |d| \leqslant \widehat{Z}}} \sum_{\substack{|c_n| \leqslant \widehat{Z} \\ d \mid c_n}} |d|^{1/2} \widehat{Z}^{1/2} \prod_{1 \leqslant i \leqslant n-2} \sum_{|c_i| \leqslant \widehat{Z}} |\mathrm{sq}(c_i)|^{1/2} \\
&\ll_\varepsilon \widehat{Z}^{n-1/2+2\varepsilon} \sum_{\substack{d \in \mathcal{O} \text{ square-full} \\ |d| \leqslant \widehat{Z}}} \frac{1}{|d|^{1/2}} \\
&\ll_\varepsilon \widehat{Z}^{n-1/2+3\varepsilon},
\end{aligned}$$

which thereby completes the proof. □

7.4. *Final touches*

We combine Propositions 7.3 and 7.6 in (7.2) to deduce that

$$\begin{aligned}
\Sigma_2 &\ll_\varepsilon \widehat{Z}^{n+\varepsilon} \left(\frac{\widehat{B}^2}{\widehat{Z}^{\eta_0}} + \frac{\widehat{B}^2}{\widehat{R}^{1/(27 \cdot 2^n)}} + \frac{1}{\widehat{Z}^{1/2}} + \frac{1}{\widehat{B}^{2/n}} \right) \\
&\ll_\varepsilon \widehat{Z}^{n+\varepsilon} \left(\frac{\widehat{B}^2}{\widehat{R}^{\min\{\eta_0/3, 1/(27 \cdot 2^n)\}}} + \frac{1}{\widehat{R}^{1/6}} + \frac{1}{\widehat{B}^{2/n}} \right),
\end{aligned}$$

since $Z \geqslant R/3$. The middle term is dominated by the first term. Moreover, we clearly have $\min\{\eta_0/3, 1/(27 \cdot 2^n)\} \geqslant \eta_0/9$. Balancing for B, we deduce that

$$\Sigma_2 \ll_\varepsilon \widehat{Z}^{n+\varepsilon} \widehat{R}^{-\eta_1}, \tag{7.8}$$

where $\eta_1 = \eta_0/(9n+9)$.

We may now deduce the statement of Theorem 7.1, recalling the notation $\Sigma_A = \Sigma_A(Z, R)$ for any $2 \leqslant A \leqslant 2 + \delta$ and any $0 \leqslant R \leqslant 3Z$. We claim that

$$\Sigma_2(Z, R) \ll \widehat{Z}^n \widehat{R}^{-\eta_1/2},$$

which will clearly suffice for the theorem, in the light of (7.1). If $Z \leqslant 2R$ then the statement follows from (7.8), on choosing $\varepsilon \leqslant \eta_1/4$. Suppose next that $Z > 2R$ and open up the square to get

$$\Sigma_2(Z, R) \ll \sum_{\substack{|r_1| = \widehat{R} \\ |r_2| = \widehat{R}}} \sum_{\substack{\boldsymbol{d} \bmod r_1 r_2 \\ \varpi \mid r_1 r_2 \Rightarrow \varpi \mid F^*(\boldsymbol{d})}} |r_1 r_2|^{-1/2} |S_{r_1}^{\natural}(\boldsymbol{d}) S_{r_2}^{\natural}(\boldsymbol{d})| \#\{\boldsymbol{c} : \boldsymbol{c} \equiv \boldsymbol{d} \bmod r_1 r_2\},$$

on noting that $S_{r_1}^{\natural}(\boldsymbol{c}) S_{r_2}^{\natural}(\boldsymbol{c})$ only depends on the value of $\boldsymbol{c}$ modulo $r_1 r_2$. The inner cardinality is $O(\widehat{Z}^n / \widehat{R}^{2n})$, whence

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$$\Sigma_2(Z,R) \ll \frac{\widehat{Z}^n}{\widehat{R}^{2n}} \sum_{\substack{|r_1|=\widehat{R}\\ |r_2|=\widehat{R}}} \sum_{\substack{\boldsymbol{d} \bmod r_1r_2\\ \varpi|r_1r_2 \Rightarrow \varpi|F^*(\boldsymbol{d})}} |r_1r_2|^{-1/2} |S^{\natural}_{r_1}(\boldsymbol{d}) S^{\natural}_{r_2}(\boldsymbol{d})|$$
$$\ll \frac{\widehat{Z}^n}{\widehat{R}^{2n}} \Sigma_2(n+2R,R)$$

by Lemma 7.7. The claim now follows on appealing to (7.8).

Lemma 7.7. *Suppose $|\boldsymbol{d}| \leqslant r_1r_2$. Then there exists $\boldsymbol{c} \in \mathcal{S}_2$ with $|\boldsymbol{c}| \leqslant q^n |r_1r_2|$ and $\boldsymbol{c} \equiv \boldsymbol{d} \bmod r_1r_2$.*

Proof. Let $H(\boldsymbol{c}) = c_1 \cdots c_n F^*(\boldsymbol{c})$. Let $S = \{\boldsymbol{d} + r_1r_2\boldsymbol{k} : |\boldsymbol{k}| \leqslant q^n\}$. We want to find $\boldsymbol{c} \in S$ with $H(\boldsymbol{c}) \neq 0$. The set S is a Cartesian product $\prod_{1\leqslant j\leqslant n} I_j$, for some sets $I_1, \dots, I_n \subseteq \mathcal{O}$ of size $|I_j| = q^{n+1}$. Since $\deg H = n + 3 \cdot 2^{n-2} < 2^n + 2^n \leqslant |I_j|$ for all j, the desired $\boldsymbol{c}$ exists by the combinatorial nullstellensatz [1, Theorem 1.2] in the field K. □

8. Endgame outside dual variety

Let w be as in § 5 with the properties specified in Hypothesis 5.3, but with parameter $L = 0$. Let

$$E_1(P) = |P|^n \sum_{\substack{\boldsymbol{c}\in\mathcal{O}^n\\ F^*(\boldsymbol{c})\neq 0}} \sum_{\substack{r\in\mathcal{O}^+\\ |r|\leqslant\widehat{Q}}} |r|^{-n} S_r(\boldsymbol{c}) I_r(\boldsymbol{c}), \tag{8.1}$$

where we recall that Q is chosen in such a way that $|P|^{3/2} \asymp \widehat{Q}$. By Lemma 5.1 we have $I_r(\boldsymbol{c}) = 0$ unless $|\boldsymbol{c}| \ll |r|\max\{1, |P|^3|r|^{-1}\widehat{Q}^{-1}\}|P|^{-1} \ll |P|^{1/2}$. In particular, the sum over $\boldsymbol{c}$ in the definition of $E_1(P)$ only runs over vectors $\boldsymbol{c} \in \mathcal{O}^n$ with $|\boldsymbol{c}| \ll |P|^{1/2}$. Moreover, a simple change of variables shows that Lemma 5.5 implies

$$I_r(\boldsymbol{c}) \ll |P|^{-3}\left(1 + \frac{|P||\boldsymbol{c}|}{|r|}\right)^{1-n/2}. \tag{8.2}$$

We also keep in mind the $L = 0$ case of Lemma 5.6, which implies the following:

$$\text{If } r_1,\ r_2 \in \mathcal{O}^+ \text{ with } |r_1| = |r_2|, \text{ then } I_{r_1}(\boldsymbol{c}) = I_{r_2}(\boldsymbol{c}).$$

The primary goal of this section is to establish the following result.

Proposition 8.1. *Assume Conjecture 4.5 holds. If $F = x_1^3 + \cdots + x_6^3$, then $E_1(P) \ll |P|^3$.*

We begin by dealing with the contribution from $c_1 \cdots c_n = 0$, which already follows from [26]. Let $\mathcal{I} \subset \{1, \dots, n\}$ with $\#\mathcal{I} = t$ and define

$$\mathcal{R}(C) := \{\boldsymbol{c} \in \mathcal{O}^n : |\boldsymbol{c}| = \widehat{C},\ \text{such that } c_i = 0 \text{ if } i \notin \mathcal{I} \text{ and } c_i \neq 0 \text{ else}\}.$$

Let us also fix $0 \leqslant Y \leqslant Q$. We will now consider the contribution to $E_1(P)$ coming from vectors $\boldsymbol{c} \in \mathcal{R}(C)$ and $r \in \mathcal{O}^+$ with $|r| = \widehat{Y}$. Since the contribution from $\widehat{C} \gg |P|^{1/2}$ vanishes, it is clear that there are at most $O(\log^2|P|)$ permissible choices for pairs (C, Y). It follows from the last display on page 21 of [26] that

$$\frac{|P|^n}{\widehat{Y}^n} \sum_{\substack{\boldsymbol{c}\in\mathcal{R}(C)\\ F^*(\boldsymbol{c})\neq 0}} \sum_{|r|=\widehat{Y}} S_r(\boldsymbol{c}) I_r(\boldsymbol{c}) \ll \frac{|P|^{n-3/2+\varepsilon}}{\widehat{Y}^{n/2}} \left(\frac{\widehat{Y}}{|P|\widehat{C}}\right)^{n/4} \widehat{Y}^{(n-t)/6} \widehat{C}^t \min\left\{1, \frac{\widehat{Y}}{|P|\widehat{C}}\right\}^{t/4}.$$

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Note that they work with a different weight function; nevertheless, the estimate continues to hold in our setting as we have the stronger integral estimate (8.2) at our disposal. Since we are assuming that $F^*(\boldsymbol{c}) \neq 0$ and $c_1 \cdots c_n = 0$, we must have $1 \leqslant t \leqslant n-1$. One now easily sees that the expression above is maximal at $t = 1$ or $t = n-1$. The contribution from $t = n-1$ is

$$\ll |P|^{n/2-5/4+\varepsilon} \widehat{Y}^{-1/12} \widehat{C}^{n/2-3/4} \ll |P|^{3n/4-13/8+\varepsilon},$$

while $t = 1$ gives

$$|P|^{3n/4-7/4+\varepsilon} \widehat{Y}^{(1-n)/12} \widehat{C}^{(3-n)/4} \ll |P|^{3n/4-7/4+\varepsilon},$$

where we used that $n \geqslant 3$ and $\widehat{C} \ll |P|^{1/2}$. We have $3n/4 - 7/4 < n-3$ and $3n/4 - 13/8 < n-3$ for $n \geqslant 6$ and since there are at most $O(\log^2 |P|)$ choices for (Y, C), this shows that the contribution from $c_1 \cdots c_n = 0$ to $E_1(P)$ is $O(|P|^{n-3-\delta})$ for some $\delta > 0$. This is satisfactory for Proposition 8.1.

Next, we factor $r = r_1 r_2$, where $r_1 \in \mathcal{R}_{\boldsymbol{c}}^{\mathrm{G}}$ and $r_2 \in \mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}}$. For $1 \leqslant \widehat{C} \ll |P|^{1/2}$ and $0 \leqslant Y_1 + Y_2 \leqslant Q$, we shall write $Y = Y_1 + Y_2$ and consider the sum

$$E_1(Y_1, Y_2, C) := \sum_{|\boldsymbol{c}|=\widehat{C}} \left| \sum_{\substack{r_1 \in \mathcal{R}_{\boldsymbol{c}}^{\mathrm{G}} \\ |r_1|=\widehat{Y}_1}} S_{r_1}^{\natural}(\boldsymbol{c}) \right| \left| \sum_{\substack{r_2 \in \mathcal{R}_{\boldsymbol{c}}^{\mathrm{B}} \\ |r_2|=\widehat{Y}_2}} S_{r_2}^{\natural}(\boldsymbol{c}) \right|,$$

where the sum over $\boldsymbol{c}$ runs over $\boldsymbol{c} \in \mathcal{O}^n$ with $c_1 \cdots c_n \neq 0$ and $F^*(\boldsymbol{c}) \neq 0$. We will now provide an upper bound for $E_1(Y_1, Y_2, C)$ depending on the size of C and the relative size of Y_2 compared to $\widehat{Q}/\widehat{Y}$. For the rest of this section, let $\delta > 0$ be a fixed small real number.

8.1. The case $\widehat{C} < |P|^{1/2-\delta}$

In this case it suffices to use the old estimates from [26] and [12]. Applying Propositions 6.9 and 6.8 in succession gives

$$E_1(Y_1, Y_2, C) \ll \widehat{Y}^{1/2+\varepsilon} \widehat{C}^{n+\varepsilon}.$$

Combining this with (8.2) shows that the contribution to $E_1(P)$ with $\widehat{C} \leqslant |P|^{1/2-\delta}$ is at most

$$\begin{aligned} \sum_{Y_1, Y_2, C} |P|^{n-3} \widehat{Y}^{1-n/2+\varepsilon} \widehat{C}^{n+\varepsilon} \left(1 + \frac{|P|\widehat{C}}{\widehat{Y}}\right)^{1-n/2} &\ll \sum_{Y_1, Y_2, C} |P|^{n/2-2} \widehat{C}^{1+n/2+\varepsilon} \widehat{Y}^{\varepsilon} \\ &\ll |P|^{3n/4-3/2+\varepsilon-\delta(1+n/2+\varepsilon)/2}, \end{aligned}$$

where we used that $\widehat{C} \ll |P|^{1/2-\delta}, \widehat{Y} \ll |P|^{3/2}$ and that the number of available (Y_1, Y_2, C) is $O(\log |P|) = O(|P|^{\varepsilon})$. Thus the contribution to $E_1(P)$ is at most $O(|P|^{n-3})$ for $n \geqslant 6$, which is satisfactory.

8.2. The case $\widehat{C} \geqslant |P|^{1/2-\delta}$

We shall divide the complementary case into two further subcases. First, let from now on $\varepsilon > 0$ be a fixed, sufficiently small real number. We then define $\alpha = (1/2 - \varepsilon)/\deg F^*$ and

$$\widehat{W} = \min\left\{ \frac{|P|^{3/2}}{\widehat{Y}}, |P|^{(1-\varepsilon)/2} \right\}. \tag{8.3}$$

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The case $\widehat{Y}_2^{n+\varepsilon} \leqslant \widehat{W}^{\alpha/2}$
In this regime we require the full strength of Conjecture 4.5 and of Lemma 5.8. To make use of Lemma 5.8, we need the following auxiliary result.

Lemma 8.2. *Let $G \in \mathcal{O}[c_1, \dots, c_n]$ be a polynomial. Let $C \geqslant 0$ and $\lambda \in \mathbb{R}$. Then*

$$\#\{\boldsymbol{c} \in \mathcal{O}^n : |\boldsymbol{c}| = \widehat{C}, |G(\boldsymbol{c})| \leqslant \widehat{C}^{\deg G}\widehat{\lambda}\} \ll_G \widehat{C}^n(\widehat{\lambda} + \widehat{C}^{-1})^{1/\deg G}.$$

Proof. By Lemma 2.7, we may assume $\deg_{c_1}(G) = \deg G$. By the $v = \infty$ case of (2.20), the number of available choices for c_1 is $\ll_G 1 + \widehat{C}\widehat{\lambda}^{1/\deg G}$, once $c_2, \dots, c_n$ are specified. Summing over $c_2, \dots, c_n$, the cardinality in Lemma 8.2 is

$$\ll_G \widehat{C}^{n-1}(1 + \widehat{C}\widehat{\lambda}^{1/\deg G}) \asymp \widehat{C}^n(\widehat{\lambda} + \widehat{C}^{-\deg G})^{1/\deg G} \ll \widehat{C}^n(\widehat{\lambda} + \widehat{C}^{-1})^{1/\deg G}.$$

(A referee pointed out to us that the middle bound $\widehat{C}^n(\widehat{\lambda}+\widehat{C}^{-\deg G})^{1/\deg G}$ might be useful for simplifying some of the subsequent estimates of § 8, as well as the choice of $\widehat{W}$ in (8.3). However, in order to maintain compatibility with the published sequel [8], we do not explore such improvements in the present paper.) □

Recall from Lemma 5.8 that $I_r(\boldsymbol{c}) = 0$ unless $|F^*(P\boldsymbol{c}/r)| \ll 1 + (|P||\boldsymbol{c}|/\widehat{Y})^{\deg F^*-1}$. After setting

$$\mathcal{S}(C) = \{\boldsymbol{c} \in \mathcal{O}^n : |\boldsymbol{c}| = \widehat{C}, 0 < |F^*(P\boldsymbol{c}/r)| \ll 1 + (|P||\boldsymbol{c}|/\widehat{Y})^{\deg F^*-1}, c_1 \cdots c_n \neq 0\},$$

it then follows from Hölder's inequality with $p_1 = 2-\varepsilon_0$, $p_2 = (2-\varepsilon_0)/(1-2\varepsilon_0)$ and $p_3 = (2-\varepsilon_0)/\varepsilon_0$ that

$$E_1(Y_1, Y_2, C) \leqslant \left(\sum_{|\boldsymbol{c}|=\widehat{C}}\left|\sum_{|r_1|=\widehat{Y}_1} S^{\natural}_{r_1}(\boldsymbol{c})\right|^{p_1}\right)^{1/p_1} (\#\mathcal{S}(C))^{1/p_2}\left(\sum_{|\boldsymbol{c}|=\widehat{C}}\left|\sum_{|r_2|=\widehat{Y}_2} S^{\natural}_{r_2}(\boldsymbol{c})\right|^{p_3}\right)^{1/p_3}.$$

We can now apply the trivial estimate $|S^{\natural}_{r_2}(\boldsymbol{c})| \leqslant |r_2|^{(n+1)/2}$ and invoke Lemma 2.3 to estimate the sum over r_2. Combining this with Conjecture 4.5 with $Z = \lceil \max(C, \frac{1}{3}Y_1)\rceil$ and $R = Y_1$ to estimate the sum over r_1, we get

$$\begin{aligned} E_1(Y_1, Y_2, C) &\ll \widehat{Y}_1^{1/2}\left(\widehat{C}^n + \widehat{Y}_1^{n/3}\right)^{1/(2-\varepsilon_0)} \#\mathcal{S}(C)^{(1-2\varepsilon_0)/(2-\varepsilon_0)}\widehat{C}^{n\varepsilon_0/(2-\varepsilon_0)}\widehat{Y}_2^{(n+1)/2+\varepsilon} \\ &\leqslant \widehat{Y}^{1/2}\widehat{Y}_2^{n/2+\varepsilon}\left(\widehat{C}^n + \widehat{Y}_1^{n/3}\right)^{1/2+\varepsilon} \#\mathcal{S}(C)^{1/2-\varepsilon}, \end{aligned}$$

where $\varepsilon := \frac{1}{2} - (1-2\varepsilon_0)/(2-\varepsilon_0) = \frac{3}{4}\varepsilon_0 + O(\varepsilon_0^2)$. Note that Lemma 8.2 implies that

$$\frac{\#\mathcal{S}(C)}{\widehat{C}^n} \ll \left(\min\left\{1, \frac{\widehat{Y}}{|P|\widehat{C}}\right\} + \widehat{C}^{-1}\right)^{1/\deg F^*} \ll \left(\frac{\widehat{Y}}{|P|\widehat{C}} + \widehat{C}^{-1}\right)^{1/\deg F^*}.$$

Together with the last display and (8.2) this shows that the contribution to $E_1(P)$ from those (Y_1, Y_2, C) under consideration is

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$$\begin{aligned}
&\ll \sum_{Y_1,Y_2,C} |P|^{n-3}\widehat{Y}^{1-n/2}\widehat{Y}_2^{n/2+\varepsilon}\widehat{C}^{n(1/2-\varepsilon)}\left(\frac{\widehat{Y}}{|P|\widehat{C}}+\widehat{C}^{-1}\right)^{\alpha}\left(\widehat{C}^n+\widehat{Y}_1^{n/3}\right)^{1/2+\varepsilon}\\
&\qquad\times\left(1+\frac{|P|\widehat{C}}{\widehat{Y}}\right)^{1-n/2}\\
&\ll \sum_{Y_1,Y_2,C} |P|^{n/2-2}\widehat{Y}_2^{n/2+\varepsilon}\left(\left(\frac{\widehat{Y}}{|P|\widehat{C}}\right)^{\alpha}+\widehat{C}^{-\alpha}\right)\left(\widehat{C}^n+\widehat{Y}_1^{n/3}\right)^{1/2+\varepsilon}\widehat{C}^{1-n\varepsilon}.
\end{aligned}$$

One can check that $-\alpha+n(1/2+\varepsilon)+1-n\varepsilon>0$ and $-\alpha+1-n\varepsilon>0$, so that upon recalling $\widehat{C}\ll|P|^{1/2}$, we get that the contribution is

$$\begin{aligned}
&\ll \sum_{Y_1,Y_2} |P|^{n/2-3/2-n\varepsilon/2}\widehat{Y}_2^{n/2+\varepsilon}\left(\left(\frac{\widehat{Y}}{|P|^{3/2}}\right)^{\alpha}+|P|^{-\alpha/2}\right)\left(|P|^{n/2}+\widehat{Y}_1^{n/3}\right)^{1/2+\varepsilon}\\
&\ll \sum_{Y_1,Y_2} |P|^{3n/4-3/2}\widehat{Y}_2^{n/2+\varepsilon}\left(\left(\frac{\widehat{Y}}{|P|^{3/2}}\right)^{\alpha}+|P|^{-\alpha/2}\right).
\end{aligned}$$

We can now use that

$$\widehat{Y}_2^{n/2+\varepsilon}\leqslant\widehat{Y}_2^{-n/2}\widehat{W}^{\alpha/2},$$

which together with the definition of $\widehat{W}$ in (8.3) implies that the contribution is

$$\begin{aligned}
&\ll \sum_{Y_1,Y_2} |P|^{3n/4-3/2}\widehat{Y}_2^{-n/2}\widehat{W}^{\alpha/2}\left(\left(\frac{\widehat{Y}}{|P|^{3/2}}\right)^{\alpha}+|P|^{-\alpha/2}\right)\\
&\ll |P|^{3n/4-3/2}\left(|P|^{-3\alpha/4}\sum_{Y_1,Y_2}\widehat{Y}_2^{-n/2}\widehat{Y}^{\alpha/2}+\sum_{Y_1,Y_2}\widehat{Y}_2^{-n/2}|P|^{-\alpha(1+\varepsilon)/4}\right)\\
&\ll |P|^{3n/4-3/2}\left(1+|P|^{\varepsilon-\alpha(1+\varepsilon)/4}\right),
\end{aligned}$$

where we used that $\sum_{Y_1,Y_2}\widehat{Y}_2^{-n/2}\widehat{Y}^{\alpha/2}\ll|P|^{3\alpha/4}$, because $\widehat{Y}\ll|P|^{3/2}$. Therefore, as $\varepsilon(1-\alpha/4)-\alpha<0$ for ε sufficiently small, the contribution from the case under consideration is again $O(|P|^{n-3})$ for $n\geqslant 6$.

The case $\widehat{Y}_2^{n+\varepsilon}>\widehat{W}^{\alpha/2}$

Applying Hölder's inequality to $E_1(Y_1,Y_2,C)$ with $p_1=2-\varepsilon$ and $p_2=(2-\varepsilon)/(1-\varepsilon)$ gives

$$E_1(Y_1,Y_2,C)\leqslant\left(\sum_{\boldsymbol{c}}\left|\sum_{r_1}S_{r_1}^{\natural}(\boldsymbol{c})\right|^{p_1}\right)^{1/p_1}\left(\sum_{\boldsymbol{c}}\left|\sum_{r_2}S_{r_2}^{\natural}(\boldsymbol{c})\right|^{p_2}\right)^{1/p_2}.$$

Next we use Conjecture 4.5 with $Z=\lceil\max(C,\frac{1}{3}Y_1)\rceil$ and $R=Y_1$ to estimate the first sum and Theorem 7.1 with $Z=\lceil\max(C,\frac{1}{3}Y_2)\rceil$ and $R=Y_2$ for the second, to obtain

$$E_1(Y_1,Y_2,C)\ll\widehat{Y}^{1/2}\widehat{Y}_2^{-\eta/p_2}\left(\widehat{C}^{n/p_1}+\widehat{Y}_1^{n/3p_1}\right)\left(\widehat{C}^{n/p_2}+\widehat{Y}_2^{n/3p_2}\right)\tag{8.4}$$

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for some $\eta > 0$. If we choose δ sufficiently small, we have $\widehat{C} > |P|^{1/2-\delta} \gg |P|^{1/4} \gg \widehat{Y}^{1/6}$. As $\widehat{Y}_1\widehat{Y}_2 = \widehat{Y}$, this implies that $\widehat{Y}_1^{1/3} \ll \widehat{C}$ or $\widehat{Y}_2^{1/3} \ll \widehat{C}$. Therefore, the bound in (8.4) gives

$$E_1(Y_1, Y_2, C) \ll \widehat{Y}^{1/2}\widehat{Y}_2^{-\eta/p_2}\widehat{C}^{n/p_2}\Big(\widehat{C}^{n/p_1} + \widehat{Y}_1^{n/3p_1} + \widehat{C}^{n(1/p_1-1/p_2)}\widehat{Y}_2^{n/3p_2}\Big).$$

Moreover, if the third term in the brackets dominates then we must have $\widehat{C}^3 \ll \widehat{Y}_2$. Thus, on noting that $1/p_1 - 1/p_2 > 0$, we obtain

$$\begin{aligned} E_1(Y_1, Y_2, C) &\ll \widehat{Y}^{1/2}\widehat{C}^{n/p_2}\widehat{Y}_2^{-\eta/p_2}\Big(\widehat{C}^{n/p_1} + \widehat{Y}_1^{n/3p_1} + \widehat{Y}_2^{n/3p_1}\Big) \\ &\ll \widehat{Y}^{1/2}\widehat{C}^{n/p_2}\widehat{Y}_2^{-\eta'}\Big(\widehat{C}^{n/p_1} + \widehat{Y}^{n/3p_1}\Big), \end{aligned}$$

where we trivially estimated $\widehat{Y}_1, \widehat{Y}_2 \leqslant \widehat{Y}$. Moreover, we have set $\eta' = \eta/p_2$, which satisfies $\eta' \geqslant \eta/4$ for ε small enough.

Observe that $n/p_2 + 1 - n/2 > 0$, so that summing over Y_1, Y_2, C and using the integral estimate (8.2) shows that the contribution from this case to $E_1(P)$ is

$$\begin{aligned} &\ll \sum_{Y_1,Y_2,C} |P|^{n-3}\widehat{C}^{n/p_2}\widehat{Y}^{1-n/2}\widehat{Y}_2^{-\eta'}\left(1 + \frac{|P|\widehat{C}}{\widehat{Y}}\right)^{1-n/2}\Big(\widehat{C}^{n/p_1} + \widehat{Y}^{n/3p_1}\Big) \\ &\ll \sum_{Y_1,Y_2} |P|^{n/2-2}\widehat{Y}_2^{-\eta'}\sum_{C}\widehat{C}^{1+n/p_2-n/2}\Big(\widehat{C}^{n/p_1} + \widehat{Y}^{n/3p_1}\Big) \\ &\ll \sum_{Y_1,Y_2} |P|^{n/2-2}\widehat{Y}_2^{-\eta'}|P|^{(1+n/p_2-n/2)/2}\Big(|P|^{n/2p_1} + \widehat{Y}^{n/3p_1}\Big) \\ &\ll \sum_{Y_1,Y_2} |P|^{3n/4-3/2}\widehat{Y}_2^{-\eta'}, \end{aligned}$$

where we used $1/p_1 + 1/p_2 = 1$ and that $\widehat{C} \ll |P|^{1/2}$, as well as the fact that $\widehat{Y} \ll |P|^{3/2}$. Since $3n/4 - 3/2 \leqslant n - 3$ for $n \geqslant 6$, to complete our treatment of $E_1(P)$ it will suffice to show that $\sum_{Y_1,Y_2}\widehat{Y}_2^{-\eta'} \ll 1$. To do so, recall that $\widehat{Y}_2^{n+\varepsilon} > \widehat{W}^{\alpha/2}$. If we set $\beta = \eta'\alpha/(4(n+\varepsilon))$, then recalling the definition of $\widehat{W}$ in (8.3) this implies

$$\begin{aligned} \widehat{Y}_2^{-\eta'} &\leqslant \widehat{Y}_2^{-\eta'/2}\widehat{W}^{-\beta} \\ &\ll_\varepsilon \widehat{Y}_2^{-\eta'/2}\left(\left(\frac{\widehat{Y}}{|P|^{3/2}}\right)^{\beta} + |P|^{\beta(\varepsilon-1)/2}\right). \end{aligned}$$

Therefore, as $\sum_{Y_1,Y_2}|P|^{\beta(\varepsilon-1)/2} \leqslant Q^2|P|^{\beta(\varepsilon-1)/2} \ll 1$, we get

$$\begin{aligned} \sum_{0\leqslant Y_2\leqslant Q}\sum_{0\leqslant Y_1\leqslant Q-Y_2}\widehat{Y}_2^{-\eta'} &\ll |P|^{-3\beta/2}\sum_{0\leqslant Y_2\leqslant Q}\widehat{Y}_2^{-\eta'/2+\beta}\sum_{0\leqslant Y_1\leqslant Q-Y_2}\widehat{Y}_1^{\beta} \\ &\ll \sum_{0\leqslant Y_2\leqslant Q}\widehat{Y}_2^{-\eta'/2} \\ &\ll 1, \end{aligned}$$

as $\eta' > 0$. This completes our proof of Proposition 8.1.

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9. Contribution from the centre

In this section we estimate the contribution from $\boldsymbol{c} = \mathbf{0}$ in (2.9). This takes the shape

$$M(P) = |P|^n \sum_{\substack{r \in \mathcal{O}^+ \\ |r| \leqslant \widehat{Q}}} |r|^{-n} S_r(\mathbf{0}) I_r(\mathbf{0}), \tag{9.1}$$

where $S_r(\mathbf{0})$ is given by (2.6), and $I_r(\mathbf{0})$ is given by (2.7) and (2.8). The following is the main goal of this section.

Proposition 9.1. *Let $n \geqslant 5$. Then $M(P) \ll |P|^{n-3}$, for any $Q \geqslant 0$.*

To begin with, it follows from (8.2) that $I_r(\mathbf{0}) \ll |P|^{-3}$, if $n \geqslant 3$. Hence

$$M(P) \ll |P|^{n-3} \sum_{r \in \mathcal{O}^+} |r|^{-n} |S_r(\mathbf{0})| = |P|^{n-3} \sum_{r \in \mathcal{O}^+} |r|^{(1-n)/2} |S_r^\natural(\mathbf{0})|, \tag{9.2}$$

if $n \geqslant 3$. We shall require the following technical result, which will also be useful in the next section.

Lemma 9.2. *If $R \geqslant 0$ and $\varepsilon > 0$, then*

$$\sum_{\substack{r \in \mathcal{O}^+ \\ |r| = \widehat{R}}} |r|^{1-n/2} \sum_{d \mid r} |d|^{-1/2} |S_d^\natural(\mathbf{0})| \ll_\varepsilon \widehat{R}^{(4-n)/3+\varepsilon}.$$

The statement of Proposition 9.1 follows on combining this result with (9.2) to deduce that

$$\begin{aligned} \sum_{r \in \mathcal{O}^+} |r|^{(1-n)/2} |S_r^\natural(\mathbf{0})| &\leqslant \sum_{R \geqslant 0} \sum_{\substack{r \in \mathcal{O}^+ \\ |r| = \widehat{R}}} |r|^{1-n/2} \sum_{d \mid r} d^{-1/2} |S_d^\natural(\mathbf{0})| \\ &\ll_\varepsilon \sum_{R \geqslant 0} \widehat{R}^{(4-n)/3+\varepsilon} \\ &\ll 1, \end{aligned}$$

if $n \geqslant 5$.

Proof of Lemma 9.2. Recalling the notation for $\mathrm{cub}(d)$ introduced in (6.1), it follows from Lemma 6.3 that $|d|^{-1/2} S_d^\natural(\mathbf{0}) \ll_\varepsilon |\mathrm{cub}(d)|^{n/6+\varepsilon}$, for any $\varepsilon > 0$. Hence

$$\sum_{\substack{r \in \mathcal{O}^+ \\ |r| = \widehat{R}}} |r|^{1-n/2} \sum_{d \mid r} d^{-1/2} |S_d^\natural(\mathbf{0})| \ll_\varepsilon \widehat{R}^{1-n/2+\varepsilon} \sum_{\substack{r \in \mathcal{O}^+ \\ |r| = \widehat{R}}} |\mathrm{cub}(r)|^{n/6},$$

on taking the trivial estimate for the divisor function. Writing $r = r_1 r_2$, where r_1 is cube-free and r_2 is cube-full, we see that

$$\begin{aligned} \sum_{\substack{r \in \mathcal{O}^+ \\ |r| = \widehat{R}}} |\mathrm{cub}(r)|^{n/6} &\leqslant \sum_{\substack{r_1 \in \mathcal{O}^+ \\ r_1 \text{ cube-free}}} \sum_{\substack{r_2 \in \mathcal{O}^+ \\ |r_2| \leqslant \widehat{R}/|r_1| \\ r_2 \text{ cube-full}}} |r_2|^{n/6} \\ &\ll \widehat{R}^{1/3+n/6} \sum_{r_1 \in \mathcal{O}^+} \frac{1}{|r_1|^{1/3+n/6}} \\ &\ll \widehat{R}^{1/3+n/6}, \end{aligned}$$

since $n \geqslant 5$. The statement of the lemma easily follows. □

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10. Contribution from dual variety

Let w be as in § 5, with the properties specified in Hypothesis 5.3, with parameter $L = 0$. The purpose of this section is to study the quantity

$$E_2(P) = |P|^n \sum_{\substack{\boldsymbol{c} \in \mathcal{O}^n \setminus \{\mathbf{0}\} \\ F^*(\boldsymbol{c}) = 0}} \sum_{\substack{r \in \mathcal{O}^+ \\ |r| \leqslant \widehat{Q}}} |r|^{-n} S_r(\boldsymbol{c}) I_r(\boldsymbol{c}), \tag{10.1}$$

where Q is chosen so that $|P|^{3/2} \asymp \widehat{Q}$ and $I_r(\boldsymbol{c}) = 0$ unless $|\boldsymbol{c}| \ll |P|^{1/2}$, by Lemma 5.1. We shall be interested in this quantity when $n = 6$ and $F = x_1^3 + \cdots + x_6^3$. Let $V \subset \mathbb{P}^5_K$ denote the cubic hypersurface defined by F.

Let Υ be the set of 3-dimensional K-vector spaces $L \subseteq K^6$ with $F|_L = 0$. (Note that the projectivisation of Υ is the K-point set of the Fano variety $F_2(V)$ of planes on V.) Since F is nonsingular, it is well-known that Υ is finite, as explained by Starr [9, Appendix]. The task of estimating $E_2(P)$ is divided into two steps. First, in Proposition 10.6 we will establish a bias in the exponential sum $S_r(\boldsymbol{c})$ whenever $\boldsymbol{c} \in L^\perp$, for some $L \in \Upsilon$. Note that the projectivisation of $L^\perp$ corresponds to a plane contained in the dual variety of V. Planes contained in the dual variety of V correspond to planes in V via bi-duality. Using Poisson summation, we shall show that the bias of the exponential sums gives a contribution that exactly matches the contribution from the points of bounded height on $\bigcup_{L \in \Upsilon} L$. The second task will be to show that the vectors $\boldsymbol{c} \in \mathcal{O}^6$ such that $F^*(\boldsymbol{c}) = 0$, but which do not lie on $L^\perp$ for any $L \in \Upsilon$, are sparse and form a negligible contribution.

The goal of this section is summarised in the following result.

Proposition 10.1. *Let $\varepsilon > 0$. If $F = x_1^3 + \cdots + x_6^3$, then*

$$E_2(P) = \sum_{L \in \Upsilon} \sum_{\boldsymbol{x} \in L \cap \mathcal{O}^6} w(\boldsymbol{x}/P) + O_\varepsilon(|P|^{3-1/4+\varepsilon}).$$

In fact, much of our argument carries over to general F in $n \in \{4, 6\}$ variables, but a restriction to diagonal F is necessary when it comes to handling square-full moduli. Throughout this section we shall therefore assume that $F = x_1^3 + \cdots + x_6^3$, unless otherwise indicated. For later convenience, we note that if $\mathrm{char}(\mathbb{F}_q) > 3$ there is a unique $L \in \Upsilon$ up to an $\mathbb{F}_q$-linear automorphism of F, given by

$$L_0 : \quad x_1 + x_4 = x_2 + x_5 = x_3 + x_6 = 0. \tag{10.2}$$

A proof of this fact is given in [52, Remark 6.3.8], based on [39,49].

10.1. Geometric preliminaries

Throughout this subsection, k denotes an arbitrary field with $\mathrm{char}(k) \neq 2$ and we shall adopt the notation $\mathbb{P}(E)$ for the projectivisation of a k-vector space E. (In particular, $\mathbb{P}^n$ denotes the projective space $\mathbb{P}(k^{n+1})$.) Suppose that $V \subset \mathbb{P}^5$ is a smooth cubic fourfold containing a k-plane $M \subset \mathbb{P}^5$, corresponding to a three-dimensional vector space $L \subset k^6$. Projection away from M, defined by the natural map $\mathbb{P}(k^6) \to \mathbb{P}(k^6/L)$, gives a morphism $V \setminus M \to \mathbb{P}(k^6/L)$. Blowing up V along M resolves the indeterminacy locus and gives a fibration $\widetilde{V} \to \mathbb{P}(k^6/L)$ whose fibres are projective quadric surfaces. Being the blow-up of a smooth variety along a smooth subvariety, $\widetilde{V}$ is also smooth. If we denote by $\mathcal{C} \subset \mathbb{P}(k^6/L)$ the zero scheme of the determinant of the matrices that define the quadratic forms in the fibres, it follows from Proposition 1.2 and Exemple 1.4.2 of Beauville [4] that $\mathcal{C}$ is a curve of degree 6 which is either smooth or has ordinary double points. Let $L^\perp$ be the orthogonal complement of L inside k^6 and note that $\mathbb{P}(L^\perp)$ parameterises hyperplanes in $\mathbb{P}(k^6)$ containing M. If $\boldsymbol{c} \in \mathbb{P}(L^\perp)$, we shall write $H_{\boldsymbol{c}}$ for the corresponding hyperplane containing M in $\mathbb{P}(k^6)$ and $H'_{\boldsymbol{c}}$ for the image of $H_{\boldsymbol{c}}$ in $\mathbb{P}(k^6/L)$, which defines a projective line.

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If we denote by $V_{\boldsymbol{c}} = H_{\boldsymbol{c}} \cap V$, then we get again a morphism $V_{\boldsymbol{c}} \setminus M \to H'_{\boldsymbol{c}}$ by projecting away from M. Blowing up $V_{\boldsymbol{c}}$ gives a fibration $\widetilde{V}_{\boldsymbol{c}} \to H'_{\boldsymbol{c}}$ into quadric surfaces and the locus of degenerate quadric surfaces is given by $\mathcal{C}_{\boldsymbol{c}} = H'_{\boldsymbol{c}} \cap \mathcal{C}$.

Note that after a suitable change of variables we may assume that L is given by $x_1 = x_2 = x_3 = 0$, so that V is defined by a cubic form of the shape

$$x_1 Q_1(\boldsymbol{x}, \boldsymbol{y}) + x_2 Q_2(\boldsymbol{x}, \boldsymbol{y}) + x_3 Q_3(\boldsymbol{x}, \boldsymbol{y}),$$

for some quadratic forms $Q_1, Q_2, Q_3 \in k[x_1, x_2, x_3, y_1, y_2, y_3]$. Upon defining

$$W := \{(\boldsymbol{s}, \boldsymbol{y}) \in \mathbb{P}^2 \times \mathbb{P}^2 \colon s_1 Q_1(\boldsymbol{0}, \boldsymbol{y}) + s_2 Q_2(\boldsymbol{0}, \boldsymbol{y}) + s_3 Q_3(\boldsymbol{0}, \boldsymbol{y}) = 0\},$$

we obtain a conic bundle $W \to \mathbb{P}^2$ by projecting to the first factor. A straightforward computation with the Jacobian criterion shows that smoothness of V along L implies smoothness of W. Hence, on appealing once again to Proposition 1.2 of Beauville [4], it follows that $f(\boldsymbol{s}) = \det(s_1 M_1 + s_2 M_2 + s_3 M_3)$ is nonzero as a polynomial in $\boldsymbol{s}$, where M_i is the 3×3 matrix underlying Q_i. The equation $f = 0$ thus defines a curve $\mathcal{D} \subset \mathbb{P}^2$.

Definition 10.2. Let $\boldsymbol{c} \in L^\perp \setminus \{\boldsymbol{0}\}$. We say that $\boldsymbol{c}$ is *good* if

(i) $\mathcal{C}_{\boldsymbol{c}}$ is smooth of dimension 0,
(ii) $H_{\boldsymbol{c}} \subset \mathbb{P}^5$ contains no plane contained in V other than M,
(iii) $H'_{\boldsymbol{c}}$ is not an irreducible component of $\mathcal{D}$,

and we say that $\boldsymbol{c}$ is *bad* otherwise.

Note that the definition depends on the choice of L. The significance of this definition is that hyperplane sections coming from good $\boldsymbol{c}$'s enjoy good geometric properties and bad vectors $\boldsymbol{c}$ only occur rarely, as we will see in the next two results.

Lemma 10.3. *Suppose $\boldsymbol{c}$ is bad. Then $\boldsymbol{c}$ lies in a subvariety of $\mathbb{P}(L^\perp)$, all of whose irreducible components have dimension at most one.*

Proof. Since $\mathcal{C}$ is reduced and has at worst ordinary double points, a generic hyperplane section is smooth of dimension 0. Indeed, as long as $H'_{\boldsymbol{c}}$ does not lie on the dual variety of $\mathcal{C}$ or contain one of the singular points of $\mathcal{C}$, the intersection $H'_{\boldsymbol{c}} \cap \mathcal{C}$ will be smooth. Moreover, it will be 0-dimensional as long as $H'_{\boldsymbol{c}}$ is not a component of $\mathcal{C}$. Each case forces $\boldsymbol{c}$ to lie on a subvariety of dimension at most one. If $H_{\boldsymbol{c}}$ contains two distinct planes contained in V, then it must also contain the linear space they span, which has dimension at least 3. Hyperplanes in $\mathbb{P}(k^6)$ containing a fixed three-dimensional linear space are parameterised by a $\mathbb{P}^1$ and since there are only finitely many planes contained in V, this shows that $\boldsymbol{c}$ lies on a union of finitely many lines. Finally, the curve $\mathcal{D}$ has at most 3 irreducible components and so there are at most 3 possibilities for $\boldsymbol{c}$ such that $H'_{\boldsymbol{c}}$ is an irreducible component of $\mathcal{D}$. □

Lemma 10.4. *Suppose that $\boldsymbol{c}$ is good. Then every fibre of the quadric surface bundle $\widetilde{V}_{\boldsymbol{c}} \to H'_{\boldsymbol{c}}$ has at worst isolated singularities.*

Proof. For $\boldsymbol{s} \in H'_{\boldsymbol{c}}$, let $Q_{\boldsymbol{s}}$ be the fibre above $\boldsymbol{s}$ in $\widetilde{V}_{\boldsymbol{c}}$, which is a projective quadric surface and hence corresponds to some matrix $M_{\boldsymbol{s}} \in \mathrm{Mat}_{4\times 4}(k)$. The claim of the lemma is now equivalent to $M_{\boldsymbol{s}}$ having rank at least 3. Note that if $\operatorname{rank} M_{\boldsymbol{s}} = 1$, then $Q_{\boldsymbol{s}}$ contains a double plane. However, $Q_{\boldsymbol{s}}$ is also a fibre of the fibration $\widetilde{V} \to \mathbb{P}(k^6/L)$ and is thus the residual intersection of a linear 3-space containing M with the cubic hypersurface V. Therefore, a result of Zak [31, Katz's Appendix, Theorem 2] shows that the singular locus of $Q_{\boldsymbol{s}}$ has dimension at most 1 and so $Q_{\boldsymbol{s}}$ cannot be a double plane. Now assume that $\operatorname{rank} M_{\boldsymbol{s}} = 2$. This implies that $Q_{\boldsymbol{s}}$ is a union of two distinct planes. However, this is impossible as at least one such plane would be distinct from M and contained in $H_{\boldsymbol{c}}$. This would contradict the assumption that $\boldsymbol{c}$ is good. □

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10.2. Exponential sums

In this section it is no harder to work with an arbitrary cubic form $F \in \mathcal{O}[x_1, \dots, x_6]$ defining a smooth hypersurface $\mathcal{V} \subset \mathbb{P}^5_{\mathcal{O}}$. Let $V := \mathcal{V} \times \operatorname{Spec} \mathbb{F}_q(t)$ and for a prime ϖ of $\mathcal{O}$ we let $\mathbb{F}_\varpi = \mathcal{O}/\varpi\mathcal{O}$ and set $\mathcal{V}_\varpi := \mathcal{V} \times \operatorname{Spec} \mathbb{F}_\varpi$. Moreover, we fix a K-vector space $L \subset K^6$ such that $F_{|L} \equiv 0$ and denote by $L^\perp$ its orthogonal complement inside K^6. We then set $\Lambda = L \cap \mathcal{O}^6$ and $\Lambda^\perp = L^\perp \cap \mathcal{O}^6$. We shall write M for the subspace of $\mathbb{P}^5$ corresponding to L. In addition, we let $L_\varpi = \Lambda/\varpi\Lambda$ and $L^\perp_\varpi = \Lambda^\perp/\varpi\Lambda^\perp$, which are both $\mathbb{F}_\varpi$-vector spaces of dimension 3.

Let $\boldsymbol{c} \in L^\perp \setminus \{\mathbf{0}\}$. Abusing the definition of the previous subsection slightly, we shall also say that $\boldsymbol{c}$ is good if the corresponding point in projective space is good. Let $g \in \mathcal{O}[y_1, y_2, y_3]$ be a degree 6 form defining the curve $\mathcal{C}$ in $\mathbb{P}(K^6/L)$. Note that if $\boldsymbol{c} \in L^\perp \setminus \{\mathbf{0}\}$ is good, then by definition $\mathcal{C}_{\boldsymbol{c}} \subset H'_{\boldsymbol{c}}$ is smooth. In particular, if we work with coordinates $(s : t)$ on $H'_{\boldsymbol{c}}$, then there exists a separable binary form $g_{\boldsymbol{c}} \in \mathcal{O}[s, t]$ of degree 6 whose coefficients are degree 6 polynomials in $\boldsymbol{c}$ that define $\mathcal{C}_{\boldsymbol{c}}$.

Example 10.5. Suppose that $L = \{x_1 = x_2 = x_3 = 0\}$. Then any $\boldsymbol{c} \in L^\perp \setminus \{\mathbf{0}\}$ takes the shape $(c_1, c_2, c_3, 0, 0, 0)$ and, without loss of generality, we may assume that $c_3 \neq 0$ and $c_1, c_2, c_3 \in \mathcal{O}$. Rational points on $H'_{\boldsymbol{c}}$ are of the form $(c_3 s : c_3 t : -(c_1 s + c_2 t))$ with $(s : t) \in \mathbb{P}^1(\mathrm{K})$ and we may take $g_{\boldsymbol{c}}(s, t) = g(c_3 s, c_3 t, -(c_1 s + c_2 t))$.

Fix coordinates z_1, z_2, z_3 on $L^\perp$. It follows from Lemma 10.3 that the collection of all bad $\boldsymbol{c} \in \mathbb{P}(L^\perp)$ is contained in a hypersurface. In particular, there exists a nonzero form $B \in \mathcal{O}[z_1, z_2, z_3]$ such that if $B(\boldsymbol{c}) \neq 0$, then $\boldsymbol{c}$ is good. Moreover, as Lemma 10.3 is independent of the base field, it follows that if $\varpi \nmid B(\boldsymbol{c})$, then the reduction of $\boldsymbol{c}$ modulo ϖ is also good with respect to L_ϖ and $\mathbb{F}_\varpi$. Recall from part (1) of Lemma 6.1 that $S^\natural_\varpi(\boldsymbol{c}) \ll |\varpi|^{1/2}$ if $F^*(\boldsymbol{c}) = 0$. The following result refines this estimate and establishes a bias in the exponential sums for primes.

Proposition 10.6. *Suppose that $\boldsymbol{c}$ is good and $\varpi \nmid B(\boldsymbol{c})$. Then*

$$S^\natural_\varpi(\boldsymbol{c}) = |\varpi|^{1/2} + O(1).$$

Proof. It follows from (2.16) that

$$\begin{aligned} S_\varpi(\boldsymbol{c}) &= |\varpi|(|\varpi| \# \mathcal{V}_{\boldsymbol{c},\varpi}(\mathbb{F}_\varpi) - \# \mathcal{V}_\varpi(\mathbb{F}_\varpi)) + |\varpi|\Big(\# \mathbb{P}^4(\mathbb{F}_\varpi) - |\varpi| \# \mathbb{P}^3(\mathbb{F}_\varpi)\Big) \\ &= |\varpi|(|\varpi| \# \mathcal{V}_{\boldsymbol{c},\varpi}(\mathbb{F}_\varpi) - \# \mathcal{V}_\varpi(\mathbb{F}_\varpi) + 1). \end{aligned} \tag{10.3}$$

After a suitable $\mathbb{F}_\varpi$-linear change of variables, we may assume that L_ϖ is given by $x_1 = x_2 = x_3 = 0$ and the cubic form F takes the shape

$$F(\boldsymbol{x}, \boldsymbol{y}) = x_1 Q_1(\boldsymbol{x}, \boldsymbol{y}) + x_2 Q_2(\boldsymbol{x}, \boldsymbol{y}) + x_3 Q_3(\boldsymbol{x}, \boldsymbol{y}),$$

for some quadratic forms $Q_1, Q_2, Q_3 \in \mathbb{F}_\varpi[x_1, x_2, x_3, y_1, y_2, y_3]$. In particular, we have $\boldsymbol{c} = (c_1, c_2, c_3, 0, 0, 0) \in \mathbb{F}^6_\varpi$. Moreover, we must have $\boldsymbol{c} \neq 0$, since $\varpi \nmid g_{\boldsymbol{c}}$. Without loss of generality, assume that $c_3 \neq 0$, so that $\mathcal{V}_{\boldsymbol{c},\varpi}$ is defined by the vanishing of the form $F_{\boldsymbol{c}}(s, t, \boldsymbol{y}) = F(c_3 s, c_3 t, -(c_1 s + c_2 t), \boldsymbol{y})$. Then we may rewrite $F_{\boldsymbol{c}}$ as $s q_1(s, t, \boldsymbol{y}) + t q_2(s, t, \boldsymbol{y})$, where

$$q_i(s, t, \boldsymbol{y}) = c_3 Q_i(c_3 s, c_3 t, -(c_1 s + c_2 t), \boldsymbol{y}) - c_i Q_3(c_3 s, c_3 t, -(c_1 s + c_2 t), \boldsymbol{y}).$$

Any $(s, t) \in \mathbb{F}^2_\varpi \setminus \{\mathbf{0}\}$ satisfies $\beta s - \alpha t = 0$ for a unique $(\alpha : \beta) \in \mathbb{P}^1(\mathbb{F}_\varpi)$ and we may then write $(s : t : y_1 : y_2 : y_3) = (\alpha u : \beta u : y_1 : y_2 : y_3)$ for some $u \in \mathbb{F}_\varpi \setminus \{0\}$. If $(\alpha u : \beta u : y_1 : y_2 : y_3) \notin M$, then $u \neq 0$ and the equation $F_{\boldsymbol{c}}(s, t, \boldsymbol{y}) = 0$ becomes

$$\alpha q_1(\alpha u, \beta u, y_1, y_2, y_3) + \beta q_2(\alpha u, \beta u, y_1, y_2, y_3) = 0. \tag{10.4}$$

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This is the defining equation for the residual quadric surface that arises when intersecting $\mathcal{V}_{\boldsymbol{c},\varpi}$ with the hyperplane spanned by L_ϖ and $(s:t)$. Upon defining the quadric surfaces

$$S_{(\alpha:\beta)} = \{(u:y_1:y_2:y_3) \in \mathbb{P}^3 \colon (10.4) \text{ holds}\}$$

and the conics

$$C_{(\alpha:\beta)} = \{(y_1:y_2:y_3) \in \mathbb{P}^2 \colon \alpha q_1(0,0,y_1,y_2,y_3) + \beta q_2(0,0,y_1,y_2,y_3) = 0\},$$

we see that $C_{(\alpha,\beta)}$ is precisely $S_{(\alpha,\beta)} \cap L_\varpi$ and we may therefore rewrite

$$\#\mathcal{V}_{\boldsymbol{c},\varpi}(\mathbb{F}_\varpi) = \#L_\varpi(\mathbb{F}_\varpi) + \sum_{(\alpha:\beta)\in\mathbb{P}^1(\mathbb{F}_\varpi)} \#S_{(\alpha:\beta)}(\mathbb{F}_\varpi) - \sum_{(\alpha:\beta)\in\mathbb{P}^1(\mathbb{F}_\varpi)} \#C_{(\alpha:\beta)}(\mathbb{F}_\varpi). \tag{10.5}$$

Since $\varpi \nmid B(\boldsymbol{c})$, the reduction of $\boldsymbol{c}$ modulo ϖ is good and hence the reduction of the binary form $g_{\boldsymbol{c}}$ modulo ϖ is separable of degree 6. Moreover, by construction $S_{(\alpha:\beta)}$ will be singular precisely when $\varpi \mid g_{\boldsymbol{c}}(\alpha,\beta)$, since $g_{\boldsymbol{c}}(\alpha,\beta)$ is just the determinant of the matrix underlying the quadratic form defining $S_{(\alpha:\beta)}$. Let $\chi\colon \mathbb{F}_\varpi^\times \to \mathbb{C}^\times$ be the unique character of order 2. It follows from Schmidt [47, §IV.2] that

$$\#S_{(\alpha:\beta)}(\mathbb{F}_\varpi) = |\varpi|^2 + |\varpi|(1+\chi(g_{\boldsymbol{c}}(\alpha,\beta))) + 1,$$

provided $\varpi \nmid g_{\boldsymbol{c}}(\alpha,\beta)$. Let $0 \leqslant N_0 \leqslant 6$ be the number of $(\alpha:\beta) \in \mathbb{P}^1(\mathbb{F}_\varpi)$ such that $\varpi \mid g_{\boldsymbol{c}}(\alpha,\beta)$. It then follows that

$$\begin{aligned}\sum_{\substack{(\alpha:\beta)\in\mathbb{P}^1(\mathbb{F}_\varpi)\\ \varpi\nmid g_{\boldsymbol{c}}(\alpha,\beta)}} \#S_{(\alpha:\beta)}(\mathbb{F}_\varpi) &= (|\varpi|^2+|\varpi|)(|\varpi|+1-N_0) + |\varpi| \sum_{\substack{t\in\mathbb{F}_\varpi\\ \varpi\nmid g_{\boldsymbol{c}}(t,1)}} \chi(g_{\boldsymbol{c}}(t,1)) + O(|\varpi|)\\ &= (|\varpi|^2+|\varpi|)(|\varpi|+1-N_0) + O(|\varpi|^{3/2}),\end{aligned}$$

where the $O(|\varpi|)$ term accounts for summing the +1 in Schmidt's formula and the contribution from those $(\alpha:\beta)$ with $\beta = 0$. The second line follows from the classical Weil bound for character sums, which is applicable since $g_{\boldsymbol{c}}(t,1)$ is square-free modulo ϖ.

As the reduction of $\boldsymbol{c}$ modulo ϖ is good, we know from Lemma 10.4 that the matrix underlying the quadratic form defining $S_{(\alpha:\beta)}$ has rank at least 3. In particular, $S_{(\alpha:\beta)}$ is irreducible and by Lang-Weil we get $\#S_{(\alpha:\beta)}(\mathbb{F}_\varpi) = |\varpi|^2 + O(|\varpi|^{3/2})$. Thus

$$\sum_{\substack{(\alpha:\beta)\in\mathbb{P}^1(\mathbb{F}_\varpi)\\ \varpi\mid g_{\boldsymbol{c}}(\alpha,\beta)}} \#S_{(\alpha:\beta)}(\mathbb{F}_\varpi) = N_0|\varpi|^2 + O(|\varpi|^{3/2}).$$

Finally, observe that since $\varpi \nmid B(\boldsymbol{c})$, property (iii) in Definition 10.2 implies that there are at most 3 possible $(\alpha:\beta) \in \mathbb{P}^1(\mathbb{F}_\varpi)$ for which $C_{(\alpha:\beta)}$ is singular. If $C_{(\alpha:\beta)}$ is smooth, then it contains $|\varpi|+1$ $\mathbb{F}_\varpi$-points, while if it is singular it is either a double line or a union of two lines, so that in either case it contains $O(|\varpi|)$ $\mathbb{F}_\varpi$-points. Therefore,

$$\sum_{(\alpha:\beta)\in\mathbb{P}^1(\mathbb{F}_\varpi)} \#C_{(\alpha:\beta)}(\mathbb{F}_\varpi) = |\varpi|^2 + O(|\varpi|).$$

Combining these estimates with the fact that $\#L_\varpi(\mathbb{F}_\varpi) = |\varpi|^2 + |\varpi| + 1$, it follows from (10.5) that

$$\#\mathcal{V}_{\boldsymbol{c},\varpi}(\mathbb{F}_\varpi) = |\varpi|^3 + 2|\varpi|^2 + O(|\varpi|^{3/2}).$$

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Moreover, by Deligne's bounds (as recorded in Equation (3.12) and the paragraph thereafter of [12]), we have $\#\mathcal{V}(\mathbb{F}_\varpi) = |\varpi|^4 + |\varpi|^3 + O(|\varpi|^2)$. Thus (10.3) yields

$$S_\varpi(\boldsymbol{c}) = |\varpi|\big(|\varpi|\#\mathcal{V}_{\boldsymbol{c},\varpi}(\mathbb{F}_\varpi) - \#\mathcal{V}_\varpi(\mathbb{F}_\varpi) + 1\big) = |\varpi|^4 + O(|\varpi|^{7/2}).$$

On dividing both sides by $|\varpi|^{7/2}$, we are led to the statement of the proposition. □

Motivated by the previous result, for good $\boldsymbol{c} \in L^\perp \setminus \{\mathbf{0}\}$ we define

$$\mathcal{N}_{\boldsymbol{c}}^{\mathrm{G}} := \{r \in \mathcal{O}^+ : \ \gcd(r, B(\boldsymbol{c})) = 1\} \quad \text{and} \quad \mathcal{N}_{\boldsymbol{c}}^{\mathrm{B}} := \{r \in \mathcal{O}^+ : \mathrm{rad}(r) \mid B(\boldsymbol{c})\}$$

and observe that these sets depend implicitly on L. Moreover, it will be convenient to introduce the notation

$$\mathcal{N}_{\geqslant}(l) := \{r \in \mathcal{O}^+ : \varpi \mid r \Rightarrow v_\varpi(r) \geqslant l\},$$

for any integer $l \geqslant 1$.

10.3. A general upper bound

Having completed the treatment of individual exponential sums, we will now focus on our case of interest and assume that

$$F(\boldsymbol{x}) = x_1^3 + \cdots + x_6^3$$

for the rest of this section. Recall the definition (2.13) of $\mathcal{S}_0$. For each set $\mathcal{T} \subseteq \mathcal{S}_0$, let

$$f(\mathcal{T}) := |P|^6 \sum_{\boldsymbol{c} \in \mathcal{T} \setminus \{\mathbf{0}\}} \sum_{\substack{r \in \mathcal{O}^+ \\ |r| \leqslant \widehat{Q}}} |I_r(\boldsymbol{c})| \cdot |r|^{-6} \cdot \sum_{s \mid r} |r/s|^4 |S_s(\boldsymbol{c})|. \tag{10.6}$$

At several points later, it will be convenient to discard $f(\mathcal{T})$ for various choices of $\mathcal{T}$. This is achieved through the following result (which is analogous to [56, Lemma 5.2]).

Lemma 10.7. *Let $\mathcal{T} \subseteq \mathcal{S}_0$ and $\delta \in \mathbb{R}$. Suppose $\{\lambda \boldsymbol{c} : (\lambda, \boldsymbol{c}) \in K^\times \times \mathcal{T}\} \cap \mathcal{O}^n = \mathcal{T}$ and $\#\{\boldsymbol{c} \in \mathcal{T} : |\boldsymbol{c}| \leqslant \widehat{C}\} = O(\widehat{C}^{n-3-\delta})$ for all $C \in \mathbb{R}_{\geqslant 0}$. Assume $\delta \leqslant \frac{5}{2}$. Then*

$$f(\mathcal{T}) \ll_\varepsilon |P|^{3-\delta/2+\varepsilon},$$

for any $\varepsilon > 0$.

Proof. The result [26, Lemma 6.1] essentially gives a version of this result with $\delta < 0$, and with $s = r$ instead of $\sum_{s|r}$. The same method extends to the present setting, with minor modifications that we now describe.

First, the integral estimate [26, Eq. (6.9)] still holds. (In fact, we have seen that the stronger estimate (8.2) is available.) Second, taking $n = 6$, we have

$$S_s(\boldsymbol{c}) \ll |s|^\varepsilon |s_1|^4 |s_2|^{14/3} |s_3|^5 |\gcd(s_2, \boldsymbol{c})|^{1/2},$$

in the notation of [26]. Therefore,

$$\sum_{s \mid r} |r/s|^4 |S_s(\boldsymbol{c})| \ll |r|^\varepsilon |r_1|^4 |r_2|^{14/3} |r_3|^5 |\gcd(r_2, \boldsymbol{c})|^{1/2},$$

since $s_2 \mid r_2$ and $s_3 \mid r_3$. This is of the exact same strength as the bound for the $s = r$ term given in [26, Lemma 6.1]. Since $-\frac{5}{2} + \delta \leqslant 0$, the rest of the argument of [26] goes through with $-\delta$ in place of η to

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give the desired upper bound for $f(\mathcal{T})$. Here our assumption on $\mathcal{T}$ guarantees that if $H\boldsymbol{c} \in \mathcal{T}$ for some $\boldsymbol{c} \in \mathcal{O}^n$ and $H \in \mathcal{O} \setminus \{0\}$, then in fact we have $\boldsymbol{c} \in \mathcal{T}$. This implies that

$$\#\{H\boldsymbol{c} \in \mathcal{T} : |\boldsymbol{c}| \leqslant \widehat{C}\} \leqslant \#\{\boldsymbol{c} \in \mathcal{T} : |\boldsymbol{c}| \leqslant \widehat{C}/|H|\},$$

which is all that is used to arrive at [26, Eq. (6.10)]. □

10.4. Factorising exponential sums

Given the bias established in Proposition 10.6, we typically expect $S_r^\natural(\boldsymbol{c})$ to be of size $|r|^{1/2}$ for $\boldsymbol{c} \in \Lambda^\perp$, at least when r is square-free. To keep track of the error, we will decompose $S_r^\natural(\boldsymbol{c})$ into smaller pieces.

For $r \in \mathcal{O}$, let $\phi_K(r) = \#(\mathcal{O}/r\mathcal{O})^\times$ be the Euler totient function. Consider the Dirichlet series

$$\Psi(s) := \sum_{r \in \mathcal{O}^+} \frac{\phi_K(r)|r|^{-1/2}}{|r|^s} = \prod_{\varpi} \frac{1 - |\varpi|^{-s-1/2}}{1 - |\varpi|^{-s+1/2}} = \frac{\zeta_K(s - \frac{1}{2})}{\zeta_K(s + \frac{1}{2})}.$$

In the notation of (4.1), it follows from Proposition 10.6 that the local factor $\Phi_\varpi(\boldsymbol{c}, s)$ should resemble the local factor $\Psi_\varpi(s)$ of $\Psi(s)$. Let $S_{r,0}(\boldsymbol{c})$ be the rth coefficient of the Euler product $\Phi(\boldsymbol{c}, s)/\Psi(s)$; then

$$\sum_{r \in \mathcal{O}^+} |r|^{-s} S_{r,0}(\boldsymbol{c}) = \Phi(\boldsymbol{c}, s)/\Psi(s) = \zeta_K(s - \tfrac{1}{2})^{-1} \cdot \zeta_K(s + \tfrac{1}{2}) \cdot \Phi(\boldsymbol{c}, s). \tag{10.7}$$

Since $\mu_K(r)$ is the rth coefficient of $\zeta_K(s)^{-1} = \prod_\varpi (1 - |\varpi|^{-s})$, it follows upon expanding products that

$$S_{r,0}(\boldsymbol{c}) = \sum_{\substack{d_0, d_1, d_2 \in \mathcal{O}^+ \\ d_0 d_1 d_2 = r}} \mu_K(d_0)|d_0|^{1/2} \cdot |d_1|^{-1/2} \cdot S_{d_2}^\natural(\boldsymbol{c}). \tag{10.8}$$

Moreover, the Euler product factorisation $\Phi = (\Phi/\Psi) \cdot \Psi$ implies

$$S_r^\natural(\boldsymbol{c}) = \sum_{\substack{r_0, r_1 \in \mathcal{O}^+ \\ r_0 r_1 = r}} S_{r_0,0}(\boldsymbol{c}) \cdot \phi_K(r_1)|r_1|^{-1/2}. \tag{10.9}$$

We will need some basic properties of $S_{r,0}(\boldsymbol{c})$ as a function of $\boldsymbol{c} \in \Lambda^\perp$ and $r \in \mathcal{O}^+$.

Lemma 10.8. *The quantity $S_{r,0}(\boldsymbol{c})$ is multiplicative in r and only depends on $\boldsymbol{c}$ mod r. Also,*

$$\mathbb{E}_{\boldsymbol{c} \in \Lambda^\perp / r\Lambda^\perp}[S_{r,0}(\boldsymbol{c})] = \begin{cases} 1 & \text{if } r = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Being the convolution of multiplicative functions, the multiplicativity is clear. Moreover, from (10.8) it follows that $S_{r,0}(\boldsymbol{c})$ depends only on the list of values $\boldsymbol{c}$ mod d for divisors d of r, which in turn implies that $S_{r,0}(\boldsymbol{c})$ only depends on $\boldsymbol{c}$ mod r. Finally, by orthogonality of characters we have

$$\mathbb{E}_{\boldsymbol{c} \in \Lambda^\perp / r\Lambda^\perp}[S_r^\natural(\boldsymbol{c})] = \phi_K(r)|r|^{-1/2},$$

so that the last assertion follows from the formal identity

$$\sum_{r \in \mathcal{O}^+} |r|^{-s} \mathbb{E}_{\boldsymbol{c} \in \Lambda^\perp / r\Lambda^\perp}[S_r^\natural(\boldsymbol{c})] = \Psi(s) \sum_{r \in \mathcal{O}^+} |r|^{-s} \mathbb{E}_{\boldsymbol{c} \in \Lambda^\perp / r\Lambda^\perp}[S_{r,0}(\boldsymbol{c})],$$

which follows from (10.7). □

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We now provide some bounds for $S_{r,0}(\boldsymbol{c})$ for $\boldsymbol{c} \in \Lambda^{\perp} \setminus \{\mathbf{0}\}$.

Lemma 10.9. *Let $r \in \mathcal{O}^+$, $\boldsymbol{c} \in \Lambda^{\perp} \setminus \{\mathbf{0}\}$ and let $\varepsilon > 0$. Then*

$$S_{r,0}(\boldsymbol{c}) \ll_{\varepsilon} |r|^{1/2+\varepsilon} \max_{\substack{d \in \mathcal{N}_{\geqslant}(3) \\ d|r}} \frac{|S_d^{\natural}(\boldsymbol{c})|}{|d|^{1/2}}.$$

Moreover, if $B(\boldsymbol{c}) \neq 0$ and r is a square-free element of $\mathcal{N}_{\boldsymbol{c}}^G$, then $S_{r,0}(\boldsymbol{c}) \ll_{\varepsilon} |r|^{\varepsilon}$.

Proof. Applying the triangle inequality in (10.8), followed by Lemma 6.1 to cube-free divisors of r, we obtain

$$|S_{r,0}(\boldsymbol{c})| \ll_{\varepsilon} |r|^{1/2+\varepsilon} \cdot \max_{d|r} \frac{|S_d^{\natural}(\boldsymbol{c})|}{|d|^{1/2}} \ll_{\varepsilon} |r|^{1/2+2\varepsilon} \max_{\substack{d \in \mathcal{N}_{\geqslant}(3) \\ d|r}} \frac{|S_d^{\natural}(\boldsymbol{c})|}{|d|^{1/2}},$$

which verifies the first part. If $\varpi \nmid B(\boldsymbol{c})$, then (10.8), Proposition 10.6 and Lemma 10.8 together imply $S_{\varpi,0}(\boldsymbol{c}) = -|\varpi|^{1/2} + |\varpi|^{-1/2} + S_{\varpi}^{\natural}(\boldsymbol{c}) \ll 1$. The second assertion now follows from multiplicativity. □

We shall also require the following estimate for averages of $S_{r,0}(\boldsymbol{c})$.

Lemma 10.10. *Let $\varepsilon > 0$. Let $\boldsymbol{c} \in \Lambda^{\perp} \setminus \{\mathbf{0}\}$ be such that $B(\boldsymbol{c}) \neq 0$ and $R \geqslant 0$. Then*

$$\sum_{\substack{r \in \mathcal{O}^+ \\ |r| = \widehat{R}}} |S_{r,0}(\boldsymbol{c})| \ll_{\varepsilon} (\widehat{R}|\boldsymbol{c}|)^{\varepsilon} \cdot \widehat{R} \sum_{\substack{d \in \mathcal{N}_{\geqslant}(2) \\ |d| \leqslant \widehat{R}}} \frac{|S_d^{\natural}(\boldsymbol{c})|}{|d|}.$$

Proof. Any $r \in \mathcal{O}^+$ can be written uniquely as $r_1 r_2 r_3$, where $r_1, r_2, r_3 \in \mathcal{O}^+$ are pairwise coprime with $r_1 \in \mathcal{N}_{\boldsymbol{c}}^{\mathrm{G}}$ square-free, $r_2 \in \mathcal{N}_{\boldsymbol{c}}^{\mathrm{B}}$ square-free and r_3 square-full. Upon writing $S_{r,0}(\boldsymbol{c}) = \prod_{1 \leqslant i \leqslant 3} S_{r_i,0}(\boldsymbol{c})$, and applying Lemma 10.9 to each factor, we get after decomposing into q-adic intervals

$$\sum_{\substack{r \in \mathcal{O}^+ \\ |r| = \widehat{R}}} |S_{r,0}(\boldsymbol{c})| \ll_{\varepsilon} \sum_{R_1 + R_2 + R_3 = R} \widehat{R}^{\varepsilon} \widehat{R}_1 \sum_{\substack{r_2 \in \mathcal{N}_{\boldsymbol{c}}^{\mathrm{B}} \\ |r_2| = \widehat{R}_2}} \widehat{R}_2^{1/2} \sum_{\substack{r_3, d \in \mathcal{N}_{\geqslant}(2) \\ |r_3| = \widehat{R}_3,\ d | r_3}} |r_3/d|^{1/2} |S_d^{\natural}(\boldsymbol{c})|.$$

Since $B(\boldsymbol{c}) \neq 0$ and $|B(\boldsymbol{c})| \ll |\boldsymbol{c}|^{O(1)}$, it follows from Lemma 2.2 that the number of $r_2 \in \mathcal{N}_{\boldsymbol{c}}^{\mathrm{B}}$ with $|r_2| = \widehat{R}_2$ is $O_{\varepsilon}((|\boldsymbol{c}|\widehat{R}_2)^{\varepsilon})$. In particular, we get, after summing over the r_i for each fixed d, that the right-hand side is

$$\ll_{\varepsilon} \sum_{R_1 + R_2 + R_3 = R} (\widehat{R}|\boldsymbol{c}|)^{\varepsilon} \widehat{R}_1 \widehat{R}_2^{1/2} \sum_{\substack{d \in \mathcal{N}_{\geqslant}(2) \\ |d| \leqslant \widehat{R}_3}} (\widehat{R}_3/|d|)^{1/2} |d|^{\varepsilon} (\widehat{R}_3/|d|)^{1/2} |S_d^{\natural}(\boldsymbol{c})|.$$

Lemma 10.10 follows, since $\widehat{R}_1 \widehat{R}_2^{1/2} \widehat{R}_3^{1/2} \widehat{R}_3^{1/2} \leqslant \widehat{R}_1 \widehat{R}_2 \widehat{R}_3 = \widehat{R}$. □

10.5. Linear space extraction

By an $\mathbb{F}_q$-linear change of variables, when studying linear spaces $L \in \Upsilon$ we may concentrate on $L = L_0$, as given by (10.2). Elements of $L_0^{\perp}$ are of the shape $(c_1, c_2, c_3, c_1, c_2, c_3)$ for $\boldsymbol{c} = (c_1, c_2, c_3) \in \mathcal{O}^3$. Motivated by this, we define $\boldsymbol{c}^* = (c_1, c_2, c_3, c_1, c_2, c_3)$ for $\boldsymbol{c} \in \mathcal{O}^3$.

Our next result will allow us to extract the main contribution from solutions on linear subspaces to our counting function. Before stating it, we need some more notation. Given $\boldsymbol{x} \in \mathcal{O}^6$, set

$$y_i = x_i + x_{i+3} \quad \text{and} \quad z_i = x_i - x_{i+3}$$

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for $i = 1, 2, 3$. Moreover, we let $M \in \mathrm{GL}_6(\mathcal{O})$ be such that $\boldsymbol{x} = M(\boldsymbol{y}, \boldsymbol{z})$. We then define the cubic form

$$\widetilde{F}(\boldsymbol{y}, \boldsymbol{z}) = F(\boldsymbol{x}) = \frac{1}{8} \sum_{i=1}^{3} \Big((y_i + z_i)^3 + (y_i - z_i)^3 \Big)$$

and the density

$$\sigma_{L_0, w} := \int_{K_\infty^3} w(M(\boldsymbol{0}, \boldsymbol{z})) \mathrm{d}\boldsymbol{z}.$$

Note that $M(\boldsymbol{0}, \boldsymbol{z}) = \frac{1}{2}(z_1, z_2, z_3, -z_1, -z_2, -z_3)$, so that $\sigma_{L_0, w}$ measures the density of K_∞-points on the lattice Λ_0 with respect to w.

Lemma 10.11. *Let $\boldsymbol{c}_0 \in \mathcal{O}^3$ and assume that $r = r_0 r_1 \in \mathcal{O}^+$. There exists a constant $C \in \mathbb{N}$ such that if $|r_1| \geqslant \widehat{C}|P|$, then*

$$\sum_{\boldsymbol{c} \in \mathcal{O}^3} I_r(\theta, \boldsymbol{c}_0^* + r_0 \boldsymbol{c}^*) = \sigma_{L_0, w} \left(\frac{|r_1|}{|P|} \right)^3.$$

Proof. Upon making the unimodular change of variables $\boldsymbol{x} \mapsto (\boldsymbol{y}, \boldsymbol{z})$ described above, we get that

$$I_r(\theta, \boldsymbol{c}_0^* + r_0 \boldsymbol{c}^*) = \int_{K_\infty^3} \int_{K_\infty^3} w(M(\boldsymbol{y}, \boldsymbol{z})) \psi \left(\theta P^3 \widetilde{F}(\boldsymbol{y}, \boldsymbol{z}) + \frac{P\boldsymbol{c} \cdot \boldsymbol{y}}{r_1} + \frac{P\boldsymbol{c}_0 \cdot \boldsymbol{y}}{r} \right) \mathrm{d}\boldsymbol{y} \mathrm{d}\boldsymbol{z}.$$

Applying another change of variables given by $\boldsymbol{y}' = P\boldsymbol{y}/r_1$, we obtain

$$\sum_{\boldsymbol{c} \in \mathcal{O}^3} I_r(\theta, \boldsymbol{c}_0^* + r_0 \boldsymbol{c}^*) = \left(\frac{|r_1|}{|P|} \right)^3 \int_{K_\infty^3} \sum_{\boldsymbol{c} \in \mathcal{O}^3} \widehat{g}_{\boldsymbol{z}}(\boldsymbol{c}) \mathrm{d}\boldsymbol{z},$$

where $\widehat{g}_{\boldsymbol{z}}$ is the Fourier transform of the function

$$g_{\boldsymbol{z}} \colon K_\infty^3 \to \mathbb{C}, \quad \boldsymbol{b} \mapsto w(M(r_1 \boldsymbol{b}/P, \boldsymbol{z})) \psi \left(\theta P^3 \widetilde{F}(r_1 \boldsymbol{b}/P, \boldsymbol{z}) + \frac{\boldsymbol{c}_0 \cdot \boldsymbol{b}}{r_0} \right).$$

In particular, using Poisson summation, in the form Lemma 2.1, we deduce that

$$\sum_{\boldsymbol{c} \in \mathcal{O}^3} I_r(\theta, \boldsymbol{c}_0^* + r_0 \boldsymbol{c}^*) = \frac{|r_1|^3}{|P|^3} \sum_{\boldsymbol{b} \in \mathcal{O}^3} \psi \left(\frac{\boldsymbol{c}_0 \cdot \boldsymbol{b}}{r_0} \right) \int_{K_\infty^3} w(M(r_1 \boldsymbol{b}/P, \boldsymbol{z})) \psi(\theta P^3 \widetilde{F}(r_1 \boldsymbol{b}/P, \boldsymbol{z})) \mathrm{d}\boldsymbol{z}.$$

Suppose that $\boldsymbol{b} \neq \boldsymbol{0}$. Then, since we may assume $|r_1| \geqslant \widehat{C}|P|$ for a sufficiently large constant C, we must have $w(M(r_1 \boldsymbol{b}/P, \boldsymbol{z})) = 0$. It follows that only the term $\boldsymbol{b} = \boldsymbol{0}$ contributes. However, we have $\widetilde{F}(\boldsymbol{0}, \boldsymbol{z}) = 0$ identically in $\boldsymbol{z}$, so that we arrive at the statement of the lemma. □

10.6. Proof of Proposition 10.1

We will now combine all the results we have obtained so far to complete the proof of Proposition 10.1. For $\mathcal{E} \subset \mathcal{O}^6$, we set

$$S(\mathcal{E}) = |P|^6 \sum_{\boldsymbol{c} \in \mathcal{E} \setminus \{\boldsymbol{0}\}} \sum_{\substack{r \in \mathcal{O}^+ \\ |r| \leqslant \widehat{Q}}} |r|^{-5/2} S_r^\natural(\boldsymbol{c}) I_r(\boldsymbol{c}).$$

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Then $S(\mathcal{S}_0) = E_2(P)$, so that Proposition 10.1 can be rephrased as

$$S(\mathcal{S}_0) = \sum_{L\in\Upsilon} \sum_{\boldsymbol{x}\in L\cap\mathcal{O}^6} w(\boldsymbol{x}/P) + O_\varepsilon(|P|^{3-1/4+\varepsilon}).$$

Let us define

$$\mathcal{E}_1 = \mathcal{S}_0 \setminus \bigcup_{L\in\Upsilon} \Lambda^\perp \quad \text{and} \quad \mathcal{E}_2(L) = \{\boldsymbol{c}\in\Lambda^\perp : B(\boldsymbol{c}) = 0 \text{ or } c_1\cdots c_6 = 0\},$$

where $B \in \mathcal{O}[x_1, x_2, x_3]$ is the form in the statement of Proposition 10.6.

Lemma 10.12. *For any $C \geqslant 1$ and $L \in \Upsilon$, we have*

$$\#\{\boldsymbol{c}\in\mathcal{E}_1\cup\mathcal{E}_2(L) : |\boldsymbol{c}| \leqslant \widehat{C}\} \ll_\varepsilon \widehat{C}^{2+\varepsilon}.$$

Proof. Since $B \in \mathcal{O}[x_1, x_2, x_3]$ is a nonzero form, we get from Lemma 2.10 that the contribution from $\boldsymbol{c} \in \mathcal{E}_2(L)$ with $B(\boldsymbol{c}) = 0$ is $O(\widehat{C}^2)$. Moreover, any $\boldsymbol{c} \in \Lambda^\perp$ takes the shape $(c_1, c_2, c_3, c_1, c_2, c_3)$ for some $(c_1, c_2, c_3) \in \mathcal{O}^3$. From this description it immediately follows that the contribution from $\boldsymbol{c} \in \Lambda^\perp$ with $c_1\cdots c_6 = 0$ is $O(\widehat{C}^2)$.

For $\boldsymbol{c} \in \mathcal{E}_1$, this is already implicit in the proof of Lemma 5.1 of [26] and so we shall be brief. Let $m_1, \dots, m_k \in \mathcal{O}$ with $1 \leqslant k \leqslant 6$ be square-free and either monic or with leading coefficient a primitive root of $\mathbb{F}_q^\times$. Suppose that $\boldsymbol{c} \in \mathcal{O}^6$ satisfies $F^*(\boldsymbol{c}) = 0$. We then partition $\{1, \dots, 6\}$ into sets $\mathcal{I}(1), \dots, \mathcal{I}(k)$, where $i \in \mathcal{I}(j)$ if and only if $c_i^3 = m_j d_i^2$ for some $d_i \in \mathcal{O}$. It then follows from the last display in the proof of Lemma 5.1 of [26] that the contribution from all $\boldsymbol{c} \in \mathcal{O}^6$ ranging over all permissible $m_1, \dots, m_k$ is $O_\varepsilon(\widehat{C}^{2+\varepsilon})$ unless $k = 3$ and $\#\mathcal{I}(1) = \#\mathcal{I}(2) = \#\mathcal{I}(3) = 2$. If the latter holds, then the display after (5.1) in the proof of Lemma 5.1 of [26] gives $d_{i_1} = -d_{i_2}$ for $\mathcal{I}(j) = \{i_1, i_2\}$ and $j = 1, 2, 3$, which in turn implies $c_{i_1}^3 = c_{i_2}^3$ and hence $\boldsymbol{c}$ lies in $\Lambda^\perp$ for some $L \in \Upsilon$. □

Note that for any distinct $L_1, L_2 \in \Upsilon$, we have

$$\{\boldsymbol{c}\in\Lambda_1^\perp\cap\Lambda_2^\perp : |\boldsymbol{c}| \leqslant \widehat{C}\} \ll \widehat{C}^2.$$

In particular, applying Lemma 10.7 twice, with $\delta = 1-\varepsilon$, in conjunction with Lemma 10.12, we obtain

$$S(\mathcal{S}_0) = |P|^6 \sum_{L\in\Upsilon} \sum_{\boldsymbol{c}\in\Lambda^\perp\setminus\mathcal{E}_2(L)} \sum_{\substack{r\in\mathcal{O}^+\\ |r|\leqslant\widehat{Q}}} |r|^{-5/2} S_r^\natural(\boldsymbol{c}) I_r(\boldsymbol{c}) + O_\varepsilon(|P|^{3-1/2+\varepsilon}).$$

Note that since any $L_1, L_2 \in \Upsilon$ are isomorphic under an $\mathbb{F}_q$-linear map, we may concentrate on the contribution from $L = L_0$. In particular, Proposition 10.1 will follow once we have established

$$|P|^6 \sum_{\boldsymbol{c}\in\Lambda_0^\perp\setminus\mathcal{E}_2} \sum_{\substack{r\in\mathcal{O}^+\\ |r|\leqslant\widehat{Q}}} |r|^{-5/2} S_r^\natural(\boldsymbol{c}) I_r(\boldsymbol{c}) = \sigma_{L_0,w}|P|^3 + O_\varepsilon(|P|^{3-1/4+\varepsilon}), \tag{10.10}$$

where $\mathcal{E}_2 = \mathcal{E}_2(L_0)$. Indeed, the only step that needs explaining is the equality

$$\sum_{\boldsymbol{x}\in\Lambda_0} w(\boldsymbol{x}/P) = \sigma_{L_0,w}|P|^3 + O_\varepsilon(|P|^{3-1/4+\varepsilon}).$$

However, this counting result is standard (with a much better error term) and we omit the proof.

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For a real number $U > 0$ and a set $\mathcal{T} \subseteq \mathcal{O}^6$, let

$$\Sigma_{<U}(P, \mathcal{T}) := |P|^6 \sum_{\boldsymbol{c} \in \mathcal{T}} \sum_{\substack{r \in \mathcal{O}^+ \\ |r| \leqslant \widehat{Q}}} |r|^{-5/2} I_r(\boldsymbol{c}) \sum_{\substack{r_0 r_1 = r \\ |r_1| < U}} S_{r_0,0}(\boldsymbol{c}) \cdot \phi_K(r_1) |r_1|^{-1/2}. \tag{10.11}$$

Similarly we define $\Sigma_{\geqslant U}(P, \mathcal{T})$ by replacing $|r_1| < U$ with $|r_1| \geqslant U$. For a parameter $W \geqslant 1$, to be chosen in due course, (10.9) allows us to rewrite the left-hand side of (10.10) as

$$\Sigma_{<\widehat{Q}/\widehat{W}}(P, \Lambda_0^\perp \setminus \mathcal{E}_2) + \Sigma_{\geqslant \widehat{Q}/\widehat{W}}(P, \Lambda_0^\perp) - \Sigma_{\geqslant \widehat{Q}/\widehat{W}}(P, \mathcal{E}_2). \tag{10.12}$$

We will complete the proof of Proposition 10.1 with the following three results.

Lemma 10.13. *For any real $W \geqslant 1$, we have*

$$\Sigma_{<\widehat{Q}/\widehat{W}}(P, \Lambda_0^\perp \setminus \mathcal{E}_2) \ll_\varepsilon |P|^{3+\varepsilon} \widehat{W}^{-1/2}.$$

Proof. Since $I_r(\boldsymbol{c}) \neq 0$ only for $|\boldsymbol{c}| \ll |P|^{1/2}$, it follows that

$$\Sigma_{<\widehat{Q}/\widehat{W}}(P, \Lambda_0^\perp \setminus \mathcal{E}_2) \ll |P|^6 \sum_{\substack{\boldsymbol{c} \in \Lambda_0^\perp \setminus \mathcal{E}_2 \\ |\boldsymbol{c}| \ll |P|^{1/2}}} \sum_{\substack{|r_0 r_1| \leqslant \widehat{Q} \\ |r_1| < \widehat{Q}/\widehat{W}}} |r_0 r_1|^{-5/2} |I_{r_0 r_1}(\boldsymbol{c})| \cdot |r_1|^{1/2} |S_{r_0,0}(\boldsymbol{c})|.$$

We now examine an individual $\boldsymbol{c} \in \Lambda_0^\perp \setminus \mathcal{E}_2$. Since $c_1 \cdots c_6 \neq 0$, it follows from (8.2) that

$$|P|^3 I_r(\boldsymbol{c}) \ll |P\boldsymbol{c}/r| \prod_{1 \leqslant i \leqslant 6} |Pc_i/r|^{-1/2}. \tag{10.13}$$

Recall that any element of $\Lambda_0^\perp$ takes the shape $(c_1, c_2, c_3, c_1, c_2, c_3)$. Let us now consider the contribution from those r_0 with $|r_0| = \widehat{R}_0$. Upon inserting (10.13) and then applying Lemma 10.10, we find that the contribution to the quantity $\Sigma_{<\widehat{Q}/\widehat{W}}(P, \Lambda_0^\perp \setminus \mathcal{E}_2)$ is

$$\ll_\varepsilon \sum_{\substack{\boldsymbol{c} \in \mathcal{O}^3 \cong \Lambda_0^\perp \\ 0 < |c_i| \ll |P|^{1/2}}} \sum_{\substack{\widehat{R}_0 |r_1| \leqslant \widehat{Q} \\ |r_1| < \widehat{Q}/\widehat{W}}} \frac{|P|^{3+\varepsilon} |P|^{-2} |\boldsymbol{c}|}{(\widehat{R}_0 |r_1|)^{1/2} |c_1 \cdots c_3|} \cdot |r_1|^{1/2} \cdot \widehat{R}_0 \sum_{\substack{d \in \mathcal{N}_{\geqslant}(2) \\ |d| \leqslant \widehat{R}_0}} \frac{|S_d^\natural(\boldsymbol{c})|}{|d|}.$$

We also have $S_d^\natural(\boldsymbol{c}) \ll_\varepsilon |d|^{1/2+\varepsilon} \prod_{1 \leqslant i \leqslant 3} |\mathrm{sq}(c_i)|^{1/2}$ by Lemma 6.2, and

$$\sum_{\substack{d \in \mathcal{N}_{\geqslant}(2) \\ |d| \leqslant \widehat{R}_0}} |d|^{-1/2} \ll_\varepsilon \widehat{R}_0^\varepsilon,$$

so that the last display is

$$\ll_\varepsilon \sum_{\substack{\boldsymbol{c} \in \mathcal{O}^{n/2} \\ 0 < |c_i| \ll |P|^{1/2}}} \sum_{|r_1| < \widehat{Q}/\widehat{W}} |P|^{1+\varepsilon} |\boldsymbol{c}| \cdot (\widehat{Q}/|r_1|)^{1/2} \prod_{1 \leqslant i \leqslant 3} \frac{|\mathrm{sq}(c_i)|^{1/2}}{|c_i|}.$$

Since $\sum_{0 < |c| \leqslant \widehat{C}} |\mathrm{sq}(c)|^{1/2}/|c| \ll_\varepsilon \widehat{C}^\varepsilon$, this becomes

$$\ll_\varepsilon \sum_{|r_1| < \widehat{Q}/\widehat{W}} |P|^{1+\varepsilon} |P|^{1/2} \cdot (\widehat{Q}/|r_1|)^{1/2} \ll_\varepsilon |P|^{3/2+\varepsilon} \widehat{Q}^{1/2} (\widehat{Q}/\widehat{W})^{1/2}.$$

Recalling that $\widehat{Q} \asymp |P|^{3/2}$ and that there are $O_\varepsilon(\widehat{Q}^\varepsilon) = O_\varepsilon(|P|^\varepsilon)$ choices for R_0 completes the proof. □

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In the light of Lemma 10.11, we make the choice

$$\widehat{W} = \widehat{C}^{-1}|P|^{1/2}. \tag{10.14}$$

We now turn to the range $|r_1| \geqslant \widehat{Q}/\widehat{W}$ in (10.12).

Lemma 10.14. *We have*

$$\Sigma_{\geqslant \widehat{Q}/\widehat{W}}(P, \Lambda_0^{\perp}) = \sigma_{L_0,w}|P|^3(1 + O(\widehat{W}^{-1})).$$

Proof. By the first part of Lemma 10.8, $S_{r_0,0}(\boldsymbol{c})$ only depends on $\boldsymbol{c}$ mod r_0. If $|r_1| \geqslant \widehat{Q}/\widehat{W} = \widehat{C}|P|$, then after splitting $\boldsymbol{c}$ into residue classes, Lemma 10.11 implies that

$$\sum_{\boldsymbol{c} \in \Lambda_0^{\perp}} S_{r_0,0}(\boldsymbol{c}) \cdot I_r(\theta, \boldsymbol{c}) = \sigma_{L_0,w}\left(\frac{|r_1|}{|P|}\right)^3 \sum_{\boldsymbol{b} \in \Lambda_0^{\perp}/r_0\Lambda_0^{\perp}} S_{r_0,0}(\boldsymbol{b}).$$

By Lemma 10.8 and the equality $\#\Lambda_0^{\perp}/r_0\Lambda_0^{\perp} = |r_0|^3$, we conclude that

$$\sum_{\boldsymbol{c} \in \Lambda_0^{\perp}} S_{r_0,0}(\boldsymbol{c}) \cdot I_r(\theta, \boldsymbol{c}) = \sigma_{L_0,w}\left(\frac{|r|}{|P|}\right)^3 \mathbf{1}_{r_0=1}.$$

We have therefore established the identity

$$\Sigma_{\geqslant \widehat{Q}/\widehat{W}}(P, \Lambda_0^{\perp}) = \sigma_{L_0,w}|P|^3 \sum_{\substack{r \in \mathcal{O}^+ \\ \widehat{Q}/\widehat{W} \leqslant |r| \leqslant \widehat{Q}}} \phi_K(r) \int_{|\theta| < |r|^{-1}\widehat{Q}^{-1}} 1 \mathrm{d}\theta.$$

However, by Dirichlet's approximation theorem (2.2) we have

$$\begin{aligned}\sum_{\substack{r \in \mathcal{O}^+ \\ \widehat{Q}/\widehat{W} \leqslant |r| \leqslant \widehat{Q}}} \phi_K(r) \int_{|\theta| < |r|^{-1}\widehat{Q}^{-1}} 1 \mathrm{d}\theta &= \sum_{\substack{r \in \mathcal{O}^+ \\ |r| \leqslant \widehat{Q}}} {\sum_{|a| < |r|}}' \int_{|\theta| < |r|^{-1}\widehat{Q}^{-1}} 1 \mathrm{d}\theta - \sum_{\substack{r \in \mathcal{O}^+ \\ |r| < \widehat{Q}/\widehat{W}}} \phi_K(r)|r|^{-1}\widehat{Q}^{-1} \\ &= 1 + O(\widehat{W}^{-1}),\end{aligned}$$

which completes the proof. □

Lemma 10.15. *We have*

$$\Sigma_{\geqslant \widehat{Q}/\widehat{W}}(P, \mathcal{E}_2) \ll_{\varepsilon} |P|^{3-2/3+\varepsilon}.$$

Proof. The main subtlety here is that we must treat $\boldsymbol{c} \neq \mathbf{0}$ and $\boldsymbol{c} = \mathbf{0}$ separately. First, given $r \in \mathcal{O}^+$, we apply the triangle inequality and the trivial divisor estimate to (10.8), giving

$$\sum_{\substack{r_0 r_1 = r \\ |r_1| \geqslant \widehat{Q}/\widehat{W}}} |S_{r_0,0}(\boldsymbol{c})|\, \phi_K(r_1)|r_1|^{-1/2} \ll_{\varepsilon} \mathbf{1}_{|r| \geqslant \widehat{Q}/\widehat{W}}|r|^{1/2+\varepsilon} \sum_{d|r} |d|^{-1/2}|S_d^{\natural}(\boldsymbol{c})|. \tag{10.15}$$

Inserting (10.15) into the definition (10.11) of $\Sigma_{\geqslant \widehat{Q}/\widehat{W}}(X, \mathcal{E}_2 \setminus \{\mathbf{0}\})$ and recalling the definition (10.6) of f, we get

$$\Sigma_{\geqslant \widehat{Q}/\widehat{W}}(P, \mathcal{E}_2 \setminus \{\mathbf{0}\}) \ll_{\varepsilon} |P|^6 \sum_{\boldsymbol{c} \in \mathcal{E}_2 \setminus \{\mathbf{0}\}} \sum_{\substack{r \in \mathcal{O}^+ \\ |r| \leqslant \widehat{Q}}} |r|^{-2+\varepsilon}|I_r(\boldsymbol{c})| \sum_{d|r} |d|^{-1/2}|S_d^{\natural}(\boldsymbol{c})| \ll_{\varepsilon} \widehat{Q}^{\varepsilon} f(\mathcal{E}_2).$$

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On combining Lemma 10.7 with the choice $\delta = 1 - \varepsilon$ in Lemma 10.12, we deduce that $f(\mathcal{E}_2) \ll_\varepsilon |P|^{3-1/2+\varepsilon}$, which is satisfactory since $\widehat{Q} = |P|^{3/2}$. Similarly, (10.15) gives

$$\Sigma_{\geqslant \widehat{Q}/\widehat{W}}(P, \{\mathbf{0}\}) \ll_\varepsilon |P|^6 \sum_{\substack{r \in \mathcal{O}^+ \\ |r| \geqslant \widehat{Q}/\widehat{W}}} |r|^{-2+\varepsilon} |I_r(\mathbf{0})| \sum_{d|r} |d|^{-1/2} |S_d^\natural(\mathbf{0})|,$$

which is $O_\varepsilon(|P|^{3+\varepsilon}(\widehat{Q}/\widehat{W})^{-2/3+\varepsilon})$ by (8.2) and Lemma 9.2 summed over q-adic ranges for $|r|$. Since $\widehat{Q}/\widehat{W} \gg |P|$, we conclude that $|P|^3(\widehat{Q}/\widehat{W})^{-2/3} \ll |P|^{3-2/3}$. □

Recalling our choice for $\widehat{W}$ in (10.14), we see that (10.10) follows from inserting Lemmas 10.13–10.15 into (10.12). This finally completes the proof of Proposition 10.1.

11. Final reckoning

11.1. *Proof of Theorem 1.1*

Let $F(\boldsymbol{x}) = x_1^3 + \cdots + x_6^3$ and let $w_{\boldsymbol{x}_0}$ be the weight function defined in Definition 5.4, associated to an appropriate $\boldsymbol{x}_0 \in (\mathbb{F}_q^\times)^6$. Then we have

$$N(P) = \sum_{\substack{\boldsymbol{x}_0 \in (\mathbb{F}_q^\times)^6 \\ F(\boldsymbol{x}_0)=0}} N_F(w_{\boldsymbol{x}_0}, P),$$

for any $P \in \mathcal{O}$, in the notation of (1.1) and (2.4). Let $L(s, \boldsymbol{c})$ be the Hasse–Weil L-function introduced in § 3, for $\boldsymbol{c} \in \mathcal{O}^6$. Theorem 1.1 relies on the Ratios Conjecture for these L-functions, as $\boldsymbol{c}$ varies. We have seen several forms of the Ratios Conjecture during the course of the paper. The following result is a stronger version of Theorem 1.1, and makes clear the various interdependencies.

Theorem 11.1. *Assume* $\mathrm{char}(\mathbb{F}_q) > 3$. *Then each of the following implies the next:*

1. *The Ratios Conjecture 3.6 (R2), for shifted second moments of* $1/L(s, \boldsymbol{c})$.
2. *Conjecture 3.7 holds.*
3. *Conjecture 3.8 holds.*
4. *Conjecture 4.5 holds.*
5. $N(P) \ll_q |P|^3$ *for any* $P \in \mathcal{O}$.
6. *If* $S \subseteq \mathcal{O}$ *has positive lower density, then so does* $\{x^3 + y^3 + z^3 : x, y, z \in S\}$.
7. *The set* $\{k : x^3 + y^3 + z^3 = k$ *is soluble in* $\mathcal{O}^+$ *with* $\max\{|x|, |y|, |z|\} = |k|^{1/3}\}$ *has positive lower density.*

Proof. We gave a proof that Conjecture 3.7 follows from the Ratios Conjecture (R2) directly after the annunciation of the conjecture, so that (1) ⇒(2). Similarly for the deduction of Conjecture 3.8 from Conjecture 3.7, whence (2) ⇒(3), and for the deduction of Conjecture 4.5 from Conjecture 3.8, whence (3) ⇒(4). These implications mostly lie in the world of L-functions.

The implication (4) ⇒(5) requires the full force of the function field circle method. Combining (2.9) with the notation introduced in (8.1), (9.1) and (10.1), we see that

$$N(P) \leqslant q^6 \max_{\substack{\boldsymbol{x}_0 \in (\mathbb{F}_q^\times)^6 \\ F(\boldsymbol{x}_0)=0}} (M(P) + E_1(P) + E_2(P)).$$

Assume that Conjecture 4.5 holds. Bearing in mind the bound (10.10) and the fact that Υ is finite, we may combine Propositions 8.1, 9.1 and 10.1 in order to conclude that $N(P) \ll |P|^3$ for any $P \in \mathcal{O}$.

The implication (6) ⇒(7) follows by taking $S = \mathcal{O}^+$ in (6). Indeed, in $\mathcal{O}^+$ we always have $\deg(x^3 + y^3 + z^3) = 3\max(\deg(x), \deg(y), \deg(z))$, since $\mathrm{char}(\mathbb{F}_q) > 3$.

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Finally, we deal with the implication (5) $\Rightarrow$(6). For any positive integer d, let

$$U_S(d) = \#\big\{k \in \mathcal{O} : |k| \leqslant q^{3d},\ \exists (x,y,z) \in S^3 \text{ such that } x^3+y^3+z^3 = k\big\}.$$

We wish to prove that

$$\liminf_{d\to\infty} \frac{\#U_S(d)}{q^{3d}} \gg_S 1,$$

where the implied constant is allowed to depend on S and q. Since $S \subseteq \mathcal{O}$ has positive lower density, there exists $A \in \mathbb{N}$ such that

$$\liminf_{d\to\infty} \frac{\#\{x \in S : |x| < q^d\}}{q^d} > \frac{1}{q^A - 1}.$$

Let $S_e = \{x \in S : |x| = q^e\}$, for any $e \in \mathbb{Z}_{\geqslant 0}$. Then, for all sufficiently large integers d, we have

$$\sum_{e<d} \#S_e \geqslant \frac{q^d}{q^A-1}.$$

Since $\sum_{e<d-A} \#S_e \leqslant q^{d-A}$, by trivially bounding $\#S_e$, we obtain

$$\sum_{d-A\leqslant e<d} \#S_e \geqslant \frac{q^d}{q^A-1} - q^{d-A} = \frac{q^{d-A}}{q^A-1}.$$

Thus, by the pigeonhole principle, there exists $e = e(d) \in [d-A, d-1]$ such that

$$\#S_e \geqslant \frac{1}{A}\cdot\frac{q^{d-A}}{q^A-1}. \tag{11.1}$$

Given $k \in \mathcal{O}$, we denote by $r_S(k)$ the number of $(x,y,z) \in S^3$ for which $x^3+y^3+z^3 = k$ and $|x| = |y| = |z| = q^e$. Then it follows from Cauchy–Schwarz that

$$\left(\sum_{|k|\leqslant q^{3e}} r_S(k)\right)^2 \leqslant U_S(e) \sum_{|k|\leqslant q^{3e}} r_S(k)^2 \leqslant U_S(e) N(t^e).$$

Part (5) implies that $N(t^e) = O(q^{3e})$. Moreover, on the left hand side we have

$$\sum_{|k|\leqslant q^{3e}} r_S(k) \geqslant \#S_e^3 \gg_A q^{3d},$$

by (11.1). It now follows that $U_S(d) \geqslant U_S(e) \gg_A q^{3d}$, which establishes part (6). □

When $F = x_1^3 + \cdots + x_6^3$, through careful use of Hölder's inequality, it would be possible to replace the upper bound in (5) by $N_F(w,P) \ll_w |P|^3$ for any $w \in S(K_\infty^n)$. However, since doing so would make the paper needlessly longer, we have decided to omit the details here.

11.2. Possible reductions to Ratios

To facilitate a deeper discussion of the Ratios Conjecture, we make the following definition.

Definition 11.2. Let $\mathcal{F}_q$ be a *geometric family* of L-functions $L(s,f)$ over $\mathbb{F}_q(t)$, with a *gauge function* $D(f)$ approximating the conductor, in the sense of [46, pp. 534–535]. Let $\mathcal{Q} \subseteq \mathbb{Z}$. Assume $\mathcal{F}_q$ varies

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naturally with $q \in \mathcal{Q}$, forming a family $\mathcal{F} = (\mathcal{F}_q)_{q\in\mathcal{Q}}$. We say that the *q-restricted Ratios Conjecture* holds if for any ratio average over $D(f) \leqslant q^Z$, one has an asymptotic with power saving $O_q(q^{-\delta Z})$ in the strip

$$\tfrac{1}{2} + q^{-\delta} \leqslant \Re(s) \leqslant \tfrac{1}{2} + \delta \tag{11.2}$$

where $\delta > 0$ depends only on $\mathcal{F}$, $\mathcal{Q}$ and the ratio in question.

Note that one can also study the Ratios Conjecture in other aspects; see work of Bui–Florea–Keating [13], and work of Florea [22]. We shall prove in Theorem 11.4 that the q-restricted Ratios Conjecture suffices in Theorem 1.1, if q is large enough. The homological stability framework of [5,43] might eventually resolve this particular form of the Ratios Conjecture for sufficiently large values of q. Evidence for this appears in work of Wang [55], in the simpler setting of (imaginary) quadratic Dirichlet L-functions.

Theorem 11.3 [55]. *The q-restricted Ratios Conjecture holds for quadratic Dirichlet L-functions over $\mathbb{F}_q(t)$, for odd q large enough in terms of the ratio in question.*

The following result illustrates how a homological stability proof might impact the work of this paper. The specific term $q^{-\delta}$ in (11.2) is unimportant; any term of the form $o_{q\to\infty}(1)$ would suffice.

Theorem 11.4. *The q-restricted Ratios Conjecture for $L(s_1, \boldsymbol{c})^{-1}L(s_2, \boldsymbol{c})^{-1}$, in the sense of Definition 11.2, implies (1)–(6) in Theorem 11.1 for all large enough q.*

Proof. Assume the q-restricted Ratios Conjecture, for $L(s_1, \boldsymbol{c})^{-1}L(s_2, \boldsymbol{c})^{-1}$. Let $\beta := 2q^{-\delta}$. If q and Z are sufficiently large in terms of δ, then $q^{-\delta} \leqslant \beta + \frac{1}{Z} \leqslant \delta$ and $6\beta < \delta$, so Conjecture 3.6 holds with constant $\beta = 2q^{-\delta}$. On the other hand, if $Z \ll 1$, then $L(s_1, \boldsymbol{c})^{-1}L(s_2, \boldsymbol{c})^{-1} \ll_q 1$ for $|\boldsymbol{c}| \leqslant q^Z$ by GRH, so Conjecture 3.6 holds trivially. This proves (1) in Theorem 11.1, and therefore (1)–(6) too, for $q \gg 1$. □

11.3. Future extensions

First, as mentioned in the introduction, a density 1 relative of Theorem 1.1 is proven in the sequel [8]. Second, most of the proof of Theorem 11.1 works when $\mathrm{char}(\mathbb{F}_q) = 2$, but the proof of the following main ingredients would need to be modified: Lemmas 5.5 and 5.8, [32, Lemma 60], Lemma 6.6, and Proposition 10.1. We leave open the challenge of extending Theorem 11.1 to $\mathrm{char}(\mathbb{F}_q) = 2$, but some general progress towards Proposition 10.6 (which is a key ingredient in Proposition 10.1) and [32, Lemma 60] can be found in [53, Theorem 1.2].

We close by discussing the task of rigorously verifying Conjecture 3.6 (R2). Thanks to a uniform homological stability result for braid groups established in [43], the Moments Conjecture for quadratic Dirichlet L-functions for large fixed odd q is now proven, even with a power saving [5]. The next questions would be (a) whether these techniques can establish the q-restricted Ratios Conjecture, say, for negative moments; and (b) whether they work for more general geometric families in the sense of [46].

The note [55] affirmatively answers (a). On the other hand, question (b) is much more technical, so we limit ourselves to a few brief remarks. The L-function of a quadratic field $K(\sqrt{d}) \cong K[y]/(y^2 - d)$ of discriminant $d \in \mathcal{O}^+$ can be interpreted in terms of the corresponding hyperelliptic curve $y^2 = d(t)$ over $\mathbb{F}_q$. In higher dimensions the situation is more complicated, but at least when $F^*(\boldsymbol{c}) \in \mathcal{O}$ is nearly square-free, one may use [55, Proposition A.1] to interpret the L-function of a smooth cubic threefold

$$x_1^3 + \cdots + x_6^3 = c_1(t)x_1 + \cdots + c_6(t)x_6 = 0$$

over K in terms of the corresponding *fibred fourfold* over $\mathbb{F}_q$.

In [5], braid groups arise as the *fundamental groups* of moduli spaces

$$\mathrm{Conf}_e(\mathbb{A}^1)(\mathbb{C}) = \{d \in \mathbb{C}[t] \text{ monic} : \deg(d) = e,\ \mathrm{disc}_t(d) \neq 0\}$$

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of hyperelliptic curves. Moreover, the moduli spaces in question are *aspherical*, allowing for a purely group-theoretic reformulation of the relevant homology groups arising from the Grothendieck–Lefschetz trace formula. As a first step towards (b), one might try to compute the (likely very rich) fundamental groups of the moduli spaces

$$X_e := \{\boldsymbol{c} \in \mathbb{C}[t]^6 : \deg(c_i) = e,\ \deg_t(F^*(\boldsymbol{c})) = e \deg F^*,\ \operatorname{disc}_t(F^*(\boldsymbol{c})) \neq 0\},$$

to bound their Betti numbers and to determine whether X_e is aspherical or not.

An interesting, yet simpler higher-dimensional task for (b) would be to study L-function statistics in quadratic or cubic twist families of elliptic curves over $\mathbb{F}_q(t)$. In this case, the base moduli spaces from [5] would most likely not need to be changed much, with the main changes occurring for the relevant *local systems* and *gluing maps*.

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