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Fluctuation-dissipation relations for a plasma-kinetic Langevin equation

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A linearised kinetic equation describing electrostatic perturbations of a Maxwellian equilibrium in a weakly collisional plasma forced by a random source is considered. The problem is treated as a kinetic analogue of the Langevin equation and the corresponding fluctuation-dissipation relations are derived. The kinetic fluctuation-dissipation relation reduces to the standard "fluid" one in the regime where the Landau damping rate is small and the system has no real frequency; in this case the simplest possible Landau-fluid closure of the kinetic equation coincides with the standard Langevin equation. Phase mixing of density fluctuations and emergence of fine scales in velocity space is diagnosed as a constant flux of free energy in Hermite space; the fluctuation-dissipation relations for the perturbations of the distribution function are derived, in the form of a universal expression for the Hermite spectrum of the free energy. Finite-collisionality effects are included. This work is aimed at establishing the simplest fluctuation-dissipation relations for a kinetic plasma, clarifying the connection between Landau and Hermite-space formalisms, and setting a benchmark case for a study of phase mixing in turbulent plasmas.

1. Introduction

Fluctuation dissipation relations (FDR) predict the response of a dynamical system to an externally applied perturbation, based on the system's internal dissipation properties. The classical Langevin equation (Kubo 1966) supplies the best known example of such FDR. The standard formulation is to consider a scalar φ forced by a Gaussian white-noise source χ and damped at the rate γ:

$$\frac{\partial \varphi}{\partial t} + \gamma \varphi = \chi, \qquad \langle \chi(t)\chi(t')\rangle = \varepsilon \delta(t - t'), \tag{1.1}$$

where angle brackets denote the ensemble average and $\varepsilon/2$ is the mean power injected into the system by the source:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\langle \varphi^2 \rangle}{2} + \gamma \langle \varphi^2 \rangle = \frac{\varepsilon}{2}. \tag{1.2}$$

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The steady-state mean square fluctuation level is then given by the FDR, linking the injection and the dissipation of the scalar fluctuations:

$$\langle \varphi^2 \rangle = \frac{\varepsilon}{2\gamma}. \tag{1.3}$$

The simplest physical example of such a system is a Brownian particle suspended in liquid, with φ the velocity of the particle and γ the frictional damping. More generally, (1.1) may be viewed as a generic model for systems where some perturbed quantity is randomly stirred and decays via some form of linear damping, a frequently encountered situation in, e.g. fluid dynamics.

Nearly every problem in plasma physics involves a system with driven and damped linear modes. Here we consider the prototypical such case: the behaviour of perturbations of a Maxwellian equilibrium in a weakly collisional plasma in one spatial and one velocity-space dimension. In such a system (and in weakly collisional or collisionless plasmas generally), damping of the perturbed electric fields occurs via the famous Landau (1946) mechanism. Landau damping, however, is different in several respects from standard "fluid" damping phenomena. It is in fact a phase mixing process: electric—and, therefore, density—perturbations are phase mixed and thus are effectively damped. Their (free) energy is transferred to perturbations of the particle distribution function that develop ever finer structure in velocity space and are eventually removed by collisions or, in a formally collisionless limit, by some suitable coarse-graining procedure. The electrostatic potential φ in such systems cannot in general be rigorously shown to satisfy a "fluid" equation of the form (1.1), with γ the Landau damping rate, although the idea that equation (1.1) or a higher-order generalisation thereof is not a bad model underlies the so-called Landau-fluid closures (Hammett and Perkins 1990; Hammett et al. 1992; Hedrick and Leboeuf 1992; Dorland and Hammett 1993; Snyder et al. 1997; Passot and Sulem 2004; Goswami et al. 2005; Passot and Sulem 2007).

It is a natural question to ask whether, despite the dynamical equations for φ (or, more generally, for the moments of the distribution function) being more complicated than equation (1.1), we should still expect the mean fluctuation level to satisfy equation (1.3), where γ is the Landau damping rate. And if that is not the case, then should the value of γ *defined* by equation (1.3) be viewed as the effective damping rate in a driven system, replacing the Landau rate? Plunk (2013) recently considered the latter question and argued that the fact that the effective damping rate defined this way differs from the Landau rate suggests a fundamental modification of Landau response in a stochastic setting. Our take on the problem at hand differs from his somewhat in that we take the kinetic version of the Langevin equation (introduced in Sec. 2) at face value and derive the appropriate kinetic generalisation of the FDR, instead of attaching a universal physical significance to the "fluid" version of it. Interestingly, the kinetic FDR does simplify to the classical fluid FDR when the Landau damping rate is small. Furthermore, we prove that in this limit (and when the system has no real frequency), the dynamics of φ is in fact described by equation (1.1) with γ equal precisely to the Landau rate (i.e. the simplest Landau fluid closure is a rigorous approximation in this limit). The latter result is obtained by treating the velocity-space dynamics of the system in Hermite space. We also show how phase mixing in our system can be treated as a free-energy flux in Hermite space, what form the FDR takes for the Hermite spectrum of the perturbations of the distribution function, and how collisional effects can be included. The intent of this treatment is

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to provide a degree of clarity as to the behaviour of a very simple plasma model and thus set the stage for modelling more complex, nonlinear phenomena.

The plan of the paper is as follows. In Sec. 2, we describe a simple model for a weakly collisional plasma, which we call the kinetic Langevin equation, and then, in Sec. 3, derive the FDR for the same, including the "fluid" limit mentioned above. In Sec. 4, Hermite-space dynamics are treated, including the limit where Landau-fluid closures hold rigorously. An itemised summary of our findings is given in Sec. 5. A version of our calculation with a different random source is presented in Appendix A.

2. Kinetic Langevin equation

We consider the following (1+1)-dimensional model of a homogeneous plasma perturbed about a Maxwellian equilibrium:

$$\frac{\partial g}{\partial t} + \underbrace{v\frac{\partial g}{\partial z}}_{\text{phase mixing}} + \underbrace{vF_0\frac{\partial \varphi}{\partial z}}_{\text{electric field}} = \underbrace{\chi(t)F_0}_{\text{source}} + \underbrace{C[g]}_{\text{collisions}}, \tag{2.1}$$

$$\varphi = \alpha\int_{-\infty}^{\infty} \mathrm{d}v\, g, \tag{2.2}$$

$$\langle \chi(t)\chi(t')\rangle = \varepsilon\delta(t-t'),$$

where $g(z, v, t)$ is the perturbed distribution function and $F_0(v)$ is the Maxwellian equilibrium distribution $F_0 = e^{-v^2}/\sqrt{\pi}$. The velocity v (in the z direction) is normalised to the thermal speed $v_{\mathrm{th}} = \sqrt{2T/m}$ (T and m are the temperature and mass of the particle species under consideration), spatial coordinate z is normalised to an arbitrary length L, and time t to L/v_{th}. Only one species (either electrons or ions) is evolved. The second species follows the density fluctuations of the first via whatever response a particular physical situation warrants: Boltzmann, isothermal, or no response—all of these possibilities are embraced by equation (2.2), which determines the (suitably normalised) scalar potential φ in terms of the perturbed density associated with g; the parameter α contains all of the specific physics. For example, if g is taken to be the perturbed ion distribution function in a low-beta magnetised plasma and electrons to have Boltzmann response, then $\alpha = ZT_e/T_i$, the ratio of the electron to ion temperatures (Z is the ion charge in units of electron charge e)—the resulting system describes (Landau-damped) ion-acoustic waves; equation (2.2) in this case is the statement of quasineutrality. Another, even more textbook example is damped Langmuir waves, the case originally considered by Landau (1946): g is the perturbed electron distribution function, ions have no response, so $\alpha = 2/k^2\lambda_D^2$, where λ_D is the Debye length and k is the wave number of the perturbation ($\partial/\partial z = ik$); equation (2.2) in this case is the Gauss-Poisson law.

A particularly astrophysically and space-physically relevant example (in the sense of being accessible to measurements in the solar wind; e.g. Celnikier et al. 1983, 1987; Marsch and Tu 1990; Bershadskii and Sreenivasan 2004; Hnat et al. 2005; Chen et al. 2011) is the compressive perturbations in a magnetised plasma—perturbations of plasma density and magnetic-field strength at scales long compared to the ion Larmor radius. These are in fact described by two equations evolving two decoupled functions g^+ and g^-, which are certain linear combinations of the zeroth and second moments of the perturbed ion distribution function with respect to the velocity perpendicular to the mean magnetic field (taken to be in the z direction). These equations are derived in Schekochihin et al. (2009, Sec. 6.2.1) and are of the form

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(2.1) with

$$\alpha^{\pm} = -\left[-\frac{T_i}{ZT_e} + \frac{1}{\beta_i} \pm A\right]^{-1}, \quad A = \sqrt{\left(1 + \frac{T_i}{ZT_e}\right)^2 + \frac{1}{\beta_i^2}} \tag{2.3}$$

for $g^{\pm}$, respectively (here $\beta_i = 8\pi n_i T_i / B^2$ is the ion beta). The physical fields, the density and magnetic-field-strength perturbations, are related to $g^{\pm}$ by

$$\frac{\delta n}{n} = \frac{1}{2A} \int \mathrm{d}v \left[\left(1 + \frac{T_i}{ZT_e} + \frac{1}{\beta_i} + A\right) g^- - \frac{T_i}{ZT_e} \frac{2}{\beta_i} g^+\right], \tag{2.4}$$

$$\frac{\delta B}{B} = \frac{1}{2A} \int \mathrm{d}v \left[\left(1 + \frac{T_i}{ZT_e} + \frac{1}{\beta_i} + A\right) g^+ - \left(1 + \frac{ZT_e}{T_i}\right) g^-\right]. \tag{2.5}$$

While these expressions are perhaps not very physically transparent, it may aid intuition to note that $\delta n/n \approx \int \mathrm{d}v g^-$ and $\delta B/B \approx \int \mathrm{d}v g^+$ either in the limit of high β_i and hot ions ($T_i \gg T_e$) or in the limit of low β_i and cold ions ($T_i \ll T_e$). At low β_i, the g^- equation describes ion-acoustic waves ($\alpha^- \approx ZT_e/T_i$; see above). At high β_i, the g^+ equation describes a kinetic version of the MHD slow mode, subject to a version of Landau damping due to Barnes (1966); in this case, $\alpha^+ \approx -1 + 1/\beta_i$.

Thus, equations (2.1) and (2.2) correspond to a variety of interesting physical situations.

The energy injection in equation (2.1) is modelled by a white-in-time, Maxwellian-in-velocity-space source $\chi(t)F_0$ supplying fixed power $\propto \varepsilon$ to the perturbations (see below). This is a direct analogue of the noise term in the "fluid" Langevin equation (1.1) and so this particular choice of forcing was made in order to enable the simplest possible comparison with the "fluid" case.† The energy injection leads to sharp gradients in the velocity space (phase mixing), which are removed by the collision operator $C[g]$. "The energy" in the context of a kinetic equation is the free energy of the perturbations (see Schekochihin et al. 2008, 2009, and references therein), given in this case by

$$W = \int \mathrm{d}v \frac{\langle g^2 \rangle}{2F_0} + \frac{\langle \varphi^2 \rangle}{2\alpha} \tag{2.6}$$

and satisfying

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{1+\alpha}{2}\varepsilon + \int \mathrm{d}v \frac{\langle gC[g] \rangle}{F_0}. \tag{2.7}$$

The first term on the right-hand side is the energy injection by the source, the second, negative definite, term is its thermalisation by collisions. Note that the variance of φ is not by itself a conserved quantity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\langle \varphi^2 \rangle}{2} + \alpha \left\langle \varphi \frac{\partial}{\partial z} \int \mathrm{d}v\, v g \right\rangle = \frac{\alpha^2}{2}\varepsilon. \tag{2.8}$$

† One might argue that this is not, however, the most physical form of forcing and that it would be better to inject energy by applying a random electric field to the plasma, rather than a source of density perturbations. In Appendix A we present a version of our calculation for such a more physical source, and show that all the key results are similar. Note that the forcing in equation (2.1) does not violate particle conservation because we assume that spatial integrals of all perturbations vanish: $\int \mathrm{d}z\, g = 0$, $\int \mathrm{d}z\, \chi = 0$.

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The power $\alpha^2\varepsilon/2$ injected into fluctuations of φ is transferred into higher moments of g via phase mixing. Landau damping is precisely this process of draining free energy from the lower moments and transferring it into higher moments of the distribution function—without collisions, this is just a redistribution of free energy within (2.6), which, in the absence of source, would look like a linear damping of φ.‡

In the presence of a source, the system described by equations (2.1) and (2.2) is a driven-damped system much like the Langevin equation (1.1). The damping of φ in the kinetic case is provided by Landau damping (phase mixing) as opposed to the explicit dissipation term in equation (1.1). It is an interesting question whether in the steady state, the second term on the left-hand side of equation (2.8) can be expressed as $\gamma_{\text{eff}}\langle\varphi^2\rangle$, leading to an analogue of the FDR (equation (1.3)), and if so, whether the "effective damping rate" γ_{eff} in this expression is equal to the Landau damping rate γ_L. The answer is that an analogue of the FDR does exist, γ_{eff} is non-zero for vanishing collisionality, but in general, $\gamma_{\text{eff}} \neq \gamma_L$.

3. Kinetic FDR

Ignoring collisions in equation (2.1) and Fourier-transforming it in space and time, we get

$$g_{k\omega} = -\varphi_{k\omega}\frac{vF_0}{v-\omega/k} - \frac{i\chi_{k\omega}}{k}\frac{F_0}{v-\omega/k}. \tag{3.1}$$

Introducing the plasma dispersion function $Z(\zeta) = \int \mathrm{d}v F_0/(v-\zeta)$, where the integration is along the Landau contour (Fried and Conte 1961), we find from equations (3.1) and (2.2):

$$\varphi_{k\omega} = -\frac{i\chi_{k\omega}}{|k|}\frac{Z(\omega/|k|)}{D_\alpha(\omega/|k|)}, \tag{3.2}$$

$$D_\alpha\left(\frac{\omega}{|k|}\right) = 1 + \frac{1}{\alpha} + \frac{\omega}{|k|}Z\left(\frac{\omega}{|k|}\right). \tag{3.3}$$

Note that $D_\alpha(\omega/|k|) = 0$ is the dispersion relation for the classic Landau (1946) problem. We now inverse Fourier transform (3.2) back into the time domain,

$$\varphi_k(t) = \int \mathrm{d}\omega\, e^{-i\omega t}\varphi_{k\omega} = -\frac{i}{|k|}\int \mathrm{d}\omega\, e^{-i\omega t}\chi_{k\omega}\frac{Z(\omega/|k|)}{D_\alpha(\omega/|k|)}, \tag{3.4}$$

and compute $\langle|\varphi_k|^2\rangle$ in the steady state. In order to do this, we use the fact that $\chi_{k\omega} \equiv \int \mathrm{d}t e^{i\omega t}\chi_k(t)/2\pi$ satisfies $\langle\chi_{k\omega}\chi^*_{k\omega'}\rangle = \varepsilon_k\delta(\omega-\omega')/2\pi$ because $\langle\chi_k(t)\chi^*_k(t')\rangle = \varepsilon_k\delta(t-t')$, where ε_k is the source power at wave number k. The result is

$$\langle|\varphi_k|^2\rangle = \frac{\varepsilon_k}{2\pi|k|}f(\alpha), \quad f(\alpha) = \int_{-\infty}^{+\infty}\mathrm{d}\zeta\left|\frac{Z(\zeta)}{D_\alpha(\zeta)}\right|^2, \tag{3.5}$$

where we have changed the integration variable to $\zeta = \omega/|k|$. This is the fluctuation-dissipation relation for our kinetic system that predicts the long-time behaviour of the electrostatic potential. The function $f(\alpha)$, computed numerically as per

‡ Note that $\alpha = -1$ corresponds to an effectively undriven system; the Landau damping rate for this case is zero (see equation (3.8)). We will see in Sec. 4.1 that in this case the driven density moment decouples from the rest of the perturbed distribution function; see equation (4.4). For $\alpha < -1$, the system is no longer a driven-damped system; this parameter regime never occurs physically.

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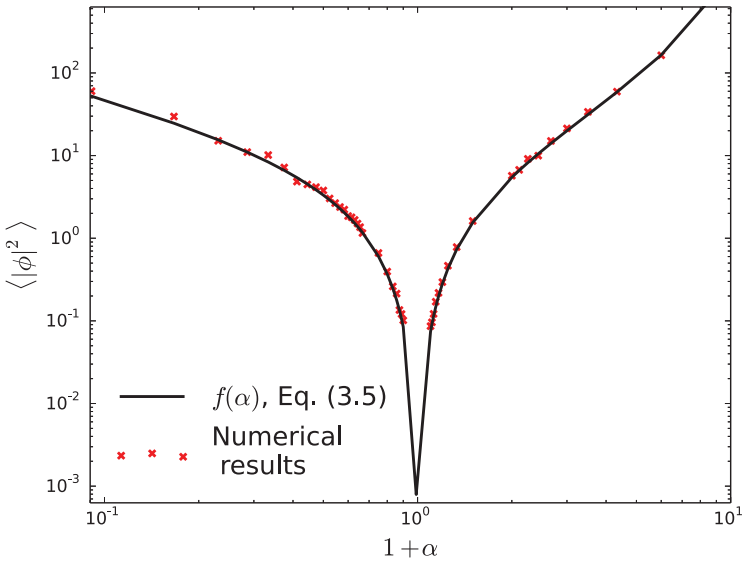


FIGURE 1. Normalised steady-state amplitude $2\pi|k|\langle|\varphi_k|^2\rangle/\varepsilon_k = f(\alpha)$ vs. $1+\alpha$: the solid line is the analytical prediction ($f(\alpha)$ as per equation (3.5)), the crosses are computed from the long-time limit of $\langle|\varphi_k|^2\rangle$ obtained via direct numerical solution of equations (2.1) and (2.2).

equation (3.5), is plotted in Fig. 1, together with the results of direct numerical solution of equations (2.1) and (2.2), in which $f(\alpha)$ is found by computing the saturated fluctuation level $\langle|\varphi_k|^2\rangle$.

Equation (3.5) can be written in the form

$$\langle|\varphi_k|^2\rangle = \frac{\alpha^2\varepsilon_k}{2\gamma_{\text{eff}}}, \quad \gamma_{\text{eff}}(\alpha) = \frac{\pi\alpha^2}{f(\alpha)}|k|, \tag{3.6}$$

but the "effective damping rate" γ_{eff} is not in general the same as the Landau damping rate γ_L. This is illustrated in Fig. 2, where we plot the real (ω_L) and imaginary ($-\gamma_L$) parts of the slowest-damped root(s) of $D_\alpha(\omega/|k|) = 0$ together with $\gamma_{\text{eff}}(\alpha)$ for $\alpha < 0$ and $\gamma_{\text{eff}}(\alpha)/2$ for $\alpha > 0$. In the latter case, the linear modes of the system have real frequencies and the analogy with the Langevin (1.1) is not apt—a better mechanical analogy is a damped oscillator, as explained at the end of Sec. 3.2; the FDR in this case acquires an extra factor of 1/2, which is why we plot $\gamma_{\text{eff}}/2$ (see equation (3.15)). Remarkably, $\gamma_{\text{eff}}(\alpha)$ does asymptote to γ_L in the limit $1+\alpha \ll 1$ and to $2\gamma_L$ in the limit $\alpha \to \infty$, i.e. when the damping is weak. These asymptotic results can be verified analytically.

3.1. *Zero real frequency, weak damping ($\alpha \to -1$)*

When $\alpha + 1 \ll 1$, the solution of the dispersion relation will satisfy $\zeta = \omega/|k| \ll 1$. In this limit,

$$Z(\zeta) \approx i\sqrt{\pi}, \quad D_\alpha(\zeta) \approx 1 + \frac{1}{\alpha} + i\zeta\sqrt{\pi} \approx i\sqrt{\pi}\left(\zeta + i\frac{1+\alpha}{\sqrt{\pi}}\right). \tag{3.7}$$

Therefore, the solution of $D_\alpha(\omega/|k|) = 0$ is

$$\omega \approx -i\gamma_L, \quad \gamma_L = \frac{1+\alpha}{\sqrt{\pi}}|k|. \tag{3.8}$$

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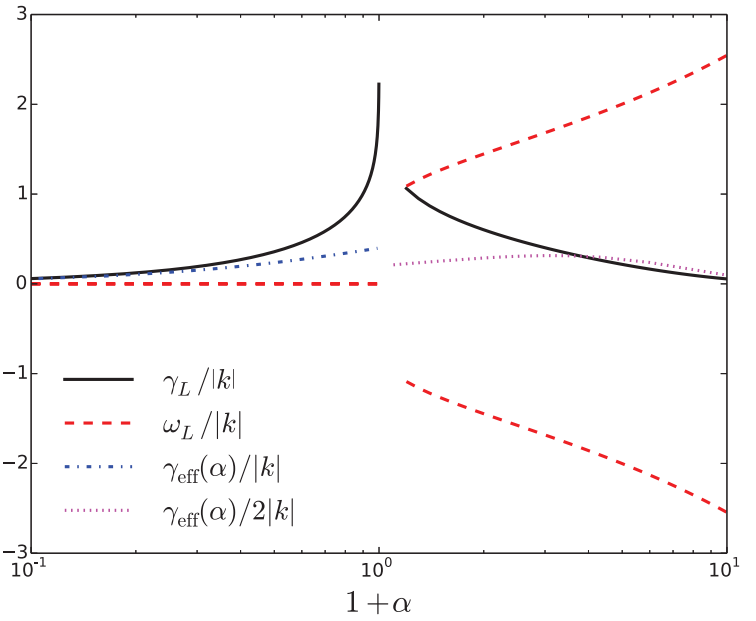


FIGURE 2. Slowest-damped solutions of the dispersion relation $D_\alpha(\omega/|k|) = 0$: normalised frequency $\omega_L/|k|$ (red dashed line) and damping rate $\gamma_L/|k|$ (black solid line) vs. $1+\alpha$. Also shown are $\gamma_{\rm eff}(\alpha)$ for $\alpha < 0$ (blue dash-dotted line) and $\gamma_{\rm eff}(\alpha)/2$ for $\alpha > 0$ (magenta dotted line), as per equation (3.6). The two asymptotic limits in which these match γ_L are discussed in Secs. 3.1 and 3.2.

A useful physical example of Landau damping in this regime is the Barnes (1966) damping of compressive fluctuations in high-beta plasmas, where $1+\alpha \approx 1/\beta_i$ (Schekochihin et al. 2009, their equation (190); see discussion in our Sec. 2).

Since the zeros of $D_\alpha(\zeta)$ and $D^*_\alpha(\zeta)$, which are poles of the integrand in the expression for $f(\alpha)$ (equation (3.5)), lie very close to the real line in this case, the integral is easily computed by using the approximate expressions (3.7) for $Z(\zeta)$ and $D_\alpha(\zeta)$ and applying Plemelj's formula, to obtain

$$f(\alpha) \approx \frac{\pi\sqrt{\pi}}{1+\alpha} = \frac{\pi|k|}{\gamma_L} \quad \Rightarrow \quad \langle|\varphi_k|^2\rangle \approx \frac{\sqrt{\pi}\,\varepsilon_k}{2(1+\alpha)|k|} = \frac{\varepsilon_k}{2\gamma_L}. \tag{3.9}$$

Since $\alpha^2 \approx 1$, this is the same as equation (3.6) with $\gamma_{\rm eff} = \gamma_L$, so the "fluid" FDR is recovered. Note, however, that this recovery of the exact form of the "fluid" FDR is a property that is not universal with respect to the exact form of energy injection: as shown in Appendix A, it breaks down for a different forcing (see equation (A 14)).

3.2. *Large real frequency, weak damping ($\alpha \to \infty$)*

Another analytically tractable limit is $\alpha \gg 1$, in which case the solutions of the dispersion relation have $\zeta = \omega/|k| \gg 1$. In this limit,

$$Z(\zeta) \approx i\sqrt{\pi}\,e^{-\zeta^2} - \frac{1}{\zeta} - \frac{1}{2\zeta^3}, \quad D_\alpha(\zeta) \approx \frac{1}{\alpha} - \frac{1}{2\zeta^2} + i\sqrt{\pi}\,\zeta e^{-\zeta^2}. \tag{3.10}$$

The solutions of $D_\alpha(\omega/|k|) = 0$ are

$$\omega \approx \pm\sqrt{\frac{\alpha}{2}}\,|k| - i\gamma_L, \quad \gamma_L = \sqrt{\pi}\,\frac{\alpha^2}{4}\,e^{-\alpha/2}|k|. \tag{3.11}$$

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Two textbook examples of Landau-damped waves in this regime are ion acoustic waves at $\beta_i \ll 1$, $T_i \ll T_e$ (cold ions), for which $\alpha = ZT_e/T_i$, and long-wavelength Langmuir waves, for which $\alpha = 2/k^2\lambda_D^2$ (Landau 1946).

In the integral in equation (3.5), the poles are again very close to the real line and so in the integrand, we may approximate, in the vicinity of one of the two solutions (3.11)

$$Z(\zeta) \approx \mp\sqrt{\frac{2}{\alpha}}, \quad D_\alpha(\zeta) \approx \pm\left(\frac{2}{\alpha}\right)^{3/2}\left(\zeta \mp \sqrt{\frac{\alpha}{2}} + i\,\frac{\gamma_L}{|k|}\right). \tag{3.12}$$

Using again Plemelj's formula and noting that equal contributions arise from each of the two roots, we find

$$f(\alpha) \approx 2\sqrt{\pi}\,e^{\alpha/2} = \frac{\pi\alpha^2|k|}{2\gamma_L} \quad \Rightarrow \quad \langle|\varphi_k|^2\rangle \approx \frac{\alpha^2\varepsilon_k}{4\gamma_L}, \tag{3.13}$$

which is the same as equation (3.6) with $\gamma_{\text{eff}} = 2\gamma_L$.

Despite the apparently discordant factor of 2, this, in fact, is again consistent with a non-kinetic, textbook FDR. However, since we are considering a system with a large frequency, the relevant mechanical analogy is not equation (1.1), but the equally standard (and more general) equation for a forced and damped oscillator:

$$\ddot{\varphi} + \gamma\dot{\varphi} + \omega^2\varphi = \dot{\chi}, \tag{3.14}$$

where overdots mean time derivatives. At the risk of outraging a mathematically fastidious reader, we continue to consider χ a Gaussian white noise satisfying $\langle\chi(t)\chi(t')\rangle = \varepsilon\delta(t-t')$. For $\omega = 0$, equation (3.14) then precisely reduces to equation (1.1). For $\omega \neq 0$, it is not hard to show (by Fourier transforming in time, solving, then inverse Fourier transforming and squaring the amplitude) that the stationary mean square amplitude $\langle\varphi^2\rangle$ for equation (3.14) still satisfies equation (1.3). However, the relationship between the actual linear damping rate γ_L of φ and the parameter γ depends on the frequency: $\gamma_L = \gamma$ when $\omega \ll \gamma$ and $\gamma_L = \gamma/2$ when $\omega \geqslant \gamma/2$. In the latter case, which is the one with which we are preoccupied here, (1.3) becomes, in terms of γ_L:

$$\langle\varphi^2\rangle = \frac{\varepsilon}{4\gamma_L}. \tag{3.15}$$

The required extra factor of 2 is manifest.†

4. Velocity-space structure

The kinetic FDR derived in the previous section was concerned with the rate of removal of free energy from the density moment of the perturbed distribution function. This free energy flows into higher moments, i.e. is "phase mixed" away. In this section, we diagnose the velocity-space structure of the fluctuations and extend the FDR to compute their amplitude.

4.1. *Kinetic equation in Hermite space*

The emergence of ever finer velocity-space scales is made explicit by recasting the kinetic equation (2.1) in Hermite space, a popular approach for many years (Armstrong 1967; Grant and Feix 1967; Hammett et al. 1993; Parker and Carati

† As in Sec. 3.1, this very simple mechanical analogy also breaks down for a different choice of forcing; see Appendix A (equation (A 15)).

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1995; Ng et al. 1999; Watanabe and Sugama 2004; Zocco and Schekochihin 2011; Hatch et al. 2013; Loureiro et al. 2013; Plunk and Parker 2014). The distribution is decomposed into Hermite moments as follows

$$g(v) = \sum_{m=0}^{\infty} \frac{H_m(v) F_0}{\sqrt{2^m m!}} g_m, \quad g_m = \int \mathrm{d}v \, \frac{H_m(v)}{\sqrt{2^m m!}} \, g(v), \tag{4.1}$$

where $H_m(v)$ is the Hermite polynomial of order m. In terms of Hermite moments, equation (2.2) becomes

$$\varphi = \alpha g_0, \tag{4.2}$$

while equation (2.1) turns into a set of equations for the Hermite moments g_m, where phase mixing is manifested by the coupling of higher-m moments to the lower-m ones:

$$\frac{\partial g_0}{\partial t} + \frac{\partial}{\partial z} \frac{g_1}{\sqrt{2}} = \chi, \tag{4.3}$$

$$\frac{\partial g_1}{\partial t} + \frac{\partial}{\partial z} \left(g_2 + \frac{1+\alpha}{\sqrt{2}} \, g_0 \right) = 0, \tag{4.4}$$

$$\frac{\partial g_m}{\partial t} + \frac{\partial}{\partial z} \left(\sqrt{\frac{m+1}{2}} \, g_{m+1} + \sqrt{\frac{m}{2}} \, g_{m-1} \right) = -\nu m g_m, \quad m \geqslant 2, \tag{4.5}$$

where ν is the collision frequency and we have used the Lenard and Bernstein (1958) collision operator, a natural modelling choice in this context because its eigenfunctions are Hermite polynomials.

The free energy (2.6) in these terms is

$$W = \frac{1+\alpha}{2} \langle g_0^2 \rangle + \frac{1}{2} \sum_{m=1}^{\infty} \langle g_m^2 \rangle \tag{4.6}$$

and satisfies

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{1+\alpha}{2} \, \varepsilon - \nu \sum_{m=2}^{\infty} m \langle g_m^2 \rangle. \tag{4.7}$$

4.2. *FDR in Hermite space*

It is an obvious generalisation of the FDR to seek a relationship between the fluctuation level in the m-th Hermite moment, $\langle |g_m|^2 \rangle$ (the "Hermite spectrum"), and the injected power ε. This can be done in exactly the same manner as the kinetic FDR was derived in Sec. 3. Hermite-transforming equation (3.1) gives

$$g_{m,k\omega} = -\frac{i \chi_{k\omega}}{|k|} \frac{1+\alpha}{\alpha} \frac{(-\operatorname{sgn} k)^m}{\sqrt{2^m m!}} \frac{Z^{(m)}(\omega/|k|)}{D_\alpha(\omega/|k|)}, \quad m \geqslant 1, \tag{4.8}$$

where we have used

$$Z^{(m)}(\zeta) \equiv \frac{\mathrm{d}^m Z}{\mathrm{d}\zeta^m} = (-1)^m \int \mathrm{d}v \, \frac{H_m(v) F_0(v)}{v - \zeta} \tag{4.9}$$

and $Z^{(m)}(\omega/k) = (\operatorname{sgn} k)^{m+1} Z^{(m)}(\omega/|k|)$. The mean square fluctuation level in the statistical steady state is then derived similarly to equation (3.5):

$$C_{m,k} \equiv \langle |g_{m,k}|^2 \rangle = \frac{\varepsilon_k}{2\pi |k|} \left(\frac{1+\alpha}{\alpha} \right)^2 \frac{1}{2^m m!} \int_{-\infty}^{+\infty} \mathrm{d}\zeta \left| \frac{Z^{(m)}(\zeta)}{D_\alpha(\zeta)} \right|^2, \quad m \geqslant 1. \tag{4.10}$$

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This is the extension of the kinetic FDR, equation (3.5), to the fluctuations of the perturbed distribution function. The "Hermite spectrum" $C_{m,k}$ characterises the distribution of free energy in phase space.

4.3. *Hermite spectrum*

It is interesting to derive the asymptotic form of this spectrum at $m \gg 1$. Using in equation (4.9) the asymptotic form of the Hermite polynomials at large m,

$$e^{-v^2/2} H_m(v) \approx \left(\frac{2m}{e}\right)^{m/2} \sqrt{2} \cos\left(v\sqrt{2m} - \pi m/2\right), \tag{4.11}$$

and remembering that the v integration is over the Landau contour (i.e. along the real line, circumnavigating the pole at $v = \zeta$ from below), we find

$$Z^{(m)}(\zeta) \approx i^{m+1} \sqrt{2\pi} \left(\frac{2m}{e}\right)^{m/2} e^{-\zeta^2/2 + i\zeta\sqrt{2m}}, \tag{4.12}$$

provided $\zeta \ll \sqrt{2m}$ (this result is obtained by expressing the cosine in equation (4.11) in terms of exponentials, completing the square in the exponential function appearing in the integral (4.9) and moving the integration contour to $v = \pm i\sqrt{2m}$; the dominant contribution comes from the Landau pole). Finally, in equation (4.10),

$$\frac{|Z^{(m)}(\zeta)|^2}{2^m m!} \approx \sqrt{\frac{2\pi}{m}}\, e^{-\zeta^2}, \tag{4.13}$$

and so the Hermite spectrum has a universal scaling at $m \gg 1$:

$$C_{m,k} \approx \left[\frac{\varepsilon_k}{\sqrt{2\pi}|k|} \left(\frac{1+\alpha}{\alpha}\right)^2 \int_{-\infty}^{+\infty} \frac{\mathrm{d}\zeta\, e^{-\zeta^2}}{|D_\alpha(\zeta)|^2}\right] \frac{1}{\sqrt{m}} = \frac{\varepsilon_k(1+\alpha)}{\sqrt{2}|k|} \frac{1}{\sqrt{m}}. \tag{4.14}$$

The universal $1/\sqrt{m}$ scaling was derived in a different way by Zocco and Schekochihin (2011) (see Sec. 4.4; cf. Watanabe and Sugama 2004; Hatch et al. 2013). The integral in (4.14) was evaluated using the Kramers–Kronig relations (Kronig 1926; Kramers 1927) for the function $h(\zeta) = 1/D_\alpha(\zeta) - \alpha$ (which is analytic in the upper half plane and decays at least as fast as $1/|\zeta|^2$ at large ζ):

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}\zeta\, e^{-\zeta^2}}{|D_\alpha(\zeta)|^2} = -\sqrt{\pi}\left[\frac{1}{\pi}\,\mathscr{P}\int_{-\infty}^{+\infty} \frac{\mathrm{d}\zeta\, \mathrm{Im}\, h(\zeta)}{\zeta - \zeta'}\right]_{\zeta'=0} = -\sqrt{\pi}\,\mathrm{Re}\, h(0) = \frac{\alpha^2}{1+\alpha}\sqrt{\pi}. \tag{4.15}$$

Note that in the limit of high frequency ($\alpha \gg 1$, Sec. 3.2), the approximation (4.12) requires $\omega_L/|k| \ll \sqrt{2m}$, or $\alpha \ll 4m$, but there is also a meaningful intermediate range of m for which $1 \leqslant m \ll \alpha/4$. In this range, we can approximate $Z(\zeta) \approx -1/\zeta$ and, since $\zeta \approx \pm\sqrt{\alpha/2}$, we have in equation (4.10):

$$\frac{|Z^{(m)}(\zeta)|^2}{2^m m!} \approx \frac{2m!}{\alpha^{m+1}} \quad \Rightarrow \quad C_{m,k} \approx \frac{\varepsilon_k}{\sqrt{\pi}|k|} \frac{m!}{\alpha^m} e^{\alpha/2}. \tag{4.16}$$

This spectrum decays with m up to $m \sim \alpha$, where it transitions into the universal spectrum (4.14)

4.4. *Free-energy flux, the effect of collisions and the FDR for the total free energy*

It could hardly have escaped a perceptive reader's notice that the total free energy in our system, with its $1/\sqrt{m}$ Hermite spectrum, is divergent. The regularisation in Hermite space (removal of fine velocity-space scales) is provided by collisions. If ν

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is infinitesimal, these are irrelevant at finite m, but eventually become important as $m \to \infty$. To take account of their effect and to understand the free-energy flow in Hermite space, we consider equation (4.5), which it is convenient to Fourier transform in z and rewrite in terms of $\tilde{g}_{m,k} \equiv (i \operatorname{sgn} k)^m g_{m,k}$:

$$\frac{\partial \tilde{g}_{m,k}}{\partial t} + \frac{|k|}{\sqrt{2}} \left(\sqrt{m+1}\, \tilde{g}_{m+1,k} - \sqrt{m}\, \tilde{g}_{m-1,k} \right) = -\nu m \tilde{g}_{m,k}. \tag{4.17}$$

The Hermite spectrum $C_{m,k} = \langle |g_{m,k}|^2 \rangle = \langle |\tilde{g}_{m,k}|^2 \rangle$ therefore satisfies

$$\frac{\partial C_{m,k}}{\partial t} + \Gamma_{m+1/2,k} - \Gamma_{m-1/2,k} = -2\nu m C_{m,k}, \tag{4.18}$$

where $\Gamma_{m+1/2,k} = |k| \sqrt{2(m+1)} \mathrm{Re} \langle \tilde{g}_{m+1,k} \tilde{g}^*_{m,k} \rangle$ is the free-energy flux in Hermite space. If we make an assumption (verified in Sec. 4.5) that for $m \gg 1$ the Hermite moments $\tilde{g}_{m,k}$ are continuous in m, i.e. $\tilde{g}_{m+1,k} \approx \tilde{g}_{m,k}$, then

$$\Gamma_{m+1/2,k} \approx |k| \sqrt{2(m+1)}\, C_{m+1,k} \tag{4.19}$$

and equation (4.18) turns into a closed evolution equation for the Hermite spectrum (Zocco and Schekochihin 2011):

$$\frac{\partial C_{m,k}}{\partial t} + |k| \frac{\partial}{\partial m} \sqrt{2m}\, C_{m,k} = -2\nu m C_{m,k}. \tag{4.20}$$

The universal $C_{m,k} \propto 1/\sqrt{m}$ spectrum derived in Sec. 4.3 is now very obviously a constant-flux spectrum, reflecting steady pumping of free energy towards higher m's (phase mixing). The full steady-state solution of equation (4.20) including the collisional cutoff is

$$C_{m,k} = \frac{A_k}{\sqrt{m}} \exp\left(-\frac{2\sqrt{2}}{3} \frac{\nu}{|k|} m^{3/2} \right), \tag{4.21}$$

where A_k is an integration constant, which must be determined by matching this high-m solution with the Hermite spectrum at low m. This we are now in a position to do: for $1 \ll m \ll (\nu/|k|)^{-2/3}$, $C_{m,k} \approx A_k/\sqrt{m}$ and comparison with equation (4.14) shows that the constant A_k is the same as the constant $A_k(\alpha)$ in that equation. Thus, equation (4.21) with A_k given by equation (4.14) provides a uniformly valid expression for the Hermite-space spectrum, including the collisional cutoff (modulo the Hermite-space continuity assumption (4.19), which we will justify in Sec. 4.5).

As a check of consistency of our treatment, let us calculate the collisional dissipation rate of the free energy. This is the second term on the right-hand side of equation (4.7). Since $C_{m,k} \propto 1/\sqrt{m}$ before the collisional cutoff is reached, the sum over m will be dominated by $m \sim (\nu/|k|)^{-2/3}$ and can be approximated by an integral:

$$\nu \sum_{m,k} m C_{m,k} \approx \sum_k \nu \int_0^\infty \mathrm{d}m\, m C_{m,k} = \sum_k \frac{A_k |k|}{\sqrt{2}}. \tag{4.22}$$

On the other hand, in steady state, equation (4.7) implies

$$\nu \sum_{m,k} m C_{m,k} = \frac{1+\alpha}{2}\, \varepsilon. \tag{4.23}$$

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If energy injection is into a single k mode, $\varepsilon = \varepsilon_k$, comparing these two expressions implies

$$A_k = \frac{\varepsilon_k(1+\alpha)}{\sqrt{2}|k|}, \tag{4.24}$$

which, of course, is consistent with equation (4.14).

Finally, we use equation (4.21) to calculate (approximately) the total steady-state amount of free energy across the phase space:

$$\frac{1}{2}\sum_{m=1}^{\infty} C_{m,k} = \frac{\Gamma(1/3)}{2\cdot 3^{2/3}}\frac{1+\alpha}{\nu^{1/3}|k|^{2/3}}\,\varepsilon_k \tag{4.25}$$

(we have again approximated the sum with an integral, assumed energy injection into a single k and used equation (4.24)). Equation (4.25) can be thought of as the FDR for the total free energy. The fact that this diverges as $\nu \to 0$ underscores the principle that the "true" dissipation (in the sense of free energy being thermalised) is always collisional—a consequence of Boltzmann's H theorem.

4.5. *Continuity in Hermite space*

In this section, we make a somewhat lengthy formal digression to justify the assumption of continuity of Hermite moments in m at large m, which we need for the approximation (4.19). The formalism required for this will have some interesting features which are useful in framing one's thinking about energy flows in Hermite space, but a reader impatient with such exercises may skip to Sec. 4.6.

Returning to equation (4.17) and considering $1 \ll m \ll (\nu/|k|)^{-2}$, we find that to lowest approximation, the $\sqrt{m}$ terms are dominant and must balance, giving $\tilde{g}_{m+1,k} \approx \tilde{g}_{m-1,k}$. This is consistent with continuity in m, viz., $\tilde{g}_{m+1,k} \approx \tilde{g}_{m,k}$, but there is also a solution allowing the consecutive Hermite moments to alternate sign: $\tilde{g}_{m+1,k} \approx -\tilde{g}_{m,k}$. Thus, there are, formally speaking, two solutions: one for which $\tilde{g}_{m,k}$ is continuous and one for which $(-1)^m\tilde{g}_{m,k}$ is. To take into account both of them, we introduce the following decomposition (Schekochihin et al. 2014):

$$\tilde{g}_{m,k} = \tilde{g}^{+}_{m,k} + (-1)^m\tilde{g}^{-}_{m,k}, \tag{4.26}$$

where the "+" ("continuous") and the "−" ("alternating") modes are

$$\tilde{g}^{+}_{m,k} = \frac{\tilde{g}_{m,k}+\tilde{g}_{m+1,k}}{2}, \quad \tilde{g}^{-}_{m,k} = (-1)^m\frac{\tilde{g}_{m,k}-\tilde{g}_{m+1,k}}{2}. \tag{4.27}$$

The Hermite spectrum and the flux of the free energy can be expressed in terms of the spectra of these modes as follows:

$$C_{m,k} \equiv \langle|\tilde{g}_{m,k}|^2\rangle = C^{+}_{m,k} + C^{-}_{m,k}, \tag{4.28}$$

$$\Gamma_{m+1/2,k} \equiv |k|\sqrt{2(m+1)}\,\mathrm{Re}\langle\tilde{g}_{m+1,k}\tilde{g}^{*}_{m,k}\rangle \approx |k|\sqrt{2m}\left(C^{+}_{m,k} - C^{-}_{m,k}\right), \tag{4.29}$$

where $C^{\pm}_{m,k} \equiv \langle|\tilde{g}^{\pm}_{m,k}|^2\rangle$ and the last expression in equation (4.29) is an approximation valid for $m \gg 1$.

The functions $\tilde{g}^{\pm}_{m,k}$ can both be safely treated as continuous in m for $m \gg 1$. Treating them so in (4.17) and working to lowest order in $1/m$, we find that they satisfy the following *decoupled* evolution equations:

$$\frac{\partial\tilde{g}^{\pm}_{m,k}}{\partial t} \pm \sqrt{2}|k|m^{1/4}\frac{\partial}{\partial m}m^{1/4}\tilde{g}^{\pm}_{m,k} = -\nu m\tilde{g}^{\pm}_{m,k}, \tag{4.30}$$

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or, for their spectra,

$$\frac{\partial C^{\pm}_{m,k}}{\partial t} \pm |k| \frac{\partial}{\partial m} \sqrt{2m}\, C^{\pm}_{m,k} = -2\nu m C^{\pm}_{m,k}. \tag{4.31}$$

Manifestly, the "+" mode propagates from lower to higher m and the "−" mode from higher to lower m—they are the "phase-mixing" and the "un-phase-mixing" collisionless solutions, respectively.†

Taking the collisional term into account and noting that energy is injected into the system at low, rather than high, m, the solution satisfying the boundary condition $\tilde{g}_{m,k} \to 0$ as $m \to \infty$ has $\tilde{g}^{-}_{m,k} = 0$ and so $\tilde{g}_{m,k} = \tilde{g}^{+}_{m,k}$. Thus, $\tilde{g}_{m,k}$ is continuous in m. With $C^{-}_{m,k} = 0$, (4.29) is the same as our earlier approximation (4.19) (to lowest order in the $m \gg 1$ expansion).

As $\tilde{g}^{+}_{m,k}$ and $\tilde{g}^{-}_{m,k}$ are decoupled at large m, if we start with a $\tilde{g}^{-}_{m,k} = 0$ solution, no $\tilde{g}^{-}_{m,k}$ will be produced. However, both the decoupling property and the interpretation of $\tilde{g}^{\pm}_{m,k}$ as the phase-mixing and un-phase-mixing modes are only valid to lowest order in $1/m$. It is useful to know how well this approximation holds.

Let us use equation (4.8) to calculate (in the collisionless limit)

$$R_{m+1} \equiv \frac{\tilde{g}_{m+1,k\omega}}{\tilde{g}_{m,k\omega}} = i \operatorname{sgn} k \, \frac{g_{m+1,k\omega}}{g_{m,k\omega}} = -\frac{i}{\sqrt{2(m+1)}} \frac{Z^{(m+1)}(\zeta)}{Z^{(m)}(\zeta)}. \tag{4.32}$$

Taking $m \gg 1$, $\zeta^2/4$ and using equation (4.12), we find‡

$$R_{m+1} = 1 + \frac{i\zeta}{\sqrt{2m}} - \frac{1}{4m} + O\left(\frac{1}{m^{3/2}}\right). \tag{4.33}$$

Therefore, to lowest order in $1/\sqrt{m}$,

$$\tilde{g}^{-}_{m,k\omega} = (-1)^m \tilde{g}_{m,k\omega} \frac{1 - R_{m+1}}{2} \approx (-1)^{m+1} \frac{i\zeta}{2\sqrt{2m}} \tilde{g}_{m,k\omega}. \tag{4.34}$$

Following the same steps as those that led to equation (4.14)¶, we get

$$C^{-}_{m,k} \approx \left[\frac{\varepsilon_k}{8\sqrt{2\pi}|k|} \left(\frac{1+\alpha}{\alpha}\right)^2 \int_{-\infty}^{+\infty} \frac{\mathrm{d}\zeta\, \zeta^2 e^{-\zeta^2}}{|D_\alpha(\zeta)|^2}\right] \frac{1}{m^{3/2}} = \frac{\varepsilon_k (1+\alpha)^2}{16\sqrt{2}|k|} \frac{1}{m^{3/2}}, \tag{4.35}$$

so both the energy (~ 1, while the total is $\sim \nu^{-1/3}$; see equation (4.25)) and the dissipation ($\sim \nu \sum_m m C^{-}_{m,k} \sim \nu^{2/3}$) associated with the "−" modes are small.

† The existence of un-phase-mixing solutions has been known for a long time: e.g. Hammett et al. (1993) treated them as forward and backward propagating waves in a mechanical analogy of (4.17) with a row of masses connected by springs. The un-phase-mixing solutions are also what allows the phenomenon of plasma echo (Gould et al. 1967), including in stochastic nonlinear systems (Schekochihin et al. 2014).

‡ The same lowest-order expression can be found by Fourier-transforming (4.17) in time, ignoring collisions, writing $R_{m+1} = R_m^{-1}\sqrt{m/(m+1)} + i\zeta\sqrt{2/(m+1)}$, approximating $R_m \approx R_{m+1}$, solving the resulting quadratic equation for R_{m+1}, expanding in powers of $1/\sqrt{m}$ and choosing the solution for which $R_{m+1} = 1$ to lowest order. This last step is the main difference between the two methods: if we work with (4.17) in the manner just described, we have to make an explicit choice between the continuous and alternating solutions ($R_{m+1} = 1$ and $R_{m+1} = -1$); on the other hand, (4.8) already contains the choice of the former (which is ultimately traceable to Landau's prescription guaranteeing damping rather than growth of the perturbations).

¶ The integral is again calculated via Kramers–Kronig relations, this time for the function $h(\zeta) = \zeta^2/D_\alpha(\zeta) - \alpha\zeta^2 - \alpha^2/2$, so $\int_{-\infty}^{+\infty} \mathrm{d}\zeta\, \zeta^2 e^{-\zeta^2}/|D_\alpha(\zeta)|^2 = \alpha^2\sqrt{\pi}/2$.

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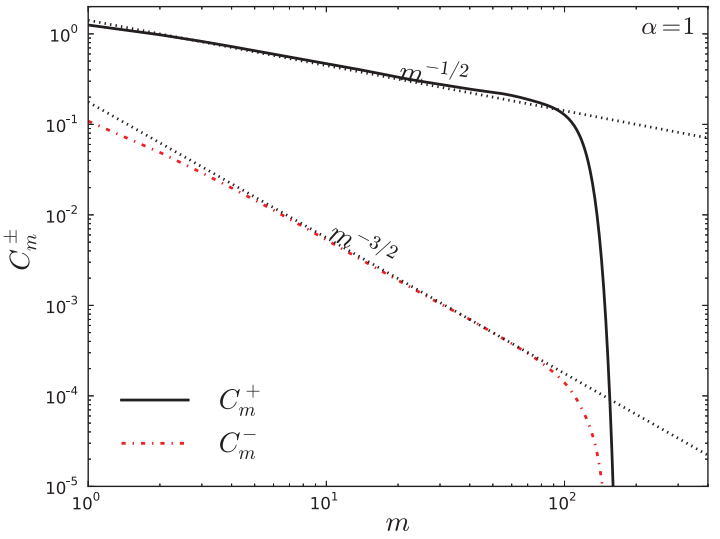


FIGURE 3. The free-energy spectra $C_m^{\pm}$ obtained via direct numerical solution of (4.3)–(4.5) with $\alpha = 1.0$ followed by decomposing the solution according to (4.27). In the code, rather than using the Lenard–Bernstein collision operator (as per (4.5)), hypercollisional regularisation (Loureiro et al. 2013), $-\nu m^6 g_{m,k}$, was used to maximise the utility of the velocity-space resolution, hence the very sharp cut off. The dotted lines show the collisionless approximation: (4.14) for $C_{m,k}^+$ (the phase-mixing "+" mode predominates, so $C_{m,k} \approx C_{m,k}^+$) and (4.35) for $C_{m,k}^-$.

The steady-state spectra $C_{m,k}^{\pm}$ obtained via direct numerical solution of equations (2.1) and (2.2) are shown in Fig. 3, where they are also compared with the analytical expressions (4.14) and (4.35).

Note that we could have, without further ado, simply taken equation (4.33) to be the proof of continuity in Hermite space. We have chosen to argue this point via the decomposition (4.26) because it provided us with a more intuitive understanding of the connection between this continuity and the direction of the free-energy flow (phase mixing rather than un-phase mixing).

4.6. *The simplest Landau-fluid closure*

Simplistically described, the idea of Landau-fluid closures is to truncate the Hermite hierarchy of (4.3)–(4.5) at some finite m and to replace in the last retained equation

$$g_{m+1,k}(t) = -(i \operatorname{sgn} k) R_{m+1} g_{m,k}(t), \tag{4.36}$$

where R_{m+1}, which in general depends on the complex frequency ζ (equation (4.32)), is approximated by some suitable frequency-independent expression leading to the correct recovery of the linear physics from the truncated system. A considerable level of sophistication has been achieved in making these choices and we are not proposing to improve on the existing literature (Hammett and Perkins 1990; Hammett et al. 1992; Hedrick and Leboeuf 1992; Dorland and Hammett 1993; Snyder et al. 1997; Passot and Sulem 2004; Goswami et al. 2005; Passot and Sulem 2007). It is, however, useful, in the context of the result of Sec. 3.1 that the "fluid" version of FDR is recovered in the limit of low frequency and weak damping, to show how the same conclusion can be arrived at via what is probably the simplest possible Landau-fluid closure.

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In the limit $\zeta \to 0$, the ratio R_{m+1}, given by equation (4.32), becomes independent of ζ and so a closure in the form (4.36) becomes a rigorous approximation. It is not hard to show that

$$Z^{(m)}(0) = \frac{i^{m+1}\sqrt{\pi}\, m!}{\Gamma(m/2+1)}. \tag{4.37}$$

Therefore, for $\zeta \ll 1$ and $m \geqslant 1$,†

$$R_{m+1} = \frac{m}{\sqrt{2(m+1)}} \frac{\Gamma(m/2)}{\Gamma((m+1)/2)}. \tag{4.38}$$

If we wish to truncate at $m = 1$, then $R_2 = \sqrt{\pi}/2$, and so in equation (4.4),

$$g_{2,k} = -i \operatorname{sgn} k \frac{\sqrt{\pi}}{2} g_{1,k}. \tag{4.39}$$

On the basis of equation (4.3), we must order $g_{1,k} \sim O(\zeta) g_{0,k}$. Thefore, $\partial g_{1,k}/\partial t \sim O(\zeta^2) g_{0,k}$ must be neglected in equation (4.4), from which we then learn that

$$g_{1,k} \approx -i \operatorname{sgn} k \sqrt{\frac{2}{\pi}} (1+\alpha)\, g_{0,k}. \tag{4.40}$$

Finally, substituting this into equation (4.3), we get

$$\frac{\partial g_{0,k}}{\partial t} + \frac{1+\alpha}{\sqrt{\pi}} |k| g_{0,k} = \chi_k. \tag{4.41}$$

This is a Langevin equation (1.1) with a damping rate that is precisely the Landau damping rate γ_L in the limit $1 + \alpha \ll 1$ (and so $\zeta \ll 1$), given by (3.8). In this limit, $\varphi = -g_0$ (equation (4.2), $\alpha \approx -1$) and we recover the standard "fluid" FDR (equation (3.9)). As we discussed in Sec. 2, a useful application of this regime is to compressive fluctuations in high-beta plasmas: in this case $1 + \alpha \approx 1/\beta_i \ll 1$ and the damping is the Barnes (1966) damping, well known in space and astrophysical contexts (Foote and Kulsrud 1979; Lithwick and Goldreich 2001; Schekochihin et al. 2009).

5. Conclusion

We have provided what in our view is a reasonably complete treatment of the simplest generalisation of the Langevin problem to plasma kinetic systems.‡ Let us itemise the main results and conclusions.

† The same result can be obtained by inferring $R_{m+1} \approx R_m^{-1}\sqrt{m/(m+1)}$ from equation (4.17) (provided $m \ll 1/\zeta^2$), then iterating this up to some Hermite number M such that $1 \ll M \ll 1/\zeta^2$, and approximating $R_M \approx 1$ (equation (4.33)). The condition $m, M \ll 1/\zeta^2$ is necessary so that the ζ terms in R_{m+1} are not just small compared to unity but also compared to the next-order $1/m$ terms (see equation (4.33)).

‡ While we have focused on the simplest Langevin problem, in which the source term is a white noise, there is an obvious route towards generalising this by considering source terms with more coherent time dependence (longer correlation times, prescribed frequency spectra; cf. Plunk 2013). One such calculation was recently undertaken by Plunk and Parker (2014), who considered a coherent oscillating source and found that when the frequency of the source is large, the amount of energy that can be absorbed by the kinetic system is exponentially small (which makes sense). Another straightforward generalisation (or variation) of our model (as treated in the main text of this paper) is energy injection into momentum, rather than density fluctuations—which can be interpreted as forcing by an externally imposed random electric field. Whereas some of the more literal parallels with the Langevin problem are lost in this case, the results are fundamentally the same (Appendix A).

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• Equation (3.5) is the FDR for the kinetic system (equations (2.1) and (2.2)), expressing the relationship between the fluctuation level $\langle|\varphi_k|^2\rangle$ and the injected power. This can be expressed in terms of an "effective" damping rate γ_{eff} in a way that resembles the standard "fluid" version of the FDR (equation (3.6)), but γ_{eff} is not in general equal to the Landau damping rate γ_L. We stress that this result is not a statement of any kind of surprising "modification" of Landau damping in a system with a random source, but rather a clarification of what the linear response in the statistical steady state of such a system actually is. The system, in general, is not mathematically equivalent to the Langevin equation (1.1) and so the FDR for it need not have the same form.

• In the limit of zero real frequency and weak Landau damping, the effective and the Landau damping rates do coincide (equation (3.9)). Another way to view this result is by noting that this is a regime in which the simplest possible Landau-fluid closure becomes a rigorous approximation and the evolution equation for the electrostatic potential can be written as a Langevin equation with the Landau damping rate γ_L (equation (4.41)). Note, however, that this direct reduction to the simplest Langevin equation (1.1) is not a universal property: it breaks down with a different choice of forcing (Appendix A).

• Another limit in which the FDR for the kinetic system can be interpreted in "fluid" (in fact, mechanical) terms is one of high real frequency and exponentially Landau small damping, although the correct analogy is not the Langevin equation but a forced-damped oscillator (Sec. 3.2; this analogy, however, ceases to hold in such a simple form for a different choice of forcing, as shown in Appendix A).

• The damping of the perturbations of φ (which are linearly proportional to the density perturbations) occurs via phase mixing, which transfers the free energy originally injected into φ away from it and into higher moments of the perturbed distribution function. This process can be described as a free-energy flow in Hermite space. The generalisation of the FDR to higher-order Hermite moments takes the form of an expression for the Hermite spectrum $C_{m,k}$ (equation (4.10)), which at high Hermite numbers $m \gg 1$ has a universal scaling $C_{m,k} \propto 1/\sqrt{m}$ (equation (4.14)). This scaling corresponds to a constant free-energy flux from low to high m (equation (4.19)). Analysis of the solutions of the kinetic equation making use of a formal decomposition of these solutions into phase-mixing and un-phase-mixing modes underscores the predominance of the former (Sec. 4.5).

• A solution for the Hermite spectrum including the collisional cutoff is derived (equation (4.21)). The FDR for the total free energy stored in the phase space (equation (4.25)) shows that it diverges $\propto \nu^{-1/3}$ in the limit of vanishing collisionality ν, a result that underscores the fact that ultimately all dissipation (i.e. all entropy production in the system) is collisional.

In the process of deriving all these results, we have made an effort to explain the simple connections between the Landau formalism (solutions of the kinetic equation expressed via the plasma dispersion function) and the Hermite-space one. This material and, indeed, most of the results described above, perhaps belong to elementary textbooks, but we are not aware of any where they are adequately explained—although implicitly they underlie the thinking behind both Landau-fluid closures (Hammett and Perkins 1990; Hammett et al. 1992; Hedrick and Leboeuf 1992; Dorland and Hammett 1993; Snyder et al. 1997; Passot and Sulem 2004; Goswami et al. 2005; Passot and Sulem 2007) and Hermite-space treatments for plasma kinetics (Armstrong 1967; Grant and Feix 1967; Hammett et al. 1993;

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Parker and Carati 1995; Ng et al. 1999; Watanabe and Sugama 2004; Zocco and Schekochihin 2011; Loureiro et al. 2013; Hatch et al. 2013; Plunk and Parker 2014).

Besides (we hope) providing a degree of clarity on an old topic in the linear theory of collisionless plasmas, our findings lay the groundwork for a study of the much more complicated nonlinear problem of the role of Landau damping and phase mixing in turbulent collisionless plasma systems (Kanekar et al. 2014; Schekochihin et al. 2014).

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Appendix A. Momentum forcing

The source term in (2.1), providing direct forcing of density perturbations, was a choice of convenience: it allowed us to compare directly the FDR for the potential field φ in a kinetic system with the FDR for the Langevin equation (1.1). If, instead, one strives for a form of energy injection with a more transparent physical interpretation, it is natural to imagine it coming from a fluctuating electric field. This changes equation (2.1) to the following:

$$\frac{\partial g}{\partial t} + v\,\frac{\partial g}{\partial z} + vF_0\frac{\partial \varphi}{\partial z} = \chi_1(t) v F_0 + C[g], \qquad \text{(A 1)}$$
$$\langle \chi_1(t)\chi_1(t')\rangle = \varepsilon\delta(t-t'),$$

where $\chi_1(t)$ is the fluctuating parallel electric field, which we model (again, for analytical convenience) as a Gaussian white noise.

The new forcing injects fluctuations of momentum, rather than density. Indeed, in terms of Hermite moments, instead of equations (4.3) and (4.4), we now have

$$\frac{\partial g_0}{\partial t} + \frac{\partial}{\partial z}\frac{g_1}{\sqrt{2}} = 0, \qquad \text{(A 2)}$$

$$\frac{\partial g_1}{\partial t} + \frac{\partial}{\partial z}\left(g_2 + \frac{1+\alpha}{\sqrt{2}}\,g_0\right) = \frac{\chi_1}{\sqrt{2}}, \qquad \text{(A 3)}$$

and equation (4.5) is unchanged. The field that is directly forced is $g_1 = \sqrt{2}\int \mathrm{d}v\, v g(v)$, which is proportional to the mean velocity associated with the perturbed distribution

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g. The new free-energy equation, an analogue of equations (2.7) and (4.7), is

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\varepsilon}{4} + \int \mathrm{d}v\, \frac{\langle gC[g]\rangle}{F_0} = \frac{\varepsilon}{4} - \nu \sum_{m=2}^{\infty} m\langle g_m^2\rangle. \tag{A 4}$$

This immediately gives us the universal Hermite spectrum and the FDR for the total free energy: we repeat the calculation in Sec. 4.4 (which is unchanged because nothing has changed at high m's) using the steady-state version of equation (A 4) instead of equation (4.23) to get

$$A_k = \frac{\varepsilon_k}{2\sqrt{2}|k|} \tag{A 5}$$

in the expression (4.21) for the Hermite spectrum. Therefore,

$$\frac{1}{2}\sum_{m=1}^{\infty} C_{m,k} = \frac{\Gamma(1/3)}{4\cdot 3^{2/3}} \frac{1}{\nu^{1/3}|k|^{2/3}}\, \varepsilon_k \tag{A 6}$$

replaces equation (4.25) as the FDR for the total free energy. The only differences are in numerical prefactors and the α dependence, which has now disappeared. This is because in our previous forcing model, the source term injected energy into g_0 (density fluctuations), which got scaled by the factor of $1+\alpha$ when passed on to g_1 (see equation (4.4)), whereas in the case we are considering now, the energy is injected directly into g_1, which is then phase mixed to higher m's, without ever encountering any α dependence.

Let us also give here the results one obtains in the collisionless limit by backtracking to equation (A 1) and solving for g explicitly, as we did in Secs. 3 and 4.2:

$$g_{k\omega} = -\left(\varphi_{k\omega} + \frac{i\chi_{1,k\omega}}{k}\right) \frac{vF_0}{v-\omega/k}. \tag{A 7}$$

This gives

$$\varphi_{k\omega} = -\frac{i\chi_{1,k\omega}}{k} \frac{1+\zeta Z(\zeta)}{D_\alpha(\zeta)}, \tag{A 8}$$

$$g_{m,k\omega} = -\frac{i\chi_{1,k\omega}}{k} \frac{1}{\alpha} \frac{(-\mathrm{sgn}\, k)^m}{\sqrt{2^m m!}} \frac{\zeta Z^{(m)}(\zeta)}{D_\alpha(\zeta)}, \quad m \geqslant 1, \tag{A 9}$$

where $\zeta = \omega/|k|$ as usual. From the last formula, proceeding in the same manner as we did to get equation (4.14), we recover again the Hermite spectrum:

$$C_{m,k} = \frac{\varepsilon_k}{2\pi|k|} \frac{1}{\alpha^2} \frac{1}{2^m m!} \int_{-\infty}^{+\infty} \mathrm{d}\zeta \left|\frac{\zeta Z^{(m)}(\zeta)}{D_\alpha(\zeta)}\right|^2 \tag{A 10}$$

$$\approx \left[\frac{\varepsilon_k}{\sqrt{2\pi}|k|} \frac{1}{\alpha^2} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\zeta\, \zeta^2 e^{-\zeta^2}}{|D_\alpha(\zeta)|^2}\right] \frac{1}{\sqrt{m}} = \frac{\varepsilon_k}{2\sqrt{2}|k|} \frac{1}{\sqrt{m}}. \tag{A 11}$$

The latter expression was obtained in the limit of $m \gg 1$ (see Sec. 4.3) and is the same result as equation (A 5). The integral is already familiar from (4.35). For completeness, the "$-$"-mode spectrum (4.35) becomes

$$C^-_{m,k} \approx \left[\frac{\varepsilon_k}{8\sqrt{2\pi}|k|} \frac{1}{\alpha^2} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\zeta\, \zeta^4 e^{-\zeta^2}}{|D_\alpha(\zeta)|^2}\right] \frac{1}{m^{3/2}} = \frac{\varepsilon_k(3+\alpha)}{32\sqrt{2}|k|} \frac{1}{m^{3/2}}. \tag{A 12}$$

The integral was done by Kramers–Kroning relations for the function $h(\zeta) = \zeta^4/D_\alpha(\zeta) - \alpha\zeta^4 - \alpha^2\zeta^2/2 - \alpha^2(3+\alpha)/4$. While again numerical prefactors and α dependence are different, none of the substative arguments in Sec. 4.5 are affected.

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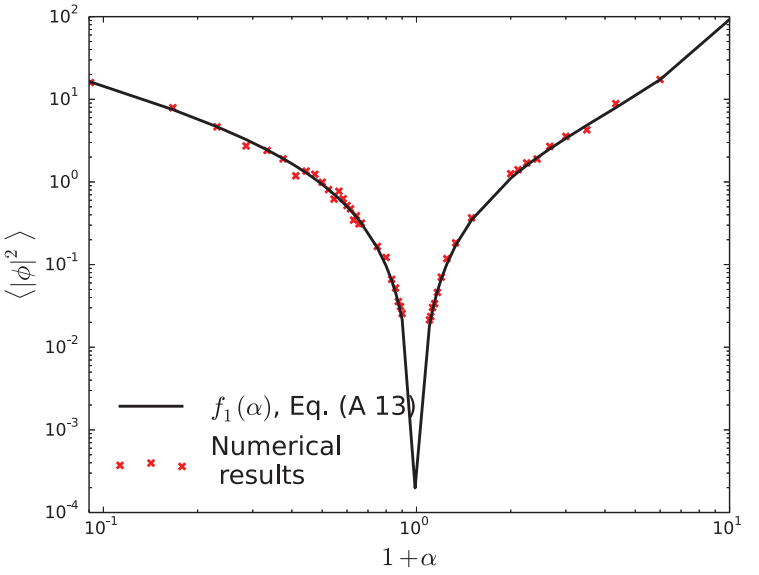


FIGURE 4. Normalised steady-state amplitude $2\pi|k|\langle|\varphi_k|^2\rangle/\varepsilon_k = f_1(\alpha)$ vs. $1+\alpha$ for the case of momentum forcing: the solid line is the analytical prediction $f_1(\alpha)$ (equation (A 13)), the crosses are computed from the long-time limit of $\langle|\varphi_k|^2\rangle$ obtained via direct numerical solution of equation (A 1).

Finally, from equation (A 8), proceeding in the same manner as in Sec. 3, we obtain the FDR relation for the mean square fluctuation amplitude of the potential:

$$\langle|\varphi_k|^2\rangle = \frac{\varepsilon_k}{2\pi|k|} f_1(\alpha), \quad f_1(\alpha) = \int_{-\infty}^{+\infty} \mathrm{d}\zeta \left|\frac{1+\zeta Z(\zeta)}{D_\alpha(\zeta)}\right|^2, \tag{A 13}$$

which is the new version of equation (3.5). The function $f_1(\alpha)$ is plotted in Fig. 4, along with the results of the direct numerical solution of equation (A 1). While formally it is a different function than $f(\alpha)$, it exhibits very similar behaviour (cf. Fig. 1). Its asymptotics are (cf. Secs. 3.1 and 3.2)

$$\alpha \to -1: \quad f_1(\alpha) \approx \frac{|k|}{\gamma_L} \quad \Rightarrow \quad \langle|\varphi_k|^2\rangle \approx \frac{\varepsilon_k}{2\pi\gamma_L}, \tag{A 14}$$

$$\alpha \to \infty: \quad f_1(\alpha) \approx \frac{\pi\alpha|k|}{4\gamma_L} \quad \Rightarrow \quad \langle|\varphi_k|^2\rangle \approx \frac{\alpha\varepsilon_k}{8\gamma_L}. \tag{A 15}$$

Whereas in both limits there is still an inverse relationship between the mean square fluctuation amplitude and the Landau damping rate γ_L, the numerical coefficients are not easily interpretable in terms of any simple "fluid" Langevin models for φ—not a surprising outcome as, already examining equations (A 2) and (A 3), we might have observed that they do not map on any obvious Langevin-like equation for $\varphi = \alpha g_0$. The elementary Landau-fluid closure that in Sec. 4.6 neatly mapped the $\alpha \to -1$ limit onto a "fluid" Langevin equation, when reworked for the case of the momentum forcing, gives

$$\frac{\partial\varphi_k}{\partial t} + \gamma_L\varphi_k = \frac{\operatorname{sgn} k}{\sqrt{\pi}}\chi_{1,k}. \tag{A 16}$$

Thus, a Langevin equation still, but with an order-unity adjusted noise term.

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