*Proc. Camb. Phil. Soc.* (1969), **66**, 31
PCPS 66–7
*Printed in Great Britain*



# Lipschitz classes on finite dimensional groups

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(*Received* 27 *May* 1968)

1. *Introduction.* In this paper $G$ will denote a compact metric Abelian group having finite topological dimension, $\dim G$. The discrete dual group of $G$ will be denoted by $\Gamma$. The metric on $G$ will be denoted by $d(x, y)$, or $|x\text{-}y|$ since translation-invariance is assumed.

The material of this paper is an extension of the author's earlier paper (1) and contains both the results of that paper, and the classical results of Bernstein, and Beurling and Pollard (see the references of (1)) as special cases. We denote by $\Lambda_\alpha(G)$ $(\alpha > 0)$ the class of all complex valued functions $f$ on $G$ for which a constant $C_f$ exists such that $|f(x)-f(x')| \leqslant C_f[d(x, x')]^\alpha$, where $x, x' \in G$. We denote by $A(G)$ the set of functions on $G$ such that $f(x) = \Sigma_\Gamma c_\gamma(x, \gamma)$ with $\Sigma_\Gamma|c_\gamma| < \infty$. The extended theorems can now be stated as follows.

THEOREM 1. *If* $\alpha > \frac{1}{2}(\dim G)$ *and* $f \in \Lambda_\alpha(G)$, *then* $f \in A(G)$.

THEOREM 2. *Let* $f \in A(G)$, *and* $E = \{x : x \in G, f(x) = 0\}$. *Then if there exists* $C_f$ *such that* $|f(x)| \leqslant C_f[d(x, E)]^\alpha$ *and* $\alpha > \frac{1}{2}(\dim G)$ *(in particular if* $f \in \Lambda_\alpha, \alpha > \frac{1}{2}(\dim G)$*), then* $f$ *satisfies spectral synthesis.*

It will be observed that Theorem 1 becomes trivial if $G = T^n$, the $n$-dimensional torus when $n \geqslant 2$, since the only functions in $\Lambda_\alpha$ with $\alpha > 1$ are constants. In this case we re-define $\Lambda_\alpha(T^n)$ with $\alpha = k + \beta$, $k$ integral, $0 < \beta \leqslant 1$, as the class of all functions on $T^n$ whose $k$th derivatives are in $\Lambda_\beta$. Then with this interpretation Theorem 1 is again both correct and non-vacuous—we give it as Theorem 3. Theorem 3 is a 'known result' in that its statement (and doubtless a proof) is known to have a number of harmonic analysts. However, no proof seems to have been published, and this leads the author to believe that the proof (which is relatively painless) is worth giving.

We shall use the following notation. $Q$ will denote the discrete group of rational numbers, and $Z$ the integers. $C$ will be used as an abbreviation for $Q/Z$ the group of all roots of unity, and $Z(p^\infty)$ for the group of all $p$th power roots of unity. The 1-dimensional solenoid, which is the compact dual of $Q$ will be denoted $\Sigma$. Its 0-dimensional subgroup, which is the compact dual of $Q/Z$ will be denoted by $\Delta$.

The method used for the proof of Theorem 1 requires the following two theorems, of which Theorem A is due to Pontryagin, and Theorem B is a well known result in Abelian group theory. Proofs of these results may be found in (2), Theorem (24·28) and Theorems (A. 14) and (A. 16) respectively.

THEOREM A. *Let* $G$ *be a compact metric Abelian group, with dual* $\Gamma$. *Then the topological dimension of* $G$ *is equal to the torsion-free rank of* $\Gamma$.

https://doi.org/10.1017/S0305004100044686 Published online by Cambridge University Press

THEOREM B. *Let $\Gamma$ be a discrete Abelian group with torsion-free rank $r_0$, and $p-$ rank $r_p$ for each prime $p$. Then $\Gamma$ can be embedded as a subgroup of a divisible group of the form $\Gamma_D = \prod_p{}^* Z(p^\infty)^{r_p} \times Q^{r_0}$, where all products are in the weak sense.*

2. *Preliminary results and special groups.*

LEMMA 1. *Let $G$ be a compact metric group with finite dimension. Then $G$ can be written in the form $G \simeq G_D/H$ where $G_D \simeq \Delta^\infty \times \Sigma^{\dim G}$, and $H$ is a 0-dimensional subgroup of $G_D$. (We use $\Delta^\infty$ for the complete product of a countably many copies of $\Delta$.)*

*Proof.* Since $G$ is metrisable, its dual $\Gamma$ is countable. Hence

$$\Gamma \subset \Gamma_D = \prod_p{}^* Z(p^\infty)^{r_p} \times Q^{r_0}$$

as in Theorem B, with $r_p \leqslant \aleph_0$, $p = 2, 3, 5, \ldots$, and $r_0 = \dim G$ by Theorem A.

Hence certainly $\Gamma \subset \prod_p{}^* Z(p^\infty)^{\aleph_0} \times Q^{r_0}$ and since $\prod_p{}^* Z(p^\infty) \simeq Q/Z = C$ we have $\Gamma \subset C^{\infty *} \times Q^{r_0}$ (where we use $C^{\infty *}$ for the weak sum of countably many copies of $C$).

It follows by duality that $G = \hat{\Gamma}$ is a quotient group of $\Delta^\infty \times \Sigma^{r_0}$, by a subgroup $H$ and since $(C^{*\infty} \times Q^{r_0})/\Gamma$ is a torsion group, $H$ is 0-dimensional.

LEMMA 2. *Suppose $G$ is a compact metric group, and $H$ is a closed subgroup.*
*Suppose also that $\Lambda_\alpha(G) \subset A(G)$ for $\alpha > \alpha_0$. Then $\Lambda_\alpha(G/H) \subset A(G/H)$ for $\alpha > \alpha_0$.*

*Proof.* Suppose $f \in \Lambda_\alpha(G/H)$. Then $f$ may be identified with a function $\phi$ on $G$, constant on the cosets of $H$. Also if $G/H$ is given the quotient metric $|x+H| = \inf(|y|, y \in x+H)$ then $\phi \in \Lambda_\alpha(G)$. Hence if $\alpha > \alpha_0$, $\phi \in A(G)$.

Suppose $K$ is the dual of $G/H$, $K$ is a subgroup of $\Gamma = \hat{G}$, $K = \{\gamma : \gamma \in \Gamma, (x, \gamma) = 1,$ all $x \in H\}$. Since $\phi$ is constant on the cosets of $H$, $\hat{\phi}$ will vanish off $K$, and $f$ will be the restriction of $\hat{\phi}$ to $K$. Hence $f \in A(G/H)$, as required.

We denote the elements of $Q$ by $m/n!$, where $m \in Z$, $n \geqslant 1$, and if $n > 1$, we assume $n$ does not divide $m$. If $m/n! \in Q/Z$, then in addition $m \geqslant 0$, $n > 1$ unless $m = 0$, and $m < n!$. The elements of $\Delta$ can be represented in the form $\sum_{j=1}^{\infty} j! x_j$, where $x_j \in \{0, 1, \ldots, j\}$, $j = 1, 2, \ldots$, with addition in the obvious sense of carrying over from one place to the next on the right, e.g. $(1 + 0.2! + 0.3! + \ldots) + (1 + 2.2! + 3.3! + \ldots) = (0 + 0 + \ldots)$. We sometimes write $\sum_{j=1}^{\infty} j! x_j$ simply as $(x_1, x_2, x_3, \ldots)$. The elements of $\Sigma$ have the form $\xi + x$ where $\xi \in [0, 1)$ and $x \in \Delta$, where

$$(\xi + x) + (\xi' + x') = (\xi + \xi' - [\xi + \xi']) + (x + x' + ([\xi + \xi'], 0, 0, \ldots)).$$

The duality between $\Delta$ and $Q/Z$ is expressed by

$$(x, \gamma) = \exp\{2\pi i m/n!(x_1 + x_2 2! + \ldots + x_{n-1}(n-1)!),$$

and that between

$$\Sigma \text{ and } Q \text{ by } \quad (x, \gamma) = \exp\{2\pi i m/n!(\xi + x_1 + x_2 2! + \ldots + x_{n-1}(n-1)!)\}.$$

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Before proceeding to the proofs of the main theorems we have to give metrics on our groups which will be compatible not only with their algebraic and topological structure, but also their dimensionality.

If $G_0$ is a 0-dimensional group, $G_n$ a base of neighbourhoods at 0 which are also open and closed subgroups of $G_0$, $G_n \supset G_{n+1}$, $\bigcap_{n=0}^{\infty} G_n = \{0\}$, then any choice of a sequence $\beta_n$ of real numbers decreasing to zero will give a metric on $G_0$ if we define

$$|x| = \beta_n \quad \text{for} \quad x \in G_n, \quad x \notin G_{n+1}.$$

If $p_n$ is the order of the finite group $G_0/G_n$, then the natural choice seems to be $\beta_n = p_{n+1}^{-1}$, and this was used in (1). However, this metric is 1-dimensional in the sense of (3); that is if $M_d(\epsilon)$ is the minimum number of balls of radius $\epsilon$ needed to cover $G_0$, then

$$\liminf_{\epsilon \to 0} \left\{ \frac{\log M_d(\epsilon)}{-\log \epsilon} \right\} = 1.$$

In order to make this limit $= 0$, we take $\beta_n = e^{-p_{n+1}}$. This difference explains the apparent discrepancy between the results of (1) where the constant $\frac{1}{2}$ occurs, and the present paper where the constant is $\frac{1}{2}(\dim G)$, or 0 in the case of 0-dimensional groups.

We now take $G_0 = \Delta$, $G_n = \{x : x_1 = x_2 = \ldots = x_n = 0\}$. Then $p_n = (n+1)!$. Hence if $x$ satisfies $x_1 = \ldots = x_{n-1} = 0$, $x_n \neq 0$, $|x| = e^{-(n+1)!}$, and this gives the required metric on $\Delta$.

Now suppose $\xi + x \in \Sigma$ with $\xi \in [0, 1), x \in \Delta$. We can suppose $0 \leqslant \xi \leqslant 4$ (otherwise consider $-(\xi + x) \in \Sigma$). Define $|\xi + x| = \max(|\xi|, |x|)$—a routine calculation shows that this gives a 1-dimensional metric on $\Sigma$.

For a finite product of metric groups $(G_i, |.|_i)$, $i = 1, 2, \ldots, n$, we metrise the product $G_1 \times \ldots \times G_n$ by $|(x_1, \ldots, x_n)| = \max(|x_1|_1, \ldots, |x_n|_n)$; and for a quotient $G/H$ of $(G, |.|)$ we metrise $G/H$ (as has been noted in Lemma 2) by $|x + H| = \inf\{|y|, y \in x + H\}$.

There remains the problem of an appropriate metric for $\Delta^\infty$. This we take in accordance with the above principles on 0-dimensional groups, using a neighbourhood base as follows. Denote elements of $\Delta^\infty$ by

$$x = (x^{(1)}, x^{(2)}, \ldots), \quad x^{(i)} \in \Delta, x^{(i)} = (x_{i1} + x_{i2}\,2! + x_{i3}\,3! + \ldots),$$

then take $\quad G_0 = \Delta^\infty, G_n = \{x : x_{i1} = x_{i2} = \ldots = x_{in} = 0, \quad i = 1, 2, \ldots, n\};$

in other words if

$$H_n = \{x : x \in \Delta, x_1 = \ldots = x_n = 0\}, \quad \text{then} \quad G_n = \prod_{i=1}^{n} H_n^{(i)} \times \prod_{i=n+1}^{\infty} \Delta.$$

Since $(n+1)!^n$ cosets of $G_n$ are required to cover $G_0$, the metric on $\Delta^\infty$ is given by

$$x = e^{-(n+1)!^n} \quad \text{if} \quad x \in G_{n-1}, x \notin G_n.$$

3. *Proofs of main theorems.*

*Proof of Theorem* 1. After Lemmas 1 and 2, we see that it is sufficient to prove the theorem when $G$ is of the form $\Delta^\infty \times \Sigma^{r_0}$, $r_0 = \dim G$. We have the formula

$$\Sigma_\Gamma |\hat{f}(y)|^2 |(h, y) - 1|^2 = \int_G |f(x+h) - f(x)|^2 \, dx = O(|h|^{2\alpha}) \quad (\alpha > \tfrac{1}{2} r_0).$$



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Recall that the dual $\Gamma$ of $G$ has the form $C^{\infty *} \times Q^{r_0}$, that is elements of $\Gamma$ have the form $y = ((y_1, y_2, \ldots), y_1', y_2', \ldots, \ldots, y_{r_0}')$ where $y_i \in C, y_i' \in Q$ and only finitely many $y_i$ are non-zero. Suppose $y_i = m_i/n_i!, y_i' = m_i'/n_i'!$.

We now divide $\Gamma$ into subsets as follows. Define $T_1 = \{y: n_1' = \ldots = n_{r_0}' = 1\}$. Evidently $T_1$ is isomorphic to $(C^{\infty *} \times) Z^{r_0}$. For $k \geqslant 2$, define

$$T_k = \{y: \max(n_1, n_2, \ldots, n_1', \ldots, n_{r_0}') = k\}.$$

(Remember that only finitely many $n_i$ are $\neq 1$.) Let

$$S_0 = \left\{y: \sum_{i=1}^{r_0} |m_i'|/n_i'! < 1\right\},$$

while for $$j = 1, 2, \ldots, S_j = \left\{y: j! \leqslant \sum_{i=0}^{r_0} |m_i'|/n_i'! < (j+1)!\right\}.$$

Let $R_0 = \{y: m_i = 0 \text{ all } i\}$, ($R_0$ is obviously isomorphic to $(C^{\infty *} \times) Q^{r_0}$) while for

$$l = 1, 2, \ldots, R_l = \{y: m_l \neq 0, m_i = 0 \quad \text{if} \quad i \geqslant l+1\}.$$

It is evident that with these definitions

$$\Gamma = \bigcup_{k=1}^{\infty} T_k = \bigcup_{j=0}^{\infty} S_j = \bigcup_{l=0}^{\infty} R_l,$$

and that $R_l \cap R_{l'} = S_j \cap S_{j'} = T_k \cap T_{k'} = \varnothing$ for $l \neq l'$, $j \neq j'$, $k \neq k'$ and $l, l', j, j'$ all $\geqslant 0, k, k' \geqslant 2$. $T_1 \cap T_k$ is not empty for any $k$.

We show separately

$$\text{(i)} \quad \Sigma_{T_1} |\hat{f}(y)| < \infty,$$

and

$$\text{(ii)} \quad \sum_{k=2}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \Sigma_{T_k \cap S_j \cap R_l} |\hat{f}(y)| < \infty,$$

which together give the required result. Consider, first, (i). On $T_1$ define

$$T_{1,q} = \{y: 2^{q-1} \leqslant |m_1'| + \ldots + |m_{r_0}'| < 2^q\}, \quad q = 1, 2, \ldots.$$

Evidently $T_1 = \{0\} \bigcup_{q=1}^{\infty} T_{1,q}$. Let $h_q \in G$ be the element $((0, 0, \ldots), \pm \frac{1}{3} 2^{-q+1}, \ldots, \pm \frac{1}{3} 2^{-q+1})$, where the sign is chosen to be the same as the sign of the corresponding $m_i'$. Then $|h_q| = \frac{1}{3} 2^{-q+1}$, and if

$$y \in T_{1,q}, |(h_q, y) - 1|^2 = |\exp\{2\pi i/(3 \cdot 2^{q-1})(|m_1'| + \ldots + |m_{r_0}'|)\} - 1|^2$$
$$= 4 \sin^2\{\pi/(3 \cdot 2^{q-1})(|m_1'| + \ldots + |m_{r_0}'|)\} \geqslant 3.$$

Hence $$\Sigma_{T_{1,q}} |\hat{f}(y)|^2 = O \Sigma_{T_{1,q}} |\hat{f}(y)|^2 |(h_q, y) - 1|^2 = O(|h_q|^{2\alpha}) = O(2^{-2\alpha q}).$$

But $T_{1,q}$ contains $2^{r_0(q-1)}$ points, and therefore by Schwarz's inequality,

$$(\Sigma_{T_{1,q}} |\hat{f}(y)|^2) = O(2^{r_0 q - 2\alpha q}).$$

Therefore $$\Sigma_{T_{1,q}} |\hat{f}(y)| = |\hat{f}(0)| + O\left(\sum_{q=1}^{\infty} 2^{(\frac{1}{2} r_0 - \alpha) q}\right)$$

which is finite since $\alpha > \frac{1}{2} r_0$.

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We now turn to (ii). On $T_k\,(k \geqslant 2)$ let $t_k = (k-1)!$ in the place where $n_i$ or $n_i' = k$, —say $n_p^{(\prime)} = k$. Then $|t_k| \leqslant e^{-k!}$ (with equality if any $n_i' = k$). If

$$y \in T_k,\ |(t_k, y) - 1|^2 = |\exp\{2\pi i\,(m_p^{(\prime)}/k)\} - 1|^2 \geqslant 4k^{-2}$$

since by our construction $k$ does not divide $m_p^{(\prime)}$.

Hence

$$\Sigma_{T_k}|\hat{f}(y)|^2 = O(k^2\Sigma_\Gamma|\hat{f}(y)|^2\,|(t_k, y) - 1|^2) = O(k^2 e^{-2\alpha k!}). \tag{A}$$

On $S_j, j = 1, 2, \ldots$, take $s_j = ((0, 0, \ldots,), \pm\frac{1}{2}(j+1)!^{-1}, \ldots, \pm\frac{1}{2}(j+1)!^{-1})$ where the $\pm 1$ are chosen as $\operatorname{sgn}(m_i')\,.\,|s_j| = \frac{1}{2}(j+1)!^{-1}$. If $y \in S_j$,

$$|(s_j, y) - 1|^2 = \left|\exp\left\{\frac{2\pi i}{2(j+1)!}\sum_{i=1}^{r_0}\frac{|m_i'|}{n_i'!}\right\} - 1\right|^2 \geqslant 4(j+1)^{-2}, \quad \text{for} \quad j = 1, 2, \ldots.$$

If $$j = 0, \quad \Sigma_{S_0}|\hat{f}(y)|^2 \leqslant \Sigma_\Gamma|\hat{f}(y)|^2 < \infty.$$

Hence for all $j$,

$$\Sigma_{S_j}|\hat{f}(y)|^2 = O[(j+1)^2\Sigma_{S_j}|\hat{f}(y)|^2|(s_j, y) - 1|^2] = O[(j+1)^2\,(j+1)!^{-2\alpha}]. \tag{B}$$

On $R_l$, if $l = 0$, $\Sigma_{R_0}|\hat{f}(y)|^2 < \infty$ as in the case of $\Sigma_{S_0}$. If $l > 0$, take $r_l = 1$ in the $l$th place, $r_l = ((0, 0, \ldots, 0, 1, 0, \ldots), 0, 0, \ldots, 0)$. Then $|r_l| = \exp\{-(l+1)!^l\}$. If

$$y \in R_l,\ |(r_l, y) - 1|^2 = |\exp(2\pi i\, m_l/n_l! - 1|^2 \geqslant 4(k!)^{-2},$$

if in addition $y \in T_k$.

Hence

$$\Sigma_{R_l \cap T_k}|\hat{f}(y)|^2 = O[k!^2 e^{-2\alpha(l+1)!^l}]. \tag{C}$$

By combining (A), (B), (C) we now have

$$\begin{aligned}\Sigma_{T_k \cap S_j \cap R_l}|\hat{f}(y)|^2 &= O[\min\{k^2 e^{-2\alpha k!}, (j+1)^2\,(j+1)!^{-2\alpha}, (k!)^2\,e^{-2\alpha(l+1)!^l}\}] \\ &= O[k^2(j+1)^2\,(k!)^2\min\{e^{-2\alpha k!}, (j+1)!^{-2\alpha}, e^{-2\alpha(l+1)!^l}\}].\end{aligned}$$

In order to be able to use Schwarz's inequality we need to know the number of points in $T_k \cap S_j \cap R_l$. The number of possibilities which arise from $C^{\infty *}$ is at most $(k!)^l$ while the number in $Q^{r_0}$ is at most $((j+1)! - j!)\,k!((j+1)!k!)^{r_0-1}$, i.e. $(j+1)!^{r_0}\,k!^{r_0+l}$ altogether. Hence by Schwarz's inequality,

$$\Sigma_{T_k \cap S_j \cap R_l}|\hat{f}(y)| = O[(j+1)!^{\frac{1}{2}r_0}k!^{\frac{1}{2}r_0+\frac{1}{2}l}k(j+1)\,k!\min\{e^{-\alpha k!}, (j+1)!^{-\alpha}, \quad e^{-\alpha(l+1)!^l}\}]. \tag{D}$$

The remainder of the proof consists of an elementary proof that the triple summation over $j, k, l$ of the series whose general term is given by the right-hand side of $(D)$ is convergent.

Let $l(k, j)$ be the greatest integer for which $e^{-\alpha(l+1)!^l} \geqslant \min(e^{-\alpha k!}, (j+1)!^{-\alpha})$, from which it follows by very crude estimates that $l(k, j) \leqslant k\log(j+1)$. The summation then becomes

$$\sum_{k=2}^{\infty}\sum_{j=0}^{\infty}(j+1)!^{\frac{1}{2}r_0}k!^{\frac{1}{2}r_0+1}k(j+1)\left[\min(e^{-\alpha k!}\,(j+1)!^{-\alpha})\sum_{l=0}^{l(k,j)}k!^{\frac{1}{2}l} + \sum_{l=l(k,j)+1}^{\infty}k!^{\frac{1}{2}l}e^{-\alpha(l+1)!^l}\right].$$



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Now the terms of the finite sum increase sufficiently rapidly for the summation to be of the order of its final term; similarly the infinite sum is of the order of its first term. Hence the summation is

$$O\left[\sum_{k=2}^{\infty}\sum_{j=0}^{\infty}(j+1)!^{\frac{1}{2}r_0}k^{\frac{1}{2}r_0+1}k(j+1)\min\left(e^{-\alpha k!},(j+1)!^{-\alpha}\right)k!^{\frac{1}{2}(l(k,j)+1)}\right]$$

$$= O\left[\sum_{k=2}^{\infty}\sum_{j=0}^{\infty}(j+1)!^{\frac{1}{2}r_0}k!^{\frac{1}{2}r_0+\frac{1}{2}k\log(j+1)+\frac{3}{2}}k(j+1)\min\left(e^{-\alpha k!},(j+1)!^{-\alpha}\right)\right].$$

Again let $k(j)$ be the greatest integer for which $e^{-\alpha k!} \geqslant (j+1)!^{-\alpha}$, whence

$$k(j) \leqslant 2\log(j+1).$$

Then the summation is

$$O\left[\sum_{j=0}^{\infty}(j+1)!^{\frac{1}{2}r_0}(j+1)\left\{[(j+1)!^{-\alpha}\sum_{k=2}^{k(j)}k.k!^{\frac{1}{2}r_0+\frac{1}{2}k\log(j+1)+\frac{3}{2}}+\sum_{k=k(j)+1}^{\infty}k.k!^{\frac{1}{2}r_0+\frac{1}{2}k\log(j+1)+\frac{3}{2}}e^{-\alpha k!}\right\}\right].$$

If we apply the same reasoning to the finite and infinite summations, we obtain

$$O\left[\sum_{j=0}^{\infty}(j+1)!^{\frac{1}{2}r_0-\alpha}(j+1)\left(k(j)+1\right)\left(k(j)+1\right)!^{\frac{1}{2}r_0+\frac{1}{2}(k(j)+1)\log(j+1)+\frac{3}{2}}\right]$$

and this is convergent by the root test since $(k(j)+1)!^{\frac{1}{2}\log(j+1)^2} \leqslant (\log(j+1)+1)^{\frac{1}{2}\log(j+1)^3}$. This completes the proof of Theorem 1.

The proofs of Theorems 2 and 3 are far less intricate—that of Theorem 2 follows closely the ideas of the classical proof in the case $G = T$.

*Proof of Theorem* 2. Suppose, as in the hypothesis, that $f \in A$, $E = f^{-1}(0)$, and that $f(x) = O[d(x,E)^{\alpha}]$ some $\alpha > \frac{1}{2}(\dim G)$, where $d$ is a metric on $G$ for which

$$\liminf_{\epsilon\to 0}\frac{\log M_d(\epsilon)}{-\log\epsilon} = \dim G = r_0$$

(see (3) for the existence of such a metric). (By using the representation of Lemma 1, one can improve this estimate to $M_d(\epsilon) = O[(1/\epsilon)^{r_0}\{\log(1/\epsilon)\}^q]$ where $q$ is a constant depending on $G$, but this still does not allow us to put $\alpha = \frac{1}{2}(\dim G)$.) We take $\epsilon_n \to 0$ such that $M_d(\epsilon_n) = O(\epsilon_n^{-r_0-o(1)})$. Now suppose $S$ is a pseudo-measure on $G$ with support in $E$. For spectral synthesis we have to show $\langle S,f\rangle = 0$.

Denote by $B_{\epsilon_n}(0)$ the open ball of radius $\epsilon_n$, centre 0, by $m_n$ the Haar measure of $B_{\epsilon_n}(0)$, and by $\chi_n$ the characteristic function of $B_{\epsilon_n}(0)$. Let $f_n = m_n^{-2}(\chi_n * \chi_n)$ ($*$ denotes convolution on $G$). Since $\chi_n \in L^2(G)$, $f_n \in A(G)$. Also we have that the support of $f$ is in $B_{2\epsilon_n}(0)$, $f_n$ is positive, $\int_G f_n = 1$, and $f_n(0) = m_n^{-1}$.

Let $S_n = S * f_n$. Since $\hat{f}_n \to 1$ as $n \to \infty$ for each point of $\Gamma$, we have $\langle S_n,f\rangle \to \langle S,f\rangle$ by dominated convergence. Hence it is sufficient to prove $\langle S_n,f\rangle \to 0$ as $n \to \infty$.

Since $f_n \in A$, $S_n$ also $\in A$, and we can write

$$\langle S_n,f\rangle = \int_G S_n(x)\bar{f}(x)\,dx.$$

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Now $f = 0$ on $E$, and $S_n$ has its support in $E_n = \{x : d(x, E) \leqslant 2\epsilon_n\}$. Hence

$$|\langle S_n, f\rangle|^2 = \left|\int_{E_n \setminus E} S_n(x)\bar{f}(x)\,dx\right|^2 \leqslant \int_{E_n\setminus E}|S_n|^2 \int_{E_n\setminus E}|f|^2.$$

Now
$$\int_{E_n\setminus E}|f|^2 \leqslant m(E_n\setminus E)\,O(\epsilon_n^{2\alpha}) = o(\epsilon_n^{2\alpha}),$$

by the hypothesis on $f$. Also

$$\int_{E_n\setminus E}|S_n|^2 \leqslant \int_G |S_n|^2 = \Sigma_\Gamma |c_\gamma|^2 |\hat{f}_n(\gamma)|^2 \quad \text{where} \quad S \sim \Sigma_\Gamma c_\gamma(x,\gamma)$$

and
$$\sup_\gamma |c_\gamma| = \|S\|_{PM} < \infty.$$

Hence
$$\int_{E_n\setminus E}|S_n|^2 = O[\Sigma_\Gamma |\hat{f}_n(\gamma)|^2] = O\left[\int_G |f_n|^2\right].$$

But
$$\int_G |f_n|^2 = \int_G f_n^2 = m_n^{-4}\int_G \left(\int_G \chi_n(x)\,\chi_n(y-x)\,dx\right)^2 dy$$

$$= m_n^{-4}\int_G\int_G \left(\int_G \chi_n(y-x)\,\chi_n(y-x')\,dy\right)\chi_n(x)\,\chi_n(x')\,dx\,dx'$$

$$= m_n^{-2}\int_G\int_G f_n(x-x')\,\chi_n(x)\,\chi_n(x')\,dx\,dx' \leqslant f_n(0) = m_n^{-1}.$$

Therefore $|\langle S_n, f\rangle|^2 = o(\epsilon_n^{2\alpha} m_n^{-1})$. But since $M_d(\epsilon_n)$ translates of $B_{\epsilon_n}(0)$ cover $G$, we have $m_n M_d(\epsilon_n) \geqslant 1$, or $m_n^{-1} = O(\epsilon_n^{-r_0 - o(1)})$.

Therefore $|\langle S_n, f\rangle|^2 = o(\epsilon_n^{2\alpha - r_0 - o(1)}) \to 0$ since $2\alpha > r_0 = \dim G$.

*Note.* In the case $G = T^n$, the estimate $M_d(\epsilon) = O(\epsilon^{-n})$ shows that $f(x) = O[d(x,E)]^{\frac{1}{2}n}$ implies $f$ satisfies spectral synthesis. In general this is not the case. The bracketed remark in the statement of the theorem (with $2\alpha = \dim G$) is trivially true as stated, but becomes false if the interpretation of Lipschitz classes in terms of derivatives is used and $n \geqslant 3$.

Finally we give

THEOREM 3. *Let be a complex valued function on $T^n$. Let $k = \frac{1}{2}n$, $\alpha > 0$ if $n$ is even, while $k = \frac{1}{2}(n-1)$, $\alpha > \frac{1}{2}$ if $n$ is odd. Then if $f$ is $k$ times differentiable and*

$$f^{(k)} \in \Lambda_\alpha \text{ (i.e. } f \in \Lambda_\beta, \beta > \tfrac{1}{2}n),$$

*where $f^{(k)}$ denotes any one of the partial derivatives of the $k$th order, then $f \in A$.*

*Proof.* Denote elements of $T^n$ by $x = (x_1, \ldots, x_n)$, and of its dual $Z^n$ by

$$y = (y_1, \ldots, y_n).$$

If
$$h = (h_1, \ldots, h_n) \in T^n \quad \text{and} \quad D_h = h_1\frac{\partial}{\partial x_1} + \ldots + h_n\frac{\partial}{\partial x_n},$$

then we have the expansion

$$f(x+h) - f(x) = \sum_{j=1}^{k-1}\frac{(D_h)^j}{j!}f(x) + \frac{1}{(k-1)!}\int_0^1 (1-t)^{k-1} D_h^k f(x+th)\,dt.$$

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It follows that the Fourier transform of

$$\frac{1}{(k-1)!}\int_0^1 (1-t)^{k-1}\{D_h^k f(x+th)-D_h^k f(x)\}\,dt$$

is equal to $$\left(e^{i\theta}-\sum_{j=0}^{k}\frac{(i\theta)^j}{j!}\right)\hat{f}(y),$$

where $\theta = 2\pi(h_1y_1+\ldots+h_ny_n) = 2\pi h.y$, say.

Hence by Plancherel's formula,

$$\Sigma_{Z^n}\left|e^{i\theta}-\sum_{j=0}^{k}\frac{(i\theta)^j}{j!}\right|^2|\hat{f}(y)|^2 = \int_{T^n}\left|\frac{1}{(k-1)!}\int_0^1(1-t)^{k-1}\{D_h^k f(x+th)-D_h^k f(x)\}\,dt\right|^2 dx$$
$$= O(|h|^{2k+2\alpha}). \tag{i}$$

For $\mu = 1,2,3,\ldots$ let $T_\mu = \{y: y\in Z^n, 2^{\mu-1}\leqslant |y_1|+\ldots+|y_n| < 2^\mu\}$. Obviously $T_\mu$ contains $\leqslant 2^{\mu n}$ points, and $Z^n = \{0\}\cup\bigcup_{\mu=1}^{\infty} T_\mu$. If $y\in T_\mu$ and

$$h_\mu = \tfrac{1}{3}(\operatorname{sgn}(y_1)2^{-\mu},\ldots,\operatorname{sgn}(y_n)2^{-\mu}),\quad |h_\mu| = O(2^{-\mu}),\quad \text{and}\quad \theta = 2\pi h_\mu.y \geqslant \tfrac{1}{3}\pi.$$

Now

$$\left|e^{i\theta}-\sum_{j=0}^{k}\frac{(i\theta)^j}{j!}\right|^2 = \left|\cos\theta-\sum_{j=0}^{l}(-1)^j\frac{\theta^{2j}}{(2j)!}\right|^2+\left|\sin\theta-\sum_{j=0}^{m}(-1)^j\frac{\theta^{2j+1}}{(2j+1)!}\right|^2, \tag{ii}$$

where $l, m$ are the largest integers with $2l\leqslant k$, $2m+1\leqslant k$.

Now it is a well known fact that both the expressions on the right-hand side of (ii) are non-zero for $\theta\neq 0$, and hence are bounded away from zero when $\theta\geqslant\frac{1}{3}\pi > 0$.

Therefore

$$\Sigma_{T_\mu}|\hat{f}(y)|^2 = O\Sigma_{T_\mu}|\hat{f}(y)|^2\left|e^{i\theta}-\sum_{j=0}^{k}\frac{(i\theta)^j}{j!}\right| = O(|h_\mu|^{2k+2\alpha}) = O(2^{-2\mu(k+\alpha)}).$$

Hence by Schwarz's inequality we have $\Sigma_{T_\mu}|\hat{f}(y)| \leqslant 2^{\frac{1}{2}\mu n}O(2^{-\mu(k+\alpha)})$. Therefore

$$\Sigma_{Z^n}|\hat{f}(y)| = |\hat{f}(0)|+\sum_{\mu=1}^{\infty}\Sigma_{T_\mu}|\hat{f}(y)| = |\hat{f}(0)|+O\sum_{\mu=1}^{\infty}2^{\mu(\frac{1}{2}n-k-\alpha)},$$

and this is convergent, since $k+\alpha > \frac{1}{2}n$, that is $f\in A$.

*Note.* The foregoing theorems evidently need to be complemented by examples of functions which show that the constants are best possible. For instance a function $f\in\Lambda_{\frac{1}{2}n}(T^n), f\notin A(T^n)$ is needed. The author hopes to rectify this omission in a future paper.

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