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103.25 An extension of Thébault's second problem

One of the things that amazes me about geometry is how *much* of it there is. Of course, one can say that about any subfield of mathematics, but geometry has been available to both the amateur and professional for such a long time, and has been pored over by so many, many eyes, that one might think there just isn't anything simple left undiscovered. So, when something elementary, yet still *new*, grabs my attention, it's always a delightful surprise to me. This Note provides another such nugget, for your enjoyment.

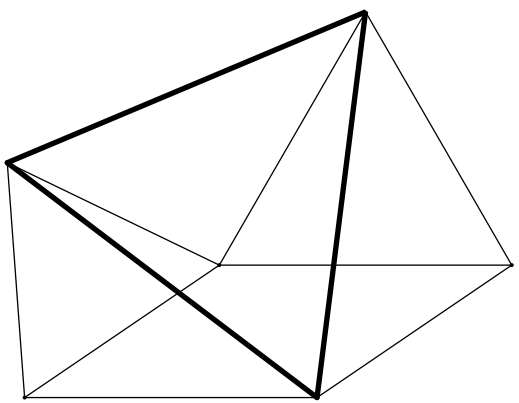


FIGURE 1

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A well-known generalisation of Thébault's celebrated second problem is as follows. Given a parallelogram with an equilateral triangle constructed outwardly on each of two consecutive sides, the free vertices of the triangles, together with the free vertex of the parallelogram, form an equilateral triangle (henceforth called a "Thébault" triangle), as we will prove later as part of our main theorem.

With the construction of two more equilateral triangles on the sides of the parallelogram, we obtain three more Thébault triangles. We then find the figure created (with dashed lines) by the centres of our four Thébault triangles, and our theorem.

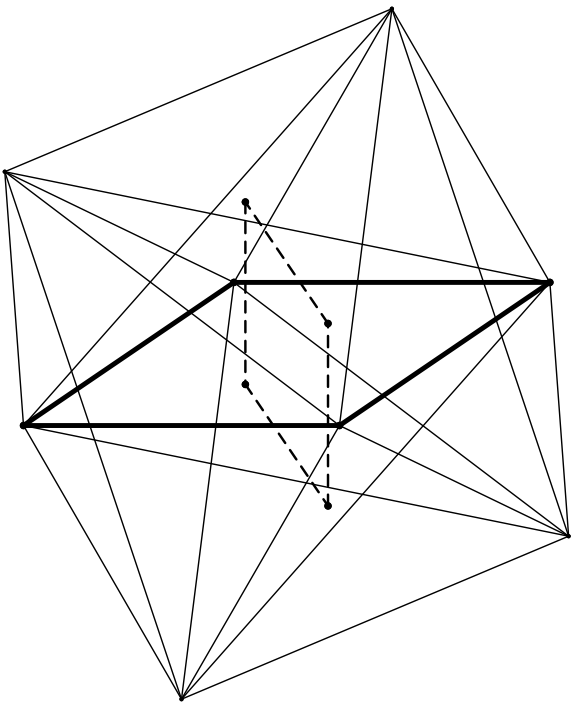


FIGURE 2

Theorem: Each Thébault triangle is equilateral, and moreover, the centres of the Thébault triangles:

(1) form another parallelogram;

(2) similar to the original parallelogram and scaled by $\frac{1}{\sqrt{3}}$;

(3) inclined by 90° to it;

(4) sharing the same centre.

Proof: We begin by situating the given parallelogram in the complex plane, with its centre at the origin. Two adjacent vertices of the parallelogram will

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be assigned affixes a and b, implying that their opposite vertices are $-a$ and $-b$, respectively. We now find the affix of several other vertices in the picture.

A triangle XYZ (with vertices labeled in counterclockwise order, and affixes x, y and z) is equilateral if, and only if, $x + y\varepsilon + z\varepsilon^2 = 0$, where $\varepsilon = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, $1 + \varepsilon + \varepsilon^2 = 0$ and $\varepsilon^3 = 1$. We apply this to determine the remaining vertices of the four triangles resting on the sides of the parallelogram, as shown in the next diagram.

Observing one of the Thébault triangles (for instance, as in our first diagram), we see that the affixes satisfy the criterion for an equilateral triangle:

$$\left(b\varepsilon + a\varepsilon^2\right) + \left(-a\varepsilon + b\varepsilon^2\right)\varepsilon + b\varepsilon^2 = b\left(1 + \varepsilon + \varepsilon^2\right) = 0.$$

We continue by using the fact that the centre of an equilateral ΔXYZ is given by its centroid $\frac{1}{3}(x + y + z)$, giving us the affixes of our purported parallelogram. These simplified affixes are shown in the diagram below, where we use $\dfrac{\varepsilon^2 - \varepsilon}{3} = \dfrac{-i}{\sqrt{3}}$.

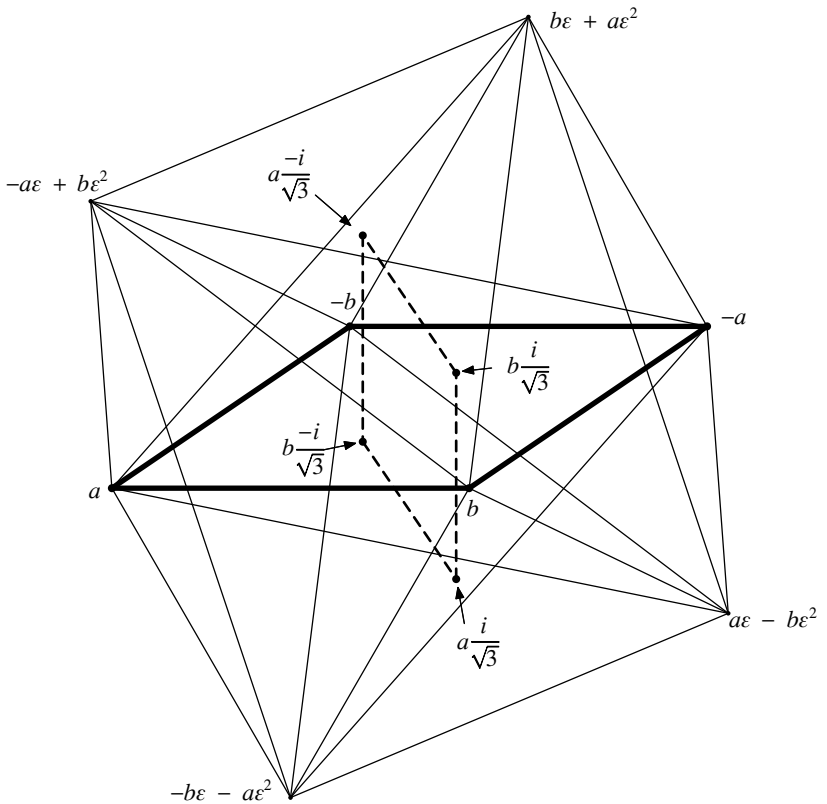


FIGURE 3

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The fact that we have a new parallelogram is evident from the symmetry in the figure. It is similar to the original as the affixes of its corresponding vertices are simply those of the original multiplied by the same constant. Indeed, multiplication by $\frac{-i}{\sqrt{3}}$ is a scaling by $\frac{1}{\sqrt{3}}$ in tandem with a clockwise rotation of 90° about the origin, which also immediately settles (3), (4).

The proof can be easily modified to show that the theorem also holds for inwardly constructed triangles.

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103.26 A proof of Clarke's conjecture

Introduction

In 2001, the late Robert J. Clarke [1] proposed the problem: Show that, for $n > m \geqslant 1$, the following identity holds:

$$\frac{(-1)^{mn}}{F_{n+m}}\prod_{r=1}^{2m}\frac{F_r}{F_{n+m-r}} = \frac{1}{F_{n+m}} + \sqrt{2}\sum_{r=1}^{2m}\left(\frac{\sin\left[2(r+1)+(-1)^m\right]\frac{\pi}{4}}{F_{n+m-r}}\prod_{s=1}^{r}\frac{F_{2m+1-s}}{F_s}\right), \quad (1)$$

where F_n is the nth Fibonacci number.

In this paper, our main goal is to give a solution for Clarke's conjecture. Our method is to rewrite the conjecture in terms of the Fibonomial coefficients. Further we will find a q-form of the conjecture satisfied by any real number q. Further we will give a proof for this q-form. Moreover, we will also provide a general case of the conjecture by choosing a particular value of q. Finally we will present a Lucas counterpart of the conjecture.

The Gibonacci polynomial sequence

For $n \geqslant 2$, define the second order linear recursive sequence $\{g_n(x)\}$ by

$$g_n(x) = a(x)\,g_{n-1}(x) + b(x)\,g_{n-2}(x),$$

where the initial polynomials are $g_0(x)$ and $g_1(x)$, and, the coefficients $a(x)$ and $b(x)$ are arbitrary polynomials. The sequence $\{g_n(x)\}$ is called the *gibonacci polynomial* sequence.

Suppose $a(x) = x$ and $b(x) = 1$. When $g_0(x) = 0$ and $g_1(x) = 1$, the sequence $\{g_n(x)\}$ is the *Fibonacci polynomial* sequence $\{f_n(x)\}$, and when $g_0(x) = 2$ and $g_1(x) = x$, the sequence $\{g_n(x)\}$ is the *Lucas polynomial* sequence $\{l_n(x)\}$. In particular, for a non-zero integer p, $f_n(p) = U_n(p)$ and $l_n(p) = V_n(p)$, the nth generalised Fibonacci and Lucas numbers, respectively. (For details, we refer to [2, 3].)

For the special case $p = 1$, the sequences $\{U_n(p)\}$ and $\{V_n(p)\}$ are reduced to the usual Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$, respectively.

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