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Quasi-standard C^*-algebras

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Abstract

A necessary and sufficient condition is given for a separable C^*-algebra to be *-isomorphic to a maximal full algebra of cross-sections over a base space such that the fibre algebras are primitive throughout a dense subset. The condition is that the relation of inseparability for pairs of points in the primitive ideal space should be an open equivalence relation.

1. *Introduction*

From an algebraic point of view the theory of C^*-algebras can be separated into the study of primitive algebras (with particular attention to the simple case) and the study of how a general C^*-algebra is related to its primitive quotients. The second of these problems may be viewed as a search for non-commutative analogues of the Gelfand–Naimark theorem for commutative C^*-algebras, with an associated requirement for non-commutative Stone–Weierstrass results. This aspect of the theory was studied by Kaplansky[**17**], and continued by Glimm[**16**], Fell[**14**], Tomiyama[**22**] and Lee[**18**], with the introduction of the notion of a maximal full algebra of cross-sections (or 'operator fields' in Fell's terminology) using varying fibre algebras over a locally compact Hausdorff base space (see [**14**], pp. 235–242 for precise details). The property of maximality may be viewed as a non-commutative generalization of the surjectivity of the Gelfand transform for a commutative C^*-algebra.

In attempting to represent a general C^*-algebra in this way, whilst also involving the primitive quotients, Fell sought a base space such that the fibre algebras are primitive throughout a dense subset. One such candidate (the τ_s-closure of the primitive ideal space Prim (A) in the notation of Section 2 below) turned out to be in some sense 'too large' in that the resulting full algebra of cross-sections is maximal if and only if Prim (A) is Hausdorff ([**14**], theorem 2·3), a severe restriction. On the other hand, Dauns and Hofmann[**9**] developed a theory which is valid for an arbitrary C^*-algebra but at the expense of possibly losing continuity of the norm of a section and also possibly losing control of the primitivity of the fibre algebras.

Fell's work indicated the existence of obstructions to the programme of representing an arbitrary C^*-algebra as a maximal full algebra of cross-sections over

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a base space such that the fibre algebras are primitive throughout a dense subset, and so attention was turned to the other strand mentioned above, namely the study of simple C^*-algebras. This is a difficult area in which a complete classification is not yet in sight. A great deal of effort has been spent in the study of specific examples together with the development of general invariants such as K-groups and Ext-groups to distinguish isomorphism classes. The survey articles [**7**, **5**] discuss the progress that has been made towards a general theory.

In this paper we return to Fell's problem with the object of clarifying the nature and extent of the obstructions referred to above. In [**3**], a study was made of the minimal primal ideals of a C^*-algebra. These had been found to arise naturally in the theory of limits of factorial states [**4**]. It was shown in theorem 5·3 of [**3**] that, under certain natural hypotheses, a C^*-algebra A is *-isomorphic to a maximal full algebra of cross-sections over the (topologized) space Min-Primal (A) of all minimal primal ideals. Moreover, if A is separable then Min-Primal (A) contains a dense subset of primitive ideals.

We show here (Theorem 3·3) that the hypotheses of theorem 5·3 of [**3**] are equivalent to the condition that the relation of inseparability for pairs of points in the primitive ideal space Prim (A) is an open equivalence relation. C^*-algebras satisfying this latter condition are said to be quasi-standard. The relevance of conditions of this type was suggested by the work of Ching [**6**]. The class of quasi-standard C^*-algebras may be viewed as a well-behaved enlargement of the sub-class consisting of those C^*-algebras with Hausdorff primitive ideal space.

Theorem 3·4 provides a useful characterization of quasi-standardness in terms of primal ideals and leads to our main result, Corollary 3·5, which gives a complete solution in the separable case to the maximality problem arising from Fell's work in [**14**], section II: a separable C^*-algebra is quasi-standard if and only if it is *-isomorphic to a maximal full algebra of cross-sections over a base space such that the fibre algebras are primitive throughout a dense subset. We also use Theorem 3·4 to show that some interesting and diverse group C^*-algebras are quasi-standard. Finally, we show in Section 4 that the class of quasi-standard C^*-algebras strictly contains the class of pre-standard C^*-algebras studied by Ching [**6**].

2. *Preliminaries*

Let A be a C^*-algebra. By an *ideal* of A we shall always mean a closed two-sided ideal. Recall that an ideal I of A is *primal* if whenever $n \geqslant 2$ and $J_1, J_2, \ldots, J_n$ are ideals of A with zero product then $J_i \subseteq I$ for at least one value of i: see [**4**]. The set Id (A) of all ideals of A can be equipped with strong and weak topologies, τ_s and τ_w. The precise definitions and the origins of these topologies are described in [**3**], section 2. The main features are that a net (I_α) is τ_s-convergent to I in Id (A) if and only if $\|a+I_\alpha\| \to \|a+I\|$ for all $a \in A$, whilst a base for τ_w is given by the family of sets of the form

$$U(F) = \{I \in \mathrm{Id}\,(A) : J \nsubseteq I \text{ for all } J \in F\},$$

where F is a finite set (possibly empty) of ideals of A. When restricted to Prim (A), the primitive ideal space of A, τ_w coincides with the Jacobson topology. The restrictions of τ_s and τ_w to the set of primal ideals are generally distinct (τ_s is Hausdorff whilst τ_w may not be), but the restrictions to Min-Primal (A) coincide. We denote by τ this common topology on Min-Primal (A).

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We now recall some properties of the complete regularization of $\mathrm{Prim}(A)$ for a C^*-algebra A (see [9] for further details). For $P, Q \in \mathrm{Prim}(A)$ let $P \approx Q$ if and only if $f(P) = f(Q)$ for all $f \in C^b(\mathrm{Prim}(A))$. Then $\approx$ is an equivalence relation on $\mathrm{Prim}(A)$ and the equivalence classes are closed subsets of $\mathrm{Prim}(A)$. It follows that there is a one-to-one correspondence between $\mathrm{Prim}(A)/\approx$ and a set of ideals of A given by

$$[P] \leftrightarrow \bigcap [P] \quad (P \in \mathrm{Prim}(A)),$$

where $[P]$ denotes the equivalence class of P. The set of ideals obtained in this way is denoted by $\mathrm{Glimm}(A)$ (see below) and we identify this set with $\mathrm{Prim}(A)/\approx$ via the above correspondence. The quotient map $\phi_A : \mathrm{Prim}(A) \to \mathrm{Glimm}(A)$ is known as the complete regularization map. Note that if $P, Q \in \mathrm{Prim}(A)$, $G \in \mathrm{Glimm}(A)$ and $P \supseteq G = \bigcap [Q]$ then, since $[Q]$ is closed, $P \in [Q]$ and so $\phi_A(P) = \phi_A(Q) = G$. It follows that if $P \in \mathrm{Prim}(A)$, $G \in \mathrm{Glimm}(A)$ and $P \supseteq G$ then $\phi_A(P) = G$.

There are two natural topologies on $\mathrm{Glimm}(A)$: the quotient topology τ_q, which is Hausdorff, and the completely regular topology τ_{cr} which is the weakest topology for which the functions on $\mathrm{Glimm}(A)$ induced by $C^b(\mathrm{Prim}(A))$ are all continuous. For a general topological space X it can happen than τ_{cr} differs from the finer topology τ_q on $X/\approx$ (see [15], 3J·3) but it is not clear whether this can occur for $X = \mathrm{Prim}(A)$. On the other hand, there are various conditions which ensure that $\tau_q = \tau_{cr}$ on $\mathrm{Glimm}(A)$. For example, if A is unital then $\mathrm{Prim}(A)$ is compact and hence $\mathrm{Glimm}(A)$ is τ_q-compact. Since the weaker topology τ_{cr} is Hausdorff, it must equal τ_q. The topologies τ_q and τ_{cr} also coincide if ϕ_A is either τ_{cr}-open or τ_q-open (and so we may speak unambiguously of ϕ_A being open). This is immediate if ϕ_A is τ_{cr}-open. On the other hand, suppose that ϕ_A is τ_q-open. Then the local compactness of $\mathrm{Prim}(A)$ passes to the Hausdorff space $(\mathrm{Glimm}(A), \tau_q)$. Thus τ_q is equal to the weak topology defined on $\mathrm{Glimm}(A)$ by $C_0(\mathrm{Glimm}(A), \tau_q)$. It follows that $\tau_{cr} \geqslant \tau_q$ and hence $\tau_{cr} = \tau_q$.

Suppose that A is unital with centre Z. It follows from the Dauns–Hofmann theorem that for $P, Q \in \mathrm{Prim}(A)$:

$$P \approx Q \Leftrightarrow P \cap Z = Q \cap Z.$$

Hence there is a bijection $\psi : \mathrm{Glimm}(A) \to \mathrm{Prim}\, Z$ such that $\psi(G) = G \cap Z$ $(G \in \mathrm{Glimm}(A))$ and $\psi^{-1}(J) = AJ$ $(J \in \mathrm{Prim}(Z))$ (AJ is closed by the Cohen factorization theorem). Thus our usage of the terminology $\mathrm{Glimm}(A)$ arises from [16], section 4. Since the mapping $P \mapsto P \cap Z$ of $\mathrm{Prim}(A)$ onto $\mathrm{Prim}(Z)$ is continuous for the Jacobson topologies, it follows that ψ is continuous for τ_q $(= \tau_{cr})$. Since $\mathrm{Glimm}(A)$ is compact and $\mathrm{Prim}\, Z$ is Hausdorff, ψ is a homeomorphism. The above results hold also in the more general case of a quasicentral C^*-algebra (see [10] for the definition). Hence the fact that the canonical isomorphism of $(G+Z)/G$ onto $Z/(Z \cap G)$ is isometric for $G \in \mathrm{Glimm}(A)$ yields a short proof of a strengthened version of the main result of [8]: if A is quasicentral and K is a compact subset of $\mathrm{Glimm}(A)$ then there exist $z \in Z$ and $\lambda > 0$ such that

$$K \subseteq \{G \in \mathrm{Glimm}(A) : \|z + G\| \geqslant \lambda\}:$$

see [21].

Returning to the case of a general C^*-algebra A, let $a \in A$. The argument in the first paragraph of the proof of theorem 3·1 of [22] shows that the function $G \mapsto \|a + G\|$ on $\mathrm{Glimm}(A)$ is upper semi-continuous for both τ_q and τ_{cr} (see also [8] and, in the unital

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case, [**20**, **16**, **23**]). However this function need not be continuous. The following result is essentially an amalgam of [**22**], theorem 3·1 and [**18**], theorem 4 (see also [**8**], corollary 1·9 and [**23**], lemma 4·1 for the equivalence of (i) and (ii)).

THEOREM 2·1. *Let A be a C^*-algebra. The following conditions are equivalent*:

(i) ϕ_A *is an open map*;

(ii) *for each* $a \in A$ *the function* $G \mapsto \|a+G\|$ *is* τ_{cr}*-continuous on* Glimm (A);

(iii) *for each* $a \in A$ *the function* $G \mapsto \|a+G\|$ *is* τ_q*-continuous on* Glimm (A);

(iv) *the topology* τ_q *is given by the hull-kernel process on* Glimm (A);

(v) $(\text{Glimm}\,(A), \tau_q)$ *is locally compact and the mapping* $a \mapsto \hat{a}$, *where* $\hat{a}(G) = a+G$ $(G \in \text{Glimm}\,(A))$, *is a* **-isomorphism of* A *onto a maximal full algebra of cross-sections over* $(\text{Glimm}\,(A), \tau_q)$.

Furthermore, when these conditions hold the topologies τ_q *and* τ_{cr} *coincide on* Glimm (A).

Proof. (i) $\Rightarrow$ (iv). This follows from the second part of the proof of [**18**], theorem 4.

(iv) $\Rightarrow$ (v). This follows from [**22**], theorem 3·1.

(v) $\Rightarrow$ (iii). This is immediate.

(iii) $\Rightarrow$ (i). This follows from the first part of the proof of [**18**], theorem 4.

This shows that (i), (iii), (iv) and (v) are equivalent. Clearly (ii) $\Rightarrow$ (iii). Finally, if (i) holds then $\tau_{cr} = \tau_q$ and (iii) holds and so (ii) holds. ∎

We note in passing that it is possible to give a direct proof of the equivalence of (i) and (iii) in a more general setting: for any equivalence relation on Prim (A) for which the equivalence classes are closed, the quotient map is open if and only if the quotient topology is given by the hull-kernel process on the quotient space (regarded as a set of ideals). Some other necessary and sufficient conditions for the openness of ϕ_A in the unital case are given in [**23**], theorem 4·3. If A is a von Neumann algebra then ϕ_A is open [**16**, **23**] but there is an example of a quotient B of a type I von Neumann algebra for which ϕ_B is not open [**21**]. Another weakly central C^*-algebra for which condition (iii) of Theorem 3·1 fails (contrary to the claim in [**20**]) is given in [**3**], example 4·8.

In the next section we shall consider in detail the possible relations between Glimm (A) and Min-Primal (A). The following result will be required.

LEMMA 2·2. *Let J be a proper primal ideal in a C^*-algebra A. Then there exists a unique Glimm ideal G such that $G \subseteq J$.*

Proof. Let $P \in \text{Prim}\,(A/J)$. By [**4**], proposition 3·2, $P \approx Q$ for each $Q \in \text{Prim}\,(A/J)$. Let $G = \phi_A(P)$. Then

$$\begin{aligned} G &= \bigcap [P] \\ &\subseteq \bigcap \{Q : Q \in \text{Prim}\,(A/J)\} \\ &= J. \end{aligned}$$

Finally, if $G_1 \in \text{Glimm}\,(A)$ and $G_1 \subseteq J$ then $G_1 \subseteq P$ and so, as noted after the definition of Glimm (A), $G_1 = \phi_A(P)$. Hence $G_1 = G$. ∎

In certain situations Glimm (A) and Min-Primal (A) may coincide, as sets. If this occurs then necessarily $\tau_{cr} \leqslant \tau_q \leqslant \tau$ (see Lemma 3·1) and in favourable circumstances the three topologies coincide (see Theorem 3·3). However, in [**9**], example 9·2,

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although Glimm (A) = Min-Primal (A) (as sets) the topology τ is locally compact whilst the topology τ_q $(= \tau_{cr})$ is not. On the other hand, by generalizing the construction in [3], example 4·8 it is possible to construct a unital C^*-algebra A (so that τ_q coincides with τ_{cr} and makes Glimm (A) a compact Hausdorff space) such that Glimm (A) = Min-Primal (A) (as sets) but τ is not locally compact.

3. *The main results*

In this section we introduce a second relation $\sim$ on Prim (A) and study its connection with primal ideals and also with the equivalence relation $\approx$ of Section 2.

For $P, Q \in \text{Prim}\,(A)$ we say that $P \sim Q$ if P and Q cannot be separated by disjoint open subsets of Prim (A). It is immediate that if $P \sim Q$ then $P \approx Q$ but it is possible for the converse to fail. For example, in the C^*-algebra of sequences $x = (x_n)_{n \geq 1}$ of 2×2 complex matrices such that, as $n \to \infty$,

$$x_{2n} \to \begin{bmatrix} \lambda(x) & 0 \\ 0 & \mu(x) \end{bmatrix} \quad \text{and} \quad x_{2n+1} \to \begin{bmatrix} \mu(x) & 0 \\ 0 & \sigma(x) \end{bmatrix}$$

we have $\ker \lambda \sim \ker \mu$ and $\ker \mu \sim \ker \sigma$ but $\ker \lambda \not\sim \ker \sigma$.

Let A be a C^*-algebra for which $\sim$ is an equivalence relation and let $q_A : \text{Prim}\,(A) \to \text{Prim}\,(A)/\sim$ be the quotient map. Since each equivalence class is a closed subset of Prim (A), Prim $(A)/\sim$ is a T_1-space. It is not clear whether Prim $(A)/\sim$ need be Hausdorff, but this is certainly the case if q_A is open (see the proof of [**6**], lemma 7). If A has a unit then Prim $(A)/\sim$ is compact since q_A is continuous. If, in addition, Prim $(A)/\sim$ is Hausdorff then the functions $f \circ q_A$ $(f \in C(\text{Prim}\,(A)/\sim))$ separate the equivalence classes in Prim (A), from which it follows that $\sim$ is equal to $\approx$. We remark that we do not know of an example in which $\sim$ is an equivalence relation but not equal to $\approx$.

LEMMA 3·1. *Let A be a C^*-algebra such that every Glimm ideal is primal. Then*

(i) *the relation $\sim$ on* Prim (A) *is equal to $\approx$ (and hence $\sim$ is an equivalence relation)*;

(ii) Glimm (A) = Min-Primal (A), *as sets*;

(iii) *the topology τ on* Min-Primal (A) *is finer than the topology τ_q on* Glimm (A).

Proof. (i) Suppose that $P \approx Q$ in Prim (A). Then $\phi_A(P) = \phi_A(Q)$ in Glimm (A). Since the primal ideal $\phi_A(P)$ is contained in both P and Q, it follows from [**4**], proposition 3·2 that there is a net (P_α) in Prim (A) which is convergent to both P and Q. Hence $P \sim Q$ as required.

(ii) Let J be a minimal primal ideal of A. Since J is primal there exists $G \in$ Glimm (A) such that $G \subseteq J$ (Lemma 2·2). By hypothesis, G is primal and so $G = J$ by minimality. Hence $J \in$ Glimm (A).

Conversely, suppose that $G \in$ Glimm (A) and suppose that J is a primal ideal such that $J \subseteq G$. By Lemma 2·2 there is a Glimm ideal G' such that $G' \subseteq J \subseteq G$. Since each primitive ideal contains a unique Glimm ideal (see Section 2), $G' = G$ and so $G = J$.

(iii) Let U be a non-empty τ_q-open subset of Glimm (A). Then

$$\phi_A^{-1}(U) = \{P \in \text{Prim}\,(A) : P \supseteq G \text{ for some } G \in U\}$$

is open in Prim (A) and so there is an ideal J of A such that

$$\phi_A^{-1}(U) = \{P \in \text{Prim}\,(A) : P \not\supseteq J\}.$$

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Suppose that $I \in \text{Min-Primal}(A)$ and $I \not\supseteq J$. Then there exists $P \in \text{Prim}(A/I)$ such that $P \not\supseteq J$. Hence $I = \phi_A(P) \in U$. Conversely, it is easy to check that if $I \in U$ then $I \not\supseteq J$. Thus

$$U = \{I \in \text{Min-Primal}(A) : I \not\supseteq J\}$$

which is τ-open. ∎

It is not necessary that every Glimm ideal should be primal in order to have $\sim$ equal to $\approx$. In [3], example 4·12, $\sim$ is equal to $\approx$, but the Glimm ideal $Q_1 \cap Q_2 \cap Q_3$ is not primal. However, the next result gives a useful partial converse to Lemma 3·1. We are grateful to Charles Batty for supplying the inductive argument in the proof below, thereby enabling us to remove a separability condition that was present in an earlier version.

PROPOSITION 3·2. *Let A be a C^*-algebra such that $\sim$ is an open equivalence relation. Then $\sim$ is equal to $\approx$ and every Glimm ideal of A is primal (and hence minimal primal).*

Proof. Since $\text{Prim}(A)$ is locally compact and $q_A : \text{Prim}(A) \to \text{Prim}(A)/\sim$ is continuous and open, $\text{Prim}(A)/\sim$ is locally compact. Furthermore, as noted above, the fact that q_A is open also implies that $\text{Prim}(A)/\sim$ is Hausdorff. Thus if $P, Q \in \text{Prim}(A)$ and $P \not\sim Q$ then there exists $f \in C^b(\text{Prim}(A)/\sim)$ such that $f \circ q_A(P) \neq f \circ q_A(Q)$ and hence $P \not\approx Q$. This shows that $\sim$ is equal to $\approx$. It follows that $\phi_A : \text{Prim}(A) \to \text{Glimm}(A)$ is τ_q-open and hence τ_q and τ_{cr} are equal on $\text{Glimm}(A)$ (see Section 2).

Let $G \in \text{Glimm}(A)$. Let $n \geqslant 2$ and suppose that $J_1, J_2, \ldots, J_n$ are ideals of A such that $J_1 J_2 \ldots J_n = \{0\}$. In order to show that G is primal we require that $J_i \subseteq G$ for at least one value of i. For each i let $V_i = \phi_A^{-1}(\phi_A(\text{Prim } J_i))$, an open subset of $\text{Prim}(A)$ since ϕ_A is open. We prove by induction on r that

$$(*) \quad V_1 \cap \ldots \cap V_r \cap \text{Prim } J_{r+1} \cap \ldots \cap \text{Prim } J_n = \varnothing.$$

For $r = 0$ this follows immediately from the fact that $J_1 J_2 \ldots J_n = \{0\}$. Let

$$U_0 = \text{Prim } J_2 \cap \ldots \cap \text{Prim } J_n,$$

$$U_k = V_1 \cap \ldots \cap V_k \cap \text{Prim } J_{k+2} \cap \ldots \cap \text{Prim } J_n \quad (1 \leqslant k \leqslant n-2),$$

$$U_{n-1} = V_1 \cap \ldots \cap V_{n-1}.$$

Suppose that (*) is true when $r = k$ for some k with $0 \leqslant k < n$. Then U_k and $\text{Prim } J_{k+1}$ are disjoint open subsets of $\text{Prim}(A)$. Thus if $P \in U_k$ and $Q \in \text{Prim } J_{k+1}$ then $P \not\sim Q$ and so $\phi_A(P) \neq \phi_A(Q)$. Hence $P \notin V_{k+1}$ and so $U_k \cap V_{k+1} = \varnothing$ i.e. (*) holds for $r = k+1$. By induction, (*) holds for $r = n$ and so $\bigcap_{i=1}^n V_i = \varnothing$.

Now suppose that $J_i \not\subseteq G$ for each $i = 1, 2, \ldots, n$. Fix $P \in \text{Prim}(A/G)$. For each i there exists $P_i \in \text{Prim}(A/G)$ such that $P_i \not\supseteq J_i$. Then $P_i \in \text{Prim } J_i$ and $\phi_A(P_i) = \phi_A(P)$ so $P \in V_i$ for each i, which contradicts the fact that $\bigcap_{i=1}^n V_i = \varnothing$. Thus $J_i \subseteq G$ for at least one value of i and so G is primal. ∎

We note that for the conclusions of Proposition 3·2 to hold it is not sufficient to assume that $\approx$, rather than $\sim$, is open. For example, in the algebra of [2], p. 676 the equivalence relation $\approx$ is open but $\sim$ is not transitive and the Glimm ideal $\{0\}$ is not primal.

We now define a C^*-algebra A to be quasi-standard if $\sim$ is an open equivalence

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relation on Prim (A). The reason for this choice of terminology is explained in Section 4. Note that, as shown in the proof of Proposition 3·2, if A is quasi-standard then ϕ_A is open and so Theorem 2·1 applies. We shall use this in the next theorem, which gives some characterizations of quasi-standard C^*-algebras. Condition (v) is the hypothesis used in [3], lemma 5·1.

THEOREM 3·3. *Let A be a C^*-algebra. The following conditions are equivalent*:

(i) *A is quasi-standard*;

(ii) *the complete regularization map ϕ_A is open and every Glimm ideal of A is primal*;

(iii) *every Glimm ideal of A is primal and the topologies τ_q and τ_{cr} on* Glimm (A) *coincide with the topology τ on* Min-Primal (A);

(iv) *every Glimm ideal of A is primal and the topology τ_q on* Glimm (A) *coincides with the topology τ on* Min-Primal (A);

(v) Min-Primal (A) *is τ_s-closed in* Primal′ (A) (*the set of proper primal ideals of A*) *and each primitive ideal of A contains a unique minimal primal ideal.*

Proof. (i) $\Rightarrow$ (ii). This follows from Proposition 3·2 and its proof.

(ii) $\Rightarrow$ (iii). Assuming (ii), it follows from Lemma 3·1 that Glimm (A) = Min-Primal (A) and $\tau \geqslant \tau_q = \tau_{cr}$. Since ϕ_A is open, it follows from Theorem 2·1 ((i) $\Rightarrow$ (iii)) that τ_q is finer than τ, and so $\tau = \tau_q = \tau_{cr}$.

(iii) $\Rightarrow$ (iv). This is immediate.

(iv) $\Rightarrow$ (v). Let (J_α) be a net of minimal primal ideals which is τ_s-convergent to a proper primal ideal I. Let $a \in A \backslash I$. By [13], 3·3·6, the set

$$X = \{G \in \text{Glimm}\,(A) : \|a+G\| \geqslant \tfrac{1}{2}\|a+I\|\},$$

is the image under ϕ_A of the compact subset

$$\{P \in \text{Prim}\,(A) : \|a+P\| \geqslant \tfrac{1}{2}\|a+I\|\}$$

of Prim (A). Thus X is τ_q-compact and, assuming (iv), contains J_α eventually. Hence there is a subnet of (J_α) which is τ_q-convergent to some $G \in X$. By (iv), this subnet is τ_s-convergent to G and so $G = I$ by uniqueness of τ_s-limits in Primal (A). Thus $I \in$ Min-Primal (A) by (iv).

Suppose that P is a primitive ideal containing minimal primal ideals J_1 and J_2. Then (iv) yields that $J_1 = \phi_A(P) = J_2$.

(v) $\Rightarrow$ (i). Assuming (v), it is shown in [3], lemma 5·1 that the map Φ, which takes each primitive ideal to the unique minimal primal ideal that it contains, is continuous and open. Let $P, Q \in \text{Prim}\,(A)$. We shall show that $P \sim Q$ if and only if $\Phi(P) = \Phi(Q)$, from which it will follow that $\sim$ is an open equivalence relation.

Suppose that $P \sim Q$. Then $P \cap Q$ is primal by [4], p. 60. Let J be a minimal primal ideal contained in $P \cap Q$. By uniqueness, $\Phi(P) = J = \Phi(Q)$. Conversely, suppose that $\Phi(P) = \Phi(Q)$. By [4], proposition 3·2 there is a net in Prim (A) which is convergent to both P and Q. Hence $P \sim Q$. ∎

Examples of quasi-standard C^*-algebras are discussed in [3], section 5, using condition (v) of Theorem 3·3. There are some interesting examples of quasi-standard group C^*-algebras. For example, the C^*-algebra of the discrete Heisenberg group is quasi-standard ([3], example 5·4), as is the C^*-algebra of the continuous Heisenberg group (described in [19], section III). Another interesting example of a quasi-

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standard C^*-algebra is given in [19], section IV. The easiest way to see that these algebras are quasi-standard is to use the characterization given below in Theorem 3·4.

If I is a non-zero ideal of a C^*-algebra A then Prim (I) is (homeomorphic to) an open subset of Prim (A) and so it follows directly from the definition that if A is quasi-standard then so is the C^*-algebra I. On the other hand, although every von Neumann algebra is quasi-standard (see [3], section 5) there is an example of a quotient of a type I von Neumann algebra which is not quasi-standard [21]. It also follows from Proposition 3·6 below that a quotient of a quasi-standard C^*-algebra need not be quasi-standard.

Recall that if a C^*-algebra A is a maximal full algebra of cross-sections over a base space X with fibre A_x at $x \in X$ then there is a one-to-one correspondence between X and a set of ideals of A given by

$$x \leftrightarrow I_x = \{a \in A : a_x = 0\} \quad (x \in X).$$

For each $x \in X$ there is an (isometric) *-isomorphism of A/I_x onto A_x given by $a + I_x \mapsto a_x$ $(a \in A)$.

THEOREM 3·4. *Let A be a C^*-algebra. Then the following conditions are equivalent:*

(i) *A is quasi-standard;*

(ii) Min-Primal (A) *is locally compact and A is *-isomorphic to a maximal full algebra of cross-sections over* Min-Primal (A);

(iii) *A is *-isomorphic to a maximal full algebra of cross-sections in which the base space contains a dense set of proper primal ideals.*

Proof. (i) $\Rightarrow$ (ii). This follows from Theorem 3·3 and either Theorem 2·1 or [3], theorem 5·3.

(ii) $\Rightarrow$ (iii). This is immediate.

(iii) $\Rightarrow$ (i). We may suppose that A is a maximal full algebra of cross-sections over a locally compact Hausdorff space X and that a dense subset of X consists of proper primal ideals. The topology on X is stronger than τ_s and hence stronger than τ_w. Since Primal (A) is τ_w-closed in Id (A) (by [3], proposition 3·1), each ideal in X is primal. By [18], theorem 4 there is a continuous open map ϕ of Prim (A) onto X such that $\phi(P) \subseteq P$ $(P \in \text{Prim}\,(A))$. By the maximality of the full algebra, for each $P \in \text{Prim}\,(A)$ the ideal $\phi(P)$ is the unique ideal of X contained in P (see, for example, the proof of [3], theorem 5·3). Let $P, Q \in \text{Prim}\,(A)$. We shall show that

$$P \approx Q \Leftrightarrow \phi(P) = \phi(Q).$$

Suppose that $\phi(P) = \phi(Q)$. Since $\phi(P)$ is primal and $P, Q \supseteq \phi(P)$ it follows from [4], proposition 3·2 that $P \sim Q$ and hence $P \approx Q$. Conversely, suppose that $\phi(P) \neq \phi(Q)$. Without loss of generality we may assume that there exists $a \in \phi(P) \backslash \phi(Q)$. For $R \in \text{Prim}\,(A)$ let $f(R) = \|a + \phi(R)\|$. Since ϕ is continuous we have $f \in C^b(\text{Prim}\,(A))$ and hence $P \not\approx Q$.

It now follows that for $P \in \text{Prim}\,(A)$

$$\begin{aligned}\phi(P) &= \bigcap \{Q \in \text{Prim}\,(A) : Q \supseteq \phi(P)\} \\ &= \bigcap \{Q \in \text{Prim}\,(A) : \phi(Q) \approx \phi(P)\} \\ &= \bigcap \{Q \in \text{Prim}\,(A) : Q \approx P\} \\ &= \phi_A(P).\end{aligned}$$

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Thus $X = \text{Glimm}(A)$, as sets, and $\phi = \phi_A$. Since ϕ is continuous and open, the given topology on X equals the quotient topology τ_q on Glimm (A). Furthermore, ϕ_A is open and so A is quasi-standard by Theorem 3·3 ((ii) ⇒ (i)). ∎

COROLLARY 3·5. *Let A be a separable C^*-algebra. The following conditions are equivalent*:

(i) *A is quasi-standard*;

(ii) *A is *-isomorphic to a maximal full algebra of cross-sections in which the base space contains a dense set of primitive ideals.*

Proof. (i) ⇒ (ii). This follows from Theorem 3·4 ((i) ⇒ (ii)) and the fact that if A is separable then Min-Primal (A) contains a dense subset of primitive ideals (see [3], corollary 4·6).

(ii) ⇒ (i). This follows from Theorem 3·4 ((iii) ⇒ (i)) since primitive ideals are primal. ∎

There is a construction due to Dixmier [12], proposition 4 of an inseparable non-unital C^*-algebra such that no minimal primal ideal is primitive [3], pp. 533–4. It can be checked directly that this algebra is quasi-standard. Thus the implication (i) ⇒ (ii) in Corollary 3·5 need not hold without the assumption of separability.

We now illustrate the application of Theorem 3·4 to three of the examples mentioned after Theorem 3·3. Let A be the C^*-algebra of the discrete Heisenberg group [1]. Then A is *-isomorphic to a maximal full algebra of cross-sections over the unit circle T. If $\lambda \in T$ is not a root of unity then the fibre at λ is the corresponding irrational rotation algebra and so the ideal of A corresponding to λ is primitive and hence primal. By Theorem 3·4 ((iii) ⇒ (i)), A is quasi-standard (see [3], example 5·4 for another proof).

Next, let A be the C^*-algebra of the continuous Heisenberg group. Then A is *-isomorphic to a maximal full algebra of cross-sections over $\mathbb{R}$ (see [19], section III). For $\lambda \in \mathbb{R}\backslash\{0\}$, the fibre at λ is *-isomorphic to the algebra of compact linear operators on a separable infinite-dimensional Hilbert space. Thus the ideal of A corresponding to λ is primitive and so A is quasi-standard by Theorem 3·4.

Finally, let A be the algebra of [19], section IV, which is *-isomorphic to a maximal full algebra of cross-sections over the one-point compactification of the positive integers. The fibre at the integer n is *-isomorphic to the algebra of $n \times n$ complex matrices and so A is quasi-standard by Theorem 3·4.

The next result is a development of [4], p. 59 and [3], proposition 4·7 which will be useful in Section 4.

PROPOSITION 3·6. *Let B be an arbitrary C^*-algebra. Then there is a quasi-standard C^*-algebra A with a minimal primal ideal J such that $B \cong A/J$. If B is separable then A can be constructed so as to be separable.*

Proof. We may assume that B is a C^*-algebra of operators on a Hilbert space H. Let A be the C^*-algebra of all sequences $x = (x_n)_{n \geq 1}$ of elements in $L(H)$ which are norm-convergent to an element $a(x)$ in A. Then A is a full algebra of cross-sections over the one-point compactification of the positive integers and A is maximal by [14], corollary on p. 242. The ideal corresponding to a positive integer is primitive and so the proof of (iii) ⇒ (i) in Theorem 3·4 shows that A is quasi-standard and the ideal J corresponding to the point at infinity (the ideal of null sequences) is a primal Glimm ideal and hence minimal primal. Clearly $B \cong A/J$.

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Suppose that B is separable. Then we can construct A to be separable as in [3], proposition 4·7 and we can apply Theorem 3·4 as in the previous paragraph. ∎

4. *Comparison with Ching's results*

In this section we compare the quasi-standard C^*-algebras of Section 3 with the pre-standard C^*-algebras introduced by Ching [6]. We show that the class of pre-standard algebras is strictly included in the quasi-standard class. Thus the theory of quasi-standard C^*-algebras, which requires no assumptions concerning the existence of minimal primitive ideals, may be regarded as an effective generalization of the pre-standard theory. We begin with some terminology from [6].

A C^*-algebra A is said to be bounded if every primitive ideal contains a minimal primitive ideal. If A is separable then every prime ideal is primitive [11] and so it follows from Zorn's lemma that A is bounded. The basic structure space A^b of a bounded C^*-algebra A is defined to be the set of all minimal primitive ideals with the restricted Jacobson topology. It is easy to check that if A is bounded and $P, Q \in A^b$ then $P \neq Q$ if and only if P and Q can be separated by disjoint open sets in A^b. The restriction of $\sim$ to A^b is not necessarily an equivalence relation, as is shown by the example for $\sim$ itself (see Section 3) in which A^b coincides with $\mathrm{Prim}\,(A)$.

A C^*-algebra A is said to be *-bounded if it is bounded and whenever S is a non-empty subset of A^b such that $\ker S \subseteq P$ for some $P \in \mathrm{Prim}\,(A)$ then there exists $P_0 \in A^b$ such that $\ker S \subseteq P_0 \subseteq P$. A C^*-algebra A is said to be normal if it is *-bounded and each primitive ideal of A contains only one minimal primitive ideal of A. To illustrate the differences between the above notions, we note that in [3], example 4·8 A is bounded but not *-bounded (consider the inclusion $J \supseteq \bigcap_{n \geqslant 1} P_{2n}$), whilst the example in [2], p. 676 is *-bounded but not normal (see [3], p. 537). This shows that a postliminal C^*-algebra need not be normal, thus answering a question raised in [6], p. 141.

We can now give the key definitions from [6]. A C^*-algebra A is standard if it is normal and A^b is a Hausdorff space. A is pre-standard if it is normal and the restriction of $\sim$ to A^b is an open equivalence relation. It is shown in [6], theorem 1 that if A is standard then A is *-isomorphic to a maximal full algebra of cross-sections with each fibre a primitive C^*-algebra. It is not difficult to show that, conversely, each such maximal full algebra of cross-sections is standard. These facts should be compared with Theorem 3·4 and Corollary 3·5.

Using Proposition 3·6 we are able to construct examples of quasi-standard C^*-algebras which are not pre-standard. Let B be any C^*-algebra with a primitive ideal containing two distinct minimal primitive ideals (a separable example is given in [2], p. 676). Then the C^*-algebra A, as constructed in the proof of Proposition 3·6, is quasi-standard but not normal. As indicated in Proposition 3·6, if B is separable the construction can be modified so as to ensure that A is separable.

We now show that every pre-standard C^*-algebra is quasi-standard.

THEOREM 4·1. *Let A be a pre-standard C^*-algebra. Then A is quasi-standard.*

Proof. Let $X = A_b/\sim$ and let $q: A^b \to X$ be the quotient map. By [6], lemma 7, X is Hausdorff. For each $P \in \mathrm{Prim}\,(A)$ let P_0 denote the unique minimal primitive ideal of A contained in P.

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We define a map $\Phi: \mathrm{Prim}\,(A) \to X$ by $\Phi(P) = q(P_0)$ $(P \in \mathrm{Prim}\,(A))$. If U is an open subset of $\mathrm{Prim}\,(A)$ then, since A is bounded,

$$\Phi(U) = \Phi(U \cap A^b) = q(U \cap A^b)$$

which is an open set of X. Thus Φ is an open mapping. We now show that Φ is continuous.

Let Y be a closed subset of X. As in [6] we may identify X with a set of ideals in A via the one-to-one correspondence:

$$x \leftrightarrow \bigcap \{P \in A^b : q(P) = x\} \quad (x \in X).$$

Then $\ker Y = \ker \Phi^{-1}(Y)$ and so for the continuity of Φ it suffices to show that

$$\Phi^{-1}(Y) = \mathrm{hull}\ker Y,$$

where the hull is taken in $\mathrm{Prim}\,(A)$. So suppose that $P \in \mathrm{Prim}\,(A)$ and $P \supseteq \ker Y$. Since $\ker Y = \ker q^{-1}(Y)$ and A is *-bounded, there exists $P_0 \in A^b$ such that

$$\ker q^{-1}(Y) \subseteq P_0 \subseteq P.$$

Thus P_0 lies in the closure of $q^{-1}(Y)$ in A^b and so $q(P_0) \in \bar{Y} = Y$ by the continuity of q. Conversely, if $P \in \mathrm{Prim}\,(A)$ and $\Phi(P) \in Y$ then $\ker Y \subseteq \Phi(P) \subseteq P$ and so $P \in \mathrm{hull}\ker Y$. Thus Φ is continuous.

Finally, we show that if $P, Q \in \mathrm{Prim}\,(A)$ then $P \sim Q$ if and only if $\Phi(P) = \Phi(Q)$. If $\Phi(P) \neq \Phi(Q)$ then $P \not\sim Q$ since X is Hausdorff and Φ is continuous. Conversely, if $\Phi(P) = \Phi(Q)$ then $q(P_0) = q(Q_0)$ and so $P_0 \sim Q_0$. Since every neighbourhood of P (respectively Q) in $\mathrm{Prim}\,(A)$ contains P_0 (respectively Q_0), it follows that $P \sim Q$.

Since $P \sim Q$ if and only if $\Phi(P) = \Phi(Q)$, it follows from the fact that Φ is a continuous open mapping that $\sim$ is an open equivalence relation as required. ∎

The results of this section show that, as contended in [3], p. 541, the notion of normality used in the definition of a pre-standard algebra is stronger than is necessary for a maximal representation by cross-sections. One may weaken the notion of normality to: every primitive ideal contains a unique ideal in the quotient space X. It can then be proved that a bounded C^*-algebra is quasi-standard if and only if it is 'pre-standard' (using the amended version of normality).

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Note added in proof. Concerning the remarks preceding Lemma 3·1, the second-named author has recently shown in [21], proposition 3·2, that if A is a quasi-central C^*-algebra then $\sim$ is an equivalence relation on $\mathrm{Prim}\,(A)$ if and only if $\sim$ is equal to $\approx$ on $\mathrm{Prim}\,(A)$.

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