# CALCULATION OF $S$-MATRIX ELEMENTS

By J. S. R. CHISHOLM

Communicated by J. Hamilton

*Received* 14 August 1951

*ABSTRACT*. A method of calculating $S$-matrix elements is described. All factors corresponding to lines in Feynman-Dyson graphs are put in the form of scalar boson line factors acted on by certain operators $\mathscr{F}$ and $\mathscr{V}_\alpha$ involving only the energy-momenta of external lines and particle masses. A general formula is given for the integral of a product of scalar boson factors, and the effect of a product of operators $\mathscr{F}$ and $\mathscr{V}_\alpha$ on this result is described. The infinities appear in a different form, and it is shown how subtraction procedure can be applied to eliminate them.

1. *Introduction.* A simple method of writing down matrix elements for processes involving two or more interacting fields has been worked out by Feynman (7, 8) and Dyson (2, 3). These processes are so far limited to collisions describable by an $S$-matrix, though Dyson (4, 5, 6) has extended the theory to include bound-state problems. The method consists of drawing graphs to represent possible sets of transitions between the initial and final states of the system considered. The $S$-matrix element is the space-time integral of the probability amplitude of the process described by the graph, one integration being required for each vertex of the graph. Each line and each vertex in a graph contribute a factor to the probability amplitude of any process; the line factors are probability amplitudes for the single transitions that they represent. The space-time integrations are easily completed by going into momentum representation, and result in energy-momentum conservation at each vertex of the graph; in practice this law is automatically taken into account when the lines of the graph are labelled. A method of integrating over the arbitrary energy-momentum variables (which we shall call 'internal momenta') has been given by Feynman ((8), Appendix), and used by Karplus and Kroll (10) to calculate the fourth-order contribution to the electron's magnetic moment. For theories involving fermion, photon and scalar meson fields, a finite $S$-matrix can be defined, and the integration over internal momenta, involving subtraction procedure, can be done by a much simpler method, described in this paper. A summary of the work has already been given (1).

2. *The matrix elements.* The vertex and line factors for interactions involving three types of field will be considered; these fields are:

(i) the fermion ($F$) field (electrons or nucleons),

(ii) the scalar or pseudoscalar meson ($M$) field, and

(iii) the photon ($P$) field.

Only scalar (and pseudoscalar) meson fields with scalar interaction are considered, since other meson theories have not yet been shown to possess a consistent $S$-matrix theory. However, to any finite order the number of types of divergence in all theories

https://doi.org/10.1017/S0305004100027651 Published online by Cambridge University Press

is finite, and a 'subtraction' procedure can be developed to give finite results to any order.

When the external lines of a graph are labelled with energy-momentum 4-vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m$ it must be remembered that the overall conservation law $\sum_{r=1}^{m} \mathbf{p}_r = 0$ holds. Apart from this the $\mathbf{p}_r$ are independent. These variables will occur in linear combinations in other lines of the graph; these linear combinations are called the 'external momenta' of the lines. It is necessary to have an external momentum $\mathbf{p}$ for each fermion line of a graph, for each meson line which connects with a meson-photon vertex and for each photon line connecting with a self-energy part. If $\mathbf{p}$ does not occur naturally in such a line, as when closed loops of fermion lines occur, a constant $\mathbf{p}$, called a 'false external momentum', is introduced and is equated to zero at the end of the work. An independent symbol $\mathbf{p}$ is used for each line, and at the end they are given their correct values in terms of the momenta $\mathbf{p}_r$.

In the following work only the fields $F$, $M$ and $P$ are considered, though calculations in all theories can be made by the following method. Line factors in matrix elements are denoted by $(F)$, $(M)$ and $(P)$, and vertex factors by $(FP)$, $(MP)$ and $(MF)$. Following the methods of Dyson the matrix element factors for the various lines and vertices are found to be

$$(FP) \quad (e/\hbar c)\,(2\pi)^4 \gamma_\mu, \tag{1a}$$

$$(MP) \quad \begin{matrix} (e/\hbar^2 c^2)\,(2\pi)^4\, i[(\mathbf{p}'+\mathbf{k}')+(\mathbf{p}''+\mathbf{k}'')]_\alpha, \\ (e/\hbar^2 c^2)\,(2\pi)^4\, i[\mathbf{p}_r+(\mathbf{p}''+\mathbf{k}'')]_\alpha, \end{matrix}$$

$$(MF) \quad (f/\hbar c)\,(2\pi)^4 \gamma_5, \tag{1c}$$

$$(F) \quad -\frac{i}{(2\pi)^4}\frac{i\boldsymbol{\gamma}(\mathbf{p}+\mathbf{k})-\kappa}{(\mathbf{p}+\mathbf{k})^2+\kappa^2},$$

$$(M) \quad -\frac{i\hbar c}{(2\pi)^4}\frac{1}{(\mathbf{p}'+\mathbf{k}')^2+\lambda^2},$$

$$(P) \quad -\frac{i\hbar c}{(2\pi)^4}\frac{\delta_{\alpha\beta}}{(\mathbf{p}'+\mathbf{k}')^2+\eta^2} \quad (\eta \to 0). \tag{2c}$$

The symbols $\mathbf{p}, \mathbf{p}', \mathbf{p}''$ denote external momenta. The momenta $\mathbf{k}, \mathbf{k}', \mathbf{k}''$ are linear combinations of the internal momenta and are determined by the choice of internal momenta and by the energy-momentum conservation law holding at each vertex. The factor in square brackets in $(MP)$ is the sum of the momenta $\mathbf{p}'+\mathbf{k}'$ and $\mathbf{p}''+\mathbf{k}''$ of the two meson lines at the vertex considered, and is placed to the left of both factors $(M)$ corresponding to the meson lines at that vertex. In $(MF)$ the $\gamma_5$ factor occurs in pseudoscalar, but not scalar theory. Other factors contributing to the $S$-matrix elements in three-field theory are: a factor $-ie^2/(\hbar c)^3\,(2\pi)^4 \delta_{\mu\nu}$ at a meson-photon vertex; a factor $(-1)^{n'-k-1}$, where $n'$ is the number of vertices connected to two nucleon lines, and $k$ is the number of closed loops of nucleon lines in the graph. These line and vertex factors have been given by several authors (11).

https://doi.org/10.1017/S0305004100027651 Published online by Cambridge University Press

In order to put all integrals in a standard form, $(MP)$, $(M)$ and $(F)$ are expressed differently; the new form of the $(F)$ factor was given by Karplus and Kroll ((10), equation (58)):

$$(MP) \quad \begin{aligned} &(e/\hbar^2c^2)\,(2\pi)^4\Big[i\partial/\partial p'_\alpha\int_{\sigma'}^{\infty}d\sigma'+i\partial/\partial p''_\alpha\int_{\sigma''}^{\infty}d\sigma''\Big],\\ &(e/\hbar^2c^2)\,(2\pi)^4\Big[i(\mathbf{p}_r)_\alpha+i\partial/\partial p''_\alpha\int_{\sigma''}^{\infty}d\sigma''\Big], \end{aligned} \qquad (1b)$$

$$(M) \quad -\frac{i\hbar c}{(2\pi)^4}\frac{1}{(\mathbf{p}'+\mathbf{k}')^2+\sigma'}, \qquad (2b)$$

$$(F) \quad -\frac{i}{(2\pi)^4}\Big[-\kappa-(i\gamma/2)\,\partial/\partial\mathbf{p}\int_{\sigma}^{\infty}d\sigma\Big]\frac{1}{(\mathbf{p}+\mathbf{k})^2+\sigma}. \qquad (2a)$$

The square bracket in $(1b)$ is denoted by $\mathscr{V}_\alpha(\mathbf{p}',\sigma';\ \mathbf{p}'',\sigma'')$ or $\mathscr{V}_\alpha(\mathbf{p}_r;\ \mathbf{p}'',\sigma'')$ and the square bracket in $(2a)$ by $\mathscr{F}(\mathbf{p},\sigma)$. The mass terms $\kappa^2$ and $\lambda^2$ in $(M)$ and $(F)$ have been replaced by symbols $\sigma,\sigma',\sigma'',\ldots$, the sigmas occurring in the same line factors as the corresponding external momenta $\mathbf{p},\mathbf{p}',\mathbf{p}'',\ldots$. When the integrations with respect to these symbols have been completed, the appropriate substitution ($\sigma=\kappa^2$, or $\sigma=\lambda^2$) is made for each sigma. The alternative forms for $(MP)$ are needed because the vertex may be connected to two internal meson lines (momenta $\mathbf{p}'+\mathbf{k}'$ and $\mathbf{p}''+\mathbf{k}''$) or to one internal and one external meson line (momenta $\mathbf{p}''+\mathbf{k}''$ and $\mathbf{p}_r$).

Using these new line and vertex factors, it is seen that the internal momenta only occur in line factors of the form $[(\mathbf{p}+\mathbf{k})^2+\sigma]^{-1}$. (3)

The character of the lines is determined by the factors

$$\gamma_\mu,\gamma_5,\mathscr{F}(\mathbf{p},\sigma) \quad\text{and}\quad \mathscr{V}_\alpha(\mathbf{p}',\sigma';\ \mathbf{p}'',\sigma'').$$

Since these operators are independent of the internal momenta they are not considered until the integration over the internal momenta has been completed.

3. *The integration over internal momenta.* When the factors independent of the internal momenta have been taken outside the integral, the integrand is a product of factors of form (3). These denominators are combined by using the formula ((8), Appendix)

$$\frac{1}{ab}=\int_0^1\frac{du}{[au+b(1-u)]^2}$$

and its derivatives with respect to $a$ and $b$. This formula holds for all values of $a$ and $b$, but when $a$ and $b$ are of opposite sign, $u$ must be considered as a complex variable, and the path of integration must deviate from the real axis so that the singularity at $u=b/(b-a)$ is avoided. The residue of the integrand at this singularity is zero, so that the path of integration can be any path in the complex plane joining $u=0$ and $u=1$, but not passing through $u=b/(b-a)$. This was pointed out by C. A. Hurst. The resulting integral is of the form

$$\int d^4\mathbf{k}_1\int d^4\mathbf{k}_2\ldots\int d^4\mathbf{k}_l[Q(\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_l)]^{-r}, \qquad (4)$$

where $Q(\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_l)=\Sigma A_{st}\mathbf{k}_s\mathbf{k}_t+\Sigma\mathbf{A}_s\mathbf{k}_s+C$ is a quadratic form in the $\mathbf{k}$, $\mathbf{A}_s\mathbf{k}_s$ and $\mathbf{k}_s\mathbf{k}_t$ denoting scalar products of 4-vectors $\mathbf{A}_s,\mathbf{k}_s$ and $\mathbf{k}_t$. $A_{st}=A_{ts},\mathbf{A}_s$ and $C$ are functions of the auxiliary variables represented by $u$ above, and of the $\mathbf{p}$ and $\sigma$. The

https://doi.org/10.1017/S0305004100027651 Published online by Cambridge University Press

inverse power $r$ of $Q$ in the integrand is equal to the number $F$ of internal lines in the graph. The number $l$ of internal momenta is $F-n+1$, where $n$ is the number of vertices in the graph. Let $c_p$ be the coefficient of a particular factor $[(\mathbf{p}+\mathbf{k})^2+\sigma]$ in $Q$. Then the quadratic form $Q$ has the following properties:

P (i) The coefficients $A_{sl}$ are independent of the $p$ and $\sigma$.

P (ii) The coefficients $\mathbf{A}_s$ are linear combinations of the external momentum vectors $\mathbf{p}$. Therefore

P (iii) $\partial\mathbf{A}_s/\partial\mathbf{p}$ are second-rank tensors, multiples of the unit metric tensor $\mathbf{1}$. If $2\partial_{pk_s}$ is the coefficient of the scalar product $(\mathbf{pk}_s)$ for a particular $\mathbf{k}_s$ in $[(\mathbf{p}+\mathbf{k})^2+\sigma]$, so that $\partial_{pk_s}=0$ or $\pm 1$; then $\partial\mathbf{A}_s/\partial\mathbf{p}=c_p\partial_{pk_s}\mathbf{1}$.

P (iv) The only place where the $\sigma$ occur is in $C$, and there they occur linearly; $\partial c/\partial\sigma=c_p$.

P (v) The $\mathbf{p}$ occur in $C$ in the form $c_p\mathbf{p}^2$, so that

$$\tfrac{1}{2}\partial c/\partial\mathbf{p}=c_p\mathbf{p} \quad \text{and} \quad \tfrac{1}{2}\partial c/\partial\mathbf{p}\,\partial\mathbf{p}'=c_p\mathbf{1}.$$

For graphs not containing self-energy parts, (4) is completely convergent, and will now be evaluated. $Q(\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_l)$ is the sum of quadratic forms occurring in (3) with positive coefficients; so if the expression (3) is termed 'positive definite', meaning that it is positive definite in the region of momentum space defined by $\mathbf{k}_1^2,\mathbf{k}_2^2,\ldots,\mathbf{k}_e^2>0$, then $Q$ is also 'positive definite'; this implies that

$$A_{11}>0, \quad \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}>0, \quad \Lambda=\begin{vmatrix} A_{11} & A_{12} & \ldots & A_{1l} \\ A_{21} & A_{22} & \ldots & A_{2l} \\ \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots \\ A_{l1} & A_{l2} & \ldots & A_{ll} \end{vmatrix}>0, \tag{5}$$

and

$$\chi=\begin{vmatrix} A_{11} & A_{12} & \ldots & A_{1l} & \mathbf{A}_1 \\ \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots & \ldots\ldots \\ A_{l1} & A_{l2} & \ldots & A_{ll} & \mathbf{A}_l \\ \mathbf{A}_1 & \mathbf{A}_2 & \ldots & \mathbf{A}_l & C \end{vmatrix} \tag{6}$$

is 'positive definite'.

$\chi$ is the discriminant of $Q$, and is called the 'characteristic determinant' of the graph; it is a scalar, a product $\mathbf{A}_r\mathbf{A}_s$ occurring in $\chi$ being a scalar product. $\Lambda$ is the discriminant of the purely quadratic part of $Q$ and is independent of the $\mathbf{p}$ and $\sigma$. Now by an orthogonal transformation and a simple translation in $(\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_l)$ space (all with Jacobian determinant $+1$), $Q$ is reducible to the diagonal form

$$Q'=A_1'\mathbf{k}_1^2+A_2'\mathbf{k}_2^2+\ldots+A_l'\mathbf{k}_l^2+c';$$

since $Q'$ is 'positive definite', $A_1',A_2',\ldots,A_l'$ are positive and $C'$ is 'positive definite'. The infinite limits of integration are unchanged by such a transformation, and the nature of the poles on the $\mathbf{k}$-axes remain the same; this is seen most easily by using the rule, given by Feynman, of adding a small negative imaginary part to each $\sigma$, so that $C$ and hence $C'$ also have small negative imaginary parts, since every $c_p\geqslant 0$. $\chi$ and $\Lambda$ are invariants under the transformation; so (4) reduces to

$$\int d^4\mathbf{k}_1\int d^4\mathbf{k}_2\ldots\int d^4\mathbf{k}_l[A_1'\mathbf{k}_1^2+A_2'\mathbf{k}_2^2+\ldots+A_l'\mathbf{k}_l^2+C']^{-r}. \tag{7}$$

https://doi.org/10.1017/S0305004100027651 Published online by Cambridge University Press

The well-known formula of Feynman (8),

$$\int d^4\mathbf{k}(\mathbf{k}^2+L)^{-3} = i\pi^2/2L,$$

gives by differentiation with respect to $L$.

$$\int d^4\mathbf{k}(\mathbf{k}^2+L)^{-q} = i\pi^2/(q-1)\,(q-2)\,L^{q-2}. \tag{8}$$

By using (8) repeatedly, the integral (7) can be evaluated; the result is

$$(i\pi^2)^l \frac{(r-2l-1)!}{(r-1)!} \frac{1}{(A_1' A_2' \ldots A_l')^2 C'^{r-2l}}. \tag{9}$$

Writing (9) in invariant form, the value of (4) is

$$(i\pi^2)^l \frac{(r-2l-1)!}{(r-1)!} \frac{\Lambda^{r-2l-2}}{\chi^{r-2l}}. \tag{10}$$

If the graph contains a self-energy part, then in electrodynamics, for instance, the procedure developed by Ward (14, 15) of inserting a zero-momentum photon line into the self-energy part is adopted. The formulae given by Ward can be taken over directly into this theory, as will be shown later; for the present it is sufficient to note that these insertions change the self-energy parts into vertex parts, and that the integral (4) is therefore absolutely convergent. Before making the insertion it is at best conditionally convergent. Therefore (10) is valid for all graphs provided that Ward's procedure is followed.

The divergence difficulties do not appear at this stage, but are encountered when we apply the operators $\mathscr{F}$ and $\mathscr{V}_\alpha$ to (10). It will be shown in §6 that the Dyson-Ward subtraction procedure gives the usual finite results for theories involving the three fields discussed.

4. *Derivatives of* $\chi$. In (10) only the denominator factor $\chi^{-m}$ $(m = r-2l)$ is dependent on the $\mathbf{p}$ and $\sigma$. Therefore operators $\mathscr{F}$ and $\mathscr{V}$ affect this factor only, and the result of applying a series of such operators to $\chi^{-m}$ is now considered. Before this is done, it is best to define a number of 'derivatives' of $\chi$, and to show how they are most easily calculated in practice. A specimen calculation is given in the appendix to demonstrate the method used. From P (iv) it follows that

$$\chi_\sigma \equiv \partial\chi/\partial\sigma = \Lambda c_p, \tag{11}$$

where $\sigma$ and $\mathbf{p}$ originate from the same factor (3). These derivatives are independent of the $\mathbf{p}$ and $\sigma$. A vector $\boldsymbol{\chi}_p$ is defined by

$$\boldsymbol{\chi}_p = (\tfrac{1}{2}\chi_\sigma^{-1})\,\partial\chi/\partial\mathbf{p}. \tag{12a}$$

Using P (ii), P (iii), P (v) and (11) it follows that

$$\boldsymbol{\chi}_p = \frac{1}{\Lambda} \begin{vmatrix} A_{11} & A_{12} & \ldots & A_{1l} & \partial_{pk_1} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ A_{l1} & A_{l2} & \ldots & A_{ll} & \partial_{pk_l} \\ \mathbf{A}_1 & \mathbf{A}_2 & \ldots & \mathbf{A}_l & \mathbf{p} \end{vmatrix}. \tag{12b}$$

https://doi.org/10.1017/S0305004100027651 Published online by Cambridge University Press

When the scalar product of $\chi_p$ with another vector $\mathbf{v}$ is formed we have

$$(\mathbf{v}\chi_p) = \frac{1}{\Lambda}\begin{vmatrix} A_{11} & A_{12} & \dots & A_{1l} & \partial_{pk_1} \\ \dots & \dots & \dots & \dots & \dots \\ A_{l1} & A_{l2} & \dots & A_{ll} & \partial_{pk_l} \\ (\mathbf{v}\mathbf{A}_1) & (\mathbf{v}\mathbf{A}_2) & \dots & (\mathbf{v}\mathbf{A}_l) & (\mathbf{v}\mathbf{p}) \end{vmatrix}. \tag{13}$$

A 'second derivative' $\chi_{pp'}$ is defined by

$$\chi_{pp'} = (4\chi_\sigma\chi_{\sigma'})^{-1}\partial^2\chi/\partial\mathbf{p}\,\partial\mathbf{p}'. \tag{14a}$$

This is a second-rank metric tensor. Clearly

$$\chi_{pp'} = (2\chi_{\sigma'})^{-1}\partial\chi/\partial\mathbf{p}' \tag{14b}$$

$$= \frac{1}{2\Lambda^2}\begin{vmatrix} A_{11} & A_{12} & \dots & A_{1l} & \partial_{pk_1} \\ A_{21} & A_{22} & \dots & A_{2l} & \partial_{pk_2} \\ \dots & \dots & \dots & \dots & \dots \\ A_{l1} & A_{l2} & \dots & A_{ll} & \partial_{pk_l} \\ \partial_{p'k_1} & \partial_{p'k_2} & \dots & \partial_{p'k_l} & c_p^{-1}\delta_{pp'} \end{vmatrix}, \tag{14c}$$

where $\delta_{pp'}$ is the Kronecker delta. When the scalar product of $\chi_{pp'}$ with two vectors $\mathbf{v}$ and $\mathbf{v}'$ is formed

$$(\mathbf{v}\chi_{pp'}\mathbf{v}') = (\mathbf{v}\mathbf{v}')\,\Omega_{pp'}, \tag{15}$$

where

$$\Omega_{pp'} = \frac{1}{2\Lambda^2}\begin{vmatrix} A_{11} & A_{12} & \dots & A_{1l} & \partial_{pk_1} \\ A_{21} & A_{22} & \dots & A_{2l} & \partial_{pk_2} \\ \dots & \dots & \dots & \dots & \dots \\ A_{l1} & A_{l2} & \dots & A_{ll} & \partial_{pk_l} \\ \partial_{p'k_1} & \partial_{p'k_2} & \dots & \partial_{p'k_l} & c_p^{-1}\delta_{pp'} \end{vmatrix}. \tag{16}$$

$\Omega_{pp'}$ is a diagonal component of the metric tensor $\chi_{pp'}$; it is easily written down in the form (16), since the $A_{rs}$ and $\partial_{pk_r}$ are all derived immediately from $Q$.

All 'derivatives' of $\chi$ of higher order than $\chi_\sigma$ and $\chi_{pp'}$ are zero since these quantities are independent of the $\mathbf{p}$ and $\sigma$. The last definition is that of a scalar

$$\Omega_p = -\kappa + i\gamma\chi_p. \tag{17a}$$

By (13),

$$\Omega_p = -\kappa + \begin{vmatrix} A_{11} & A_{12} & \dots & A_{1l} & \partial_{pk_1} \\ A_{21} & A_{22} & \dots & A_{2l} & \partial_{pk_2} \\ \dots & \dots & \dots & \dots & \dots \\ A_{l1} & A_{l2} & \dots & A_{ll} & \partial_{pk_l} \\ (i\gamma\mathbf{A}_1) & (i\gamma\mathbf{A}_2) & \dots & (i\gamma\mathbf{A}_l) & (i\gamma\mathbf{p}) \end{vmatrix}. \tag{17b}$$

The simplest method of calculating $\Omega_p$ is to remember that $\chi_p$ is a homogeneous linear function of the $\mathbf{p}$; so using (11), (13) and (14*b*)

$$\Omega_p + \kappa = \sum_{\mathbf{p}'} i\gamma(\mathbf{p}'\partial\chi_p/\partial\mathbf{p}') = \Lambda^{-1}\sum_{\mathbf{p}'}(2\Lambda^2\Omega_{pp'})\,(i\gamma\mathbf{p}'c_{p'}). \tag{18}$$

This is written as a ($\mathbf{p}$-) matrix equation with $\mathbf{p}$ and $\mathbf{p}'$ labelling the rows and columns of $\Omega_{pp'}$.



https://doi.org/10.1017/S0305004100027651 Published online by Cambridge University Press

Similarly the best way of calculating $\chi$ itself in general is to split off the part containing $C$, defining a scalar $\chi_{(C)}$ by

$$\chi = \Lambda . C + \chi_{(C)}, \tag{19}$$

so that $\chi_{(C)}$ is $\chi$ with $C$ equated to zero. $\chi_{(C)}$ is a homogeneous second-degree function of the $\mathbf{p}$. Corresponding to (14$a$) and (16), $\Omega_{pp'(C)}$ is a diagonal component of the tensor

$$\boldsymbol{\chi}_{pp'(C)} = (4\chi_\sigma \chi_{\sigma'})^{-1} \partial^2 \chi_{(C)} / \partial \mathbf{p} \gamma \mathbf{p}'. \tag{20}$$

Then $\Omega_{pp'(C)}$ is given simply by the $\mathbf{p}$-matrix equation

$$\Omega_{pp'(C)} = \Omega_{pp'} - (2\Lambda)^{-1} \delta_{pp'} c_p^{-1}. \tag{21}$$

The last term in (21) is just the multiple of the unit $\mathbf{p}$-matrix obtained from the Kronecker delta term in (14$c$) and (16). Using (11), (20) and the modified form of (15) it follows that

$$\chi_{(C)} = \tfrac{1}{2} \sum_{\mathbf{p},\mathbf{p}'} \left( \mathbf{p} \frac{\partial^2 \chi}{\partial \mathbf{p} \partial \mathbf{p}'} \mathbf{p}' \right) = \sum_{\mathbf{p},\mathbf{p}'} (\mathbf{p}\mathbf{p}') (c_p) (2\Lambda^2 \Omega_{pp'(C)}) (c_{p'}). \tag{22}$$

This is another $\mathbf{p}$-matrix equation, and with (19) gives the most useful expression of $\chi$.

5. *The operators $\mathscr{F}$ and $\mathscr{V}_\alpha$.* All the material is now at hand for considering the effect of a series of operators $\mathscr{F}$ and $\mathscr{V}_\alpha$, perhaps interspersed by operators $\gamma_\mu$ or $\gamma_5$ from vertex factors, on $\chi^{-m}$. At first the effect of operators $\mathscr{F}$ only is considered; quantum electrodynamics does not require consideration of the operators $\mathscr{V}_\alpha$, and the first results are therefore sufficient to deal with electromagnetic interactions.

Consider

$$\mathscr{F}(\mathbf{p}, \sigma) [F(\chi) \chi^{-m}] = \left[ -\kappa - (i\boldsymbol{\gamma}/2) \partial / \partial \mathbf{p} \int_\sigma^\infty d\sigma \right] [F(\chi) \chi^{-m}], \tag{23}$$

where $F(\chi)$ is a function of the derivatives of $\chi$, so that $F(\chi)$ is independent of $\sigma$. The integration over $\sigma$ is carried out using (11), to give for (23)

$$-\kappa F(\chi) \chi^{-m} - (i\boldsymbol{\gamma}/2\chi_\sigma) \partial / \partial \mathbf{p} [F(\chi) I(\chi) \chi^{-m}],$$

where $I(\chi)$ is defined by

$$I(\chi) \chi^{-m} = \left[ \int_\beta^\infty (\chi + \alpha)^{-m} d\alpha \right]_{\beta=0}. \tag{24}$$

After subtractions have been performed, the value of $I(\chi) \chi^{-m}$ is required for $m \geqslant 2$ only. For these values of $m$, $[(m-1) I(\chi)]$ operating on $\chi^{-m}$ is equivalent to multiplication by $\chi$. Using (12$a$), expression (23) becomes

$$-\kappa F(\chi) \chi^{-m} - (i\boldsymbol{\gamma}/2\chi_\sigma) (\partial F(\chi)/\partial \mathbf{p}) I(\chi) \chi^{-m} + i\boldsymbol{\gamma} \boldsymbol{\chi}_p F(\chi) \chi^{-m},$$

i.e.

$$\mathscr{F}(\mathbf{p}, \sigma) [F(\chi) \chi^{-m}] = \Omega_p F(\chi) \chi^{-m} - (i\boldsymbol{\gamma}/2\chi_\sigma) (\partial F(\chi)/\partial \mathbf{p}) I(\chi) \chi^{-m}. \tag{25}$$

Consider a term of the form

$$\mathscr{F}(\mathbf{p}, \sigma) \gamma_\mu \mathscr{F}(\mathbf{p}', \sigma') \ldots \mathscr{F}(\mathbf{p}^{(n-1)}, \sigma^{(n-1)}) \gamma_\tau \mathscr{F}(\mathbf{p}^{(n)}, \sigma^{(n)}) \chi^{-m}, \tag{26}$$

where $\mathbf{p}, \mathbf{p}', \ldots, \mathbf{p}^{(n-1)}, \mathbf{p}^{(n)}$ are external momenta of fermion lines, and

$$\sigma, \sigma', \ldots, \sigma^{(n-1)}, \sigma^{(n)}$$

the corresponding mass terms. Using (25) with $F(\chi) = 1$,

$$\mathscr{F}(\mathbf{p}^{(n)}, \sigma^{(n)}) \chi^{-m} = \Omega_{p^{(n)}} \chi^{-m}.$$

https://doi.org/10.1017/S0305004100027651 Published online by Cambridge University Press

Using (14*b*) and the same equality with $F(\chi) = \Omega_{p^{(n)}}$,

$$\mathscr{F}(\mathbf{p}^{(n-1)}, \sigma^{(n-1)})\,\gamma_\nu\,\mathscr{F}(\mathbf{p}^{(n)}, \sigma^{(n)})\,\chi^{-m} = \mathscr{F}(\mathbf{p}^{(n-1)}, \sigma^{(n-1)})\,\gamma_\nu\,\Omega_{p^{(n)}}\,\chi^{-m}$$

$$= \Omega_{p^{(n-1)}}\gamma_\nu\,\Omega_{p^{(n)}}\chi^{-m} - (i\boldsymbol{\gamma})\,\gamma_\nu(i\boldsymbol{\gamma}\boldsymbol{\chi}_{p^{(n)}p^{(n-1)}})\,I(\chi)\,\chi^{-m}$$

(the terms in the brackets forming a scalar product)

$$= \Omega_{p^{(n-1)}}\gamma_\nu\,\Omega_{p^{(n)}}\chi^{-m} + \gamma_\alpha\gamma_\nu\gamma_\alpha\,\Omega_{p^{(n-1)}p^{(n)}}\,I(\chi)\,\chi^{-m}. \tag{27}$$

In the last line equation (15) with $\mathbf{v} = \mathbf{v}' = i\boldsymbol{\gamma}$ has been used, and suffix notation for this vector has been introduced. In this simple case $\Omega_{p^{(n-1)}}\gamma_\nu\,\Omega_{p^{(n)}}\chi^{-m}$ is defined to be the 'basic term'. The other term is derived from this by replacing $\Omega_{p^{(n-1)}} \ldots \Omega_{p^{(n)}}$ by $\gamma_\alpha \ldots \gamma_\alpha\,\Omega_{p^{(n)}p^{(n-1)}}\,I(\chi)$. In general, the basic term of (26) is

$$\Omega_p\,\gamma_\mu\,\Omega_{p'} \ldots \Omega_{p^{(n-1)}}\gamma_\nu\,\Omega_{p^{(n)}}\chi^{-m}. \tag{28}$$

Other terms are derived by replacing $\Omega_{p^{(i)}} \ldots \Omega_{p^{(j)}}$ by $\gamma_\alpha \ldots \gamma_\alpha\,\Omega_{p^{(i)}p^{(j)}}\,I(\chi)$, the $\gamma_\alpha$ terms entering in the derived terms in the places which the $\Omega_p$ terms occupied in the basic term. The value of (26) is the sum of the expression (28) and all possible terms formed by making any number of these replacements in (28); this class of replacements is denoted by $R_1$. This rule can be proved inductively starting from (27), remembering that derivatives of $\chi$ higher than second order are all zero. By this rule (26) can be evaluated at once in terms of the derivatives defined in §4.

When there are in addition a number of factors

$$\mathscr{V}_\alpha(\mathbf{p}^{(a)}, \sigma^{(a)};\ \mathbf{p}^{(b)}, \sigma^{(b)}) = \left[i\partial/\partial p_\alpha^{(a)}\int_{\sigma^{(a)}}^{\infty} d\sigma^{(a)} + i\partial/\partial p_\alpha^{(b)}\int_{\sigma^{(b)}}^{\infty} d\sigma^{(b)}\right] \tag{29}$$

corresponding to $(MP)$ vertices in the graph, a factor $-2[i\boldsymbol{\chi}_{p^{(a)}} + i\boldsymbol{\chi}_{p^{(b)}}]_\alpha$ occurs in the term which contains no second derivatives of $\chi$. This term is regarded as the sum of $2^x$ basic terms (where $x$ equals the number of $(MP)$ vertices) formed by choosing for each vertex either the first or second operator in expression (29). If some $(MP)$ vertices are connected to an external meson line, then $\mathscr{V}_\alpha$ must be replaced by the alternate form given by equation (1*b*), and the basic terms modified correspondingly. Other terms are derived from the basic terms by making replacements in the basic term (of class $R_2$) of $i(\boldsymbol{\chi}_{p^{(a)}})_\alpha \ldots \Omega_{p^{(i)}}$ by $\gamma_\alpha\,\Omega_{p^{(a)}p^{(i)}}\,I(\chi)$, the matrix $\gamma_\alpha$ occurring in the derived term where $\Omega_{p^{(i)}}$ occurred in the basic term, or a replacement (of class $R_3$) of $i(\boldsymbol{\chi}_{p^{(a)}})_\alpha \ldots i(\boldsymbol{\chi}_{p^{(c)}})_\beta$ by $\delta_{\alpha\beta}\,\Omega_{p^{(a)}p^{(c)}}\,I(\chi)$. In all these terms replacements of class $R_1$ must then be made; owing to the close similarity between the two types of replacement, it is simplest to write down all the basic terms and make any number of replacements of the combined class $R_1 + R_2 + R_3$ to derive all other terms.

The general $S$-matrix element due to interaction of fields $F$, $M$ and $P$ has therefore been evaluated in terms of the quantities $\Lambda, \chi, \chi_p, \Omega_p, \Omega_{pp'}$ and $\gamma$-matrices. It is at this stage that the external momenta $\mathbf{p}, \mathbf{p}', \mathbf{p}'', \ldots$ and the mass terms $\sigma, \sigma', \sigma'', \ldots$ are expressed in terms of the momenta $\mathbf{p}_r$ of the external lines and the particle masses $\kappa$ and $\lambda$. The external line factors are $\overline{\psi}(\mathbf{p}_r)$ and $\psi(\mathbf{p}_r)$ for fermions, $\phi(\mathbf{p}_r)$ for mesons



https://doi.org/10.1017/S0305004100027651 Published online by Cambridge University Press

and $A_\mu(\mathbf{p}_r)$ for photons. The matrix element must now be reduced by using the free-particle equations for the momenta $\mathbf{p}_r$; for example,

$$\left.\begin{aligned}(i\gamma\mathbf{p}_r+\kappa)\,\psi(\mathbf{p}_r)&=0\\ (\mathbf{p}_r^2+\lambda^2)\,\phi(\mathbf{p}_r)&=0.\end{aligned}\right\} \quad (30)$$

and

The most difficult part of this reduction is the calculation of the matrix elements formed by a product of the $\Omega_p$ and $\gamma$-matrices occurring between a pair of external fermion line factors $\overline{\psi}(\mathbf{p}_r)$ and $\psi(\mathbf{p}_s)$. The extraction of magnetic moments from graphs with three external lines has been studied, and can be performed systematically to any order.

6. *Subtraction procedure.* By arranging that the part of all propagation factors which depends on internal momenta is of the form of a boson-line factor and by using Ward's methods of reducing self-energy parts to vertex parts we have avoided all divergence difficulties in deriving (10). The divergences are concealed in the operators $\mathscr{F}$ and $\mathscr{V}_\alpha$, and difficulties in interpreting $I(\chi)$ arise when more than $(m-1)$ operators $I(\chi)$ are applied to $\chi^{-m}$ in evaluating the matrix elements corresponding to certain graphs. It will now be shown that by subtracting the usual terms from any divergent part, difficulties in interpreting $I(\chi)$ disappear. The terms to be subtracted from any part are functions only of the external momenta of that part, so that they take the same mathematical form before and after the $k$-integration, and the same formulae for them can be used as have been used hitherto. A simple method of calculating the subtraction terms is given, and in the appendix is applied to a particular example. The methods of Ward(14,15) will be used, and only electrodynamics will be discussed fully, though the ideas are equally applicable to three-field theory, now that Ward has extended his work to cover the interactions of $F$, $M$ and $P$ fields (not yet published).

In order to prove that subtraction procedure gives finite results, it is necessary to prove that after subtraction terms such as $I(\chi)\,\chi^{-1}$ and $I(\chi)\,I(\chi)\,\chi^{-1}$ do not arise. Let $p_\chi$ be defined for any graph as the inverse power $m$ of $\chi$ in (10) minus the maximum number of $I(\chi)$ introduced by replacements of class $R_1+R_2+R_3$. Subtraction procedure must be applied to those graphs for which $p_\chi<1$. Let $E_f$, $E_p$ and $E_m$ be the number of external fermion, photon and meson lines in a graph and let

$$E_f+E_p+E_m=E.$$

Let the corresponding numbers of internal lines be $F_f, F_p$ and $F_m$ (total $F$) and let $n_{fp}, n_{fm}$ and $n_{pm}$ be the numbers of vertices of the three types (total $n$). Then

$$E_f+2F_f=2n_{fp}+2n_{fm} \quad (31)$$

and

$$E+2F=3n. \quad (32)$$

The inverse power $r$ of $Q$ in (4) is equal to $F$, and the number of $\mathbf{k}$-integration variables is $l=F-n+1$. So the inverse power of $\chi$ in (10) is $r-2l=2n-F-2$. The number of factors $\Omega_p$ (due to $\mathscr{F}$) and $i(\boldsymbol{\chi}_p)_\alpha$ (due to $\mathscr{V}_\alpha$) in the basic term is $F_f+n_{pm}$. Any two of these can be replaced by a term containing $I(\chi)$; so the maximum number of $I(\chi)$ is $[\frac{1}{2}(F_f+n_{pm})]$, the square brackets denoting 'integral part of'. Therefore

$$p_\chi=2n-F-2-[\tfrac{1}{2}(F_f+n_{pm})].$$

By (31) and (32)

$$p_\chi=\tfrac{1}{2}(E+n)-[\tfrac{1}{2}(n-\tfrac{1}{2}E_f)]-2. \quad (33)$$

https://doi.org/10.1017/S0305004100027651 Published online by Cambridge University Press

For given $E$ and $E_f$ this is independent of $n$, since the parities of $E$ and $n$ are the same. The only values of $E$ and $E_f$ giving $p_\chi < 1$ are listed in the following table:

| $E$ | Parity of $n$ | $E_f$ | $p_\chi$ | Type of divergent part |
|---|---|---|---|---|
| 2 | Even | 0 | $-1$ | Photon or meson self-energy |
| | | 2 | 0 | Fermion self-energy |
| 3 | Odd | 0 | 0 | Vertex parts |
| | | 2 | 0 | Vertex parts (2 types) |
| 4 | Even | 0 | 0 | Scattering of { lights by light; mesons by mesons; photons by mesons } |

These are the divergences already considered by Feynman, Dyson, Ward, Salam and others (11, 12, 13). When a photon line is inserted in one of the self-energy parts, one of the vertex parts results; it will be shown that for boson self-energy parts and vertex parts with $E_f = 0$ two terms have to be subtracted, and for the other divergent parts one term has to be subtracted to obtain finite results.

Consider first a vertex part in electrodynamics; let the momenta of the external fermion lines be $\mathbf{p}_1 + \mathbf{k}_1$ and $\mathbf{p}_2 + \mathbf{k}_2$. Then, using a notation similar to that of Ward ((15), equations A), if the unsubtracted matrix element is $\Lambda_\mu(\mathbf{p}_1 + \mathbf{k}_1, \mathbf{p}_2 + \mathbf{k}_2)$, the subtraction term is $\Lambda_\mu(\mathbf{p}_0, \mathbf{p}_0)$, where $\mathbf{p}_0$ is any 'free particle momentum'. Since there is only one external photon line in a vertex part graph, the number of internal fermion lines is even, or else the matrix element vanishes by Furry's theorem (9). So if $\Lambda_\mu(\mathbf{p}_1 + \mathbf{k}_1, \mathbf{p}_2 + \mathbf{k}_2)$ were calculated, the basic term would contain an even number of $\Omega_p$ factors in the numerator, and the term with the greatest number of $I(\chi)$ would have no $\Omega_p$ factor, i.e. it would be a multiple of $\gamma_\mu I(\chi)\chi^{-1}$, since $p_x = 0$. This is the only term in which the interpretation of $I(\chi)$ is difficult. If $\chi_0$ is the 'subtraction equivalent' of $\chi$, formed by replacing both $\mathbf{p}_1 + \mathbf{k}_1$ and $\mathbf{p}_2 + \mathbf{k}_2$ by $\mathbf{p}_0$, this difficult term becomes after subtraction a constant multiple of

$$\gamma_\mu I(\chi)[\chi^{-1} - \chi_0^{-1}] = \gamma_\mu \left[ \int_\beta^\infty \{(\chi+\alpha)^{-1} - (\chi_0+\alpha)^{-1}\} d\alpha \right]_{\beta=0}.$$

This is well-defined and finite, and the difficulty in interpreting $I(\chi)$ is resolved. In practice the subtraction is performed by 'relabelling' the graph, so that wherever momenta equal to $\mathbf{p}_1 + \mathbf{k}_1$ or $\mathbf{p}_2 + \mathbf{k}_2$ occur inside the vertex part in the original graph, momenta equal to $\mathbf{p}_0$ occur in the 'subtraction graph'; the subtraction term derived from this graph is calculated in the same way as the original term. There is a close connexion between $\chi$ and $\chi_0$, and therefore between the 'derivatives' $\Omega_p, \Omega_{pp'}$ and their subtraction equivalents, which are denoted by $\Omega_{p0}, \Omega_{pp'0}$. Consider a fermion line factor in the vertex part, for instance

$$\mathscr{F}(\mathbf{p}, \sigma)[(\mathbf{p} + \mathbf{k} + \mathbf{k}_1)^2 + \sigma] \quad \text{with } \mathbf{p} = \mathbf{p}_1,$$

where $\mathbf{k}$ is a linear combination of the internal momenta which occur only in lines of the vertex part considered, each of these momenta occurring in each line of some closed cycle in the vertex part; it is replaced in the subtraction term by

$$\mathscr{F}(\mathbf{p}, \sigma)[(\mathbf{p} + \mathbf{k})^2 + \sigma] \quad \text{with } \mathbf{p} = \mathbf{p}_0.$$

https://doi.org/10.1017/S0305004100027651 Published online by Cambridge University Press

Thus the product of operators $\mathscr{F}(\mathbf{p}, \sigma)$ and $\gamma_\mu$ occurring in the original and subtraction terms is the same; therefore the subtraction term is the same function of $\chi_0, \Omega_{p0}$, etc., as the original term is of $\chi, \Omega_p$, etc. It is therefore quite simple to calculate.

Similar methods can be used to calculate subtraction terms arising from fermion and photon self-energy parts in electrodynamics. Since the calculation of these terms involves consideration of graphs with zero-momentum photon lines inserted, a theorem concerning the insertion of such lines into graphs will be quoted. When a zero-momentum photon line is inserted into a graph, yielding a 'complicated graph', let the labelling of the graph be unaltered except for the line into which the insertion is made; let the external momentum of this line be $\mathbf{q}$ originally, let $\mathbf{q}'$ and $\mathbf{q}''$ be the external momenta of the two lines in the 'complicated graph' derived from this line, and let $\mathbf{p}, \mathbf{p}', \ldots$ be the other external momenta. Then the theorem states that when the external momenta have been given their ultimate values in terms of the momenta $\mathbf{p}_r$ of the external lines of the graph, the characteristic determinant $\chi$ is the same for the original and complicated graphs; and, in addition, $\Omega_p, \Omega_{pp'}$, etc., are the same, $\Omega_{q'} = \Omega_{q''} = \Omega_q$, $\Omega_{q'p} = \Omega_{q''p} = \Omega_{qp}$, $\Omega_{q'q'} = \Omega_{q''q''} = \Omega_{qq}$, and $\Omega_{q'q''} = \Omega_{qq(C)}$. Thus the matrix element for the 'complicated graph' can be written down in terms of quantities derivable from the original graph by the methods of §4. Since the complicated graph has an extra line factor, one more auxiliary variable must be introduced when the factors (3) are combined, but the integration over this variable is trivial, as the variable does not occur in the integrand after the momenta $\mathbf{p}, \mathbf{p}', \ldots$ have been expressed in terms of the $\mathbf{p}_r$.

Let the unsubtracted term derived from a fermion self-energy part be $S_F$, and the matrix element after subtraction be $S_{F1}$. Then if the momentum of the lines joining part to the rest of the graph is $\mathbf{p}_1 + \mathbf{k}'$, and if

$$(\mathbf{p}_1 + \mathbf{k}')^\lambda = (\mathbf{p}_1 + \mathbf{k}')\lambda + \mathbf{p}_0(1-\lambda) = \lambda\mathbf{k}' + \mathbf{p},$$

where $\mathbf{p}_\lambda = \lambda\mathbf{p}_1 + (1-\lambda)\mathbf{p}_0$, then (15)

$$S_{F1} = S_F - 2\pi S_F \int_0^1 d\lambda (\mathbf{p}_1 + \mathbf{k}' - \mathbf{p}_0)_\mu \{\Lambda_\mu((\mathbf{p}_1 + \mathbf{k}')^\lambda, (\mathbf{p}_1 + \mathbf{k}')^\lambda) - \Lambda_\mu(\mathbf{p}_0, \mathbf{p}_0)\} S_{F1}. \quad (34)$$

Since by Furry's theorem (9) the vertex part obtained from a fermion self-energy part with non-zero matrix element has an even number of fermion line factors, it follows (as for ordinary vertex parts) that $S_{F1}$ is finite. The term containing

$$\Lambda_\mu((\mathbf{p}_1 + \mathbf{k}')^\lambda, (\mathbf{p}_1 + k')^\lambda)$$

can be calculated by replacing terms in the self-energy part which are equal to $\mathbf{p}_1 + \mathbf{k}'$ by $(\mathbf{p}_1 + \mathbf{k}')^\lambda$ and inserting a zero-momentum photon line into the graph; the insertion is made into every possible fermion line, in the manner described by Ward (15). The subtraction term is exactly the same, but with $\mathbf{p}_0$ replacing $(\mathbf{p} + \mathbf{k}')^\lambda$. The factor $(\mathbf{p}_1 + \mathbf{k}' - \mathbf{p}_0)_\mu$ is written in terms of an operator acting on the line factor of one of the lines joining the self-energy part to the rest of the graph. Thus

$$(\mathbf{p}_1 + \mathbf{k}' - \mathbf{p}_0)_\mu [\mathscr{F}(\mathbf{p}, \sigma)\{(\mathbf{p} + \mathbf{k}')^2 + \sigma\}^{-1}]_{\mathbf{p}=\mathbf{p}_1}$$
$$= \left[\left\{-\tfrac{1}{2}\partial/\partial\mathbf{p} \int_\sigma^\infty d\sigma - \mathbf{p}_0\right\}_\mu \mathscr{F}(\mathbf{p}, \sigma)\{(\mathbf{p} + \mathbf{k}')^2 + \sigma\}^{-1}\right]_{\mathbf{p}=\mathbf{p}_1}, \quad (35)$$

https://doi.org/10.1017/S0305004100027651 Published online by Cambridge University Press

where the two operator factors act independently. Then the **k**-integration is of the standard form treated in § 3. The operator in curly brackets forms a scalar product with the $\gamma_\mu$ at the zero-momentum vertex to give an operator factor $\mathscr{G}(\mathbf{p}, \sigma)$ say. If $\Omega_{p1}$ corresponds to $\mathscr{G}(\mathbf{p}, \sigma)$ as $\Omega_p$ corresponds to $\mathscr{F}(\mathbf{p}, \sigma)$, then

$$\mathscr{G}(\mathbf{p}, \sigma) \dots \mathscr{F}(\mathbf{p}, \sigma) \chi^{-m} = \Omega_{p1} \dots \Omega_p \chi^{-m}.$$

There is no term like $\gamma_\alpha \dots \gamma_\alpha \Omega_{pp} I(\chi) \chi^{-m}$, since $\mathscr{G}$ and $\mathscr{F}$ act independently. As for vertex parts the original and subtracted terms are exactly of the same form.

If a photon self-energy part contains an odd number of fermion lines, the matrix element of the whole graph is zero, by Furry's theorem. So when we insert a zero-momentum photon line into the self-energy part of a graph corresponding to a non-zero matrix element, the resulting 'vertex part' contains an odd number of fermion lines. Hence the argument showing that the subtraction of one term gives finite matrix elements fails; it is not difficult to show directly that finite results follow when we subtract two terms in accordance with the usual subtraction rules. The set of matrix elements corresponding to the vertex parts derived by the zero-momentum line insertions are denoted by $\Delta_\mu(\mathbf{k}', \mathbf{k}')$, where $\mathbf{k}'$ is the momentum of the lines connecting the part to the rest of the graph. If $D_F$ is the self-energy matrix element before subtraction, and $D_{F1}$ the element after subtraction, then

$$D_{F1} = D_F - 2\pi D_F \int_0^1 d\lambda\, k'_\mu \left\{ \Delta_\mu(\lambda \mathbf{k}', \lambda \mathbf{k}') - \Delta_\mu(\mathbf{0}, \mathbf{0}) - \lambda k'_\nu \left( \frac{\partial \Delta_\mu(\lambda \mathbf{k}', \lambda k')}{\partial(\lambda k'_\nu)} \right)_0 \right\} D_{F1}. \quad (36)$$

The terms are treated exactly the same as in the fermion self-energy work. The lines with momentum $\mathbf{k}'$ are labelled $\mathbf{p} + \mathbf{k}'$ and $\mathbf{p}' + \mathbf{k}' (\mathbf{p} = \mathbf{p}' = \mathbf{0})$, so that

$$k'_\mu [\{(\mathbf{p} + \mathbf{k}')^2 + \sigma\}^{-1}]_{\mathbf{p}=\mathbf{0}} = \left[ \left( -\tfrac{1}{2} \partial / \partial \mathbf{p} \int_\sigma^\infty d\sigma \right)_\mu \{(\mathbf{p} + \mathbf{k}')^2 + \sigma\}^{-1} \right]_{\mathbf{p}=\mathbf{0}}. \quad (37)$$

Momenta $\mathbf{k}'$ inside the self-energy part are replaced by $\lambda \mathbf{k}'$ or $\mathbf{0}$, according to which term in the curly bracket in (36) is being calculated. The third term on the right-hand side of (36) is derived from a graph into which a further zero-momentum line has been inserted; in calculating this the theorem quoted above is very useful. In practice, the first term in the curly brackets in equations (34) and (36) can be most easily calculated in its original form, without introduction of the zero-momentum photon line and the parameter $\lambda$. In the subtraction term, the zero-momentum photon line is needed for defining the subtraction term explicitly when 'overlapping' divergences occur, but in these terms the parameter $\lambda$ only appears in very simple expressions, and the extra integration can be performed without difficulty.

Thus suitable relabelling of graphs allows the terms to be subtracted in virtue of self-energy and vertex part divergences to be calculated quite simply. It may happen that more than one divergent part occurs in a graph, as in the example in the appendix. Then from the original term, terms derived from the graphs with one divergent part relabelled are subtracted; then terms from the graphs with two divergent parts relabelled are added, and so on.

7. *Conclusion.* This paper has given an account of a simple method of performing integrations over internal momenta of matrix elements. The calculation of these

https://doi.org/10.1017/S0305004100027651 Published online by Cambridge University Press

matrix elements is thereby simplified, but before calculations of high-order terms are possible, several major difficulties have to be overcome. They are (i) evaluation of products of $\Omega_p$ factors, interspersed by $\gamma$-matrices; (ii) performing the integrations over the Feynman auxiliary variables; (iii) summing terms derived from graphs of either the same or different order; and (iv) finding a method of performing subtractions without referring to the particular graph concerned. Some progress has been made in solving some of these problems; until they are solved calculations of terms of order higher than the sixth become extremely complicated by present methods.

The author wishes to acknowledge invaluable help and advice given throughout this work by Mr J. Hamilton, earlier teaching by Dr N. Kemmer, and much useful information gained in discussions with Mr A. Salam, Mr C. A. Hurst and Dr R. J. Eden.

## REFERENCES

(1) CHISHOLM, J. S. R. *Phys. Rev.* (2), 82 (1951), 448.
(2) DYSON, F. J. *Phys. Rev.* (2), 75 (1949), 486.
(3) DYSON, F. J. *Phys. Rev.* (2), 75 (1949), 1736.
(4) DYSON, F. J. *Phys. Rev.* (2), 82 (1951), 428.
(5) DYSON, F. J. *Phys. Rev.* (2), 83 (1951), 608.
(6) DYSON, F. J. *Proc. roy. Soc.* A, 207 (1951), 395.
(7) FEYNMAN, R. P. *Phys. Rev.* (2), 76 (1949), 749.
(8) FEYNMAN, R. P. *Phys. Rev.* (2), 76 (1949), 769.
(9) FURRY, W. H. *Phys. Rev.* (2), 51 (1937), 125.
(10) KARPLUS, R. and KROLL, N. M. *Phys. Rev.* (2), 77 (1950), 536.
(11) MATTHEWS, P. T. *Phil. Mag.* (7), 42 (1951), 221.
(12) RORLICH, F. *Phys. Rev.* (2), 80 (1950), 666.
(13) SALAM, A. *Phys. Rev.* (2), 82 (1951), 217.
(14) WARD, J. C. *Phys. Rev.* (2), 78 (1950), 182.
(15) WARD, J. C. *Proc. phys. Soc. Lond.* 64 (1951), 54.

## APPENDIX

As an example of the use of the methods described above, the calculation of one term contributing to the fourth-order nuclear magnetic moments is given below. The graph for the process is shown in Fig. 1. The constant factors from vertices and lines contribute

$$-(e/\hbar c)(\alpha^2/16\pi^6)(2\pi)^4, \tag{i}$$

where $\alpha = (f^2/4\pi\hbar c)$. The operator factors from vertices and fermion lines, assuming a pseudoscalar meson field, are

$$\gamma_5 \mathscr{F}(\mathbf{p}, \sigma)\, \gamma_\mu \mathscr{F}(\mathbf{p}', \sigma')\, \gamma_5 \mathscr{F}(\mathbf{p}'', \sigma'')\, \gamma_5 \mathscr{F}(\mathbf{p}''', \sigma''')\, \gamma_5. \tag{ii}$$

The external lines factors are $\overline{\psi}(\mathbf{p}_1)$, $\psi(\mathbf{p}_2)$ and $A_\mu(\mathbf{p}_1 - \mathbf{p}_2)$. The remaining factors are all 'boson line factors', and involve the internal momenta $\mathbf{k}$ and $\mathbf{k}'$. The integral to be evaluated is

$$\int d^4\mathbf{k}\, d^4\mathbf{k}' [(\mathbf{k}^2 + \lambda^2)((\mathbf{p} + \mathbf{k})^2 + \sigma)((\mathbf{p}' + \mathbf{k})^2 + \sigma') \times ((\mathbf{p}'' + \mathbf{k} + \mathbf{k}')^2 + \sigma'')((\mathbf{p}''' + \mathbf{k}')^2 + \sigma''')(\mathbf{k}'^2 + \lambda^2)]^{-1}. \tag{iii}$$

https://doi.org/10.1017/S0305004100027651 Published online by Cambridge University Press

The denominators are combined by an extended form of the usual Feynman formula using coefficients $e, c, c', c'', c'''$ and $e'$ (in that order) to multiply the denominator factors. Then if

$$c+c'+c''=c_k, \quad c''+c'''=c_{k'}, \quad c''=c_{kk'}, \tag{iv}$$

the integral (3) becomes

$$\int_0^1 dc \int_0^1 dc' \int_0^1 dc'' \int_0^1 dc''' \int_0^1 de \int_0^1 de' \int d^4\mathbf{k}\, d^4\mathbf{k}' [Q(\mathbf{k}, \mathbf{k}')]^{-6}, \tag{v}$$

subject to $$c+c'+c''+c'''+e+e'=1,$$

with $$Q(\mathbf{k}, \mathbf{k}') = A\mathbf{k}^2 + 2H\mathbf{k}\mathbf{k}' + B\mathbf{k}'^2 + 2\mathbf{G}\mathbf{k} + 2\mathbf{F}\mathbf{k}' + c$$

and $$A = c_k + e, \quad H = c_{kk'}, \quad B = c_{k'} + e',$$

$$\mathbf{G} = c\mathbf{p} + c'\mathbf{p}' + c''\mathbf{p}'' = \sum_k (c\mathbf{p}), \text{ say},$$

$$\mathbf{F} = c'\mathbf{p}' + c''\mathbf{p}'' + c'''\mathbf{p}''' = \sum_{k'} (c\mathbf{p}), \text{ say},$$

$$C = c(\mathbf{p}^2+\sigma) + c'(\mathbf{p}'^2+\sigma') + c''(\mathbf{p}''^2+\sigma'') + c'''(\mathbf{p}'''^2+\sigma''') + \lambda^2(e+e').$$

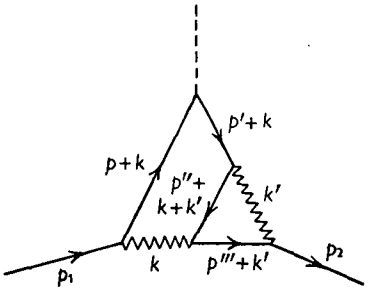



Fig. 1. $\mathbf{p} = \mathbf{p}_1$ and $\mathbf{p}' = \mathbf{p}'' = \mathbf{p}''' = \mathbf{p}_2$ eventually.

Now that the quadratic form $Q$ is defined, the values of $\Lambda$, $\chi$ and all the 'derivatives' of $\chi$ can be written down:

$$\Lambda = AB - H^2 = (c_k+e)(c_{k'}+e') - c_{kk'}^2, \tag{vi}$$

$$\chi = \begin{vmatrix} A & H & \mathbf{G} \\ H & B & \mathbf{F} \\ \mathbf{G} & \mathbf{F} & C \end{vmatrix}$$

$\Omega_{pp}$, etc., are easily written down; for instance

$$\Omega_{pp} = \frac{1}{2\Lambda^2} \begin{vmatrix} A & H & 1 \\ H & B & 0 \\ 1 & 0 & c_p^{-1} \end{vmatrix} = \frac{(\Lambda/c)-B}{2\Lambda^2},$$

and

$$\Omega_{pp'} = \frac{1}{2\Lambda^2} \begin{vmatrix} A & H & 1 \\ H & B & 0 \\ 1 & 1 & 0 \end{vmatrix} = \frac{H-B}{2\Lambda^2}.$$

The **p**-matrix $(\Omega_{pp'})$ is therefore

$$\frac{1}{2\Lambda^2} \begin{pmatrix} (\Lambda/c)-B & -B & H-B & H \\ -B & (\Lambda/c')-B & H-B & H \\ H-B & H-B & (\Lambda/c'')+2H-A-B & H-A \\ H & H & H-A & (\Lambda/c''')-A \end{pmatrix},$$

https://doi.org/10.1017/S0305004100027651 Published online by Cambridge University Press

and the $p$-matrix $(\Omega_{pp'(C)})$ is given by equation (21); that is, by omitting the $\Lambda$ terms from the diagonal elements of $(\Omega_{pp'})$.

To calculate the 'first derivative' $\Omega_p$, etc., equation (18) gives

$$\begin{pmatrix} \Omega_p+\kappa \\ \Omega_{p'}+\kappa \\ \Omega_{p''}+\kappa \\ \Omega_{p'''}+\kappa \end{pmatrix} = \Lambda^{-1}\begin{pmatrix} (\Lambda/c)-B & -B & H-B & H \\ -B & (\Lambda/c')-B & H-B & H \\ H-B, & H-B & (\Lambda/c'')+2H-A-B & H-A \\ H & H & H-A & (\Lambda/c''')-A \end{pmatrix}\begin{pmatrix} ci\gamma\mathbf{p} \\ c'i\gamma\mathbf{p}' \\ c''i\gamma\mathbf{p}'' \\ c'''i\gamma\mathbf{p}''' \end{pmatrix},$$

Putting in the final values of $\mathbf{p}, \mathbf{p}'$, etc., that is, $\mathbf{p}=\mathbf{p}_1$ and $\mathbf{p}'=\mathbf{p}''=\mathbf{p}'''=\mathbf{p}_2$,

$$\Omega_p+\kappa = R_{pp_1}i\gamma\mathbf{p}_1+R_{pp_2}i\gamma\mathbf{p}_2, \quad \Omega_{p'}+\kappa = R_{p'p_1}i\gamma\mathbf{p}_1+R_{p'p_2}i\gamma\mathbf{p}_2, \quad \text{etc.,} \quad \text{(vii)}$$

where

$$R_{pp_1} = 1-\Lambda^{-1}cB, \quad R_{pp_2} = \Lambda^{-1}[-c'B+c''(H-B)+c'''H],$$
$$R_{p'p_1} = -\Lambda^{-1}cB, \quad R_{p'p_2} = 1+\Lambda^{-1}[-Bc'+c''(H-B)+c'''H],$$
$$R_{p''p_1} = \Lambda^{-1}c(H-B), \quad R_{p''p_2} = 1+\Lambda^{-1}[c'(H-B)+c''(2H-A-B)+c'''(H-A)],$$
$$R_{p'''p_1} = \Lambda^{-1}cH, \quad R_{p'''p_2} = 1+\Lambda^{-1}[c'H+c''(H-A)-c'''A].$$

The 'free-particle' value of $\chi$, $\chi_f$, is found by using (19):

$$\chi_f = \chi_{cf}+\Lambda.C_f,$$

where

$$\Lambda.C_f = \Lambda\lambda^2(e+e'), \quad \text{(viii)}$$

and $\chi_{cf}$ is the 'free-particle' value of

$$\chi_c = \begin{vmatrix} A & H & \mathbf{G} \\ H & B & \mathbf{F} \\ \mathbf{G} & \mathbf{F} & O \end{vmatrix}.$$

Now for working out the contribution to the nuclear magnetic moments, $\mathbf{p}_1^2=\mathbf{p}_2^2=-\kappa^2$ and $(\mathbf{p}_1-\mathbf{p}_2)^2=0$, so that $\mathbf{p}^2=\mathbf{p}'^2=\ldots=-\kappa^2$ and $(\mathbf{p}.\mathbf{p}')=(\mathbf{p}.\mathbf{p}'')=\ldots=-\kappa^2$. Thus

$$(\mathbf{F}.\mathbf{F})_f = \sum_k(c\mathbf{p}).\sum_k(c\mathbf{p}) = -\kappa^2c_k^2, \quad (\mathbf{G}.\mathbf{G})_f = -\kappa^2c_{k'}^2 \quad \text{and} \quad (\mathbf{F}.\mathbf{G})_f = -\kappa^2c_kc_{k'}. \quad \text{(ix)}$$

It follows that

$$\chi_{cf} = \kappa^2[Ac_{k'}^2+Bc_k^2-2Hc_kc_{k'}].$$

Therefore

$$\chi_f = \Lambda\lambda^2(e+e')+\kappa^2[c_kc_{k'}(c_k+c_{k'})-2c_{kk'}c_kc_{k'}+ec_{k'}^2+e'c_k^2]. \quad \text{(x)}$$

The value of (v) is, by (10), $-(\pi^4/120\chi^2)$. The basic term is therefore, bringing in factors (i) and (ii) and the external fermion line factors,

$$(e/\hbar c)(\alpha^2/16\pi^2)(2\pi)^4\int_0^1dc\int_0^1dc'\int_0^1dc''\int_0^1dc'''\int_0^1de\int_0^1de'$$
$$\times\overline{\psi}(\mathbf{p}_1)\gamma_5\Omega_p\gamma_\mu\Omega_{p'}\gamma_5\Omega_{p''}\gamma_5\Omega_{p'''}\gamma_5\psi(\mathbf{p}_2)\chi_f^{-2}. \quad \text{(xi)}$$

The terms derived by one substitution of class $R_1$ are six in number. The integrand in equation (xi) is replaced by

$$\left.\begin{aligned} &\overline{\psi}(\mathbf{p}_1)\gamma_5\gamma_\alpha\gamma_\mu\gamma_\alpha\gamma_5\Omega_{p''}\gamma_5\Omega_{p'''}\gamma_5\psi(\mathbf{p}_2)\,\Omega_{pp'}I(\chi)\chi_f^{-2}, \\ &\overline{\psi}(\mathbf{p}_1)\gamma_5\Omega_p\gamma_\mu\gamma_\alpha\gamma_5\gamma_\alpha\gamma_5\Omega_{p'''}\gamma_5\psi(\mathbf{p}_2)\,\Omega_{p'p''}I(\chi)\chi_f^{-2}, \end{aligned}\right\} \quad \text{(xii)}$$

and so on. The terms derived by two substitutions are just constant multiples of $\overline{\psi}(p_1)\gamma_\mu\psi(p_2)I(\chi)I(\chi)\chi_f^{-2}$, and vanish on subtraction ($p_\chi=0$).

https://doi.org/10.1017/S0305004100027651 Published online by Cambridge University Press

*Subtraction terms.* These are calculated by using exactly the same method as used for the original term. There are three 'subtraction graphs'; the terms, denoted by (*a*) and (*b*), derived from the first two are subtracted from the original term, and that from the last, denoted by (*d*), is added. All 'subtraction' graphs are of the same form as Fig. 1. That for (*a*) is labelled exactly the same as Fig. 1, but the final values of the external momenta are different, being

$$(a) \quad \mathbf{p} = \mathbf{p}' = \mathbf{p}'' = \mathbf{p}''' = \mathbf{p}_0,$$

where $\mathbf{p}_0$ is any 'free-particle momentum'. The graphs for (*b*) and (*d*) are labelled the same as Fig. 1, except that the fermion line momentum $\mathbf{p}''+\mathbf{k}+\mathbf{k}'$ is replaced by $\mathbf{p}''+\mathbf{k}'$. The final values of $\mathbf{p}, \ldots, \mathbf{p}'''$ are

$$(b) \quad \mathbf{p} = \mathbf{p}_1; \quad \mathbf{p}' = \mathbf{p}_2; \quad \mathbf{p}'' = \mathbf{p}''' = \mathbf{p}_0,$$
$$(d) \quad \mathbf{p} = \mathbf{p}' = \mathbf{p}'' = \mathbf{p}''' = \mathbf{p}_0.$$

The terms from these graphs corresponding to (xi) and (xii) are easily written down. Since the subtraction terms are the same functions of the subtraction equivalents $\chi_0$, $\Omega_{p0}$, etc. as the original term is of $\chi$, $\Omega_p$, etc. (given by (xi) and (xii)), it only remains to write down $\chi_0$, $\Omega_{p0}$, etc. Graph (*a*) has subtraction equivalents of exactly the same form as the original $\chi$, $\Omega_p$, etc., except for the final values of the momenta $\mathbf{p}, \mathbf{p}', \mathbf{p}''$ and $\mathbf{p}'''$. Similarly graphs (*b*) and (*d*) lead to the same functions apart from the values of the external momenta. We therefore need only deal with graph (*b*). The subtraction equivalent of $Q$ is

$$Q_0(\mathbf{k}, \mathbf{k}') = A_0\mathbf{k}^2 + 2H_0\mathbf{k}\mathbf{k}' + \ldots + c_0,$$

where $A_0 = c + c' + e = c_k + e - c_{kk'}$, $H_0 = 0$, $B_0 = c'' + c''' + e' = c_{k'} + e'$,

$\mathbf{G}_0 = c\mathbf{p} + c'\mathbf{p}'$, $\mathbf{F}_0 = c''\mathbf{p}'' + c'''\mathbf{p}'''$, $C_0 = c(\mathbf{p}^2+\sigma) + \ldots + c'''(\mathbf{p}'''^2+\sigma''') + \lambda^2(e+e')$.

So

$$\Lambda_0 = A_0B_0,$$

$$\chi_0 = \begin{vmatrix} A_0 & 0 & c\mathbf{p}+c'\mathbf{p}' \\ 0 & B_0 & c''\mathbf{p}''+c'''\mathbf{p}''' \\ c\mathbf{p}+c'\mathbf{p}' & c''\mathbf{p}''+c'''\mathbf{p}''' & C_0 \end{vmatrix}$$

$$= A_0B_0[c(\mathbf{p}^2+\sigma) + \ldots + c'''(\mathbf{p}'''^2+\sigma''') + \lambda^2(c+e')] - A_0(c''\mathbf{p}''+c'''\mathbf{p}''')^2 - B_0(c\mathbf{p}+c'\mathbf{p}')^2;$$

$$(\Omega_{pp'})_0 = (2\Lambda_0^2)^{-1}\begin{pmatrix} (\Lambda_0/c)-B_0 & -B_0 & 0 & 0 \\ -B_0 & (\Lambda_0/c')-B_0 & 0 & 0 \\ 0 & 0 & (\Lambda_0/c'')-A_0 & -A_0 \\ 0 & 0 & -A_0 & (\Lambda_0/c''')-A_0 \end{pmatrix};$$

$$(\Omega_{pp'(C)})_0 = (2\Lambda_0^2)^{-1}\begin{pmatrix} -B_0 & -B_0 & 0 & 0 \\ -B_0 & -B_0 & 0 & 0 \\ 0 & 0 & -A_0 & -A_0 \\ 0 & 0 & -A_0 & -A_0 \end{pmatrix};$$

from these $(\Omega_{p0})$ can be written down at once.

CHRIST'S COLLEGE
CAMBRIDGE*

* Present address: Department of Natural Philosophy, University of Glasgow.

https://doi.org/10.1017/S0305004100027651 Published online by Cambridge University Press