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Multi-valued mappings and Lefschetz fixed point theorems

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1. *Introduction.* By a *multi-valued map* from a space X to a space Y we mean a map which assigns to each point x in X a non-empty subset $F(x)$ of Y. When $X = Y$, a point x in X is a *fixed point for* F if x is in $F(x)$.

'Lefschetz' fixed-point theorems for multi-valued maps of compact spaces were first studied by Eilenberg and Montgomery (5). Motivated by the Leray–Schauder fixed point results, the concept of Lefschetz space has been generalized to that of Λ-space (see (8)). A space X is a Λ-space if, for every compact map $f: X \to X$, the Lefschetz number is defined and the Lefschetz theorem holds. The purpose of this note is to extend the results of (8) to multi-valued maps. At the same time the Eilenberg–Montgomery results are generalized.

2. *Notation and definitions.* The image of a subset A of X under F is $F(A) = \bigcup_{x \in A} F(x)$. The *graph* of F is

$$\Gamma(F) = \{(x, y): y \in F(x)\} \subset X \times Y.$$

The projections $p: \Gamma(F) \to X$ and $q: \Gamma(F) \to Y$ are called the *projections associated with* F.

F is said to be *upper semi-continuous* (*u.s.c.*) if:

(i) $F(x)$ is compact for each $x \in X$ and

(ii) for each $x \in X$ and each open set V containing $F(x)$, there is an open neighbourhood U of x such that $F(U) \subset V$.

The usual results involving upper semi-continuity are well known and easily proved. We list the following results without proof.

(2·1) Proposition. Let $F: X \to Y$ be a multi-valued map.

(*a*) If F is u.s.c. and Y is Hausdorff, then $\Gamma(F)$ is closed in $X \times Y$.

(*b*) If F is u.s.c., the image of each compact set under F is compact.

(*c*) When X and Y are first countable Hausdorff spaces, F is u.s.c. if and only if $\Gamma(F)$ is closed and the image of each compact set under F is compact.

If $F: X \to Y$ and $G: Y \to Z$ are multi-valued maps, the *composition* of F and G is denoted $G \circ F: X \to Z$ and is defined by $(G \circ F)(x) = \bigcup_{y \in F(x)} G(y)$. It is an easy exercise to verify that $(G \circ F)$ is u.s.c. when F and G are u.s.c.

3. *The induced homomorphism.* Let $\mathscr{S}$ denote some topological category and $\mathscr{H}$ a homology functor. For X, Y in $\mathscr{S}$ and $F: X \to Y$ a multi-valued map we are able to

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define, in certain cases, an induced homomorphism $\mathscr{H}(F)\colon \mathscr{H}(X) \to \mathscr{H}(Y)$. We also show that after $\mathscr{H}$ is extended in this way it remains functorial. The basis for the definition is the well-known Vietoris Mapping Theorem (see Begle (1, 2)).

(3·1) DEFINITION. A continuous (single-valued) map is *inverse acyclic* (with respect to $\mathscr{H}$) if $f^{-1}(y)$ is acyclic (i.e. has the homology of a point) for each y in Y. (In particular, $f^{-1}(y) \neq \varnothing$ and hence f is onto.)

(3·2) THEOREM. (Vietoris Mapping Theorem.) *Let X, Y be compact, Hausdorff spaces and let $\check{H}$ denote Čech homology theory with coefficients in a field. Let $f\colon X \to Y$ be a continuous map which is inverse acyclic. Then the induced homomorphism*

$$\check{H}(f)_k\colon \check{H}_k(X) \to H_k(Y)$$

is an isomorphism for each k.

(3·3) DEFINITION. Let X and Y be topological spaces and $F\colon X \to Y$ a multi-valued map. Then if $F(x)$ is acyclic (with respect to $\mathscr{H}$) for all x in X, F is said to be *acyclic* (with respect to $\mathscr{H}$).

Eilenberg and Montgomery defined an induced homomorphism for an u.s.c. acyclic map $F\colon X \to Y$ when X and Y are compact, Hausdorff spaces. They put

$$\check{H}(F) = \check{H}(q) \circ \check{H}(p)^{-1},$$

where p, q are the projections associated with F. This makes good sense since $\Gamma(F) \subset X \times Y$ is compact and for each x in X, $p^{-1}(x)$ is homeomorphic to $F(x)$ and hence is acyclic. Thus by (3·2) $\check{H}(p)$ is an isomorphism.

In order to have an induced map for more general spaces, it is necessary to extend the Vietoris theorem. We exhibit only one such extension here. However, the extension question is interesting in its own right. Moreover, all the theory presented here can be developed axiomatically by placing certain (completely natural) restrictions on $\mathscr{S}$ and $\mathscr{H}$ (see (10)).

Let $\mathscr{A}$ denote the category of graded vector spaces over the field Q of rational numbers and homomorphisms of degree zero. If $\mathscr{S}$ is some topological category and $\mathscr{H}$ is a functor from $\mathscr{S}$ to $\mathscr{A}$, the image of objects X in $\mathscr{S}$ will be denoted

$$\mathscr{H}(X) = \{\mathscr{H}_k(X)\}_{k=0}^{\infty}.$$

Usually the only additional requirement that will be necessary on the functor $\mathscr{H}$ is that it satisfy the homotopy axiom. The only exceptions to this are (5·10) and (5·11) where $\mathscr{H}$ must agree with $\check{H}$ on compact polyhedra and (6·4) where $\mathscr{H}$ must satisfy the dimension axiom.

Let $\mathscr{S}$ denote the category of all Hausdorff spaces and continuous maps. Take X in $\mathscr{S}$ and let $\mathscr{K}$ denote the collection of compact subsets of X, partially ordered by inclusion. If $K > K'$ in $\mathscr{K}$, $\pi_{K'}^{K}\colon K' \to K$ is the inclusion map. Then $(\mathscr{K}, \pi)$ is a directed system. Moreover, for each K in $\mathscr{K}$, $\check{H}(K)$ is defined and for each $\pi_{K'}^{K}$,

$$(\pi_{K'}^{K})_*\colon \check{H}(K) \to \check{H}(K')$$

is defined. Then $\{\check{H}(K), \pi_*\}$ is a directed system in $\mathscr{A}$. Let X, Y and $f\colon X \to Y$ be in

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$\mathscr{S}$ and let $\mathscr{K}$, $\mathscr{K}'$ denote the collection of compact subsets of X, Y. Then $K \mapsto f(K)$ determines an order preserving map $\mathscr{K} \to \mathscr{K}'$. Let $f_K\colon K \to f(K)$ be defined by f. This determines a map of direct systems

$$\{\check{H}(K), \pi_*\}_{\mathscr{K}} \to \{\check{H}(K'), \pi'_*\}_{\mathscr{K}'}.$$

A functor $\vec{H}\colon \mathscr{S} \to \mathscr{A}$ is defined by taking direct limits:

$$\vec{H}(X) = \operatorname{dir\,lim}\{\check{H}(K)\}_{\mathscr{K}}$$

for X in $\mathscr{S}$, and

$$\vec{H}(f) = f_* = \operatorname{dir\,lim}\{(f_K)_*\}_{\mathscr{K}}$$

for $f\colon X \to Y$ in $\mathscr{S}$. It is easy to verify that H is a functor satisfying the homotopy axiom.

(3·4) DEFINITION. A continuous map $f\colon X \to Y$ of topological spaces is *proper* if for each compact subset of K of Y, $f^{-1}(K)$ is compact. (See (11), p. 319.)

(3·5) THEOREM. *Let $\mathscr{S}$ be the category of Hausdorff spaces and continuous maps. If $f\colon X \to Y$ in $\mathscr{S}$ is inverse acyclic and proper, then $\vec{H}(f)$ is an isomorphism in $\mathscr{A}$.*

Proof. It is shown that in each dimension the map $f_*\colon \vec{H}(X) \to \vec{H}(Y)$ by

$$f_* = \operatorname{dir\,lim}\{(f_K)_*\}_{\mathscr{K}}$$

is an isomorphism. (Let $\mathscr{K}$, $\mathscr{K}'$ denote the collections of compact subsets of X, Y.) If $\mathscr{K}_1 = \{f^{-1}f(K)\colon K \text{ in } \mathscr{K}\}$, then $\mathscr{K}_1$ is cofinal in $\mathscr{K}$. Each f_* is defined by first considering the map $\mathscr{K} \to \mathscr{K}'$ by $K \to f(K)$. Since f is onto, the image of $\mathscr{K}_1$ under this map is $\mathscr{K}'$. For each $K' = f^{-1}f(K)$ in $\mathscr{K}_1$, $f_{K'}\colon K' \to f(K')$ is a continuous map of compact spaces. And for y in $f(K')$, $f_{K'}^{-1}(y) = f^{-1}(y)$, since $f(K') = f(K)$. Then $f_{K'}$ is inverse acyclic and by the Vietoris Theorem, $(f_{K'})_*\colon \check{H}(K') \to \check{H}(f(K'))$ is an isomorphism. Finally ((6), p. 223) f_* is an isomorphism.

(3·6) DEFINITION. Let $\mathscr{S}$ be the category of Hausdorff spaces and continuous maps. Let X, Y be in $\mathscr{S}$ and $F\colon X \to Y$ an u.s.c. acyclic multi-valued map. The *homomorphism induced by* F (with respect to $\vec{H}$) is defined

$$\vec{H}(F) = F_* = q_* \circ p_*^{-1} = [\operatorname{dir\,lim}\{(q_K)_*\}_{\mathscr{K}'}] \circ [\operatorname{dir\,lim}(p_K)_{*\mathscr{K}'}]^{-1},$$

where p, q are the projections associated with F, $\mathscr{K}'$ is the collection of compact subsets of $\Gamma(F)$, and p_K, q_K are defined by p, q.

We have already noted that $p\colon \Gamma(F) \to X$ is continuous and inverse acyclic. Moreover, if K is compact in X, $p^{-1}(K) = [K \times F(K)] \cap \Gamma(F)$. Since $F(K)$ is compact and $\Gamma(F)$ is closed, $p^{-1}(K)$ is compact and p is proper. Thus (3·6) is justified. Note that letting $\mathscr{K}$ denote the collection of compact subsets of X and using F to define $F_K\colon K \to F(K)$, the homomorphism $(F_K)_*\colon \check{H}(K) \to \check{H}(F(K))$ is defined for each K in $\mathscr{K}$ and

$$F_* = \operatorname{dir\,lim}\{(F_K)_*\}_{\mathscr{K}}.$$

In what follows $\mathscr{S}$, $\mathscr{H}$ will represent either the category of compact, Hausdorff spaces with $\check{H}$ or the category of all Hausdorff spaces with $\vec{H}$.

We now have $\mathscr{H}$ extended to certain multi-valued maps of spaces in $\mathscr{S}$ and we wish to consider to what extent $\mathscr{H}$ remains functorial. The composition $G \circ F$ of two acyclic

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maps is not necessarily acyclic. However, we will show in (3·8) that if $G \circ F$ is acyclic, then $(G \circ F)_* = G_* \circ F_*$. Even when $G \circ F$ fails to be acyclic, there may well be cross-sections of $G \circ F$ which are acyclic. (Given H, $K: X \to Y$ with $H(x) \subset K(x)$ for all x, H is said to be a cross-section of K.)

Evidently there is no natural cross-section of $G \circ F$ which will always be acyclic. A logical candidate would be $H(x) = \bigcap_{y \in F(x)} G(y)$. But it is easy to exhibit an acyclic u.s.c. map F for which the cross-section $H(x) = \bigcap_{y \in F(x)} F(y)$ of $F \circ F$ is defined and u.s.c., but not acyclic. Let X be the closed triangle in R^2 with vertices $a = (\frac{1}{2}, 1)$, $b = (0, 0)$, and $c = (1, 0)$. Define $F: X \to X$ by sending (x, y) to the union in R^2 of the line segment from b to $(\frac{1}{2}, x - \frac{1}{2}y)$ with the line segment from $(\frac{1}{2}, x - \frac{1}{2}y)$ to c. Then

$$F(a) = \{(x, 0): 0 \leqslant x \leqslant 1\} \quad \text{and} \quad \bigcap_{0 \leqslant x \leqslant 1} F(x, 0) = \{b, c\}.$$

The main result on compositions is proceeded by a lemma which is easily proved.

(3·7) LEMMA. *Let $F, G: X \to Y$ be acyclic and u.s.c. with X, Y in $\mathscr{S}$. If G is a cross-section of F, then $G_* = F_*$.*

(3·8) THEOREM. *Let $F: X \to Y$ and $G: Y \to Z$ be acyclic u.s.c. maps of spaces in $\mathscr{S}$.*

(*a*) *If $(G \circ F)$ is acyclic, then $(G \circ F)_* = G_* \circ F_*$.*

(*b*) *If $H: X \to Z$ is an acyclic u.s.c. cross-section of $(G \circ F)$ satisfying*

$$H(x) \subseteq \bigcap_{y \in F(x)} G(y),$$

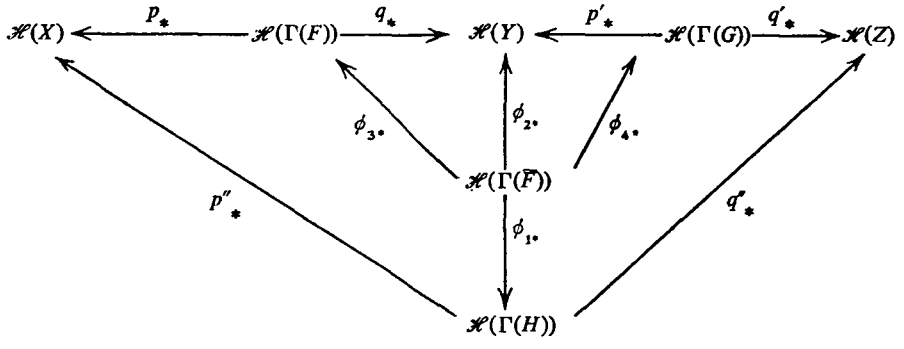
then $H_* = G_* \circ F_*$.

Proof. We first prove (*b*). Denote the projections associated with F, G, H by p, q; p', q'; and p'', q'' (resp.). Define $\overline{F}: \Gamma(H) \to Y$ by $\overline{F}(x, z) = F(x)$. Then $\overline{F}$ is an acyclic u.s.c. map and we denote its projections by ϕ_1, ϕ_2. It suffices to prove

(i) $\phi_{2*} \circ (\phi_{1*})^{-1} \circ (p''_*)^{-1} = q_* \circ (p_*)^{-1}$ and

(ii) $q''_* = q'_* \circ (p'_*)^{-1} \circ \phi_{2*} \circ (\phi_{1*})^{-1}$.

To this end define $\phi_3: \Gamma(\overline{F}) \to \Gamma(F)$ by $\phi_3(x, z, y) = (x, y)$ and $\phi_4: \Gamma(\overline{F}) \to \Gamma(G)$ by $\phi_4(x, z, y) = (y, z)$. The following diagram is commutative and both (i) and (ii) can be verified by diagram chasing.



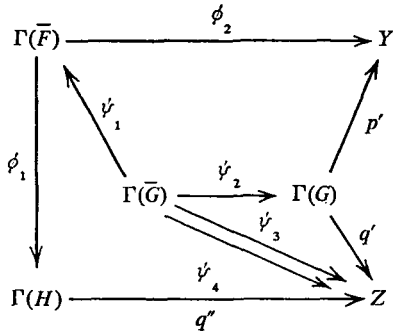
For part (*a*), let H denote $G \circ F$ and proceed as in the proof of (*b*). Again it suffices to prove (i) and (ii). Using the projection ϕ_3 and the above diagram, (i) can be verified.

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However, ϕ_4 is no longer defined. In its place define $\bar{G}\colon \Gamma(\bar{F}) \to Y$ by $\bar{G}(x,z,y) = \bar{G}(y)$. Then $\bar{G}$ is acyclic and u.s.c. and $\Gamma(\bar{G}) = \{(x,z,y,z')\colon z' \in G(y), z \in H(x), y \in F(x)\}$. Define projection maps

$$\begin{aligned}
&\psi_1\colon \Gamma(\bar{G}) \to \Gamma(\bar{F}) \quad \text{by} \quad \psi_1(x,z,y,z') = (x,z,y),\\
&\psi_2\colon \Gamma(\bar{G}) \to \Gamma(G) \qquad\quad \psi_2(x,z,y,z') = (y,z'),\\
&\psi_3\colon \Gamma(\bar{G}) \to Z \qquad\quad \psi_3(x,z,y,z') = z',\\
&\psi_4\colon \Gamma(\bar{G}) \to Z \qquad\quad \psi_4(x,z,y,z') = z.
\end{aligned}$$

In the following diagram we have $\phi_2 \circ \psi_1 = p' \circ \psi_2, q' \circ \psi_2 = \psi_3$, and $q'' \circ \phi_1 \circ \psi_1 = \psi_4$.



Since ψ_{1*} is an isomorphism condition (ii) can be proved if $\psi_{3*} = \psi_{4*}$. Define $\bar{H}\colon \Gamma(\bar{G}) \to Z$ by $\bar{H}(x,z,y,z') = H(x) = G \circ F(x)$. Then $\bar{H}$ is an acyclic u.s.c. map with cross-sections ψ_3 and ψ_4. By (3·7), $\psi_{3*} = \psi_{4*}$. This completes the proof.

In most of the applications of (3·8) one of the maps will be single-valued. Then $(G \circ f)$ is acyclic whenever G is acyclic and $(g \circ F)$ is acyclic when F is acyclic and g is 'acyclic on compact sets' (i.e. sends compact acyclic sets to acyclic sets).

Note that when $(G \circ F)$ is not acyclic the additional condition on the cross-section H in (3·8)(*b*) is necessary. Indeed, consider $F\colon S^1 \to S^1$ defined by letting $F(x)$ be the half-circle with centre x. Then F is acyclic and u.s.c. and for each x, $F \circ F(x) = S^1$. Then, without the condition, $H\colon S^1 \to S^1$ could be *any* acyclic u.s.c. map.

4. *Lefschetz number.* In (9) Leray extended the definition of trace for f in End (V), where V is a finite-dimensional vector space, to certain endomorphisms f of infinite-dimensional spaces. This in turn permits the Lefschetz number to be defined for a larger class of maps. We briefly recall Leray's results. (They are more general than those given in (8).)

Let V be a vector space and $f \in \text{End}\,(V)$; define $N_f = \{v | f^p(v) = 0 \text{ for some } p \geqslant 1\}$. Suppose that the space V/N_f has finite dimension. Then the *trace of* f, $\text{tr}\,(f)$ or $\text{tr}_V(f)$, is defined by $\text{tr}_V(f) = \text{tr}_{V/N_f}(f')$, where $f' \in \text{End}\,(V/N_f)$ is induced by f and $\text{tr}\,(f')$ is the classical trace.

(4·1) Definition. Let $f = \{f_k\}\colon V \to V$ be an endomorphism (of degree zero) in $\mathscr{A}$. Then f is said to be of *finite type* if

(i) V_k/N_{f_k} has finite dimension for each k, and

(ii) $N_{fk} = V_k$ in all but a finite number of dimensions k.

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(4·2) LEMMA. *Let V, W be graded vector spaces in $\mathscr{A}$ and let $f\colon V \to W$, $g\colon W \to V$ be homomorphisms of degree zero Suppose that $g \circ f\colon V \to V$ is of finite type. Then $f \circ g\colon W \to W$ is of finite type and in each dimension k,* $\operatorname{tr}(g_k \circ f_k) = \operatorname{tr}(f_k \circ g_k)$. (See Leray's prop. c (9); p. 224.)

(4·3) DEFINITION. Take X in $\mathscr{S}$ and let $F\colon X \to X$ be an acyclic u.s.c. multi-valued map. Suppose that $\mathscr{H}(F) = F_*$ is of finite type. Then $\operatorname{tr}(F_{*k})$ is defined for each k and the sum

$$\Lambda_{\mathscr{H}}(F) = \sum_{k=0}^{\infty} (-1)^k \operatorname{tr}(F_{*k})$$

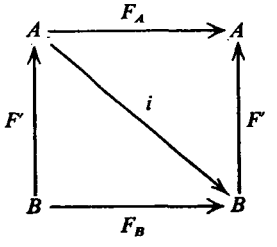
is called the *Lefschetz number of F (with respect to $\mathscr{H}$)*. (When no confusion is possible we simply write $\Lambda(F)$.) If $\mathscr{H}(F)$ is of finite type and if F has a fixed point whenever $\Lambda(F) \neq 0$, then F is called a *Lefschetz map (with respect to $\mathscr{H}$)*.

It is possible to define $\Lambda(F)$ in a somewhat more general situation, relaxing the condition that $\mathscr{H}(F)$ be of the finite type. We could agree that the Lefschetz number of F is defined whenever $\mathscr{H}(F)$ satisfies (4·1) (i) and the sum $\Lambda(F)$ in (4·3) converges. Having done this, all the theory in section 5 remains valid.

A situation may arise where $F\colon X \to X$, but X is not in $\mathscr{S}$ or $\mathscr{H}(F)$ is not defined. There may, however, be a subset A of X containing $F(X)$ and with A in $\mathscr{S}$. Defining $F_A\colon A \to A$ by $F_A(x) = F(x)$, we have an acyclic u.s.c. map which has a fixed point if and only if F does. If $(F_A)_*$ were defined, $\Lambda(F_A)$ could be studied in place of $\Lambda(F)$. The following theorem shows that this can be done in a rather nice manner.

(4·4) THEOREM. *Let $F\colon X \to X$ be an acyclic u.s.c. multi-valued map and suppose that A and B are spaces in $\mathscr{S}$ such that $F(X) \subset A \subset B \subset X$. Then $\mathscr{H}(F_A)$ is of finite type if and only if $\mathscr{H}(F_B)$ is of finite type and $\Lambda_{\mathscr{H}}(F_A) = \Lambda_{\mathscr{H}}(F_B)$ when defined.*

Proof. Suppose $\Lambda_{\mathscr{H}}(F_A)$ exists. Consider the commutative diagram:



where F' is the co-restriction of F_B and i is the inclusion. Then $i_*\colon \mathscr{H}(A) \to \mathscr{H}(B)$ and $F'_*\colon \mathscr{H}(B) \to \mathscr{H}(A)$ are maps in $\mathscr{A}$ with $F'_* \circ i_*$ of finite type. Then by (4·2) $i_* \circ F'_*$ is of finite type and in each dimension $\operatorname{tr}(F'_* \circ i_*) = \operatorname{tr}(i_* \circ F'_*)$. Thus, since the diagram commutes at the homology level,

$$\Lambda_{\mathscr{H}}(F_A) = \Lambda_{\mathscr{H}}(F_B).$$

(4·5) COROLLARY. *Let $F\colon X \to X$ be an acyclic u.s.c. multi-valued map with $X \in \mathscr{S}$. Let*

$$\mathfrak{A} = \{A\colon A \in \mathscr{S} \text{ and } F(X) \subset A \subset X\}.$$

Then

(*a*) *if F_* is of finite type, so is F_{A*} for each A in $\mathfrak{A}$ and $\Lambda(F) = \Lambda(F_A)$,*

(*b*) *if F_{A*} is of finite type for some A in $\mathfrak{A}$, so is F_* and $\Lambda(F_A) = \Lambda(F)$.*

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In particular

(*a'*) *if F is a Lefschetz map, F_A is a Lefschetz map for all A in $\mathfrak{A}$,*

(*b'*) *if F_A is a Lefschetz map for some A in $\mathfrak{A}$, F is a Lefschetz map.*

(4·6) COROLLARY. *Let X be any Hausdorff space and let $F: X \to X$ be an acyclic u.s.c. map which is compact (i.e. $\overline{F(X)}$ is compact). Let K, K' be any compact subsets of X containing $F(X)$. Then $\check{H}(F_K)$ is of finite type if and only if $\check{H}(F_{K'})$ is of finite type and $\Lambda_{\check{H}}(F_K) = \Lambda_{\check{H}}(F_{K'})$ when defined.*

5. *$M\Lambda$-spaces.*

(5·1) DEFINITION. A space X is a *Lefschetz* [*M-Lefschetz*] *space* (with respect to $\mathscr{H}$) if every continuous [acyclic, u.s.c., multi-valued] self-mapping of X is a Lefschetz map.

A space X is a *Λ-space* [*$M\Lambda$-space*] (with respect to $\mathscr{H}$) if every continuous, compact [acyclic, u.s.c., compact, multi-valued] self-mapping of X is a Lefschetz map.

Obviously the class of $M\Lambda$-spaces is contained in the class of Λ-spaces. However, whether these classes coincide is an open question. It is shown in (8) that (metric) absolute neighbourhood retracts are Λ-spaces. But there is not yet a proof that such spaces are $M\Lambda$-spaces.

It is assumed throughout this section that all single-valued maps are continuous and all multi-valued maps are u.s.c.

(5·2) DEFINITION. A continuous map $f: X \to Y$ is said to be *acyclic on sets* if it sends compact acyclic sets to acyclic sets.

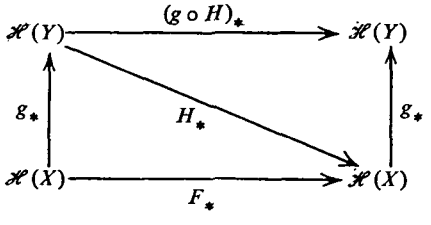
(5·3) THEOREM. *Let X be in $\mathscr{S}$ and $F: X \to X$ a compact, acyclic, multi-valued map. Then if one of the following conditions holds, F is a Lefschetz map:*

(*a*) *there is a Λ-space Y in $\mathscr{S}$ and there are maps $g: X \to Y$, $h: Y \to X$ such that h is compact and $h \circ g$ is a cross-section of F;*

(*b*) *there is an $M\Lambda$-space Y in $\mathscr{S}$ and there are maps $g: X \to Y$, $H: Y \to X$ such that H is compact acyclic, g is acyclic on sets, and $(H \circ g)$ is a cross-section of F;*

(*c*) *there is an $M\Lambda$-space Y in $\mathscr{S}$ and there are maps $G: X \to Y$, $h: Y \to X$ such that G is acyclic, h is acyclic on sets and compact, and $(h \circ G)$ is a cross-section of F.*

Proof. We prove only (*b*); the other parts are similar. Since $(g \circ H)_*: Y \to Y$ is compact and acyclic, it is a Lefschetz map. Since $(H \circ g)$ is a cross-section of F, $F_* = (H \circ g)_*$. By (3·7) the diagram commutes.



Now g_* and H_* are linear maps of graded vector spaces in $\mathscr{A}$ and $g_* \circ H_*$ is of finite type. Then by (4·2), F_* is of finite type and $\Lambda(F) = \Lambda(g \circ H)$. Suppose $\Lambda(F) \neq 0$. Then there

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is a point y in Y with y in $g(H(y))$. Choose x in $H(y)$ so that $g(x) = y$. Then

$$x \in H(y) = (H \circ g)(x) \subset F(x)$$

and F has a fixed point.

(5·4) THEOREM. *Let X be in $\mathscr{S}$ and let Y be an $M\Lambda$-space in $\mathscr{S}$. Suppose that there are maps $g\colon X \to Y$, $h\colon Y \to X$ such that $h \circ g = 1_X$. (h is not necessarily compact.) Then X is an $M\Lambda$-space.*

Proof. Let $F\colon X \to X$ be compact and acyclic. Then $(F \circ h)\colon Y \to X$ is compact and acyclic. Since g is an embedding, g is acyclic on sets. Moreover $((F \circ h) \circ g)$ is a cross-section of F; thus (5·3)(*b*) implies that F is a Lefschetz map.

(5·5) COROLLARY. *A retract of an $M\Lambda$-space is again an $M\Lambda$-space.*

The hypothesis of (5·3) is far too strong. For example, while it is essential that $F_* = h_* \circ g_*$, it is certainly not necessary that $h \circ g$ be a cross-section of F. By placing a suitable restriction on the space X, (5·3) can be generalized.

(5·6) DEFINITION. Let F, $G\colon X \to Y$ be multi-valued maps and take α in $\mathrm{Cov}(Y)$. Then F *is α-near G* if for each x in X and y in $F(x)$, there is a U in α containing y and meeting $G(x)$. (Note that in general this relation is symmetric only when F and G are single-valued.)

(5·7) LEMMA. *Let F, $G\colon X \to Y$ be acyclic maps of spaces in $\mathscr{S}$. If F and G are homotopic under an acyclic homotopy H, then $F_* = G_*$.*

(5·8) THEOREM. *Let X be a T_3-space in $\mathscr{S}$ and $F\colon X \to X$ a compact, acyclic multi-valued map. Then F is a Lefschetz map if for each open cover α of X, one of the following conditions holds (all homotopies are acyclic):*

(*a*) *there is a Λ-space Y_α in $\mathscr{S}$ and there are maps $g_\alpha\colon X \to Y_\alpha$, $h_\alpha\colon Y_\alpha \to X$ such that h_α is compact, $h_\alpha \circ g_\alpha$ is homotopic to F, and $h_\alpha \circ g_\alpha$ is α-near F;*

(*b*) *there is an $M\Lambda$-space Y_α in $\mathscr{S}$ and there are maps $g_\alpha\colon X \to Y_\alpha$, $H_\alpha\colon Y_\alpha \to X$ such that H_α is compact acyclic, g_α is acyclic on sets, $(H_\alpha \circ g_\alpha)$ is homotopic to F, and $(H_\alpha \circ g_\alpha)$ is α-near F;*

(*c*) *there is an $M\Lambda$-space Y_α in $\mathscr{S}$ and there are maps $G_\alpha\colon X \to Y_\alpha$, $h_\alpha\colon Y_\alpha \to X$ such that G_α is acyclic, h_α is acyclic on sets and compact, $(h_\alpha \circ G_\alpha)$ is homotopic to F, and $(h_\alpha \circ G_\alpha)$ is α-near F.*

Proof. We prove only (*a*). Take any α in $\mathrm{Cov}\, X$ and the corresponding Y_α, g_α, h_α. Since $g_\alpha \circ h_\alpha\colon Y_\alpha \to Y_\alpha$ is compact, it is a Lefschetz map. Since $h_\alpha \circ g_\alpha$ is homotopic to F, $F_* = h_{\alpha *} \circ g_{\alpha *}$. Then as before we see that F_* is of finite type and $\Lambda(F) = \Lambda(g_\alpha \circ h_\alpha)$.

Suppose $\Lambda(F) \neq 0$. Then for each α in $\mathrm{Cov}\, X$, there is a y_α in Y_α with $g_\alpha \circ h_\alpha(y_\alpha) = y_\alpha$. Let $x_\alpha = h(y_\alpha)$ in X. Let K be a compact set in X containing $F(X)$. Then $F(x_\alpha) \subset K$ for each α. Now since $h_\alpha \circ g_\alpha(x_\alpha) = x_\alpha$ and since $h_\alpha \circ g_\alpha$ is α-near F, there is a U_α in α such that x_α, $\bar{x}_\alpha$ are in U_α for some $\bar{x}_\alpha$ in $F(x_\alpha)$. Then $\{\bar{x}_\alpha, \alpha \in \mathrm{Cov}\, X\}$ is a net in K. Since K is compact, this net has a subnet $T = \{T(m), m \in E\}$ converging to some point x_0 in K. (Then there is a map $N\colon E \to \mathrm{Cov}\, X$ of directed sets satisfying $T(m) = \bar{x}_{N(m)}$ and for each $\alpha \in \mathrm{Cov}\, X$, there is an $n \in E$ such that if $m > n$, then $N(m) > \alpha$.)

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Consider the net $S = \{S(m), m \in E\}$ defined by $S(m) = x_{N(m)}$. Suppose that this net converges to x_0. Then $S \times T$ is a net in $\Gamma(F)$ converging to (x_0, x_0). Thus by (2·1), (x_0, x_0) is an element of $\Gamma(F)$ and x_0 is a fixed point for F.

Let V be any open neighbourhood of x_0. We show that S is eventually in V. First recall that x_α and $\bar{x}_\alpha$ are both in U_α for each α in $\operatorname{Cov} X$. Since X is regular, there is an open neighbourhood W of x_0 such that $\overline{W} \subset V$. Let $\alpha_0 = \{V, X - \overline{W}\}$ in $\operatorname{Cov} X$. Since the net T converges to x_0, there exists an element n_W in E such that if $m > n_W$, then $T(m) \in W$. Moreover there is an element n_0 of E such that $N(m) > \alpha_0$ when $m > n_0$. Since E is directed, there exists an element n of E such that $n > n_0, n_W$. Now take any $m > n$. It is easy to verify that $S(m)$ is in V. For $T(m) = \bar{x}_{N(m)} \in W$ and $\bar{x}_{N(m)}$, $x_{N(m)}$ are both in $U_{N(m)}$. Then $U_{N(m)} \cap W \neq \varnothing$ and hence $U_{N(m)} \notin X - \overline{W}$. Since $m > n_0$, $N(m) > \alpha_0$ and $U_{N(m)} \subset V$. Thus, $x_{N(m)} = S(m) \in V$.

(5·9) THEOREM. *Let X be a T_3-space in $\mathscr{S}$. Suppose that for each α in* $\operatorname{Cov} X$ *there is an $M\Lambda$-space Y_α in $\mathscr{S}$ and maps $g_\alpha: X \to Y_\alpha$, $h_\alpha: Y_\alpha \to X$ such that g_α is acyclic on sets and $h_\alpha \circ g_\alpha$ and 1_X are α-near and homotopic. (h_α is not necessarily compact.) Then X is an $M\Lambda$-space.*

Proof. Let $F: X \to X$ be compact and acyclic and take α in $\operatorname{Cov} X$. Consider the corresponding Y_α, g_α, h_α. Let $k: X \times I \to X$ be a (single-valued) homotopy from 1_X to $h_\alpha \circ g_\alpha$. Then $(F \circ k): X \times I \to X$ is an acyclic homotopy from F to $((F \circ h_\alpha) \circ g_\alpha)$. Hence $F_* = (F \circ h_\alpha)_* \circ g_{\alpha *}$ and as before it can be shown that F_* is of finite type and $\Lambda(F) = \Lambda([g_\alpha \circ (F \circ h_\alpha)])$.

If $\Lambda(F) \neq 0$, then for each α in $\operatorname{Cov} X$, there is a y_α in Y_α such that y_α is in $[g_\alpha \circ (F \circ h_\alpha)](y_\alpha)$. Let $x_\alpha = h_\alpha(y_\alpha)$ in X and choose $\bar{x}_\alpha$ in $F(h_\alpha(y_\alpha))$ so that $g_\alpha(\bar{x}_\alpha) = y_\alpha$. Let K be a compact set in X containing $F(X)$. Then $\{\bar{x}_\alpha, \alpha \text{ in } \operatorname{Cov} X\}$ is a net in K. Moreover, $\bar{x}_\alpha \in F(x_\alpha)$ and there is an element U_α in α which contains both x_α and $\bar{x}_\alpha$ (for $h_\alpha g_\alpha(\bar{x}_\alpha) = x_\alpha$ and $h_\alpha \circ g_\alpha$ and 1_X are α-near). Given these two facts the proof of Theorem (5·8) above can be used to show that x_0 is a fixed point for F. (Note that (5·8) (*b*) does not apply here since we do not have $(F \circ h) \circ g$ α-near F.)

Remark. The above theorems also hold for M-Lefschetz spaces in the sense that '$M\Lambda$-space' can be replaced by 'M-Lefschetz space' throughout. When this is done the compactness conditions on the maps can be dropped.

An application to absolute neighbourhood retracts is now made. In (5) Eilenberg and Montgomery proved that every compact (metric) ANR is a M-Lefschetz space. This result is needed in the following theorem.

(5·10) THEOREM. *Every* CW*-complex is an $M\Lambda$-space (with respect to $\mathscr{H}$).*

Proof. Let X be a CW-complex and $F: X \to X$ a compact acyclic map. Then there is a finite subcomplex X' of X containing $\overline{F(X)}$. Let $F_{X'}: X' \to X'$ be defined by F. Since X' is a compact ANR, the Eilenberg–Montgomery result implies that $F_{X'}$ is a Lefschetz map. Then by Corollary (4·5) (*b'*), F is a Lefschetz map.

(5·11) COROLLARY. *Every polyhedron with the Whitehead topology (see* (7), *p.* 99) *is an $M\Lambda$-space (with respect to $\mathscr{H}$).*

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The following theorem (see (8)) is a corollary of (5·9) and (5·11) stated for single-valued maps. Unfortunately, a corresponding theorem in the multi-valued setting cannot be proved because of the assumption in (5·9) that the maps g be acyclic on sets. It is conjectured, however, that such a theorem is valid.

(5·12) THEOREM. *Every (metric)* ANR *is a* Λ*-space (with respect to* $\mathscr{H}$*).*

Proof. For each open cover α of X there is a polyhedron P_α (with the Whitehead topology) and maps $g_\alpha: X \to P_\alpha$, $h_\alpha: P_\alpha \to X$ such that $h_\alpha \circ g_\alpha$ is α-homotopic to 1_X (see (7), p. 138). In particular, $h_\alpha \circ g_\alpha$ and 1_X are homotopic and are α-near. Then by (5·9), X is a Λ-space.

A metric space is an absolute neighbourhood retract if it is homeomorphic to a neighbourhood retract of a convex subset of a Banach space ((7), p. 95). While there is not yet a proof that such spaces are $M\Lambda$-spaces, a partial result can be obtained using a different method. Theorem (5·13) below implies that *a neighbourhood retract of a Banach space is an* $M\Lambda$*-space.* This argument is due (in the single-valued setting) to Browder (4).

(5·13) THEOREM. *Every open subset of a Banach space is an* $M\Lambda$*-space (with respect to* $\mathscr{H}$*).*

Proof. Let U be an open subset of a Banach space B. Let $F: U \to U$ be a compact, acyclic, multi-valued map. Then $\overline{F(U)}$ is compact and can be covered by a finite number of closed balls $\{C_i\}_{i=1}^n$ which are contained in U. For each i, $C_i \cap \overline{F(U)}$ is compact. Thus the convex closure K_i of $C_i \cap \overline{F(U)}$ is compact. Since C_i is closed and convex, $K_i \subset C_i \subset U$. Then $K = \bigcup_{i=1}^n K_i$ is compact and $\overline{F(U)} \subset K \subset U$. Moreover, K is an absolute neighbourhood retract (see (4), Lemma 3). If $F_K: K \to K$ is defined by F, the Eilenberg–Montgomery Theorem implies that F_K is a Lefschetz map. Finally, by (4·5), F is a Lefschetz map.

6. *Applications.* Theorems (6·3) and (6·4) are motivated by a lemma of Browder (3) for compact ANR's.

(6·1) SCHAUDER THEOREM. (α) *If X is any convex subset of a Banach space B, then every continuous compact map of X has a fixed point.*

(b) *If X is either an open or a closed convex subset of a Banach space B, then every compact, acyclic, u.s.c., multi-valued map of X has a fixed point.*

Proof. (a) X is an AR and hence a Λ-space. If $f: X \to X$ is a compact map, $\Lambda(f) = 1$ and f has a fixed point.

(b) If X is open, it is an $M\Lambda$-space (5·13). And if X is closed in B, there is an open set U in B and a retraction $r: U \to X$. Then by (5·13) and (5·5), X is an $M\Lambda$-space. Since X is contractible X is acyclic (with respect to H). Thus $\Lambda(F) = \operatorname{tr}(F_{*0})$. But the following composition is the identity

$$\vec{H}_0(\{x\}) \xrightarrow{j_*} \vec{H}_0(X) \xrightarrow{F_*} \vec{H}_0(X) \xrightarrow{c_*} \vec{H}_0(\{x\});$$

thus $F_{*0} \neq 0$ and $\Lambda(F) \neq 0$.

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(6·2) COROLLARY. *Every compact (single-valued) self-mapping of a metric* AR *has a fixed point.*

Proof. If X is a metric AR it can be imbedded in a convex subset C of a Banach space and there is a retraction $r: C \to X$. If $f: X \to X$ is a compact map, then $g = f \circ r: C \to C$ has a fixed point which must also be a fixed point for f.

(6·3) THEOREM. *Let $\mathscr{T}$ be some topological category and $\mathscr{H}: \mathscr{T} \to \mathscr{A}$ a functor satisfying the homotopy and dimension axioms. Let $X \in \mathscr{T}$ be a Λ-space (with regard to $\mathscr{H}$) and $f: X \to X$ a compact map. Suppose that there is some integer m such that $f^m(X) \subset A$, where A is acyclic (with respect to $\mathscr{H}$). Then f has a fixed point.*

Proof. Define $g: X \to A$ by f^m and let $i: A \to X$ be the inclusion. Then $(f^m)_{*k} = i_{*k} \circ g_{*k}$ in each dimension k. Thus $(f_{*k})^m = (f^m)_{*k} = 0$ for $k \geqslant 1$. And

$$\operatorname{tr}(f_{*k}) = \operatorname{tr}(f_{*k} \mid (f_{*k})^m(\mathscr{H}_k(X))) = 0, \quad \text{for} \quad k \geqslant 1.$$

Then $\Lambda(f) = \operatorname{tr}(f_{*0})$ and it suffices to show that $\operatorname{tr}(f_{*0}) \neq 0$. First note that $i: A \to X$ induces a monomorphism and hence $\mathscr{H}_0(A)$ can be considered a subspace of $\mathscr{H}_0(X)$. Indeed, the following composition,

$$\mathscr{H}_0(\{x\}) \xrightarrow{j_*} \mathscr{H}_0(A) \xrightarrow{i_*} \mathscr{H}_0(X) \xrightarrow{c_*} \mathscr{H}_0(\{x\}),$$

is the identity and j_* is an isomorphism since A is acyclic. Thus i_* must be a monomorphism.

Next note that $(f_*)^m(\mathscr{H}_0(X)) = \mathscr{H}_0(A)$. For

$$(f_*)^m(\mathscr{H}_0(X)) = i_* \circ g_*(\mathscr{H}_0(X)) \subset i_*(\mathscr{H}_0(A))$$

—which is identified with $\mathscr{H}_0(A)$. Then since $(f_*)^m(\mathscr{H}_0(X))$ is a subspace of $\mathscr{H}_0(X)$ and is contained in $\mathscr{H}_0(A) \cong Q$, it is either 0 or $\mathscr{H}_0(A)$. But $(f_*)^m(\mathscr{H}_0(X))$ cannot be 0, since the following composition is the identity.

$$\mathscr{H}_0(\{x\}) \xrightarrow{j_*} \mathscr{H}_0(X) \xrightarrow{(f^m)_*} \mathscr{H}_0(X) \xrightarrow{c_*} \mathscr{H}_0(\{x\}).$$

Finally, $\operatorname{tr}(f_{*0}) = \operatorname{tr}(h)$, where $h: (f_*)^m(\mathscr{H}_0(X)) \to (f_*)^m(\mathscr{H}_0(X))$ is defined by f_{*0}. Since $(f_*)^m(\mathscr{H}_0(X)) = \mathscr{H}_0(A) \cong 0$, it is only necessary to show that h is not the zero map. But if $h = 0$, $(f_*)^{m+1}(\mathscr{H}_0(X)) = f_*((f_*)^m(\mathscr{H}_0(X))) = 0$; this is seen to be impossible by replacing m by $m+1$ in the above composition.

The proof of the next theorem is analogous to the proof of (6·3).

(6·4) THEOREM. *Let X be an $M\Lambda$-space (with respect to $\overrightarrow{H}$). Let $F: X \to X$ be a compact, acyclic, u.s.c., multi-valued map. If either of the following conditions hold, F has a fixed point.*

(a) There is an integer m such that the compositions F^j are acyclic for $j = 1, \ldots, m+1$ and $F^m(X) \subset A$, where A is acyclic (with respect to $\overrightarrow{H}$).

(b) There is an integer m and acyclic u.s.c. maps $G_j: X \to X$ $(j = 2, \ldots, m+1)$ such that $G_2(x) \subseteq \bigcap_{y \in F(x)} F(y)$ for each x in X and $G_j(x) \subseteq \bigcap_{y \in G_{j-1}(x)} F(y)$ for each x in X $(j = 3, \ldots, m+1)$. Also $G_m(X) \subseteq A$, where A is acyclic (with respect to $\overrightarrow{H}$).

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