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Asymptotic structure in substitution tiling spaces

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Abstract. Every sufficiently regular non-periodic space of tilings of $\mathbb{R}^d$ has at least one pair of distinct tilings that are asymptotic under translation in all the directions of some open $(d-1)$-dimensional hemisphere. If the tiling space comes from a substitution, there is a way of defining a location on such tilings at which asymptoticity 'starts'. This leads to the definition of the branch locus of the tiling space: this is a subspace of the tiling space, of dimension at most $d-1$, that summarizes the 'asymptotic in at least a half-space' behavior in the tiling space. We prove that if a d-dimensional self-similar substitution tiling space has a pair of distinct tilings that are asymptotic in a set of directions that contains a closed $(d-1)$-hemisphere in its interior, then the branch locus is a topological invariant of the tiling space. If the tiling space is a two-dimensional self-similar Pisot substitution tiling space, the branch locus has a description as an inverse limit of an expanding Markov map on a zero- or one-dimensional simplicial complex.

1. *Introduction*

Tilings of $\mathbb{R}^d$, and the tiling spaces associated with them, play a fundamental role in recent investigations in number theory, physics, logic, computer science, and dynamical systems. For a given tiling T of $\mathbb{R}^d$, one would like to understand the recurrence properties of the various patterns made by the tiles of T. These properties are encoded in the topology of the tiling space Ω associated with T and in the dynamics of the action of $\mathbb{R}^d$ by translation on Ω. Under standard assumptions on the nature of T, the space Ω is locally homeomorphic with a product of a d-dimensional disk and a Cantor set, the arc-components of Ω coincide with the orbits of the $\mathbb{R}^d$-action, and recurrences of patterns in a tiling correspond to returns under the $\mathbb{R}^d$-action to local neighborhoods.

Except in the degenerate periodic case, the $\mathbb{R}^d$-action on a tiling space is not equicontinuous. That is, there are distinct tilings that become arbitrarily close under translation. It is exactly this phenomenon that distinguishes tiling spaces from the globally

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homogeneous structure of a compact abelian group. To understand the manner in which the aperiodic order of an individual tiling is expressed in the topology of the tiling space we are led to consider structural properties of the collection of pairs of tilings that become arbitrarily close under translation.

In particular, we are interested in the *asymptotics* of the translation action on tiling spaces. We will put sufficient conditions on the tilings we consider so that the associated tiling spaces are compact metric spaces: under the metric, d, two tilings are close if a small translate of one agrees with the other in a large neighborhood of the origin. Tilings $T, T' \in \Omega$ are then asymptotic in direction $v \in \mathbb{S}^{d-1}$ if $d(T - tv, T' - tv) \to 0$ as $t \to \infty$. That is to say, the tilings T and T' look more and more the same as we move out along the ray in direction v (see Figure 1). We prove under rather general hypotheses that there are $T \neq T' \in \Omega$ that are asymptotic in an entire open hemisphere's worth of directions. To say more, we must restrict a bit.

There are three common procedures for constructing tilings of $\mathbb{R}^d$: cut-and-project; matching rules; and substitutions. With the cut-and-project method, the asymptotic tilings and directions can be immediately deduced from the geometry of the boundary of the defining window (see, for example, [**FHK**]). Global properties of matching rule tilings are, on the other hand, very difficult to ascertain—unless the rules enforce the sort of hierarchical structure enjoyed by substitution tilings (see [**M, G-S**]). We are thus led to consider substitutions Φ and their associated tiling spaces Ω_Φ.

A substitution Φ induces a self-map, also denoted by Φ, of Ω_Φ. This will, among other things, provide a way of pinning down a specific location for the start of asymptoticity. For example, in the one-dimensional ($d = 1$) case, there are a finite and non-zero number of asymptotic $\mathbb{R}$-orbits (at most n^2 if there are n distinct tiles—see [**BDH**]). Each of these orbits contains a unique tiling that is periodic under Φ and we may reasonably designate these Φ-periodic, $\mathbb{R}$-asymptotic, orbits as the exact locations on their arc-components where asymptoticity starts. Let us say this more precisely. If $T \neq T' \in \Omega$ are such that $d(T - t, T' - t) \to 0$ as $t \to \infty$ (or as $t \to -\infty$), then there is a unique $t_0 \in \mathbb{R}$ so that, if $T_0 := T - t_0$ and $T_0' := T' - t_0$, then T_0 and T_0' are Φ-periodic. Moreover, $T_0 \notin W^s(T_0')$ and $T_0 - t \in W^s(T_0' - t)$, for all $t > 0$ (respectively, $t < 0$), where W^s denotes the Φ-stable 'manifold'. This choice for the initial point of asymptoticity is made less arbitrary by the rigidity result of [**BS**]: if Ω_Φ and Ω_Ψ are homeomorphic one-dimensional substitution tiling spaces, then there is a homeomorphism of Ω_Φ with Ω_Ψ that not only conjugates the $\mathbb{R}$-actions (perhaps with rescaling), but also conjugates some power of Φ with a power of Ψ and hence takes Φ-periodic orbits to Ψ-periodic orbits. Thus, the finite collection of *asymptotic pairs* $\{(T_0, T_0')\}$, as above, is a topological invariant for Ω_Φ. A complete topological invariant for one-dimensional substitution tiling spaces is derived from asymptotic pairs in [**BD**], and a primary motivation of the current article is to lay the groundwork for an extension of that result to higher dimensions (see Problem 6.2 in §6 and the related discussion).

We will call a pair of tilings $(T, T') \in \Omega_\Phi \times \Omega_\Phi$ a *branch pair* if $T \notin W^s(T')$ and there is an open hemisphere $H \subset \mathbb{S}^{d-1}$ so that $T - tv \in W^s(T' - tv)$ for all $v \in H$ and all $t > 0$. We will see that such a pair is asymptotic in the directions H. The substitution takes branch pairs to branch pairs and if the inflation associated with the substitution is scalar (we term

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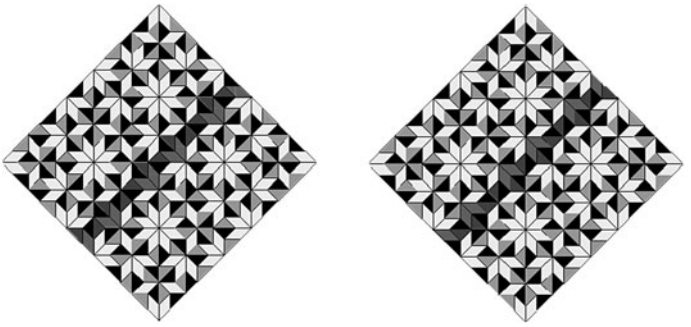


FIGURE 1. Patches of a branch pair in the octagonal tiling space. The tilings differ only along the dark diagonal strip.

such a substitution *self-similar*) then the substitution preserves asymptotic directions. For this reason (and also so that the results of [**K**] apply) we will mostly restrict attention to self-similar substitutions. The terminology 'branch pair' is inspired by the fact that, if the tiles are polyhedral, then the space Ω_Φ is obtained as an inverse limit of a map on a branched manifold. The asymptotic-in-a-half-space pairs are then a ghostly remnant of the branching in the approximating branched manifolds.

The general situation is quite complicated when $d > 1$. Members of branch pairs are not necessarily periodic under the substitution and there are typically infinitely many branch pairs. However, there are at most finitely many branch pairs that are asymptotic in a set of directions that contains a given closed hemisphere in its interior, and they are all substitution-periodic (Proposition 3.19). To capture the asymptotic structure of a substitution tiling space, we define the *branch locus* (essentially, this consists of all the members of branch pairs, together with all the tilings obtained by translating these members in directions in the boundary of their maximal asymptotic sectors, and all tilings accumulated on by such). We prove (Theorem 3.32) that if a self-similar substitution tiling space has a branch pair whose set of asymptotic directions contains a closed hemisphere in its interior, then the branch locus is a topological invariant.

If we restrict further to two-dimensional self-similar substitution tiling spaces, the structure of the branch locus becomes manageable: there are only finitely many lines along which branching occurs (Theorem 4.2) and if the substitution is also Pisot, then the branch locus is the inverse limit of an expanding Markov map on a compact one-dimensional simplicial complex, together with finitely many isolated points. In the latter case, it follows from the rigidity result of Kwapisz [**K**] that the branch locus is also a topological invariant (Theorem 4.11).

It frequently occurs that asymptotic structure in a tiling space can be held responsible for the appearance of certain subgroups in the cohomology of the space. For example, in the one-dimensional period doubling tiling space ($a \mapsto ab$, $b \mapsto aa$), there is a branch pair (T, T') with $d(T - t, T' - t) \to 0$ as $t \to \pm\infty$: this pair contributes a $\mathbb{Z}$ in H^1. Similarly, the branch locus of the two-dimensional half-hex tiling space (see Example 4.6) consists of three tilings, each asymptotic to the other two in all directions: this contributes $\mathbb{Z}^2$ to H^2. For the chair tiling (Example 4.7), there are two 'tubes' each made of $\mathbb{R}^2$ orbits of

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four 2-adic solenoids in the branch locus that contribute $\mathbb{Z}[1/2]^2$ to H^2 (the asymptotic structure also is responsible for a $\mathbb{Z}/3\mathbb{Z}$ that, while it doesn't show up in H^2 of the full tiling space, is, none-the-less, a topological invariant—see [**BDHS**]). Calculations along these lines can be found in the dissertation, [**O**], of the second author, as can preliminary versions of many of the results contained in this article. A different perspective on the contribution of branching to cohomology can be found in [**BSa**]. For a connection between branching in one-dimensional substitution tiling spaces and the Matsumoto K_0-group, see [**BDS**]. We do not pursue these topics in the current paper.

The existence of tilings that are asymptotic in a hemisphere's worth of directions is closely related to lack of expansiveness of translation subdynamics. In [**BL**], Boyle and Lind consider subdynamics of expansive $\mathbb{Z}^d$-actions on compact metric spaces. In particular, they show that, if the space is infinite, there is always a $(d-1)$-dimensional subspace of $\mathbb{R}^d$ so that the restriction of the $\mathbb{Z}^d$-action to elements within a bounded distance of the subspace is not expansive. The corresponding notion of expansiveness for the translation action of $\mathbb{R}^d$ on a d-dimensional tiling space Ω would be: the action is *transversely expansive* provided there is an $\alpha > 0$ so that if $T \neq T' \in \Omega$ are any two tilings that share a tile, then there is a $v \in \mathbb{R}^d$ with $d(T - v, T' - v) \geq \alpha$. Under the assumption of finite local complexity (see §2), the translation action is transversely expansive. Under additional (very mild) hypotheses, Proposition 3.2 implies that there is always a $(d-1)$-dimensional subspace of $\mathbb{R}^d$ restricted to which the translation action on Ω is not transversely expansive. If Ω is a self-similar substitution tiling space and either $d = 2$ or the scalar inflation λ is not a root of an integer, it is a consequence of Theorem 3.26 that there are at least two distinct hyperspaces restricted to which the translation is not transversely expansive. In particular, this shows that the answer to [**BL**, Problem 9.2], *Given a direction in* $\mathbb{R}^2$*, is there a* $\mathbb{Z}^2$*-action on an infinite metric space which is not expansive only in that single direction?*, is 'no' in the setting of self-similar substitution tiling spaces. (The answer in general is 'yes'—see [**H**].)

In §2 we review basic definitions and facts about tiling spaces. Section 3 introduces the terminology related to asymptotic structures and establishes some general results. In §4 we restrict to the case of two-dimensional self-similar substitution tiling spaces. Proofs of some of the more technical results are presented in §5 and in §6 we list a few open problems.

2. *Background*

By a *tile* τ in $\mathbb{R}^d$ we mean an ordered pair $\tau = (\mathrm{spt}(\tau), m)$, where $\mathrm{spt}(\tau)$, the *support* of τ, is a compact subset of $\mathbb{R}^d$, and m is a *mark* taken from some finite set of marks. A tile τ is *topologically regular* if $\mathrm{cl}(\mathrm{int}(\mathrm{spt}(\tau))) = \mathrm{spt}(\tau)$ and tiles $\tau = (\mathrm{spt}(\tau), m)$ and σ are *translationally equivalent* if there is a $v \in \mathbb{R}^d$ with $\tau + v := ((\mathrm{spt}(\tau) + v, m) = \sigma$. In this article all tiles will be assumed to be topologically regular. By the *interior of a tile* we will mean the interior of its support: $\mathring{\tau} := \mathrm{int}(\mathrm{spt}(\tau))$.

A *patch* is a collection of tiles with pairwise disjoint interiors, the *support* of a patch P, $\mathrm{spt}(P)$, is the union of the supports of its constituent tiles, the *diameter* of P, $\mathrm{diam}(P)$, is the diameter of its support, and a *tiling* of $\mathbb{R}^d$ is a patch with support $\mathbb{R}^d$. A collection Ω of tilings of $\mathbb{R}^d$ has *translationally finite local complexity* (FLC) if it is the case that for

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each R there are only finitely many translational equivalence classes of patches $P \subset T \in \Omega$ with $\text{diam}(P) \leq R$. Given a tiling T, let $B_0[T] := \{\tau \in T : 0 \in \text{spt}(\tau)\}$, and, for $R > 0$, $B_R[T] := \{\tau \in T : B_R(0) \cap \text{spt}(\tau) \neq \emptyset\}$. Here $B_R(0)$ denotes the open ball of radius R centered at $0 \in \mathbb{R}^d$. If Ω is a collection of tilings of $\mathbb{R}^d$ with FLC, there is a metric d on Ω with the property: $d(T, T') < \epsilon$ if there are $v, v' \in \mathbb{R}^n$ with $|v|, |v'| < \epsilon/2$, so that $B_{1/\epsilon}[T' - v'] = B_{1/\epsilon}[T - v]$. In other words, in this metric two tilings are close if a small translate of one agrees with the other in a large neighborhood of the origin. (See [**AP**] for details.) We will call a collection Ω of tilings of $\mathbb{R}^d$ a *d-dimensional tiling space* if Ω has FLC, is closed under translation ($T \in \Omega$ and $v \in \mathbb{R}^d \Rightarrow T - v \in \Omega$), and is compact in the metric d. (All tiling spaces in this article are assumed to have FLC, but we will occasionally include the FLC hypothesis for emphasis.) For example, if T is an FLC tiling of $\mathbb{R}^d$, then $\Omega = \{T' : T'$ is a tiling of $\mathbb{R}^d$ and every patch of T' is a translate of a patch of $T\}$ is a d-dimensional tiling space, called the *hull* of T [**AP**].

Suppose that $\mathcal{A} = \{\rho_1, \ldots, \rho_k\}$ is a set of translationally inequivalent tiles (called *prototiles*) in $\mathbb{R}^d$ and Λ is an expanding linear isomorphism of $\mathbb{R}^d$. A *substitution* on $\mathcal{A}$ with expansion Λ is a function $\Phi : \mathcal{A} \to \{P : P$ is a patch in $\mathbb{R}^d\}$ with the properties that, for each $i \in \{1, \ldots, k\}$, every tile in $\Phi(\rho_i)$ is a translate of an element of $\mathcal{A}$, and $\text{spt}(\Phi(\rho_i)) = \Lambda(\text{spt}(\rho_i))$. Such a substitution naturally extends to patches whose elements are translates of prototiles by $\Phi(\{\rho_{i(j)} + v_j : j \in J\}) := \bigcup_{j \in J}(\Phi(\rho_{i(j)}) + \Lambda v_j)$. A patch P is *allowed* for Φ if there is an $m \geq 1$, an $i \in \{1, \ldots, k\}$, and a $v \in \mathbb{R}^d$, with $P \subset \Phi^m(\rho_i) - v$. The *substitution tiling space* associated with Φ is the collection $\Omega_\Phi := \{T : T$ is a tiling of $\mathbb{R}^d$ and every finite patch in T is allowed for $\Phi\}$. If Λ is multiplication by the scalar $\lambda > 1$, Ω_Φ is called *self-similar*.

A d-dimensional tiling space Ω is *repetitive* if for each patch P with compact support that occurs in some tiling in Ω there is an R so that, for all $T \in \Omega$ and all $x \in \mathbb{R}^d$, there is a $v \in \mathbb{R}^d$ so that $P - v \subset B_R[T - x]$. It is clear that if Ω is repetitive, then the action of $\mathbb{R}^d$ on Ω by translation is minimal. The substitution Φ is *primitive* if for each pair $\{\rho_i, \rho_j\}$ of prototiles there is an $n \in \mathbb{N}$ so that a translate of ρ_i occurs in $\Phi^n(\rho_j)$. If Φ is primitive then Ω_Φ is repetitive.

If the translation action on Ω is free (i.e., $T - v = T \Rightarrow v = 0$), Ω is said to be *non-periodic*. If Φ is primitive and Ω_Φ is FLC and non-periodic then Ω_Φ is compact in the metric described above, $\Phi : \Omega_\Phi \to \Omega_\Phi$ is a homeomorphism, and the translation action on Ω_Φ is minimal and uniquely ergodic [**AP, So1, So2**].

A real number λ is a *Pisot number* if it is an algebraic integer and all of its algebraic conjugates lie strictly inside the unit circle. That is, there is a monic integer polynomial p, the *minimal polynomial* of λ, that is irreducible over $\mathbb{Q}$, has λ as a root, and all other roots of p have absolute value less than 1. The *degree* of λ is the degree of p. A self-similar substitution is a *Pisot substitution* if the expansion λ for Φ is a Pisot number.

3. *Definitions and preliminary results*

Suppose that Ω is a d-dimensional tiling space, $v \in \mathbb{S}^{d-1}$, and $S \subset \mathbb{S}^{d-1}$.

Definition 3.1. Tilings $T, T' \in \Omega$ are *asymptotic in direction* v provided $d(T - tv, T' - tv) \to 0$ as $t \to \infty$, and are *uniformly asymptotic in directions* S provided, for each $\epsilon > 0$, there is an R so that $d(T - tw, T' - tw) < \epsilon$ for all $w \in S$ and all $t \geq R$.

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We'll say that T and T' *agree on* $X \subset \mathbb{R}^d$ provided $B_0[T - v] = B_0[T' - v]$, for all $v \in X$.

PROPOSITION 3.2. *If* Ω *is a non-periodic, repetitive,* FLC *tiling space then there are tilings* $\bar{T} \neq \bar{T}'$ *in* Ω *that agree on an open half-space.*

Proof. Let T be any tiling in Ω. By repetitivity, there are $x_n \neq 0$ with $|x_n| \to \infty$ so that $B_n[T - x_n] = B_n[T]$. By non-periodicity, $T - x_n \neq T$ so $r_n := \sup\{r : B_r[T - x_n] = B_r[T]\} < \infty$. Pick $y_n \in B_{r_n}(0)$ with $B_1[T - x_n - y_n] \neq B_1[T - y_n]$. Since $B_1[T - x_n - y_n] \cap B_1[T - y_n] \neq \emptyset$, finite local complexity insures that, up to translation, there are only finitely many pairs $(B_1[T - x_n - y_n], B_1[T - y_n])$. Passing to a subsequence we may assume that $(B_1[T - x_n - y_n], B_1[T - y_n]) = (P' - z_n, P - z_n)$, with $z_n \to 0$, for some pair of distinct patches (P', P). By compactness, there is a subsequence n_i, tilings $\bar{T}', \bar{T} \in \Omega$, and u, so that $T - x_{n_i} - y_{n_i} \to \bar{T}'$, $T - y_{n_i} \to \bar{T}$, and $y_{n_i}/|y_{n_i}| \to u$. Then $\bar{T} \neq \bar{T}'$ (since $\bar{T}' \supset P'$, $\bar{T} \supset P$) and $\bar{T}$, $\bar{T}'$ agree on the half-space $\{v : \langle v, u\rangle < 0\}$. Indeed, if $\langle v, u\rangle < 0$, then $\langle v, y_{n_i}\rangle = |y_{n_i}|\langle v, y_{n_i}/|y_{n_i}|\rangle \to \infty$ as $i \to \infty$. Thus such v is in $B_{|y_{n_i}|}(-y_{n_i}) \subset B_{r_{n_i}}(-y_{n_i})$, on which $T - x_{n_i} - y_{n_i}$ and $T - y_{n_i}$ agree for all large i. □

The translation action on a tiling space is not expansive: if $|w|$ is small, then $d(T - v, T - w - v)$ remains small for all v. But, assuming FLC, the $\mathbb{R}^d$-action on a d-dimensional tiling space is *transversely expansive* in the following sense: there is an $\alpha > 0$ so that if $T \neq T' \in \Omega$ and $T \cap T' \neq \emptyset$ then there is a $v \in \mathbb{R}^d$ so that $d(T - v, T' - v) \geq \alpha$. This notion of transversely expansive is the natural way of transferring expansivity of a $\mathbb{Z}^d$-action to 'expansivity' of the $\mathbb{R}^d$-action obtained by suspension. More generally, if V is a subspace of $\mathbb{R}^d$, then the translation action on Ω is *transversely expansive in the direction of* V if, in the definition of transversely expansive, the v can be taken from V. The following is analogous to a result of Boyle and Lind ([**BL**]) for expansive $\mathbb{Z}^d$-actions.

COROLLARY 3.3. *If* Ω *is a* d*-dimensional, non-periodic, repetitive,* FLC *tiling space then there is a* $(d-1)$*-dimensional subspace* V *of* $\mathbb{R}^d$ *so that the translation action is not transversely expansive in the direction of* V*. Moreover, if* V *is such a subspace, then there are tilings* $\bar{T} \neq \bar{T}' \in \Omega$ *that agree on one of the components of* $\mathbb{R}^d \backslash V$.

Proof. By Proposition 3.2 there is a $w \neq 0$ and tilings $\bar{T} \neq \bar{T}' \in \Omega$ so that $\bar{T}$ and $\bar{T}'$ agree on $\{x : \langle x, w\rangle > 0\}$. Let $V := \{w\}^{\perp}$. Then, for any $\alpha > 0$, there is a $t > 0$ so that, if $T := \bar{T} - tw$ and $T' := \bar{T}' - tw$, $\sup\{d(T - v, T' - v) : v \in V\} < \alpha$.

Suppose now that V is a $(d-1)$-dimensional subspace of $\mathbb{R}^d$ so that the translation action on Ω is not transversely expansive in the direction of V. There are then $T_n \neq T_n' \in \Omega$ so that $T_n \cap T_n' \neq \emptyset$ and $d(T_n - v, T_n' - v) < 1/n$, for each $n \in \mathbb{N}$, and all $v \in V$. There are then $x_n \to 0$ so that $B_0[T_n - x_n] = B_0[T_n']$, for all $n \in \mathbb{N}$. It follows that there are $t_n \to \infty$ so that $T_n - x_n - v$ agrees with $T_n' - v$ on the strip $V + \{tw : |t| \leq t_n\}$, where $w \in \mathbb{S}^{d-1}$ is perpendicular to V. Notice that since $T_n \cap T_n' \neq \emptyset$, $T_n - x_n \neq T_n'$ for large n. By appropriately translating $T_n - x_n$ and T_n' we obtain tilings $S_n \neq S_n' \in \Omega$ with the properties: $B_1[S_n] \neq B_1[S_n']$ and S_n and S_n' agree on $V + \{tw : 0 \leq t \leq 2t_n\}$ or on $V - \{tw : 0 \leq t \leq 2t_n\}$. Let $n_i \to \infty$ be such that $S_{n_i} \to \bar{T} \in \Omega$ and $S_{n_i}' \to \bar{T}' \in \Omega$. Then $\bar{T} \neq \bar{T}'$ and $\bar{T}$ and $\bar{T}'$ agree on one of the components of $\mathbb{R}^d \backslash V$. □

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It is clear that if tilings T and T' agree on the open half-space $\{v \in \mathbb{R}^d : \langle v, u\rangle > 0\}$, some $u \in \mathbb{S}^{d-1}$, then T and T' are uniformly asymptotic in directions $\{v \in \mathbb{S}^{d-1} : \langle v, u\rangle > \epsilon\}$ for each $\epsilon > 0$.

LEMMA 3.4. *If Ω is a d-dimensional* FLC *tiling space and $T, T' \in \Omega$ are asymptotic in direction $v \in \mathbb{S}^{d-1}$ then there is R so that T and T' agree on $\{tv : t \geq R\}$.*

Proof. As a consequence of finite local complexity, there is $\epsilon > 0$ so that: (1) if $T_1, T_2 \in \Omega$ satisfy $B_1[T_1] = B_1[T_2]$ and $|x| < \epsilon$, then $B_0[T_1 - x] = B_0[T_2 - x]$; and (2) if $B_0[T_1] = B_0[T_1 - x]$ for some $T_1 \in \Omega$ and x with $|x| < \epsilon$, then $x = 0$. Now if T and T' are asymptotic in direction v, there is $x(t)$ so that $B_1[T - tv] = B_1[T' - tv - x(t)]$ and $x(t) \to 0$ as $t \to \infty$. Let R be large enough so that $|x(t)| < \epsilon/2$ for $t \geq R$. For $t_0 > R$ we have $B_1[T - t_0 v] = B_1[T' - t_0 v - x(t_0)]$, so $B_0[T - tv] = B_0[T' - tv - x(t_0)]$, for all t with $|t - t_0| < \epsilon$, by (1). Also, $B_0[T - tv] = B_0[T' - tv - x(t)]$. Thus, if $|t - t_0| < \epsilon$ and $t, t_0 \geq R$, we have $B_0[T' - tv - x(t_0)] = B_0[T' - tv - x(t)]$ and $|x(t) - x(t_0)| < \epsilon$, so that $x(t) = x(t_0)$, by (2). That is, $x(t)$ is constant for $t \geq R$, and hence $x(t) = 0$ for $t \geq R$. □

Note that from the assumption of finite local complexity it follows that, for each $\epsilon > 0$, there is an $r = r(\epsilon) \to \infty$ as $\epsilon \to 0$ so that if T and T' agree at x and $d(T, T') < \epsilon$, then T and T' agree on $B_r(x)$. Hence, from the above lemma, T and T' are asymptotic in direction v if and only if T and T' agree on larger and larger balls about tv as $t \to \infty$.

LEMMA 3.5. *If Ω is a d-dimensional* FLC *tiling space and $T, T' \in \Omega$ are uniformly asymptotic in directions $S \subset \mathbb{S}^{d-1}$ then there is R so that T and T' agree on $\{tv : v \in S, t \geq R\}$.*

Proof. Let ϵ be as in the proof of Lemma 3.4. Then, by uniform asymptoticity, there is an R so that, for each $v \in S$, there is $x(t, v)$ with $|x(t, v)| < \epsilon/2$ for $t \geq R$ and $B_1[T - tv] = B_1[T' - tv - x(t, v)]$. As in the proof of Lemma 3.4, this implies that $x(t, v) = 0$, for all $v \in S$ and $t \geq R$. Thus T and T' agree on $\{tv : v \in S, t \geq R\}$. □

If tilings T and T' are asymptotic in direction v then so are $T - x$ and $T' - x$, for all $x \in \mathbb{R}^d$. Thus it makes no sense to say that T and T' start being asymptotic in direction v at any particular place. But in the setting of a substitution tiling space with substitution Φ, we will see that, for such T, T' and v, there is a smallest $t_0 \in [-\infty, \infty)$ so that $T - tv$ and $T' - tv$ are asymptotic under iteration of Φ (that is, they are in the same Φ-stable manifold) for all $t > t_0$. It will follow that, if T and T' are asymptotic in direction v, for all v with $\langle v, u\rangle > 0$, for some $u \in \mathbb{S}^{d-1}$, then there is a smallest $t_0 \in (-\infty, \infty)$ so that $T - x$ and $T' - x$ are in the same Φ-stable manifold for all x with $\langle x, u\rangle > t_0$. It is thus possible to make sense of locations where asymptoticity in a half-space begins and this will be the basis for the definition of the *branch locus* in a substitution tiling space. The rest of this section is devoted to making this idea precise.

If Φ is the induced homeomorphism on a substitution tiling space Ω_Φ and $T \in \Omega_\Phi$, the *stable manifold* of T is the set $W^s(T) := \{T' \in \Omega_\Phi : d(\Phi^n(T'), \Phi^n(T)) \to 0 \text{ as } n \to \infty\}$.

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LEMMA 3.6. *Suppose that $\Omega = \Omega_\Phi$ is an FLC substitution tiling space and $T \in \Omega$. Then $W^s(T) = \{T' \in \Omega : \text{for all } r, \text{ there exists } n \text{ such that } B_r[\Phi^n(T')] = B_r[\Phi^n(T)]\} = \{T' \in \Omega : B_0[\Phi^n(T')] = B_0[\Phi^n(T)] \text{ for some } n\}$.*

Proof. The equality of the last two sets, and their containment in $W^s(T)$, is clear. For the opposite containment, note that, by finite local complexity, there is $\epsilon_1 > 0$ so that, for all ϵ_2 with $0 < \epsilon_2 \le \epsilon_1$, $\inf\{d(T_1, T_2 - x) : T_1, T_2 \in \Omega, B_1[T_1] = B_1[T_2], \epsilon_2 \le |x| \le \epsilon_1\} =: \delta(\epsilon_2) > 0$. Choose $\epsilon_1 > 0$ also small enough so that if T_1 and T_2 are any two tilings in Ω with $B_1[T_1] = B_1[T_2]$, then $B_1[T_1 - x] = B_1[T_2 - y]$ and $|x|, |y| \le \epsilon_1$ implies that $x = y$. Set $\epsilon_2 = \epsilon_1/|\Lambda|_{\max}$, where $|\Lambda|_{\max} := \max\{|\Lambda v| : |v| = 1\}$ and Λ is the inflation matrix for Φ. Now if $T' \in W^s(T)$, there are N and x_n with $|x_n| < \epsilon_2$, so that $B_1[\Phi^n(T')] = B_1[\Phi^n(T) - x_n]$ and $d(\Phi^n(T'), \Phi^n(T)) < \delta(\epsilon_2)$ for all $n \ge N$. If $x_N \ne 0$, there is $k > 0$ so that $\epsilon_2 \le |\Lambda^k x_N| \le \epsilon_1$. We would have

$$\begin{aligned} B_1[\Phi^{N+k}(T) - x_{N+k}] &= B_1[\Phi^{N+k}(T')] = B_1[\Phi^k(\Phi^N(T) - x_N)] \\ &= B_1[\Phi^{N+k}(T) - \Lambda^k x_N] \end{aligned}$$

so that $x_{N+k} = \Lambda^k x_N$, from which it would follow that $d(\Phi^{N+k}(T'), \Phi^{N+k}(T)) \ge \delta(\epsilon_2)$, in contradiction to the choice of N. Thus $B_1[\Phi^N(T')] = B_1[\Phi^N(T)]$. □

If $\Omega = \Omega_\Phi$ is a d-dimensional substitution tiling space with $T, T' \in \Omega$ and $T \in W^s(T')$ we will say that T and T' are *stably related* and write $T \sim_s T'$. Note that as a consequence of Lemma 3.6 the relation $\nsim_s$, the negation of $\sim_s$, has the property: $T - x_n \nsim_s T' - x_n$ and $x_n \to x \implies T - x \nsim_s T' - x$. If $X \subset \mathbb{R}^d$ and $T - x \sim_s T' - x$, for all $x \in X$, we'll say that T and T' are *stably related on* X. Given $u \in \mathbb{S}^{d-1}$, H_u will denote the open hemisphere $H_u := \{v \in \mathbb{S}^{d-1} : \langle v, u\rangle > 0\}$.

Definition 3.7. The tiling $T \in \Omega$ is a *branch point* if there is a tiling $T' \in \Omega$ and $u \in \mathbb{S}^{d-1}$ so that $T \nsim_s T'$ but T and T' are stably related on the positive cone $\{tv : v \in H_u, t > 0\}$ over H_u. In this case, (T, T') is called a *branch pair*.

We denote the collection of all branch pairs of Ω_Φ by $\mathcal{BP}(\Phi)$. For $(T, T') \in \mathcal{BP}(\Phi)$, let $\Delta(T, T') := \{u \in \mathbb{S}^{d-1} : T \text{ and } T' \text{ are stably related on the positive cone over } H_u\}$.

Definition 3.8. The *asymptotic sector* of the branch pair (T, T') is the set

$$S(T, T') := \bigcup_{u \in \Delta(T, T')} H_u.$$

Thus, in order for v to be in the asymptotic sector $S(T, T')$, it is necessary that T and T' be asymptotic in an entire open hemisphere's worth of directions containing v. Note that a self-similar substitution Φ preserves asymptotic directions, so such Φ takes branch pairs to branch pairs, preserving asymptotic sectors.

For a given substitution tiling space Ω, patches P, P', and $X \subset \text{int}(\text{spt}(P) \cap \text{spt}(P'))$, we say that P and P' are stably related on X provided, whenever $T, T' \in \Omega$ are such that $P \subset T$ and $P' \subset T'$, it is the case that T and T' are stably related on X. If P and P' are stably related on $X = \text{int}(\text{spt}(P) \cap \text{spt}(P')) \ne \emptyset$, we say that P and P' are *stably related on overlap*. The following is an adaptation of a lemma in [**So2**].

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LEMMA 3.9. *If P and P' are patches for a non-periodic, FLC, self-similar substitution tiling space and P and P' are stably related on overlap, there is an $\epsilon > 0$ so that if $0 < |x| < \epsilon$ then $P - x$ and P' are not stably related on overlap.*

Proof. We prove the result in case $P = P' = \{\tau\}$ for some tile τ, from which the general result follows easily. Let λ be the expansion factor for Φ. Pick $y \in \text{int}(\tau)$ and let $r > 0$ be small enough so that $\text{cl}(B_{2r}(y)) \subset \text{int}(\tau)$. By [**So2**, Lemma 2.4], there is an $N > 1$ so that if Q and $Q - x_1$ are both patches in some tiling $T \in \Omega$, $B_{r_1}(y) \subset \text{spt}(Q)$, and $|x_1| < r_1/N$, then $x_1 = 0$. Let $\epsilon := r/N$, let $|x| < \epsilon$, and suppose that $\{\tau\}$ and $\{\tau\} - x$ are stably related on overlap. Since the compact ball $\text{cl}(B_r(y))$ is contained in $\text{int}(\tau \cap (\tau - x))$, there is an M so that $B_0[\Phi^M(\{\tau\} - z)] = B_0[\Phi^M(\{\tau\} - x - z)]$, for all $z \in \text{cl}(B_r(y))$ (Lemma 3.6). That is, both the patch $Q := B_{\lambda^M r}[\Phi^M(\{\tau\})]$ and the patch $Q - \lambda^M x = B_{\lambda^M r}[\Phi^M(\{\tau\} - x)]$ occur in the patch $\Phi^M(\{\tau\})$. Moreover, letting $x_1 = \lambda^M x$ and $r_1 = \lambda^M r$, we have $B_{r_1}(y) \subset \text{spt}(Q)$ and $|x_1| < r_1/N$. Thus $x_1 = 0$ and hence $x = 0$. □

LEMMA 3.10. *If $\Omega = \Omega_\Phi$ is a d-dimensional non-periodic, FLC, self-similar substitution tiling space and T, $T' \in \Omega$ are such that T and T' are stably related on $\mathbb{R}^d$, then $T = T'$.*

Proof. Let $\Delta := \max\{\text{diam}(\rho) : \rho \text{ is a prototile for } \Omega\}$. Then, for all $\bar{T} \in \Omega$, $\text{spt}(B_0[\bar{T}]) \subset \text{int}(\text{spt}(B_{\Delta+1}[\bar{T}]))$, and there are, up to translation, just finitely many pairs of the form $(B_0[\bar{T}], B_{\Delta+1}[\bar{T}])$. It follows from Lemma 3.9 that, up to translation, there are only finitely many pairs of pairs of the form $((B_0[T - x], B_{\Delta+1}[T - x]), (B_0[T' - x], B_{\Delta+1}[T' - x]))$, say for each x there is an $i \in \{1, \ldots, n\}$ and a $y = y(x)$ with

$$\begin{aligned}&((B_0[T - x], B_{\Delta+1}[T - x]), (B_0[T' - x], B_{\Delta+1}[T' - x]))\\ &\quad = ((P_i - y, Q_i - y), (P_i' - y, Q_i' - y)).\end{aligned}$$

Let x, y, i be as in the last sentence. For each $z \in \text{spt}(P_i) \cap \text{spt}(P_i')$, there is an $m = m(x, y, z)$ so that

$$B_1[\Phi^m(T - x + y - z)] = B_1[\Phi^m(T' - x + y - z)],$$

by Lemma 3.6, and in fact we can choose this m to depend only on z (and i), since if x', y', i are also such that

$$\begin{aligned}&((B_0[T - x'], B_{\Delta+1}[T - x']), (B_0[T' - x'], B_{\Delta+1}[T' - x']))\\ &\quad = ((P_i - y', Q_i - y'), (P_i' - y', Q_i' - y')),\end{aligned}$$

then

$$\begin{aligned}&(B_0[T - x' + y' - z], B_0[T' - x' + y' - z])\\ &\quad = (B_0[T - x + y - z], B_0[T' - x + y - z])\end{aligned}$$

(this is because $\text{spt}(P_i) \cap \text{spt}(P_i') \subset \text{int}(\text{spt}(Q_i) \cap \text{spt}(Q_i'))$). Now, if z' is sufficiently close to z, then

$$B_0[\Phi^m(T - x + y - z')] = B_0[\Phi^m(T' - x + y - z')].$$

Thus, by compactness of $\text{spt}(P_i) \cap \text{spt}(P_i')$, there is an $M = M(i)$ so that

$$B_0[\Phi^M(T - x + y - z)] = B_0[\Phi^M(T' - x + y - z)] \quad \text{for all } z \in \text{spt}(P_i) \cap \text{spt}(P_i')$$

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and x, y so that

$$\begin{aligned}&((B_0[T-x],\, B_{\Delta+1}[T-x]),\, (B_0[T'-x],\, B_{\Delta+1}[T'-x]))\\&\quad=((P_i-y,\, Q_i-y),\, (P_i'-y,\, Q_i'-y)).\end{aligned}$$

Let $K=\max\{M(1),\ldots,M(n)\}$. Then

$$B_0[\Phi^K(T-x+y-z)]=B_0[\Phi^K(T'-x+y-z)]$$

for all appropriate x, y, z: that is, $\Phi^K(T)=\Phi^K(T')$, and since Φ is a homeomorphism, $T=T'$. □

LEMMA 3.11. *If $\Omega=\Omega_\Phi$ is a d-dimensional, non-periodic, FLC, self-similar substitution tiling space, there is an R so that if T, $T'\in\Omega$ are stably related on $B_R(0)$, then $B_0[T]=B_0[T']$.*

Proof. Otherwise, for each $n\in\mathbb{N}$, there are T_n, $T_n'\in\Omega$ with $T_n-x\sim_s T_n'-x$ for all x with $|x|\le n$ and $B_0[T_n]\ne B_0[T_n']$. There is a smallest m_n so that $B_0[\Phi^{m_n}(T_n)]=B_0[\Phi^{m_n}(T_n')]$: replace T_n by $\Phi^{m_n-1}(T_n)$ and T_n' by $\Phi^{m_n-1}(T_n')$. By finite local complexity, there are only finitely many pairs $(B_0[T_n],\, B_0[T_n'])$, up to translation: say $(B_0[T_{n_i}],\, B_0[T_{n_i}'])=(P-x_i,\, P'-x_i)$ with $x_i\to 0$, $T_{n_i}\to T\in\Omega$, and $T_{n_i}'\to T'\in\Omega$. Since $P\subset T$ and $P'\subset T'$, $T\ne T'$. Fix $r>0$. Since the tilings $T_{n_i}+x_i$ all share the patch P and the tilings $T_{n_i}'+x_i$ all share the patch P', finite local complexity guarantees that there are just finitely many pairs $(B_r[T_{n_i}+x_i],\, B_r[T_{n_i}'+x_i])$. It follows that $B_r[T_{n_i}+x_i]\subset T$ and $B_r[T_{n_i}'+x_i]\subset T'$ for large i. If i is also large enough so that $n_i+|x_i|>r$, then $T_{n_i}+x_i-y\sim_s T_{n_i}'+x_i-y$ for all $y\in B_r(0)$ and hence $T-y\sim_s T'-y$ for all $y\in B_r(0)$. Thus $T-y\sim_s T'-y$ for all y and by Lemma 3.10 we arrive at the contradiction $T=T'$. □

LEMMA 3.12. *If $\Omega=\Omega_\Phi$ is a d-dimensional, non-periodic, FLC, self-similar substitution tiling space, then given r and $\epsilon>0$, there are, up to translation, only finitely many pairs of (allowed) patches $(P,\, P')$ with the properties:* $\operatorname{diam}(P),\operatorname{diam}(P')\le r$*; P and P' are stably related on $B_\epsilon(0)$.*

Proof. First, there is a $k=k(\epsilon)$ so that for any such $(P,\, P')$, $\Phi^k(P)\cap\Phi^k(P')\ne\emptyset$. Otherwise, there are $(P_k,\, P_k')$ and tilings $T_k\supset\Phi^k(P_k)$, $T_k'\supset\Phi^k(P_k')$ with $T_k-x\sim_s T_k'-x$, for all $x\in B_{\lambda^k\epsilon}(0)$, and with $B_0[T_k]\ne B_0[T_k']$. This violates Lemma 3.11. Now, for this k and for all such pairs $(P,\, P')$, the elements of the pairs $(\Phi^k(P),\, \Phi^k(P'))$ share a tile and have diameters bounded by $\lambda^k r$. By finite local complexity, there are, up to translation, only finitely many such $(\Phi^k(P),\, \Phi^k(P'))$ and hence, only finitely many $(P,\, P')$. □

PROPOSITION 3.13. *If Ω is a non-periodic, FLC, self-similar substitution tiling space and $T\in\Omega$ is a branch point then there are finitely many $T'\in\Omega$ such that $(T,\, T')$ is a branch pair.*

Proof. Let Φ be the substitution with associated inflation λ. Suppose $(T,\, T_k')$, $k\in\mathbb{N}$, are branch pairs. It follows from Lemma 3.11 and finite local complexity that there is a subsequence $\{T_{k_i}'\}$ and an R so that, for each $m\in\mathbb{N}$, the $B_R[\Phi^{-m}(T_{k_i}')]$ all share a tile.

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There is M so that if $\{T_1, \ldots, T_M\}$ are any M tilings that satisfy $\bigcap_{i+1}^{M} B_R[T_i] \neq \emptyset$, then $B_1[T_i] = B_1[T_j]$ for some $i \neq j \in \{1, \ldots, M\}$. If the T'_{k_i}, $i = 1, \ldots, M$, are all distinct, then there is r so that the patches $B_r[T'_{k_i}]$, $i = 1, \ldots, M$, are all distinct. Let $m \in \mathbb{N}$ be large enough so that $\lambda^m > r$. There are then $i \neq j \in \{1, \ldots, M\}$ so that $B_1[\Phi^{-m}(T'_{k_i})] = B_1[\Phi^{-m}(T'_{k_j})]$. But then $B_r[T'_{k_i}] = B_r[T'_{k_j}]$, contradicting the choice of r. □

COROLLARY 3.14. *If (T, T') is a branch pair and T is Φ-periodic, then so is T'.*

Definition 3.15. If T is a branch point and T is Φ-periodic, T is called a *periodic branch point* and a branch pair (T, T') is called a *periodic branch pair*. The collection of all periodic branch pairs is denoted by $\mathcal{PBP}(\Phi)$.

LEMMA 3.16. *Suppose that $\Omega = \Omega_\Phi$ is a d-dimensional, non-periodic, FLC, self-similar substitution tiling space. If $(T, T') \in \mathcal{PBP}(\Phi)$ is a periodic branch pair, then $\{v \in \mathbb{S}^{d-1} :$ T and T' are stably related on the ray $\{tv : t > 0\}\}$ is open in $\mathbb{S}^{d-1}$.*

Proof. With no loss of generality, we may assume that T and T' are fixed by Φ. Let λ be the expansion factor for Φ. Given $v \in \mathbb{S}^{d-1}$ so that T and T' are stably related on the ray $\{tv : t > 0\}$ and $t \in [1, \lambda]$ there is $\epsilon(t) > 0$ so that $T - tv - x \sim_s T' - tv - x$, for all $x \in B_{\epsilon(t)}(0)$ (Lemma 3.6). From compactness of $[1, \lambda]$, there is $\epsilon > 0$ so that if $w \in \mathbb{S}^{d-1} \cap B_\epsilon(v)$ then $T - tw \sim_s T' - tw$, for all $t \in [1, \lambda]$. For such w, any $n \in \mathbb{Z}$, and $t \in [1, \lambda]$, $T - \lambda^n tw = \Phi^n(T - tw) \sim_s \Phi^n(T' - tw) = T' - \lambda^n tw$. Thus $T - tw \sim_s T' - tw$ for all $t \in \bigcup_{n\in\mathbb{Z}} \lambda^n[1, \lambda] = \mathbb{R}^+$. □

LEMMA 3.17. *Suppose that $\Omega = \Omega_\Phi$ is a non-periodic, FLC, self-similar substitution tiling space. Then for each r there are, up to translation, only finitely many pairs of patches of the form $(B_r[T], B_r[T'])$ with $(T, T') \in \mathcal{BP}(\Phi)$. Furthermore, if $(T_n, T'_n) \in \mathcal{BP}(\Phi)$ and $(T_n, T'_n) \to (T, T')$ in $\Omega \times \Omega$, then, for sufficiently large n, there are $x_n \to 0$ so that $(B_r[T_n], B_r[T'_n]) = (B_r[T - x_n], B_r[T' - x_n])$.*

Proof. The first statement follows immediately from Lemma 3.12.

Suppose that $(T_n, T'_n) \in \mathcal{BP}(\Phi)$ and $(T_n, T'_n) \to (T, T')$ in $\Omega \times \Omega$, and let r be given. There are then x_n and y_n so that $B_r[T_n] = B_r[T - x_n]$ and $B_r[T'_n] = B_r[T' - y_n]$, for all sufficiently large n, with $x_n, y_n \to 0$. If $x_n - y_n \to 0$ takes on infinitely many different values, we would have infinitely many translationally independent pairs $(B_r[T_n], B_r[T'_n])$. Thus $x_n = y_n$ for sufficiently large n. □

LEMMA 3.18. *Suppose that $\Omega = \Omega_\Phi$ is a non-periodic, FLC, self-similar substitution tiling space and $T \in \Omega$ has the property that $B_r[\Phi^{-n}(T)] = B_r[\Phi^{-n_1}(T)]$ for some $r \geq 0$, $n_1 \in \mathbb{N}$, and for infinitely many $n \in \mathbb{N}$. Then T is Φ-periodic.*

Proof. Say $B_r[\Phi^{-n_i}(T)] = B_r[\Phi^{-n_1}(T)] =: P$ for $i \in \mathbb{N}$. Let $m = \min\{n \in \mathbb{N} : P \subset \Phi^n(P)\}$. Since $\Phi(B_r[T']) \supset B_{\lambda r'}[\Phi(T')] \supset B_r[\Phi(T')]$ for any $T' \in \Omega$, where λ is the expansion factor of Φ and $r' = r$ if $r > 0$ and, if $r = 0$ then $r' > 0$ is such that $B_{r'}(0) \subset \mathrm{spt}(B_0[T'])$, we see that m divides $n_i - n_1$ for all i. Say $n_i = n_1 + mk_i$ and let $\bar{T} \in \Omega$ be defined by

$$\bar{T} := \Phi^{n_1}\left(\bigcup_{k\in\mathbb{N}} \Phi^{mk}(B_r[\Phi^{-n_1}(T)])\right) = \bigcup_{k\in\mathbb{N}} \Phi^{mk}(B_r[T]).$$

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Then $\Phi^m(\bar{T}) = \bar{T}$. Since the union in the definition of $\bar{T}$ is nested and $k_i \to \infty$,

$$\begin{aligned}\bar{T} &= \Phi^{n_1}\left(\bigcup_{i\in\mathbb{N}} \Phi^{mk_i}(B_r[\Phi^{-n_1}(T)])\right)\\ &= \Phi^{n_1}\left(\bigcup_{i\in\mathbb{N}} \Phi^{mk_i}(B_r[\Phi^{-n_1-mk_i}(T)])\right)\\ &\supset \Phi^{n_1}\left(\bigcup_{i\in\mathbb{N}} B_{\lambda^{mk_i}r'}[\Phi^{-n_1}(T)]\right)\\ &= \Phi^{n_1}(\Phi^{-n_1}(T)) = T.\end{aligned}$$

Thus $T = \bar{T}$ is periodic. □

PROPOSITION 3.19. *Suppose that $\Omega = \Omega_\Phi$ is a d-dimensional, non-periodic, FLC, self-similar substitution tiling space and suppose that $\bar{H} \subset \mathbb{S}^{d-1}$ is a closed hemisphere. Then any branch pair (T, T') with $\bar{H} \subset S(T, T')$ is $\Phi \times \Phi$-periodic and there are at most finitely many such.*

Proof. Let $\tilde{H} := \{tv : v \in \bar{H}, t > 0\}$ and suppose that (T_n, T_n') are branch pairs with $\bar{H} \subset S(T_n, T_n')$ for $n \in \mathbb{N}$. By Lemma 3.17, there are, up to translation, only finitely many pairs $(B_1[\Phi^k(T_n)], B_1[\Phi^k(T_n')])$ for $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Suppose that n_1, n_2, k_1, k_2 and x are such that

$$(B_1[\Phi^{k_1}(T_{n_1})], B_1[\Phi^{k_1}(T_{n_1}')]) = (B_1[\Phi^{k_2}(T_{n_2})] - x, B_1[\Phi^{k_2}(T_{n_2}')] - x).$$

The fact that $B_1[\Phi^{k_1}(T_{n_1})]$ is stably related to $B_1[\Phi^{k_1}(T_{n_1}')]$ on

$$\operatorname{int}(\operatorname{spt}(B_1[\Phi^{k_1}(T_{n_1})])) \cap \operatorname{int}(\operatorname{spt}(B_1[\Phi^{k_1}(T_{n_1}')])) \cap (\tilde{H})$$

while $B_1[\Phi^{k_2}(T_{n_2})] - x$ is not stably related to $B_1[\Phi^{k_2}(T_{n_2}')] - x$ on $\{x\}$ means that $x \notin \tilde{H}$. The preceding statement remains valid if n_1 and n_2 are switched, k_1 and k_2 are switched, and x is replaced by $-x$. We have that $x \notin \tilde{H}$ and $-x \notin \tilde{H}$; hence $x = 0$.

Now fix n. There must be $m > 0$ and $l_i \to -\infty$ so that

$$(B_1[\Phi^{l_i}(T_n)], B_1[\Phi^{l_i}(T_n')]) \quad \text{and} \quad (B_1[\Phi^{l_i+m}(T_n)], B_1[\Phi^{l_i+m}(T_n')])$$

are the same, up to translation, for each i. From the above,

$$(B_1[\Phi^{l_i}(T_n)], B_1[\Phi^{l_i}(T_n')]) = (B_1[\Phi^{l_i+m}(T_n)], B_1[\Phi^{l_i+m}(T_n')])$$

for each i. By Lemma 3.18, T_n and T_n' are Φ-periodic.

Since we have only finitely many distinct pairs $(B_1[T_n], B_1[T_n'])$ and since $B_1[T_n] = B_1[T_m]$ and T_n, T_m Φ-periodic implies that $T_n = T_m$, we have only finitely many branch pairs with asymptotic sector containing $\bar{H}$, all of which are $\Phi \times \Phi$-periodic. □

Example 4.8 will show the necessity of the assumption that $\bar{H}$ is a closed hemisphere in the proposition above. In that example, there are no branch pairs with asymptotic sector containing a closed hemisphere, but there are uncountably many with asymptotic sector containing an open hemisphere, infinitely many of which are $\Phi \times \Phi$-periodic.

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Remark 3.20. In the next section we will see that in dimension $d = 2$ (as in $d = 1$), even without first fixing the closed hemisphere $\bar{H} \subset \mathbb{S}^{d-1}$, there are only finitely many branch pairs whose asymptotic sector contains a closed hemisphere. The argument relies on the rather trivial geometry of connected open sets in $\mathbb{S}^1$. The situation is potentially much more complicated for $d > 2$ and we have been unable to extend the argument. (See Problem 6.4.)

PROPOSITION 3.21. *If $\Omega = \Omega_\Phi$ is a d-dimensional, non-periodic, FLC, self-similar substitution tiling space and $T \neq T' \in \Omega$ agree on a set $X = \{v \in \mathbb{R}^d : \langle v/|v|, u\rangle > \alpha\}$ for some $u \in \mathbb{S}^{d-1}$ and $\alpha \leq 0$, then there is a branch pair $(\bar{T}, \bar{T}') \in \mathcal{BP}(\Phi)$ with $X \cap \mathbb{S}^{d-1} \subset S(\bar{T}, \bar{T}')$. Moreover, if $\alpha < 0$, we may take $\bar{T} = T - z$ and $\bar{T}' = T' - z$ for some $z \in \mathbb{R}^d$.*

Proof. Let λ be the expansion factor for Φ. For each $n \in \mathbb{N}$, let $T_n := \Phi^{-n}(T)$ and $T'_n := \Phi^{-n}(T')$. Then $T_n - x \sim_s T'_n - x$, for all $x \in X$ and all $n \in \mathbb{N}$. It follows from Lemma 3.11 that there is an $R > 0$ so that T_n and T'_n agree on $\{Ru\}$ for all n. Just as in the proof of Lemma 3.17 there are, up to translation, only finitely many pairs of patches $(B_r[T_n], B_r[T'_n])$ for any particular r. Choose a subsequence $\{n_i\}$ so that $T_{n_i} \to \bar{T} \in \Omega$ and $T'_{n_i} \to \bar{T}' \in \Omega$ as $i \to \infty$. By Lemma 3.10, there is a y so that $T - y \nsim_s T' - y$. Then $T_{n_i} - y_{n_i} \nsim_s T'_{n_i} - y_{n_i}$ for all i where $y_k := \lambda^{-k} y$, since $T_{n_i} - y_{n_i} \to \bar{T}$ and $T'_{n_i} - y_{n_i} \to \bar{T}'$, $\bar{T} \nsim_s \bar{T}'$. Suppose v is such that $\bar{T} - v \nsim_s \bar{T}' - v$. Let $r = |v| + 1$. For large enough i there are x_{n_i} and x'_{n_i}, with $|x_{n_i}| < 1$, $|x'_{n_i}| < 1$, $x_{n_i} \to 0$ and $x'_{n_i} \to 0$, so that $B_r[T_{n_i}] \subset B_{r+1}[\bar{T}] - x_{n_i}$ and $B_r[T'_{n_i}] \subset B_{r+1}[\bar{T}'] - x'_{n_i}$. Since there are only finitely many pairs $(B_r[T_{n_i}], B_r[T'_{n_i}])$, up to translation, there are arbitrarily large i for which $x_{n_i} = x'_{n_i}$. For such i,

$$v - x_{n_i} \in \operatorname{int}(\operatorname{spt}(B_r[T_{n_i}])) \cap \operatorname{int}(\operatorname{spt}(B_r[T'_{n_i}]))$$

so that $T_{n_i} - (v - x_{n_i}) \sim_s T'_{n_i} - (v - x_{n_i})$ if and only if $\bar{T} - v \sim_s \bar{T}' - v$. Since $\bar{T} - v \nsim_s \bar{T}' - v$, $T_{n_i} - (v - x_{n_i}) \nsim_s T'_{n_i} - (v - x_{n_i})$, and hence $\langle (v - x_{n_i})/|v - x_{n_i}|, u\rangle \leq \alpha$. Since $x_{n_i} \to 0$, $\langle v/|v|, u\rangle \leq \alpha$. In other words, $\bar{T} - x \sim_s \bar{T}' - x$ for all $x \in X$.

Suppose now that $\alpha < 0$. If $j_i \to \infty$ is any sequence so that $T_{j_i} \to \bar{T}_j$ and $T'_{j_i} \to \bar{T}'_j$, the forgoing argument shows that $(\bar{T}_j, \bar{T}'_j)$ is a branch pair with $X \cap \mathbb{S}^{d-1} \subset S(\bar{T}_j, \bar{T}'_j)$. By Proposition 3.19, there are only finitely many such branch pairs and they are all periodic. It must be the case that all these branch pairs are on the same periodic orbit under $\Phi \times \Phi$ (in any invertible dynamical system, a finite alpha-limit set is a single periodic orbit). Say this periodic orbit is $\{(\Phi^k(\bar{T}), \Phi^k(\bar{T}')) : k = 1, \ldots, m\}$. There are then an l and $x_n \to 0$ so that $(B_1[T_n], B_1[T'_n]) = (B_1[\Phi^{-n+l}(\bar{T}) + x_n], B_1[\Phi^{-n+l}(\bar{T}') + x_n])$ for large n. From this we see that $x_n = \lambda x_{n+1}$ for sufficiently large n, say, for $n \geq N$. We have

$$\begin{aligned} T_N \supset \Phi^k(B_1[T_{N+k}]) &= \Phi^k(B_1[\Phi^{-N-k+l}(\bar{T}) + x_{N+k}]) \\ &\supset B_{\lambda^k}[\Phi^{-N+l}(\bar{T}) + x_N] \quad \text{for all } k \in \mathbb{N}. \end{aligned}$$

That is, $T_N = \Phi^{-N+l}(\bar{T}) + x_N$. Similarly, $T'_N = \Phi^{-N+l}(\bar{T}') + x_n$. Thus $(T - z, T' - z) = (\Phi^l(\bar{T}), \Phi^l(\bar{T}'))$ for $z = \lambda^N x_N$. □

LEMMA 3.22. *Suppose that $\Omega = \Omega_\Phi$ is a d-dimensional, non-periodic, FLC, self-similar substitution tiling space and suppose that there are $T, T' \in \Omega_\Phi$, $r > 0$, and $u \in \mathbb{S}^{d-1}$ so that T and T' are stably related on $B_r(0)$ but $T - ru \nsim_s T' - ru$. Then there is a pair $(\bar{T}, \bar{T}') \in \mathcal{BP}(\Phi)$ with $-u \in \Delta(T, T')$ (that is, $H_{-u} \subset S(T, T')$).*

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Proof. For $n \in \mathbb{N}$, let $T_n = \Phi^n(T - ru)$ and $T'_n = \Phi^n(T' - ru)$. By Lemma 3.11, there is an R so that $B_0[T_n - y] = B_0[T'_n - y]$, for all $y \in B_{\lambda^n r - R}(\lambda^n ru)$ for each n. Let $\{n_i\}$ be a subsequence so that $T_{n_i} \to \tilde{T}$ and $T'_{n_i} \to \tilde{T}'$. Since $B_{R+1}[T_n]$ and $B_{R+1}[T'_n]$ share a tile, finite local complexity ensures that there are only finitely many pairs of patches $(B_{R+1}[T_n], B_{R+1}[T'_n])$, up to translation. It follows from $T_{n_i} \nsim_s T'_{n_i}$, that $\tilde{T} \neq \tilde{T}'$. Now $\tilde{T} + (R+1)u \neq \tilde{T}' + (R+1)u$ and $B_0[\tilde{T} + (R+1)u - y] = B_0[\tilde{T}' + (R+1)u - y]$ for all $y \in \mathbb{R}^d$ with $\langle y, -u\rangle > 0$. The desired $(\bar{T}, \bar{T}')$ exists by Proposition 3.21. □

LEMMA 3.23. *Suppose that $\Omega = \Omega_\Phi$ is a d-dimensional, non-periodic, FLC, self-similar substitution tiling space and that $(T, T') \in \mathcal{BP}(\Phi)$. Suppose also that there is a vector subspace W of $\mathbb{R}^d$ with $T - w \nsim_s T' - w$ for all $w \in W$. Then, if there are natural numbers $n_2 > n_1$, and a vector $w \in W$ such that $(B_1[\Phi^{n_2}(T)], B_1[\Phi^{n_2}(T')]) = (B_1[\Phi^{n_1}(T)], B_1[\Phi^{n_1}(T')]) - w$, there is $(\bar{T}, \bar{T}') \in \mathcal{PBP}(\Phi)$. Moreover, if $H_u \subset S(T, T')$ and $\langle w, u\rangle = 0$ for all $w \in W$ then there is such $(\bar{T}, \bar{T}')$ with $H_u \subset S(\bar{T}, \bar{T}')$.*

Proof. Let $m = n_1$ and $k = n_2 - n_1$. We have

$$\begin{aligned}(\Phi^k(B_1[\Phi^m(T)]), \Phi^k(B_1[\Phi^m(T')])) &\supset (B_1[\Phi^k(\Phi^m(T))], B_1[\Phi^k(\Phi^m(T'))]) \\ &= (B_1[\Phi^m(T)], B_1[\Phi^m(T')]) - w.\end{aligned}$$

Set $w' := (1/(1-\lambda^k))w$. Then

$$\begin{aligned}&(\Phi^k(B_1[\Phi^m(T)] - w'), \Phi^k(B_1[\Phi^m(T')] - w')) \\ &\quad \supset (B_1[\Phi^k(\Phi^m(T))], B_1[\Phi^k(\Phi^m(T'))]) - \lambda^k w' \\ &\quad = (B_1[\Phi^m(T)], B_1[\Phi^m(T')]) - \lambda^k w' - w \\ &\quad = (B_1[\Phi^m(T)] - w', B_1[\Phi^m(T')] - w').\end{aligned}$$

Let

$$\bar{T} := \bigcup_{l=1}^{\infty} \Phi^{lk}(B_1[\Phi^m(T)] - w')$$

and

$$\bar{T}' := \bigcup_{l=1}^{\infty} \Phi^{lk}(B_1[\Phi^m(T')] - w').$$

Then $\bar{T} \nsim_s \bar{T}'$ (since $w' \in W$), $\bar{T}$ and $\bar{T}'$ are fixed by Φ^k, and, if $H_u \subset S(T, T')$, then $H_u \subset S(\bar{T}, \bar{T}')$. To see this latter, note that $T - w \nsim_s T' - w$ for all $w \in W$ means that W is orthogonal to u. Let $\epsilon > 0$ be small enough so that

$$B_\epsilon(0) \subset (\text{int}(\text{spt}(B_1[\Phi^m(T)] - w')) \cap \text{int}(\text{spt}(B_1[\Phi^m(T')] - w'))).$$

Then $H_u \subset S(T, T')$ implies $H_u \subset S(\Phi^m(T), \Phi^m(T'))$ implies $B_1[\Phi^m(T)]$ and $B_1[\Phi^m(T')]$ are stably related on their overlap with H_u^+, the open half-space containing H_u, which implies that $B_1[\Phi^m(T)] - w'$ and $B_1[\Phi^m(T')] - w'$ are stably related on $B_\epsilon(0) \cap H_u^+$, since $\langle w', u\rangle = 0$. Now for $x \in H_u$ take l large enough so that $x' := (1/\lambda^{lm})x \in B_\epsilon(0)$. Then $B_1[\Phi^m(T)] - w'$ and $B_1[\Phi^m(T')] - w'$ are stably related at x', so that $\bar{T} - x' \sim_s \bar{T}' - x'$, and hence $\Phi^{lm}(\bar{T} - x') = \bar{T} - x$ is stably related to $\Phi^{lm}(\bar{T}' - x') = \bar{T}' - x$. Thus $(\bar{T}, \bar{T}') \in \mathcal{PBP}(\Phi)$ and $H_u \subset S(\bar{T}, \bar{T}')$. □

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THEOREM 3.24. *If $\Omega = \Omega_\Phi$ is a d-dimensional, non-periodic, FLC, self-similar substitution tiling space and $(T, T') \in \mathcal{BP}(\Phi)$, with $H_u \subset S(T, T')$, for some $u \in \mathbb{S}^{d-1}$, there is $(\bar{T}, \bar{T}') \in \mathcal{PBP}(\Phi)$ with $H_u \subset S(\bar{T}, \bar{T}')$.*

Proof. Suppose that $(T, T') \in \mathcal{BP}(\Phi)$ and $H_u \subset S(T, T')$. If $T - w \nsim_s T' - w$ for all w with $\langle w, u\rangle = 0$, let $n_2 > n_1$ in $\mathbb{N}$ be such that $(B_1[\Phi^{n_2}(T)], B_1[\Phi^{n_2}(T')]) = (B_1[\Phi^{n_1}(T)], B_1[\Phi^{n_1}(T')]) - w$ for some w (up to translation, there are only finitely many pairs of patches $(B_1[\Phi^n(T)], B_1[\Phi^n(T')])$ by Lemma 3.17). As in the proof of Proposition 3.19 (the argument is also repeated below) it must be that $\langle w, u\rangle = 0$. The hypotheses of Lemma 3.23 are then satisfied with $W = W_0 := \{v \in \mathbb{R}^d : \langle v, u\rangle = 0\}$, and we have the desired $(\bar{T}, \bar{T}')$.

Assume, then, that $T - w \sim_s T' - w$ for some $w \in W_0$. Let $r_0 = \sup\{r : |x| < r$ and $x \in W_0 \implies T - w - x \sim_s T' - w - x\}$. Then $r_0 > 0$ and there is $x_0 \in W_0$ with $|x_0 - w| = r_0$ and $T - x_0 \nsim_s T' - x_0$. Let $u_0 := (1/r_0)(w - x_0)$. Then $(T - x_0, T' - x_0)$ is a branch pair with $H_u \subset S(T - x_0, T' - x_0)$. Furthermore, $T - x_0 - v \sim_s T' - x_0 - v$ for all v in the open $(d-1)$-dimensional ball of radius r_0 centered at $r_0 u_0$ in the subspace W_0.

Now let $n_i \to \infty$ be such that $\Phi^{n_i}(T - x_0) \to U_0$ and $\Phi^{n_i}(T' - x_0) \to U_0'$. Then (U_0, U_0') is a branch pair with $H_u \subset S(U_0, U_0')$. What's more, we claim that $U_0 - x \sim_s U_0' - x$, for all $x \in W_0$ with $\langle x, u_0\rangle > 0$. Indeed, given R, for sufficiently large i there are y_i so that

$$(B_R[\Phi^{n_i}(T - x_0)], B_R[\Phi^{n_i}(T' - x_0)]) = (B_R[U_0 - y_i], B_R[U_0' - y_i])$$

(Lemma 3.17). These y_i converge to 0 and lie in W_0. Furthermore, $\Phi^{n_i}(T - x_0) - x \sim_s \Phi^{n_i}(T' - x_0) - x$ for all x in the open $(d-1)$-ball of radius $\lambda^{n_i} r_0$ centered at $\lambda^{n_i} r_0 u_0$ in W_0. Thus, $B_R[\Phi^{n_i}(T - x_0)] = B_R[U_0 - y_i]$ is stably related to $B_R[\Phi^{n_i}(T' - x_0)] = B_R[U_0' - y_i]$ on that $\lambda^{n_i} r_0$-ball. It follows that $B_R[U_0]$ is stably related with $B_R[U_0']$ at all $x \in W_0 \cap (\text{int}(\text{spt}(B_R[U_0])) \cap \text{int}(\text{spt}(B_R[U_0'])))$ with $\langle x, u_0\rangle > 0$. As R is arbitrary, $U_0 - x \sim_s U_0' - x$, for all $x \in W_0$ with $\langle x, u_0\rangle > 0$, as claimed.

Let W_1 be the $(d-2)$-dimensional subspace of $\mathbb{R}^d$

$$W_1 := \{w \in W_0 : \langle w, u_0\rangle = 0\}.$$

Suppose that $U_0 - w \nsim_s U_0' - w$ for all $w \in W_1$. We may now proceed just as with T, T' at the beginning of this proof. To be sure, let us check that if

$$(B_1[\Phi^{n_2}(U_0)], B_1[\Phi^{n_2}(U_0')]) = (B_1[\Phi^{n_1}(U_0)], B_1[\Phi^{n_1}(U_0')]) - w,$$

then $w \in W_1$. To simplify notation, let $P_i = B_1[\Phi^{n_i}(U_0)]$ and $P_i' = B_1[\Phi^{n_i}(U_0')]$, $i = 1, 2$. The fact that $(P_2, P_2') = (P_1, P_1') - w$ and 0 is in the interior of the support of all the patches implies that $w \in \text{int}(\text{spt}(P_1)) \cap \text{int}(\text{spt}(P_1'))$ and $-w \in \text{int}(\text{spt}(P_2)) \cap \text{int}(\text{spt}(P_2'))$. We have that P_2 and P_2' are not stably related at 0, so P_1 and P_1' are not stably related at w. Since P_1 and P_1' are stably related at all x with $\langle x, u\rangle > 0$ and $x \in \text{int}(\text{spt}(P_1)) \cap \text{int}(\text{spt}(P_1'))$, it must be that $\langle w, u\rangle \le 0$. Likewise, interchanging the roles of 1 and 2 and replacing w by $-w$, we have $\langle -w, u\rangle \le 0$. Thus $w \in W_0$. But we also have that P_1 and P_1' are stably related at all x with $x \in W_0$, $\langle x, u_0\rangle > 0$ and $x \in \text{int}(\text{spt}(P_1)) \cap \text{int}(\text{spt}(P_1'))$, so the same argument yields $\langle w, u_0\rangle \le 0$ and, again, interchanging the roles of 1 and 2

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and replacing w by $-w$, $\langle -w, u_0\rangle \le 0$. Thus $w \in W_1$ and if $U_0 - w \nsim_s U_0' - w$ for all $w \in W_1$, there is a periodic branch pair as desired (Lemma 3.23).

If it is not the case that $U_0 - w \nsim_s U_0' - w$ for all $w \in W_1$, then we find, as above, $x_1 \in W_1$, $r_1 > 0$, and $u_1 \in \mathbb{S}^{d-1} \cap W_1$ with $U_0 - x_1 \nsim_s U_0' - x_1$ but $U_0 - x \sim_s U_0' - x$, for all x in the open $(d-2)$-ball of radius r_1 centered at $r_1 u_1$ in W_1. Let m_i be such that $\Phi^{m_i}(U_0 - x_1) \to U_1$ and $\Phi^{m_i}(U_0' - x_1) \to U_1'$. Then (U_1, U_1') is a branch pair, $H_u \in S(U_1, U_1')$, $U_1 - x \sim_s U_1' - x$ for all $x \in W_i$ with $\langle x, u_i\rangle > 0$ for $i = 1, 2$.

Continuing in this manner, we find the desired periodic branch pair, or, after d steps, we arrive at a branch pair, $(\bar{T}, \bar{T}') = (U_{d-1}, U_{d-1}')$, with $H_u \subset S(\bar{T}, \bar{T}')$, and an orthonormal basis $\{u, u_0, \ldots, u_{d-1}\}$ of $\mathbb{R}^d$ with the property that $\bar{T} - x \sim_s \bar{T}' - x$ if: $\langle x, u\rangle > 0$; or $\langle x, u\rangle = 0$ and $\langle x, u_0\rangle > 0$; or $\langle x, u\rangle = 0$, $\langle x, u_0\rangle = 0$, and $\langle x, u_1\rangle > 0$; or, ...; or $\langle x, u\rangle = 0$, $\langle x, u_0\rangle = 0, \ldots,$ and $\langle x, u_{d-1}\rangle > 0$. Such a pair must be $\Phi \times \Phi$-periodic: pick $n, k > 0$ so that

$$(B_1[\Phi^{n+k}(\bar{T})],\, B_1[\Phi^{n+k}(\bar{T}')]) = (B_1[\Phi^n(\bar{T})],\, B_1[\Phi^n(\bar{T}')]) - w.$$

Then, as above, $w \in W_{d-1} = \{0\}$, from which it follows that $\Phi^k(\bar{T}) = \bar{T}$ and $\Phi^k(\bar{T}') = \bar{T}'$. □

COROLLARY 3.25. *If $\Omega = \Omega_\Phi$ is a d-dimensional, non-periodic, FLC, self-similar substitution tiling space then $\mathcal{PBP}(\Phi) \neq \emptyset$.*

Proof. This follows from Propositions 3.2 and 3.21 and Theorem 3.24. □

THEOREM 3.26. *If $\Omega = \Omega_\Phi$ is a d-dimensional, non-periodic, FLC, self-similar substitution tiling space with expansion λ, and either $d = 2$ or no power of λ is an integer, then there are periodic branch pairs $(T_1, T_1'), (T_2, T_2') \in \mathcal{PBP}(\Phi)$ and $u_1, u_2 \in \mathbb{S}^{d-1}$, u_1 not parallel with u_2, so that $H_{u_i} \subset S(T_i, T_i')$, $i = 1, 2$.*

It may well be that the above is true without restriction on d or λ (see Problem 6.4 in §6). The somewhat technical proof of Theorem 3.26 is deferred to §5.

Let us briefly return to consideration of non-expansive directions. The following proposition completely characterizes these in terms of asymptotic sectors.

PROPOSITION 3.27. *Suppose that $\Omega = \Omega_\Phi$ is a d-dimensional, non-periodic, FLC, self-similar substitution tiling space and V is a $(d-1)$-dimensional subspace of $\mathbb{R}^d$. Then the translation action on Ω_Φ is not transversely expansive in directions V if and only if there is a branch pair $(T, T') \in \mathcal{BP}(\Phi)$ with $V \cap \mathbb{S}^{d-1} \subset \mathrm{cl}(S(T, T'))$.*

Proof. If $V \cap \mathbb{S}^{d-1} \subset \mathrm{cl}(S(T, T'))$ and $S(T, T') \supset H_u$, $u \in \mathbb{S}^{d-1}$, then, given $\epsilon > 0$, there is big enough r so that $B_0[T - ru] = B_0[T' - ru]$ and $d(T - ru - v, T' - ru - v) < \epsilon$ for all $v \in V$. Thus the translation action in directions V is not transversely expansive.

Conversely, suppose that the translation action is not transversely expansive in directions V. For each $n \in \mathbb{N}$, let $T_n \neq T_n' \in \Omega_\Phi$ be such that $T_n \cap T_n' \neq \emptyset$ and $d(T_n - v, T_n' - v) < 1/n$, for all $v \in V$. There are then $r_n \to \infty$ and $w_n \to 0$ so that $T_n - w_n$ and T_n' agree on the strip $\{tu + v : |t| < r_n, v \in V\}$, where u is a unit vector orthogonal to V (as in the proof of Lemma 3.4). Note that, for sufficiently small w_n, $T_n - w_n \neq T_n'$, since $T_n \cap T_n' \neq \emptyset$. Replace T_n by $T_n - w_n$. By appropriately translating the pairs (T_n, T_n') we

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may assume that T_n and T_n' agree on the strip $\{tu + v : L < t < r_n,\ v \in V\}$, or $\{tu + v : L < -t < r_n,\ v \in V\}$, for some fixed L, and $B_0[T_n] \neq B_0[T_n']$. Without loss of generality, assume they agree on $\{tu + v : L < t < r_n,\ v \in V\}$, for all n. Pick a subsequence $n_i \to \infty$ so that $(T_{n_i}, T_{n_i}') \to (\bar{T}, \bar{T}') \in \Omega_\Phi \times \Omega_\Phi$. Then $B_0[\bar{T}] \neq B_0[\bar{T}']$, by FLC, and $\bar{T} - Lu$, $\bar{T}' - Lu$ agree on the half-space $\{w : \langle w, u\rangle > 0\}$. By Proposition 3.21, there is a pair $(U, U') \in \mathcal{BP}(\Phi)$ with $H_u \subset S(U, U')$. In particular, $V \cap \mathbb{S}^{d-1} \subset \mathrm{cl}(S(U, U'))$. □

As an immediate consequence of Theorem 3.26 and Proposition 3.27 we have the following corollary.

COROLLARY 3.28. *If $\Omega = \Omega_\Phi$ is as in Theorem 3.26 then there are $(d-1)$-dimensional subspaces $V_1 \neq V_2$ of $\mathbb{R}^d$ so that the translation action is not transversely expansive in either directions V_1 or V_2.*

Let Φ be a substitution with periodic branch pairs $\mathcal{PBP} = \mathcal{PBP}(\Phi)$. Given $(T, T') \in \mathcal{PBP}$ and $u \in \Delta(T, T')$, a point $v \in \partial H_u$ is an *extreme point* for (T, T') at u if there is $t > 0$ so that $T - tv \nsim_s T' - tv$. Let $E_u(T, T') \subset \partial S(T, T')$ denote the set of all extreme points for (T, T') at u; note that $E_u(T, T')$ is closed, and non-empty for some u if $S(T, T') \neq \mathbb{S}^{d-1}$, by Lemma 3.16. By the same lemma, we see that $u \in \partial\Delta(T, T')$ if and only if $E_u(T, T') \neq \emptyset$. Let

$$C_u(T, T') := \left\{ \sum_{i=1}^{k} t_i v_i : k \in \mathbb{N},\ t_i \geq 0,\ v_i \in E_u(T, T') \right\} \cup \{0\}$$

denote the cone over the convex hull of $E_u(T, T')$.

Definition 3.29. The *branch locus* of Φ (or of Ω_Φ) is the subset of Ω_Φ:

$$\mathcal{BL}(\Phi) := \bigcup_{(T,T') \in \mathcal{PBP}} \bigcup_{u \in \Delta(T,T')} \mathrm{cl}\left(\bigcup_{x \in C_u(T,T')} \{T - x\} \right).$$

By Corollary 3.25, the branch locus of a non-periodic, FLC, self-similar substitution tiling space is non-empty and, as is easily checked, invariant under Φ.

As the branch locus is our central object of study, let us take a few minutes to explain the definition. First, consider the one-dimensional 'period doubling' substitution, Φ, on two prototiles labeled a and b, each having support $[0, 1]$. The substitution is given by $\Phi(a) = \{a, b + 1\}$ and $\Phi(b) = \{a, a + 1\}$. There are tilings $T, T' \in \Omega_\Phi$ of period two under Φ with 0-patches $B_0[T] = \{a - 1, a\}$ and $B_0[T'] = \{b - 1, a\}$. One can check that $\lim_{t \to \pm\infty} d(T - t, T' - t) = 0$, so $(T, T') \in \mathcal{PBP}(\Phi)$ and $S(T, T') = \{\pm 1\} = \mathbb{S}^0$. (With a little work, one can show that this is the only branch pair for Φ so $\mathcal{BL}(\Phi) = \{T, T'\}$). Let X be the (uncollared) Anderson–Putnam complex for Φ: X is the wedge of two circles, one labeled a, the other b, which we think of as the prototiles with their endpoints identified. Let $f : X \to X$ be the Φ-induced map that wraps the a circle first around the a circle, then around the b circle, and wraps the b circle twice around the a circle, everywhere stretching arc length by a factor of two. The inverse limit $\varprojlim(f : X \to X)$ is almost the same thing as Ω_Φ. Indeed, $\varprojlim(f : X \to X)$ is obtained as the quotient of Ω_Φ by identifying $T - t$ with $T' - t$, (T, T') the branch pair above, for all $t \geq 0$. (Identifying $T - t$ with $T' - t$ for all $t \in \mathbb{R}$ would produce the (unbranched) 2-adic solenoid.) The

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branching in $\varprojlim(f : X \to X)$, which occurs at the point where T and T' are identified, clearly comes from the fixed under f branch point of X.

Now imagine a d-dimensional substitution Φ whose prototiles are polytopes. The Anderson–Putnam complex X is formed by gluing (some of) the prototiles along their $(d-1)$-faces. Let $f : X \to X$ be the Φ-induced map. Suppose there are three prototiles ρ_1, ρ_2, ρ_3 all glued along a $(d-1)$-face δ in X, ρ_1 on one side of δ, ρ_2 and ρ_3 on the other, and that $x_0 \in \mathring{\delta}$ is fixed by f. Further, imagine that f is injective on a neighborhood of x_0 in $\bigcup \rho_i$, mapping that neighborhood over itself. The branching of X along δ then persists to branching along a $(d-1)$-dimensional subspace in $\varprojlim(f : X \to X)$ and there is a branch pair $(T, T') \in \mathcal{PBP}(\Phi)$ with $H_u \subset S(T, T')$, where $u \in \mathbb{S}^{d-1}$ is perpendicular to δ and points into ρ_1. Other periodic points of f in δ that are similarly locally 'non-collapsing', also correspond to periodic branch pairs. If (T_1, T_1') is such a pair, there will be $v \in \mathbb{S}^{d-1}$ parallel with δ and $t > 0$ so that $T - tv \in W^s(T_1)$ and $T' - tv \in W^s(T_1')$. Then $v \in E_u(T, T')$ is an extreme point for (T, T') and the closures of the rays $\{T - tv : t \geq 0\}$ and $\{T' - tv : t \geq 0\}$ (which, incidentally, contain T_1 and T_1', respectively) are contained in the branch locus. If x_0 is in the interior of the convex hull of such periodic points, then $S(T, T') = H_u$ and the entire $(d-1)$-dimensional sheets $\{T - v : \langle v, u\rangle = 0\}$ and $\{T' - v : \langle v, u\rangle = 0\}$ are contained in the branch locus (just as $\varprojlim(f : X \to X)$ is truly branched along a corresponding sheet).

Of course it may happen that the inverse limit of the Anderson–Putnam complex has no true branching—if the prototiles are first collared then resulting substitution 'forces its border' and the inverse limit on the collared Anderson–Putnam complex is the tiling space. None-the-less, as we have seen, there is the analog of branching (tilings that are asymptotic in a half-space) in the tiling space. In many cases (and perhaps always), if there is a branch pair (T, T'), with $C_u(T, T')$ a $(d-1)$-dimensional hyperspace (these are the *line pairs* in $d = 2$, see the next section), translates of this hyperspace can be used to cut up the tiles so that, when the cut up tiles are appropriately reassembled into a new prototile set, the original branch pair corresponds to true branching in the inverse limit of the new Anderson–Putnam complex, exactly as in the preceding paragraph. (For more on this, see §6.)

Remark 3.30. For one-dimensional substitutions Φ, $\mathcal{BL}(\Phi)$ is the finite subset of Ω_Φ consisting of the branch points (there are no extreme points). The term branch locus has a different meaning in [**BDS**], where it refers to the image of the set of branch points in the 'geometric realization' (maximal equicontinuous factor) for a Pisot substitution tiling space.

LEMMA 3.31. *If $\Omega = \Omega_\Phi$ is a d-dimensional, non-periodic, FLC, self-similar substitution tiling space, then $\mathcal{BP}(\Phi)$ is closed in $\Omega \times \Omega$. Moreover, the asymptotic sectors $S(T, T')$, $(T, T') \in \mathcal{BP}(\Phi)$, are upper semicontinuous: if $(T_n, T_n') \in \mathcal{BP}(\Phi)$ and $u_n \in \mathbb{S}^{d-1}$, $n \in \mathbb{N}$, are such that $H_{u_n} \subset S(T_n, T_n')$; u is an accumulation point of $\{u_n : n \in \mathbb{N}\}$; and $(T_n, T_n') \to (T, T')$; then $H_u \subset S(T, T')$.*

Proof. Suppose $(T_n, T_n') \in \mathcal{BP}(\Phi)$ with $(T_n, T_n') \to (T, T') \in \Omega \times \Omega$. Let $u_n \in \mathbb{S}^{d-1}$ be such that $H_{u_n} \subset S(T_n, T_n')$. By passing to a subsequence, we may assume that $u_n \to u$

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as $n \to \infty$. As in the proof of Proposition 3.21, there are $x_n \to 0$ so that $(B_1[T_n], B_1[T_n']) = (B_1[T - x_n], B_1[T' - x_n])$ for all sufficiently large n. If $B_1[T]$ and $B_1[T']$ are stably related on $\{0\}$ then they are stably related on $B_\epsilon(0)$ for some $\epsilon > 0$ (Lemma 3.6). But then $B_1[T_n]$ would be stably related to $B_1[T_n']$ on $\{0\}$ for n large enough so that $|x_n| < \epsilon$, contradicting $T_n \not\sim_s T_n'$. Thus $T \not\sim_s T'$.

Let $z \in \mathbb{R}^n$ be such that $\langle z, u\rangle > 0$ and let $r = |z| + 1$. For sufficiently large n,

$$(B_r[T_n], B_r[T_n']) = (B_r[T - x_n], B_r[T' - x_n]).$$

Then $B_r[T]$ is stably related to $B_r[T']$ on $\{z\}$ if and only if $B_r[T_n]$ is stably related to $B_r[T_n']$ on $\{z - x_n\}$ (for large n). Since $u_n \to u$, $x_n \to 0$, and $\langle z, u\rangle > 0$, $\langle z - x_n, u_n\rangle > 0$ for sufficiently large n. For such n, $(z - x_n)/|z - x_n| \in S(T_n, T_n')$ so $T_n - (z - x_n) \sim_s T' - (z - x_n)$; hence $B_r[T_n]$ is stably related to $B_r[T_n']$ on $\{z - x_n\}$, $B_r[T]$ is stably related to $B_r[T']$ on $\{z\}$, and $T - z \sim_s T' - z$. So $(T, T') \in \mathcal{BP}(\Phi)$ with $H_u \subset S(T, T')$. □

It is quite possible that our results to this point can be extended to substitutions with more general linear inflation. But the proof of the following theorem, which asserts the topological invariance of the branch locus, depends crucially on the inflation being scalar. We use the remarkable rigidity results of [**K**]: if Ω_Φ and Ω_Ψ are homeomorphic d-dimensional, self-similar, non-periodic, FLC substitution tiling spaces, then there is a homeomorphism that preserves, up to linear isomorphism, their $\mathbb{R}^d$-actions; and, if such a homeomorphism takes some Φ-periodic point to a Ψ-periodic point, then Φ^m and Ψ^n are conjugate for some $m, n > 0$. For a simple example showing that this topological rigidity does not extend to substitutions with non-scalar inflation, consider the two-dimensional substitutions $\Phi = \sigma \times \tau$ and $\Psi = \sigma^2 \times \tau$, where σ is the Thue–Morse substitution (symbolically, $1 \mapsto 12$ and $2 \mapsto 21$) and τ is the Fibonacci substitution ($1 \mapsto 12$ and $2 \mapsto 1$). Then Ω_Φ and Ω_Ψ are both homeomorphic with $\Omega_\sigma \times \Omega_\tau$, but the topological entropies, $h(\Phi) = \log(1 + \sqrt{5})$ and $h(\Psi) = \log(2) + \log(1 + \sqrt{5})$, of Φ and Ψ are rationally independent, so no positive power of Φ can be topologically conjugate with a power of Ψ.

THEOREM 3.32. *Suppose that Ω_Φ and Ω_Ψ are homeomorphic d-dimensional, non-periodic, FLC, self-similar substitution tiling spaces and suppose that there are distinct tilings $T_0, T_0' \in \Omega_\Phi$ for which there is $S \subset \mathbb{S}^{d-1}$ that contains a closed hemisphere in its interior such that T_0 and T_0' are uniformly asymptotic in directions S. There is then a homeomorphism $h : \Omega_\Phi \to \Omega_\Psi$, a linear isomorphism $L : \mathbb{R}^d \to \mathbb{R}^d$, and $m, n > 0$, so that $h(\mathcal{BL}(\Phi)) = \mathcal{BL}(\Psi)$, $h(\mathcal{BP}(\Phi)) = \mathcal{BP}(\Psi)$, $h \circ \Phi^m = \Psi^n \circ h$ and $S(h(T), h(T')) = L(S(T, T'))$ for all $(T, T') \in \mathcal{BP}(\Phi)$.*

Proof. By [**K**, Theorem 1.1], there is a homeomorphism $h_0 : \Omega_\Phi \to \Omega_\Psi$ and a linear isomorphism L of $\mathbb{R}^d$ with $h_0(T - x) = h_0(T) - Lx$ for all $T \in \Omega_\Phi$ and all $x \in \mathbb{R}^d$. Let $u \in \mathbb{S}^{d-1}$ and $\alpha < 0$ be such that $\{v \in \mathbb{S}^{d-1} : \langle v, u\rangle > \alpha\} \subset S$. By Lemma 3.5 there is $R > 0$ so that T_0 and T_0' agree on $\{tv : \langle v, u\rangle > \alpha, v \in \mathbb{S}^{d-1}, t \geq R\}$. There is then β, $\alpha < \beta < 0$, so that $T_0 - 2Ru$ and $T_0' - 2Ru$ agree on $\{tv : \langle v, u\rangle > \beta, v \in \mathbb{S}^{d-1}, t > 0\}$, and then by Proposition 3.21 there is z such that

$$(T_0 - 2Ru - z, T' - 2Ru - z) \in \mathcal{BP}(\Phi)$$

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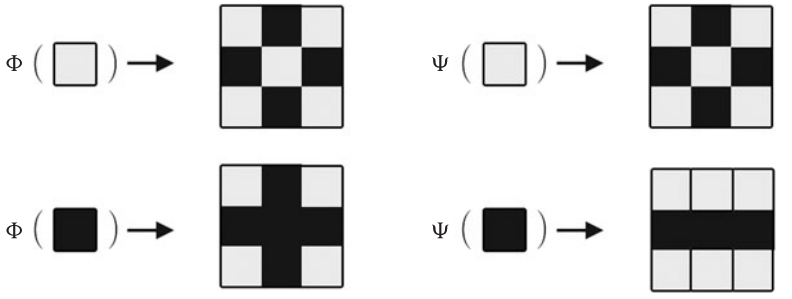


FIGURE 2. Example 3.33 substitution rule.

and

$$S(T_0 - 2Ru - z, T_0' - 2Ru - z) \supset \{v \in \mathbb{S}^{d-1} : \langle v, u\rangle > \beta\}.$$

Let $x_0 := 2Ru + z$. By Proposition 3.19, $(T_0 - x_0, T_0' - x_0) \in \mathcal{PBP}(\Phi)$. Since $T_0 - x_0$ and $T_0' - x_0$ are uniformly asymptotic in directions S, it follows (from the 'linear' nature of h_0) that $h_0(T_0 - x_0)$ and $h_0(T_0' - x_0)$ are uniformly asymptotic in directions $S' := \{Lv/|Lv| : v \in S\}$. The set of directions S' must also contain a closed hemisphere in its interior, so, again by Lemma 3.5 and Propositions 3.19 and 3.21, there is y_0 so that $h_0(T_0 - x_0) - y_0$ and $h_0(T_0' - x_0) - y_0$ are Ψ-periodic. Let $h_1 : \Omega_\Phi \to \Omega_\Psi$ be defined by $h_1(T) := h(T - x_0) - y_0$. Since the homeomorphism h_1 is now 'pinned down' in that it takes a particular Φ-periodic point to a Ψ-periodic point, [**K**, Corollary 1.5] guarantees that there is a homeomorphism $h : \Omega_\Phi \to \Omega_\Psi$ together with positive integers m, n so that $h \circ \Phi^m = \Psi^n \circ h$ (and h still satisfies $h(T - v) = h(T) - Lv$). Such an h necessarily takes Φ-periodic points to Ψ-periodic points, Φ-stable manifolds to Ψ-stable manifolds and L takes asymptotic sectors to asymptotic sectors and extreme points to extreme points. The conclusions of the theorem follow easily. □

Here is a simple example that illustrates the use of Theorem 3.32 to distinguish the topological types of two tiling spaces.

Example 3.33. Similar substitutions with non-homeomorphic branch loci.

Let Φ be the two-dimensional substitution with two prototiles, each with support $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$, one called W and the other B. Let $\lambda = 3$,

$$\Phi(W) = \{W + (-1, 1), W + (-1, -1), W, W + (1, -1), W + (1, 1)\}$$
$$\cup \{B + (-1, 0), B + (0, -1), B + (1, 0), B + (0, 1)\},$$

and $\Phi(B) = (\Phi(W)\backslash\{W\}) \cup \{B\}$. (See Figure 2.) There are then tilings T, T', fixed by Φ, with $B_0[T] = \{W\}$ and $B_0[T'] = \{B\}$. The pair (T, T') is a periodic branch pair for Φ and $S(T, T') = \mathbb{S}^1$. If one sets $T - v \sim T' - v$ for all $v \in \mathbb{R}^2$, the quotient space $\Omega_\Phi/\sim$ is a solenoid (the inverse limit of the map $v + \mathbb{Z}^2 \mapsto 3v + \mathbb{Z}^2$ on the torus $\mathbb{R}^2/\mathbb{Z}^2$). Since the $\mathbb{R}^2$-action on the solenoid is equicontinuous, there are no pairs in $\Omega_\Phi \times \Omega_\Phi\backslash\Delta$, other than (T, T') and its translates, that are asymptotic in any direction. Thus $\mathcal{BL}(\Phi) = \{T, T'\}$.

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Now let Ψ be the substitution on the same prototiles, with $\Psi(W) = \Phi(W)$ and

$$\Psi(B) = \{W + (-1, 1), W + (-1, -1), W + (0, -1), W + (1, -1), \\ W + (1, 1), W + (0, 1)\} \cup \{B + (-1, 0), B, B + (1, 0)\}.$$

Let T, T' be the Ψ-fixed tilings with $B_0[T] = \{W\}$ and $B_0[T'] = \{B\}$. Then $(T, T') \in \mathcal{PBP}(\Psi)$ and $S(T, T') = H_{(-1,0)} \cup H_{(1,0)}$. The points $(0, 1)$ and $(0, -1)$ are extreme points for (T, T'), so

$$\mathcal{S} := \mathrm{cl}\{T - t(1, 0) : t \in \mathbb{R}\} \subset \mathcal{BL}(\Psi).$$

One sees that $\Psi|_{\mathcal{S}}$ is conjugate with the shift homeomorphism on the one-dimensional 3-adic solenoid. All points in $\mathcal{S}$ are branch points belonging to branch pairs having the same asymptotic sector as (T, T'), so by Theorem 3.26 there must be more to the branch locus than $\mathcal{S}$. Let $T_1, T_1' \in \Omega_\Psi$ be tilings with $B_0[T_1] = \{W + (0, 1/2), W + (0, -1/2)\}$ and $B_0[T_1'] = \{B + (0, 1/2), B + (0, -1/2)\}$ that are 2-periodic under Ψ (so $\Psi(T_1) = T_1'$ and $\Psi(T_1') = T_1$). Then $(T_1, T_1') \in \mathcal{PBP}(\Psi)$, (T_1, T_1') has the same asymptotic sector and extreme points as (T, T'), and the arcs $\{T_1 - t(0, 1) : t \in \mathbb{R}\}$ and $\{T_1' - t(0, 1) : t \in \mathbb{R}\}$ spiral densely onto $\mathcal{S}$ in both directions. The branch locus is the union of $\mathcal{S}$ with these two arcs. The periodic branch pairs with asymptotic sector different from $S(T, T')$ are as follows. Let $T_2, T_2' \in \mathcal{S}$ have $B_0[T_2] = \{W + (0, 1/2), B + (0, -1/2)\}$ and $B_0[T_2'] = \{B + (0, 1/2), W + (0, -1/2)\}$ and be of period 2 under Φ (so (T_2, T_2') is a periodic branch pair with $S(T_2, T_2') = H_{(-1,0)} \cup H_{(1,0)}$). Then (T_1, T_2), (T_1', T_2'), (T_1, T_2'), and (T_1', T_2) are periodic branch pairs with asymptotic sectors $\mathbb{S}^1\backslash\{(0, -1)\}$, $\mathbb{S}^1\backslash\{(0, -1)\}$, $\mathbb{S}^1\backslash\{(0, 1)\}$, and $\mathbb{S}^1\backslash\{(0, 1)\}$, respectively.

By Theorem 3.32, Ω_Φ and Ω_Ψ are not homeomorphic. It is not hard to compute (say, by [**BDHS**] or [**BSa**]) that Ω_Φ and Ω_Ψ have the same integer Čech cohomology ($\mathbb{Z}$, $\mathbb{Z}[1/3] \oplus \mathbb{Z}[1/3]$, and $\mathbb{Z}[1/3] \oplus \mathbb{Z}$ in dimensions 0, 1, and 2), so (traditional) cohomology does not distinguish these spaces.

The branch locus may be rather difficult to describe in general. But in dimension two (as in dimension one) we will see (as above) that there are only finitely many sectors $S(T, T')$ for a given Φ, which considerably simplifies the situation. We turn to that case now.

4. *Two-dimensional tilings*

All two-dimensional substitution tiling spaces in this section will be assumed to be non-periodic, primitive, self-similar, and have finite translational local complexity.

LEMMA 4.1. *Suppose that $\Omega = \Omega_\Phi$ is a two-dimensional, non-periodic, FLC, self-similar substitution tiling space and $(T, T') \in \mathcal{BP}(\Phi)$. If $S(T, T')$ contains a closed hemisphere and $v \in \partial S(T, T')$, then there are $t_n \to \infty$ so that $T - t_n v \nsim_s T' - t_n v$. Otherwise, for $v \in \partial S(T, T')$, there are $s_{-1} < 0 < s_1$ so that $T - s_i v \nsim_s T' - s_i v$ for $i = \pm 1$ and there are $t_n \to \infty$ such that either $T - t_n v \nsim_s T' - t_n v$ or $T + t_n v \nsim_s T' + t_n v$ for all n.*

Proof. If $S(T, T')$ contains a closed hemisphere, then (T, T') is $\Phi \times \Phi$-periodic (Proposition 3.19) and the result follows from Lemma 3.16.

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Otherwise, let $u \in \mathbb{S}^1$ be such that $H_u \subset S(T, T')$. For $n \in \mathbb{Z}$, let $(T_n, T'_n) := (\Phi^{-n}(T), \Phi^{-n}(T'))$. Choose a subsequence n_i so that $(T_{n_i}, T'_{n_i}) \to (\bar{T}, \bar{T}')$ as $i \to \infty$. Then $(\bar{T}, \bar{T}') \in \mathcal{BP}(\Phi)$, $H_u \subset S(\bar{T}, \bar{T}')$, and there are $x_i \to 0$ so that $(B_1[T_{n_i}], B_1[T'_{n_i}]) = (B_1[\bar{T} - x_i], B_1[\bar{T}' - x_i])$ for sufficiently large i (Lemmas 3.31 and 3.17).

Let $\mathbf{l}$ be the line $\mathbf{l} := \{tv : t \in \mathbb{R}\}$ and let D^+ be the component of $B_1(0)\backslash\mathbf{l}$ that contains $\frac{1}{2}u$. Then $B_1[\bar{T}]$ is stably related to $B_1[\bar{T}']$ on D^+, so it must be the case that $x_i \notin D^+$. Now, if $x_i \notin \mathbf{l}$, then $\langle -x_i, u\rangle > 0$ so $T_{n_i} \sim_s T'_{n_i}$. But then $\bar{T} \sim_s \bar{T}'$, which is not the case. Thus $x_i \in \mathbf{l}$ for all large i. If $x_i = 0$ for infinitely many i, then (T, T') is $\Phi \times \Phi$-periodic by Lemma 3.18 and the result follows from Lemma 3.16. Otherwise, after passing to a subsequence, we have $x_i = t'_i v \in \mathbf{l}$, with $i \geq L$, and $t'_i \to 0$ strictly monotonically. Without loss of generality, t'_i is increasing. We have $T_{n_L} + x_L \nsim_s T'_{n_L} + x_L$ and $T_{n_i} - (x_L - x_i) \nsim_s T'_{n_i} - (x_L - x_i)$ for $i > L$. Thus $T + \lambda^{N_L} t'_L v \nsim_s T' + \lambda^{N_L} t'_L v$ and $T - \lambda^{n_i}(t'_i - t'_L)v \nsim_s T' - \lambda^{n_i}(t'_i - t'_L)v$ for $i > L$. Let $s_{-1} := -\lambda^{n_L} t'_L v$, $s_1 := \lambda^{n_{L+1}}(t'_{L+1} - t'_L)$ and $t_i := \lambda^{n_{L+i}}(t'_{L+i} - t'_L)$. □

In particular, for two-dimensional Φ, all points $v \in \partial S(T, T')$, $S(T, T') \in \mathcal{PBP}(\Phi)$ are extreme points so that

$$\mathcal{BL}(\Phi) = \bigcup_{(T,T')\in\mathcal{PBP}(\Phi)} \{T\} \cup \left(\bigcup_{v\in\partial S(T,T')} \mathrm{cl}\left(\bigcup_{t\geq 0} \{T - tv\} \right) \right).$$

THEOREM 4.2. *If $\Omega = \Omega(\Phi)$ is a two-dimensional, non-periodic, FLC, self-similar substitution tiling space, then the collection $\{S(T, T') : (T, T') \in \mathcal{BP}(\Phi)\}$ is finite.*

Proof. Suppose instead that there are $(T_n, T'_n) \in \mathcal{BP}(\Phi)$, $n \in \mathbb{N}$, so that $\partial S(T_n, T'_n) = \{v_n, w_n\}$ and $\{v_i, w_i\} \neq \{v_j, w_j\}$ for all $i \neq j$. Passing to a subsequence, we may assume that $v_n \to v$ and $w_n \to w$ as $n \to \infty$. By Lemma 4.1 there are $s_n \in \mathbb{R}^+$ so that $T_n - s_n v_n \nsim_s T'_n - s_n v_n$. Let $k_n \in \mathbb{Z}$ be so that $\lambda^{k_n} s_n \in [\lambda^{-2}, \lambda^{-1})$. Replace (T_n, T'_n) by $(\Phi^{k_n}(T_n), \Phi^{k_n}(T'_n))$ and s_n by $\lambda^{k_n} s_n$. We still have $\partial S(T_n, T'_n) = \{v_n, w_n\}$ since Φ is self-similar. Let $u_n \in \mathbb{S}^1$ be such that $H_{u_n} \subset S(T_n, T'_n)$. By passing to a subsequence, we may assume that $(T_n, T'_n) \to (T, T') \in \Omega \times \Omega$, $s_n \to s \in [\lambda^{-2}, \lambda^{-1}]$, and $u_n \to u \in \mathbb{S}^1$. From Lemma 3.31 we have $(T, T') \in \mathcal{BP}(\Phi)$, $H_u \subset S(T, T')$, and $T - sv \nsim_s T' - sv$ (the latter because $(T_n - s_n v_n, T'_n - s_n v_n) \in \mathcal{BP}(\Phi)$ and $(T_n - s_n v_n, T'_n - s_n v_n) \to (T - sv, T' - sv)$). By Lemma 3.17 there is an N and there are $x_n \to 0$ so that $(B_1[T_n], B_1[T'_n]) = (B_1[T - x_n], B_1[T' - x_n])$ for all $n \geq N$. By taking N large enough we can assure that $|x_n| < \epsilon$ and $|x_n + s_n v_n - sv| < \epsilon$ for all $n \geq N$, where $\epsilon > 0$ is small enough so that the ϵ balls centered at sv and 0 are disjoint and contained in $B_1(0)$. Let $\mathbf{l}$ and $\mathbf{l}_n$ be the lines $\{tv : t \in \mathbb{R}\}$ and $\{tv_n + x_n : t \in \mathbb{R}\}$, respectively. Let $D^\pm$ and $D_n^\pm$ be the components of $B_1(0)\backslash\mathbf{l}$ and $B_1(0)\backslash\mathbf{l}_n$, respectively, with $\frac{1}{2}u \in D^+$ and $\frac{1}{2}u_n + x_n \in D_n^+$. Then $B_1[T]$ is stably related to $B_1[T']$ on D^+. Pick $n \geq N$. Since $(B_1[T_n], B_1[T'_n]) = (B_1[T - x_n], B_1[T' - x_n])$ and $T_n \nsim_s T'_n$, $B_1[T]$ and $B_1[T']$ are not stably related on $\{x_n\}$; that is $x_n \notin D^+$. Similarly, $x_n + s_n v_n \notin D^+$. Suppose that $v_n \neq v$. Then the lines $\mathbf{l}_n$ and $\mathbf{l}$ intersect in a single point $\{p = tv\}$. If $t \geq s$, then $0 \in D_n^+$; if $t \leq 0$, then $sv \in D_n^+$; and if $0 < t < s$ then exactly one of 0, sv is in D_n^+. So if $v_n \neq v$, $\{0, sv\} \cap D_n^+ \neq \emptyset$. If $0 \in D_n^+$ then $\langle u_n, -x_n\rangle > 0$ so $T_n + x_n \sim_s T' + x_n$. But then $T \sim_s T'$, which is not the case. Similarly, if $sv \in D_n^+$ then $\langle u_n, sv - x_n\rangle > 0$ so that $T_n - (sv - x_n) \sim_s T' - (sv - x_n)$.

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But then $T - sv \sim_s T' - sv$, which is also not the case. Thus it must be that $v_n = v$ for all $n \geq N$.

Now, for each $n \geq N$ there are $t_n > 0$ so that $T_n - t_n w_n \nsim_s T'_n - t_n w_n$. Let $m_n \in \mathbb{Z}$ be such that $\lambda^{m_n} t_n \in [\lambda^{-2}, \lambda^{-1})$: replace (T_n, T'_n) by $(\Phi^{m_n}(T_n), \Phi^{m_n}(T'_n))$ and t_n by $\lambda^{m_n} t_n$. Let $\bar{u}_n \in \mathbb{S}^1$ be such that $H_{\bar{u}_n} \subset S(T_n, T'_n)$ for $n \geq N$. Passing to a subsequence, we may assume that $(T_n, T'_n) \to (\bar{T}, \bar{T}') \in \Omega \times \Omega$, $t_n \to \bar{t}$, and $\bar{u}_n \to \bar{u}$ as $n \to \infty$. Note that for the new T_n, T'_n, it's still the case that $\partial S(T_n, T'_n) = \{v, w_n\}$ for $n \geq N$, with $w_i \neq w_j$ for $i \neq j$. Proceeding just as above, we may conclude that $w_n \equiv w$ for all sufficiently large n. That is, there could only have been finitely many asymptotic sectors to begin with. □

Definition 4.3. $\mathcal{BP}(\Phi)$ is *non-collapsing* if, whenever $(T, T') \in \mathcal{BP}(\Phi)$, $v \in \partial S(T, T')$, and $0 \leq t_0 < t_1$ are such that $T - t_0 v \nsim_s T' - t_0 v$ and $T - t_1 v \nsim_s T' - t_1 v$, then $T - tv \nsim_s T' - tv$ for all $t \in [t_0, t_1]$.

The above is equivalent to saying that the sets $\{t : T - tv \nsim_s T' - tv : t \geq 0\}$, $v \in \partial S(T, T')$, $(T, T') \in \mathcal{BP}(\Phi)$, are closed intervals. The examples of the last section, as well as all of the examples below, are non-collapsing. We will see that the collection of branch pairs is non-collapsing if and only if the branch locus coincides with the collection of branch points.

Definition 4.4. If (T, T') is a branch pair in the two-dimensional substitution tiling space Ω, then (T, T') is:

- an *isolated pair* if $S(T, T') = \mathbb{S}^1$;
- a *corner pair* if $S(T, T')$ is proper in $\mathbb{S}^1$ but contains a closed semicircle in its interior;
- a *line pair* if $\partial S(T, T') = \{\pm v\}$ for some $v \in \mathbb{S}^1$.

Every branch pair is of one of the above types. Furthermore, there are only finitely many isolated and corner pairs and they are all periodic (Lemma 4.1 and Theorem 4.2).

PROPOSITION 4.5. *If $\mathcal{BP}(\Phi)$ is non-collapsing then $\mathcal{BL}(\Phi) = \bigcup_{(T,T') \in \mathcal{BP}(\Phi)} \{T\}$. In this case, $\mathcal{BL}(\Phi)$ is compact.*

Proof. If $\mathcal{BP}(\Phi)$ is non-collapsing, $(T, T') \in \mathcal{PBP}(\Phi)$, and $v \in \partial S(T, T')$, then $(T - tv, T' - tv) \in \mathcal{BP}(\Phi)$ for all $t \geq 0$. Thus, since the collection of branch points is closed, $\mathcal{BL}(\Phi) \subset \bigcup_{(T,T') \in \mathcal{BP}(\Phi)} \{T\}$.

Now let $(T, T') \in \mathcal{BP}(\Phi)$. Suppose there are $v \in \partial S(T, T')$ and $t > 0$ with $T - tv \sim_s T' - tv$. Let $t_0 := \inf\{t > 0 : T - tv \sim_s T' - tv\}$. Then $(T - t_0 v, T' - t_0 v)$ is either an isolated or a corner pair and hence is in $\mathcal{PBP}(\Phi)$. Moreover, $-v \in \partial S(T - t_0 v, T' - t_0 v)$ and we see that $T \in \mathcal{BL}(\Phi)$. If there are no such v and t, then either (T, T') is an isolated pair, hence in $\mathcal{PBP}$, and $T \in \mathcal{BL}(\Phi)$, or (T, T') is a line pair with $v \in \partial S(T, T')$ and $T - tv \nsim_s T' - tv$ for all $t \in \mathbb{R}$. In the latter case, and using Lemma 3.17, there is a $k \in \mathbb{N}$ and a sequence $n_i \to \infty$ so that all of the pairs of patches $(B_1[\Phi^{-n_i}(T)], B_1[\Phi^{-n_i}(T')])$ and $(B_1[\Phi^{-n_i - k}(T)], B_1[\Phi^{-n_i - k}(T')])$ are translates of the same pair of patches (P, P'). Moreover, we may take $n_i \equiv n_j \bmod(k)$, and, without loss of generality, $n_i \equiv 0 \bmod(k)$. As in the proof of Proposition 3.19, the vectors translating the pairs of patches $(B_1[\Phi^{-n_i}(T)], B_1[\Phi^{-n_i}(T')])$ to the pairs of

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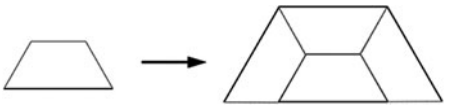


FIGURE 3. Half-hex substitution rule.

patches $(B_1[\Phi^{-n_i-k}(T)], B_1[\Phi^{-n_i-k}(T')])$ are all parallel with v. Passing to a subsequence, we may assume that $(B_1[\Phi^{-n_i-k}(T)], B_1[\Phi^{-n_i-k}(T')]) \to (Q, Q')$. We have that $\Phi^k(Q) \supset Q - sv$ and $\Phi^k(Q') \supset Q' - sv$ for some s. Then

$$\Phi^k\left(Q - \left(\frac{s}{1-\lambda^k}\right)v\right) \supset Q - \left(\frac{s}{1-\lambda^k}\right)v$$

and

$$\Phi^k\left(Q' - \left(\frac{s}{1-\lambda^k}\right)v\right) \supset Q' - \left(\frac{s}{1-\lambda^k}\right)v.$$

Let

$$\bar{T} := \bigcup_{n\geq 0} \Phi^{nk}\left(Q - \left(\frac{s}{1-\lambda^k}\right)v\right)$$

and

$$\bar{T}' := \bigcup_{n\geq 0} \Phi^{nk}\left(Q' - \left(\frac{s}{1-\lambda^k}\right)v\right).$$

Then $(\bar{T}, \bar{T}')$ is $\Phi \times \Phi$-periodic of period k and is a line branch pair with $\partial S(\bar{T}, \bar{T}') = \{\pm v\}$ by the non-collapsing assumption. Moreover, there are bounded s_i so that $B_1[\Phi^{-n_i}(T)] = B_1[\bar{T} - s_i v]$. Let $n_i = m_i k$. Then $B_{\lambda^{m_i k}}[T] = B_{\lambda^{m_i k}}[\bar{T} - \lambda^{m_i k} s_i v]$ for all i, so that $\bar{T} - \lambda^{m_i k} s_i v \to T$ as $i \to \infty$. Thus $T \in \mathcal{BL}(\Phi)$. Now that $\mathcal{BL}(\Phi) = \bigcup_{(T,T')\in\mathcal{BP}(\Phi)}\{T\}$, it follows from Lemma 3.31 that $\mathcal{BL}(\Phi)$ is compact. □

The branch loci in the examples below can be found experimentally (and then verified exhaustively) as follows. Take $T \in \Omega_\Phi$ and let $v \in \mathbb{R}^n$, $v \neq 0$, be such that $T' := T - v$ and T share a tile. For $k \geq 0$, let $T_k := \lambda^{-k}\Phi^k(T)$ and $T'_k := \lambda^{-k}\Phi^k(T')$. That is, T_k and T'_k are obtained from T and T' by subdividing, but not inflating, T and T' k times. Let $\tilde{T}_k := T_k \backslash (T_k \cap T'_k)$ and $\tilde{T}'_k := T'_k \backslash (T_k \cap T'_k)$. The supports of $\tilde{T}_k$ and $\tilde{T}'_k$ are nested decreasing. In fact, if the translation action on Ω_Φ has pure discrete spectrum and the tiling space has no non-zero translation periods , as is the case for the examples below, then these supports, in any compact region, have measure tending to 0. The branch locus then emerges from the overlap of $\tilde{T}_k$ with $\tilde{T}'_k$ on the boundary of their supports for large k (see [**BSW**]).

Example 4.6. The Half-hex. This is a self-similar Pisot substitution tiling space with $\lambda = 2$ and substitution rule shown in Figure 3. There are six prototiles, consisting of a half hexagon tile and its rotations through multiples of $\pi/6$.

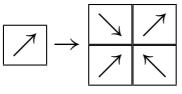
Six branch pairs are created from three distinct tilings, T_1, T_2, and T_3, with patches at the origin shown in Figure 4. These tilings are fixed by the substitution and are identical off the support of the pictured patches; hence $S(T_i, T_j) = \mathbb{S}^1$, $i, j \in \{1, 2, 3\}$, $i \neq j$. That is, these are isolated pairs. There are no other branch pairs so the branch locus is $\mathcal{BL} = \{T_1, T_2, T_3\}$.

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FIGURE 4. The 1-patches of the three elements of the half-hex branch locus.

Example 4.7. (The Chair) This is also a self-similar Pisot substitution tiling space with $\lambda = 2$. There are four square prototiles for the chair substitution Φ_C. (We will describe the 'square chair'—see [**S**] for its equivalence with the more familiar substitution with chair-shaped tiles.) Let ρ_0 be the unit square tile with the origin at the center and marked with a northeast pointing arrow, and let $\rho_i := r^i(\rho_0)$, where r denotes rotation through $\pi/2$. By this we mean, for example, that ρ_1 has the same support as ρ_0 but is marked with a northwest pointing arrow. The substitution on ρ_0 inflates by a factor of 2 and fills in the 2×2 square with translates of the ρ_i as pictured.



Let $\Phi_C(\rho_i) := r^i(\Phi_C(\rho_0))$. The tilings $T_i := \bigcup_{n\geq 1} \Phi_C^n(\rho_i)$ are fixed by Φ_C and are members of corner pairs: $S(T_i, T_{i+1}) = r^i(\{v \in \mathbb{S}^1 : \langle v, (0,-1)\rangle > -\sqrt{2}/2\})$, with the subscripts on the T's taken mod(4). These are all the corner pairs, there are no isolated pairs, and all members of line pairs have zero-patch equal to: a translate of ρ_0 or ρ_2 in a northeast–southwest direction; a translate of ρ_1 or ρ_3 in a northwest–southeast direction; or a collection of four prototiles with all arrows pointing towards, or away from, the origin. There are five zero-patches of the latter form: $B_0[T_i]$, $i = 0, \ldots, 3$, and the patch P pictured below.

$P =$ 

Let $T^* := \bigcup_{n\in\mathbb{N}} \Phi_C^n(P)$ be the tiling, fixed under Φ_C with $B_0[T^*] = P$. The branch locus of Φ_C then consists of the disjoint union of four dyadic solenoids, $S_i := \mathrm{cl}(\{T_i - tr^i((\sqrt{2}/2, \sqrt{2}/2)) : t \in \mathbb{R}\})$, $i = 0, \ldots, 3$, together with four rays, $R_i := \{T^* - tr^i((\sqrt{2}/2, \sqrt{2}/2)) : t \geq 0\}$, $i = 0, \ldots, 3$, joined at their common endpoint T^*. Each ray R_i winds densely on S_i. In addition, $r^{i+1}((\sqrt{2}/2, \sqrt{2}/2)) \in \partial S(T_i, T_{i+1})$ and $r^{i+3}((\sqrt{2}/2, \sqrt{2}/2)) \in \partial S(T_i, T_{i+3})$, so the rays $Q_i^{\pm} := \{T_i - tr^i((\sqrt{2}/2, \sqrt{2}/2)) : \pm t \geq 0\}$, which are attached to S_i at T_i and wind densely on S_{i+3} and S_{i+1}, are in the branch locus. $\mathcal{BP}(\Phi_C)$ is non-collapsing, so

$$\mathcal{BL}(\Phi_C) = \bigcup_{(T,T')\in\mathcal{BP}(\Phi_C)} \{T\} = \left(\bigcup_{i=0}^{3} S_i\right) \cup \left(\bigcup_{i=0}^{3} R_i\right) \cup \left(\bigcup_{i=0}^{3} Q_i^{\pm}\right).$$

The branch locus for the chair tiling can be described as an inverse limit of an expanding map on a branched 1-manifold (see Theorem 4.10 below) as follows. We first construct (a modification of) the Anderson–Putnam complex for Φ_C. Let $\mathbb{T}_i$ be the torus formed by identifying opposite edges of the prototile ρ_i, and sew these four tori together along

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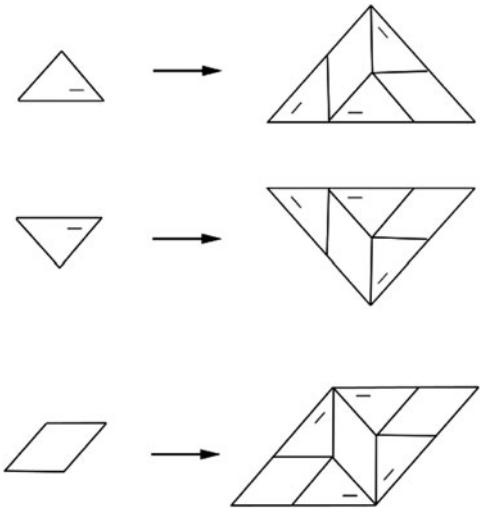


FIGURE 5. Octagonal substitution rule.

their 1-skeleta. So far, we have the uncollared A–P complex for Φ_C. The map induced by Φ_C on this complex has the flattening property everywhere along the edges, except at the single vertex. We remedy this by blowing up the vertex. To accomplish this, remove a small square neighborhood U_i of the vertex in each torus and sew in square disks, one cut from the center of each patch $B_0[T]$, $T \in \Omega_{\Phi_C}$, having a tile with the origin at a corner. There are nine distinct such patches and the square cut from each is marked, in each of its quadrants, by the arrow of the tile occupying that quadrant of the patch. These squares are sewn into $\bigcup \mathbb{T}_i \backslash U_i$ so that the arrows in the quadrants of the squares agree with the arrows in the $\mathbb{T}_i$. (For example, the square from the center of $B_0[T^*]$ has its boundary component in the first quadrant sewn into the southwest corners of $\mathbb{T}_0 \backslash U_0$, $\mathbb{T}_1 \backslash U_1$, and $\mathbb{T}_3 \backslash U_3$.) Call the resulting complex Y. The substitution Φ_C induces a map $F : Y \to Y$ with inverse limit homeomorphic with Ω_{Φ_C} (see [**BDHS**]). Consider the 1-complex $X \subset Y$ consisting of the union of the four diagonal line segments, aligned with the arrow, one in each $\mathbb{T}_i \backslash U_i$, and the five diagonal $X's$, one from each of the sewn in squares cut from $B_0[T_i]$, $i = 0, \ldots, 3$, and $B_0[T^*]$. With $f := F|_X$, Φ_C restricted to $\mathcal{BL}(\Phi_C)$ is topologically conjugate with the shift $\hat{f}$ on $\varprojlim f : X \to X$.

Example 4.8. (The Octagon) The (undecorated) octagonal tiling space is a self-similar Pisot substitution tiling space with $\lambda = 1 + \sqrt{2}$. There are 20 prototiles: four unmarked rhombi, $\rho_i = r^i(\rho_0)$, $i = 0, \ldots, 3$, with $r^4(\rho_0) = \rho_0$, r being rotation through $\pi/4$, with the origin at their centers; and sixteen marked isosceles right triangles $\tau_i = r^i(\tau_0)$, $\tau_i' = r^i(\tau_0')$, $i = 0, \ldots, 7$, with τ_i and τ_i' having the same support but bearing different marks. The origin is at the midpoint of the hypotenuse of the triangular prototiles. The octagonal substitution, Φ_O, is described in Figure 5.

If ρ_0' denotes the translate of ρ_0 with the origin slightly off-center (see Figure 6), then $\rho_0' \in \Phi_O^2(\rho_0')$ so that the tilings $T_{1,i} := \bigcup_{n \in \mathbb{N}} \Phi_O^{2n}(r^i(\rho_0'))$, $i = 0, \ldots, 7$, are fixed

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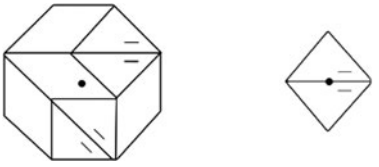


FIGURE 6. 1-patches of the tilings $T_{1,0}$ and $T_{2,0}$. Black dots indicate the placement of the origin.

by Φ_O^2. Also, $\tau_i \in \Phi_O^2(\tau_i)$ and $\tau_i' \in \Phi_O^2(\tau_i')$, so the tilings $T_{2,i} := \bigcup_{n\in\mathbb{N}} \Phi_O^{2n}(\{\tau_i, \tau_{i+4}'\})$, $i = 0, \ldots, 7$, and $i + 4$ taken mod(8), are fixed by Φ_O^2. Note that $\Phi_O(T_{j,i}) = T_{j,i+4}$, for $j = 1, 2$ and $i = 0, \ldots, 7$, with subscripts taken mod(8).

The tilings $T_{1,i}$ and $T_{1,i+3}$ are members of a branch line pair, as are $T_{1,i}$ and $T_{1,i+5}$ (subscripts taken mod(8)), and the corresponding sectors have boundaries: $\partial S(T_{1,i}, T_{1,i+3}) = \{\pm r^i((\sqrt{2}/2, \sqrt{2}/2))\}$; and $\partial S(T_{1,i}, T_{1,i+5}) = \{\pm r^i((0, 1))\}$. Let $\Omega_i := \mathrm{cl}(\{T_{1,i} - tr^i((\sqrt{2}/2, \sqrt{2}/2)) : t \in \mathbb{R}\})$. $\mathcal{BP}(\Phi_O)$ is non-collapsing, and the branch locus of the octagonal tiling is $\mathcal{BL}(\Phi_O) = \bigcup_{(T,T')\in\mathcal{BP}(\Phi_O)}\{T\} = \bigcup_{i=0}^{7} \Omega_i$ with the Ω_i intersecting as follows:

$$\Omega_0 \cap \Omega_j = \begin{cases} \{T_{1,0}\}, & j = 1, \\ \{T_{2,0}\}, & j = 2, \\ \emptyset, & j = 3, \\ \emptyset, & j = 4, \\ \emptyset, & j = 5, \\ \{T_{2,6}\}, & j = 6, \\ \{T_{1,7}\}, & j = 7. \end{cases}$$

Note that r determines a homeomorphism of Ω_0 by $r(\{\sigma_i + v_i\}) := \{r(\sigma_i) + r(v_i)\}$ for $v_i \in \mathbb{R}^2$ and prototiles σ_i. Then $\Omega_i = r^i(\Omega_0)$ and the intersections of Ω_i with the various Ω_j are obtained by applying r^i to the above. Ω_0 is itself (homeomorphic with) a one-dimensional substitution tiling space which can be described as follows. Let X be a wedge of three circles, labeled by 1, 2 and 3, and let $f : X \to X$ be the Markov map that maps these circles according to the pattern of the substitution ϕ:

$$1 \mapsto 23131$$
$$2 \mapsto 23231$$
$$3 \mapsto 2323131.$$

Then $\Phi_O^2|_{\Omega_0}$ is conjugate with the shift $\hat{f}$ on $\Omega_\phi \simeq \varprojlim f \simeq \Omega_0$. To describe all of $\mathcal{BL}(\Phi_O)$, let x_0, x_1, x_2, and x_3 be the fixed points of $\hat{f}$ corresponding to the last occurrence of 3 in $\phi(3)$, the first occurrence of 3 in $\phi(3)$, the first occurrence of 1 in $\phi(1)$, and the second occurrence of 2 in $\phi(2)$, respectively. (These points correspond to $T_{1,0}$, $T_{1,7}$, $T_{2,0}$, and $T_{2,6}$, respectively.) Let $B := (\bigcup_{i=0}^{7} \varprojlim f \times \{i\})/\sim$, in which $\sim$ identifies points as follows:

$$(x_0, i) \sim (x_1, i + 1),\ (x_1, i) \sim (x_0, i + 7),\ (x_2, i) \sim (x_3, i + 2),$$

and $(x_3, i) \sim (x_2, i + 6)$. $\Phi_O^2|_{\mathcal{BL}(\Phi_O)}$ is then conjugate with $F : B \to B$, where $F([(x, i)]) := [(\hat{f}(x), i)]$.

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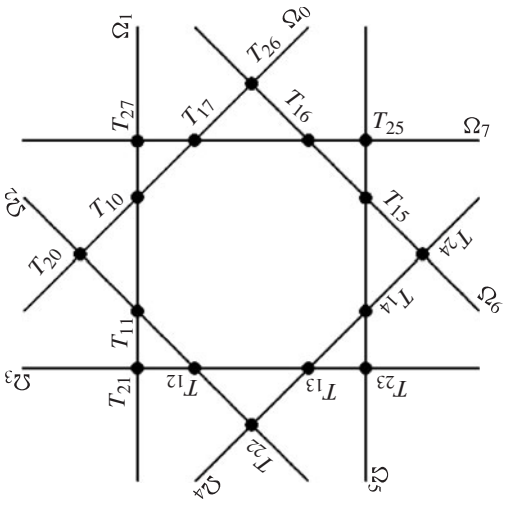


FIGURE 7. Schematic of the branch locus of the octagonal tiling space.

One sees from the inverse limit description that the Φ_O-periodic points in $\mathcal{BL}(\Phi_O)$ are dense. It follows from the non-collapsing property of $\mathcal{BP}(\Phi_O)$ that each of these points is actually a periodic branch point (Proposition 4.5) and a member of a line pair.

The octagon tilings can also be obtained from a cut-and-project scheme (see [**FHK**]) in which the 'window' is an octagon. In this presentation, the ambiguity introduced by each of the eight boundary segments of the octagon is responsible for each of the eight pieces Ω_i of the branch locus.

The following Theorem guarantees that, unless the branch locus consists of just a finite number of points (as for the half-hex), the branch locus is rather complicated.

THEOREM 4.9. *If Ω_Φ is a two-dimensional, non-periodic, FLC, self-similar substitution tiling space that has no isolated branch pairs, then there are branch pairs (T_1, T_1'), $(T_2, T_2') \in \Omega_\Phi \times \Omega_\Phi$ with $\partial S(T_1, T_1') \neq \partial S(T_2, T_2')$. Moreover, if Ω_Φ also has no corner branch pairs, then all line branch pairs (T, T') are non-collapsing in the sense that $T - tv \nsim_s T' - tv$ for all $t \in \mathbb{R}$, $v \in \partial S(T, T')$.*

Proof. The first assertion follows immediately from Theorem 3.26. For the 'moreover' part, suppose that (T, T') is a branch pair, $v \in \partial S(T, T')$, and there is $t > 0$ with $T - tv \sim_s T' - tv$. There is then $t_0 \geq 0$ and $\epsilon > 0$ so that $T - t_0 v \nsim_s T' - t_0 v$ and $T - tv \sim_s T' - tv$ for $t_0 < t < t_0 + \epsilon$. Pick $n_i \to \infty$ so that $(\Phi \times \Phi)^{n_i} \to (\bar{T}, \bar{T}') \in \Omega_\Phi \times \Omega_\Phi$. Then $(\bar{T}, \bar{T}')$ is a corner, or isolated, branch pair. That is, $\mathcal{BP}(\Phi)$ is *not* non-collapsing. □

We'll say that a continuous map $f : K \to K$ on a finite one-dimensional simplicial complex K with vertices V is an *expanding Markov* map provided $f(V) \subset V$, $f^{-1}(V)$ is finite, and f is one-to-one and stretches length by a factor of at least λ, for some $\lambda > 1$, on each component of $K \backslash (V \cup f^{-1}(V))$. Such maps are quite similar to the Williams presentations of one-dimensional hyperbolic attractors [**W**]. In particular, at all but finitely

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many points, the inverse limit space $\varprojlim f$ is homeomorphic with the product of a zero-dimensional set with an arc.

THEOREM 4.10. *Suppose that Ω_Φ is a two-dimensional, non-periodic, FLC, self-similar substitution tiling space whose tiles have polygonal support. Suppose also that either $\mathcal{BP}(\Phi)$ is non-collapsing or Φ is Pisot. Then there is a finite one-dimensional simplicial complex K and an expanding Markov map $f : K \to K$ so that $\Phi|_{\mathcal{BL}(\Phi)}$ is topologically conjugate with $\hat{f}$ on $\varprojlim f$.*

Sketch of proof. Let X be the 1-collared Anderson–Putnam complex for Ω with $\pi : \Omega \to X$ and $F : X \to X$ so that $\pi \circ \Phi = F \circ \pi$. Then $\hat{\pi} : \Omega \to \hat{X} := \varprojlim F$ is a homeomorphism that conjugates Φ with $\hat{F}$ (see [**AP**]). We will prove in the next section that, under the hypotheses of Theorem 4.10, the collection $\{(\tau, l)\}$ consisting of all pairs with $\tau \in T$ and $\tau \cap l \neq \emptyset$, where either (T, T') is a periodic line pair for which $\partial S(T, T') = \partial S(T, T') = \{\pm v\}$ and $l = \{tv : t \in \mathbb{R}\}$, or (T, T') is a corner pair with $v \in \partial S(T, T')$ and $l = \{tv : t \geq 0\}$, is finite, up to translation. From this it will follow that the set $K := \pi(\mathcal{BL}(\Phi)) \subset X$ is a finite one-dimensional simplicial complex invariant under F. Then $f = F|_K$ is expanding Markov and $\Phi|_{\mathcal{BL}(\Phi)}$ is topologically conjugate with $\hat{f}$ on $\varprojlim f$. A complete proof appears in the next section.

THEOREM 4.11. *Suppose that Ω and Ω' are homeomorphic two-dimensional, non-periodic, FLC, self-similar substitution tiling spaces and at least one of the following conditions is met:*

- *Ω has an isolated or a corner branch pair;*
- *Ω has line pairs (T_1, T_2) and (T_2, T_3) with $\partial S(T_1, T_2) \neq \partial S(T_2, T_3)$;*
- *Ω is Pisot.*

Then there is a homeomorphism, from Ω to Ω', that conjugates some powers of the substitution homeomorphisms, that takes branch pairs to branch pairs, preserving type, and that restricts to a homeomorphism of the branch loci.

Sketch of proof. As was the case for the proof of Theorem 3.32, the rigidity results of [**K**] will provide the key to the proof of Theorem 4.11. First, we may replace an arbitrary homeomorphism of Ω and Ω' by a 'linear' homeomorphism that preserves asymptoticity. To further improve the linear homeomorphism to a conjugacy (between some powers of the substitution maps) it is necessary to show that the linear homeomorphism can be made to be 'pointed', that is, to take some self-similar (i.e., periodic under substitution) tiling to a self-similar tiling. This is accomplished by establishing the existence of tilings with such special asymptotic properties, that any tiling with these properties must be self-similar (at least up to translation). This existence is established by fiat in the first two of the alternative hypotheses of Theorem 4.11. From Theorem 4.9 it follows that if Ω has neither isolated nor corner branch pairs, then Ω has non-parallel line branch pairs. Intersection of the corresponding lines would yield the second of the alternative hypotheses. In case Ω is Pisot we will see that these lines have a virtual intersection when viewed in the maximal equicontinuous factor of Ω. This virtual intersection will prove sufficient to pin down a linear homeomorphism of Ω and Ω' to a pointed homeomorphism. Once powers of the substitutions on Ω and Ω' are conjugated, Theorem 4.11 will follow easily.

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5. *Proofs of the main theorems*

Throughout this section, we assume that Ω_Φ and Ω_Ψ are d-dimensional, self-similar, non-periodic, FLC, substitution tiling spaces. When we say *let T be fixed under Φ* we really mean: *let T be of period $k \geq 1$ under Φ and replace Φ by Φ^k.*

Given $u \in \mathbb{S}^{d-1}$ and $y \in \mathbb{R}^d$, let $\{u\}^\perp := \{v \in \mathbb{R}^d : \langle v, u\rangle = 0\}$ and let $A(y, u) := \{y + v : v \in \{u\}^\perp\}$. If $T \in \Omega_\Phi$ and $\epsilon > 0$, we will say that $A(y, u)$ is an *ϵ-characteristic* for T if there is $0 \neq w \in \mathbb{R}^d$ so that $T - x \nsim_s T - w - x$ and $T - x - su \sim_s T - w - x - su$ for all $x \in A(y, u)$ and all $s \in (0, \epsilon]$.

LEMMA 5.1. *Given $T \in \Omega_\Phi$, $\epsilon > 0$, $r \geq 0$, and $u \in \mathbb{S}^{d-1}$, there are, up to translation in directions of $\{u\}^\perp$, only finitely many patches of the form $B_r[T - x]$ for x in any ϵ-characteristic $A(y, u)$ for T.*

Proof. If the lemma is true for $r = 0$ then it's also true for any $r \geq 0$ by FLC. So let $r = 0$ and first suppose that ϵ is larger than the diameter of any tile. By Lemma 3.11, there are only finitely many pairs of patches of the form $(B_0[T - x], B_0[T - w - x])$, up to translation (by elements of $\mathbb{R}^d$), with x in an ϵ-characteristic $A(y, u)$ for T, w as in the definition of ϵ-characteristic. Suppose that for such patch pairs we have

$$(B_0[T - x], B_0[T - w - x]) = (B_0[T - x'], B_0[T - w' - x']) - v.$$

Using the fact that $(B_0[T - x], B_0[T - w - x])$ and $(B_0[T - x'], B_0[T - w' - x'])$ are stably related on their overlap with H_u, $T - x \nsim_s T - w - x$, and $T - x' \nsim_s T - w' - x'$, just as in the proof of Proposition 3.19, it follows that $v \in \{u\}^\perp$. This establishes the lemma for large ϵ. For general ϵ, let N be large enough so that $\lambda^N \epsilon$ is bigger than the diameter of any tile. There are then just finitely many patches $B_0[\Phi^N(T) - \lambda^N x]$, x in an ϵ-characteristic $A(y, u)$, up to translation in directions in $\{u\}^\perp$, by the foregoing. From this it follows that the collection of $\{u\}^\perp$-translational equivalence classes of the $B_0[T - x]$ is also finite. □

It follows from the above that there is a $\delta = \delta(\epsilon) > 0$ so that if $A(y_1, u)$ and $A(y_2, u)$ are two distinct, parallel ϵ-characteristics for T, then the distance between $A(y_1, u)$ and $A(y_2, u)$ is at least δ.

LEMMA 5.2. *Suppose that U is a proper open subset of $\mathbb{R}^d$ and $u \in \mathbb{S}^{d-1}$ are such that: for all $x \in U$ and $r = r(x) := \operatorname{dist}(x, \partial U)$, $\partial B_r(x) \cap \partial U \subset \{x \pm ru\}$. Then U is a union of strips of the form $\{y : \langle y_1, u\rangle < \langle y, u\rangle < \langle y_2, u\rangle\}$ parallel to $\{u\}^\perp$.*

Proof. Let $x \in U$ and let $r_0 > 0$ be small enough so that $\operatorname{cl}(B_{r_0}(x)) \subset U$. For each $r > 0$ let

$$E_r(x) := \left\{ x + su + tv : \frac{s^2}{r_0^2} + \frac{t^2}{r^2} = 1;\ s, t \in \mathbb{R},\ v \in \{u\}^\perp \right\}$$

be the ellipsoidal ball at x with semi-major and -minor axes of length r and r_0. Then $\partial E_r(x)$ is a smooth $(d-1)$-dimensional ellipsoid whose normal vector is parallel with u only at $\{x \pm r_0 u\}$. Let $R := \sup\{r : E_r(x) \subset U\}$. If $R < \infty$, then there is $y \in \partial E_R(x) \cap \partial U$, while $E_R(x) \subset U$. Since $\partial(B_{r_0}(x)) \cap \partial U = \emptyset$, $y \neq \pm r_0 u$, so the inward unit normal w to $\partial E_R(x)$ at y is not parallel with u. Let $r > 0$, $x_0 \in E_R(x)$ be such

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that $\partial B_r(x_0) \cap \partial E_R(x) = \{y\}$ and $B_r(x_0) \subset E_R(x) \subset U$. Then, $x_0 \in U$, $r = \mathrm{dist}(x_0, \partial U)$ but $\partial B_r(x_0) \cap \partial U \not\subset \{x_0 \pm ru\}$, since $y = x_0 - rw$ and $w \neq \pm u$. Thus $R = \infty$ and x is an element of the strip

$$\bigcup_{r>o} E_r(x) = \{y : \langle x - ru, u\rangle < \langle y, u\rangle < \langle x + ru, u\rangle\},$$

which is contained in U. □

Given $u \in \mathbb{S}^{d-1}$ and $y, y' \in \mathbb{R}^d$ with $\langle y, u\rangle < \langle y', u\rangle$, by the *open strip between y and y'* we will mean

$$\mathcal{S} = \{x \in \mathbb{R}^d : \langle y, u\rangle < \langle x, u\rangle < \langle y', u\rangle\}.$$

If $T \in \Omega_\Phi$, the *group of s-periods* of T on $\mathcal{S}$ is the group $\{v \in \{u\}^\perp : T - x - v \sim_s T - x$, for all $x \in \mathcal{S}\}$. It follows from Lemma 3.9 that the group of s-periods of T on $\mathcal{S}$ is a discrete lattice, i.e., is topologically isomorphic with $\mathbb{Z}^k$ for some $k \leq d - 1$.

LEMMA 5.3. *Suppose that $S(T, T') \subset H_u \cup H_{-u}$ for some $u \in \mathbb{S}^{d-1}$ and all $(T, T') \in \mathcal{PBP}(\Phi)$. Then every tiling in Ω_Φ has an ϵ-characteristic. In fact, if $T \in \Omega_\Phi$ and $0 \neq x \in \mathbb{R}^d$ is such that $T - x \cap T \neq \emptyset$, then the connected components of $\{y \in \mathbb{R}^d : T - y - x \sim_s T - x\}$ are proper open strips, or half-spaces, parallel to $\{u\}^\perp$, whose boundary components are ϵ-characteristics.*

Proof. By Theorem 3.24, we also have that $S(T, T') \subset H_u \cup H_{-u}$ for some $u \in \mathbb{S}^{d-1}$ and all $(T, T') \in \mathcal{BP}(\Phi)$. Pick $T \in \Omega_\Phi$ and $0 \neq y \in \mathbb{R}^d$ so that $B_1[T - y] = B_1[T]$. Let $U := \{v \in \mathbb{R}^d : T - y - v \sim_s T - v\}$. U is open, and proper by Lemma 3.10 and the non-periodicity of Ω_Φ. If $x \in U$, $\mathrm{dist}(x, \partial U) = r$, and $\partial B_r(x) \cap \partial U$ is not a subset of $\{x \pm ru\}$ then there is a branch pair with asymptotic sector not contained in $H_u \cup H_{-u}$ (Lemma 3.22). Thus U is a union of strips, or half-spaces parallel with $\{u\}^\perp$, by Lemma 5.2. Any boundary component of a component of U is an ϵ-characteristic. □

LEMMA 5.4. *Assume the hypotheses of Lemma 5.3. Given $T \in \Omega_\Phi$, there are open strips U_i, $i \in \mathbb{Z}$, parallel with $\{u\}^\perp$, and a finite set of non-zero elements W of $\{u\}^\perp$ so that $\mathbb{R}^d = \bigcup_{i=-\infty}^{\infty} U_i$ and each U_i is s-periodic of period w_i for some $w_i \in W$.*

Proof. By Lemma 5.3, there is an ϵ-characteristic $A(x_0, u)$ for T. By Lemma 5.1, there is a finite set W of non-zero elements of $\{u\}^\perp$ so that if $A(y, u)$ is any $\epsilon = 1$-characteristic, then $B_1[T - x - w] = B_1[T - x]$ for some $x \in A(y, u)$ and $w \in W$. Pick $w_0 \in W$ and $x \in A(x_0, u)$ so that $B_1[T - x - w] = B_1[T - x]$. Let $U = \{y \in \mathbb{R}^d : T - y - w \sim_s T - y\}$. Then the connected component U_0 of U containing $T - x_0$ is an open strip, or half-space, parallel with $\{u\}^\perp$, whose boundary components are ϵ-characteristics with $\epsilon = 1$. Note that w_0 is an s-period of T on U_0. Suppose that U_0 is a half-space, say $U_0 = \{y \in \mathbb{R}^d : \langle y_0, u\rangle < \langle y, u\rangle\}$. Pick $t_n \to \infty$ so that $T - t_n u \to \bar{T} \in \Omega_\Phi$. Then $T - w - t_n u \to \bar{T} - w$ and $\bar{T} - y \sim_s \bar{T} - y - w$, for all $y \in \mathbb{R}^d$, in contradiction to Lemma 3.10. Thus U_0 is a strip containing $A(x_0, u)$, bounded by $\epsilon = 1$-characteristics $A(y_0, u)$ and $A(y_1, u)$. Now repeat this process, replacing $A(x_0, u)$ with $A(y_0, u)$ and $A(y_1, u)$, to obtain strips $U_{-1} \supset A(y_0, u)$, on which T is s-periodic of period $w_{-1} \in W$, and $U_1 \supset A(y_1, u)$, on which T is s-periodic of period $w_1 \in W$. Now get strips U_{-2} and U_2 containing the 'left' boundary component of U_{-1} and the 'right' boundary component of U_1, respectively, etc.

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Note that since any two distinct $\epsilon = 1$-characteristics are at least δ apart for some uniform δ, $\bigcup_{i=-\infty}^{\infty} U_i = \mathbb{R}^d$. □

Proof of Theorem 3.26. Suppose there are no such periodic branch pairs. Let T be fixed by Φ and let U_i be as in Lemma 5.4. We may assume that $0 \in U_0$. Let G_i be the group of all s-periods of T on U_i. Since $\lambda U_0 \supset U_0$, λw_0 is also an s-period of T on U_0. If λ is not an integer, then λ is irrational (it's an algebraic integer). But G_0 is discrete, so λ must be an integer, establishing the theorem under the hypothesis that λ^k is not an integer for any integer $k \geq 0$.

Suppose now that $d = 2$. Then all G_i have rank one, that is, all G_i are isomorphic with $\mathbb{Z}$. Let M be large enough so that $\{w_{-M}, \ldots, w_M\} = W$ and let N be large enough so that $\lambda^N U_0 \supset \bigcup_{i=-M}^{M} U_i$. Then $w_i, \lambda^N w_0 \in G_i \simeq \mathbb{Z}$ for $-M \leq i \leq M$, so there are integers m_i, n_i so that $n_i w_i = m_i \lambda^N w_0$. Let $m := \prod_{i=-M}^{M} m_i$. Then T is s-periodic of period $m\lambda^N w_0$ on U_i, for all $i \in \mathbb{Z}$. That is $T - m\lambda^N w_0 \sim_s T - x$, for all $x \in \mathbb{R}^2$, contradicting Lemma 3.10. □

Proof of Theorem 4.10. As indicated in the 'sketch of proof' following the statement of Theorem 4.10, the key is to establish the finiteness condition: there are finitely many pairs (τ_i, l_i), l_i a line segment in the tile τ_i, so that if $T \in \mathcal{BL}(\Phi)$, then there is $x \in \mathbb{R}^2$ and i with $0 \in l_i + x \subset \tau_i + x \in T$. The collection of the l_i will then define a one-dimensional simplicial sub-complex K of the collared Anderson–Putnam complex and the branch locus is obtained as the inverse limit of the substitution induced map on K. The finiteness condition will be verified easily in case $\mathcal{BP}(\Phi)$ is non-collapsing. To establish the finiteness condition under the assumption that the inflation is Pisot, we will compare the dynamics on Ω_Φ to those on its maximal equicontinuous factor, a solenoid obtained as the inverse limit of a hyperbolic torus endomorphism. We will show that if $(T, T') \in \mathcal{PBP}(\Phi)$ and $v \in S(T, T')$, then there is $t_0 > 0$ and Φ-periodic $\bar{T}$ so that $T - t_0 v \in W^s_{\text{loc}}(\bar{T})$. From the algebraic nature of the dynamics on the solenoid, we will see that $g(T - tv)$ returns to $W^s_{\text{loc}}(g(\bar{T}))$ with bounded gap (g being the semi-conjugacy onto the maximal equicontinuous factor). Invoking finite-to-oneness of g, $T - tv$ then returns to $W^s_{\text{loc}}(\bar{T})$ with finite gap, and the desired finiteness condition follows from this.

So let us first suppose that $\mathcal{BP}(\Phi)$ is non-collapsing. Then $\mathcal{BL}(\Phi) = \bigcup_{(T,T') \in \mathcal{BP}(\Phi)} \{T\}$ and the branch locus is compact (Proposition 4.5). Pick $(T, T') \in \mathcal{PBP}(\Phi)$ and $v \in \partial S(T, T')$, and let $u \in \mathbb{S}^1$ be such that $H_u \subset S(T, T')$. Suppose that $\tau_1, \tau_2 \in T$, $t_1, t_2 \in \mathbb{R}^+$, and $x \in \mathbb{R}^2$ are such that $t_i v \in \text{int}(\tau_i)$ for $i = 1, 2$, $\tau_1 + x = \tau_2$, and $B_0[T' - t_1 v] + x = B_0[T' - t_2 v]$. Since $B_0[T - t_i v]$ is stably related to $B_0[T' - t_i v]$ on $\text{int}(\text{spt}(B_0[T - t_i v]) \cap \text{spt}(B_0[T' - t_i v])) \cap \{y \in \mathbb{R}^2 : \langle y, u \rangle > 0\}$, while $B_0[T - t_i v]$ is not stably related to $B_0[T' - t_i v]$ on $\{t_i v\}$ for $i = 1, 2$, it must be the case that x is parallel with v. That is, the pairs $(\tau_1, l \cap \tau_1)$ and $(\tau_2, l \cap \tau_2)$ are the same up to translation, where $l = l_v := \{tv : t \in \mathbb{R}\}$. Since there are only finitely many translationally inequivalent pairs of patches of the form $(B_0[\bar{T}], B_0[\bar{T}'])$ with $(\bar{T}, \bar{T}') \in \mathcal{BP}(\Phi)$ (Lemma 3.17—note that, in the above, $(T - t_i v, T' - t_i v) \in \mathcal{BP}(\Phi)$ from the non-collapsing assumption), and since the set $\{v : v \in \partial S(T, T'), (T, T') \in \mathcal{PBP}(\Phi)\}$ is finite (Theorem 4.2), the collection $\{[(\tau, \tau \cap l)]\}$ of translation equivalence classes of pairs $(\tau, \tau \cap l)$, where $0 \in \tau \in T$ and $l = l_v$ for some T with $(T, T') \in \mathcal{PBP}(\Phi)$ and $v \in \partial S(T, T')$, is finite.

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By first collaring tiles if necessary, we may assume that Φ forces the border so that the map $\hat{\pi} : \Omega_\Phi \to \hat{X}_\Phi := \varprojlim F_\Phi$ onto the inverse limit of the substitution map F_Φ on the Anderson–Putnam complex X_Φ, is a homeomorphism [**AP**]. The two-dimensional CW-complex X_Φ is the quotient of $\mathbb{R}^2$ formed from $T \in \Omega_\Phi$ by identifying x_1 and x_2 if there are τ_1 and τ_2 in T with $x_i \in \tau_i$ and $\tau_2 = \tau_1 + (x_2 - x_1)$. Equivalently, $X_\Phi := \{[(\tau, x)] : x \in \tau,\ \tau \in T,\ T \in \Omega_\Phi\}/\sim$, with $\sim$ the transitive closure of the relation defined by $[(\tau_1, x_1)] \sim [(\tau_2, x_2)]$ if there are $T \in \Omega_\Phi$, $\bar{\tau}_1, \bar{\tau}_2 \in T$, and y_1, y_2 so that $\bar{\tau}_i = \tau_i + y_i$, $i = 1, 2$, and $x_1 + y_1 = x_2 + y_2$. With this description of X_Φ, $\pi : \Omega_\phi \to X_\Phi$ is given by $\pi(T) := [[(\tau, 0)]]$, where $0 \in \tau \in T$ and $[[(*, *)]]$ denotes the $\sim$-equivalence class of the translational equivalence class of $(*, *)$. We see that $K := \pi(\mathcal{BL}(\Phi))$ is a one-dimensional simplicial complex (or finite collection of points, if $\mathcal{BL}(\Phi)$ consists only of isolated pairs) contained in the finite union of line segments $\{[[(\tau, x)]] : x \in \tau \cap l_v\}$, with $0 \in \tau \in T$, $(T, T') \in \mathcal{PBP}(\Phi)$, and $v \in \partial S(T, T')$, and points $\pi(T)$, where (T, T') is an isolated pair. With $f := F_\Phi \mid_K$, we have that $\hat{\pi} \mid_{\mathcal{BL}(\Phi)}$ maps $\mathcal{BL}(\Phi)$ homeomorphically onto $\varprojlim f$, conjugating $\Phi \mid_{\mathcal{BL}(\Phi)}$ with $\hat{f}$. The set V of vertices of K is defined as follows. Let

$$V_1 := \{\pi(T) : (T, T') \in \mathcal{PBP}(\Phi) \text{ is an isolated or corner pair}\}.$$

V_1 is finite by Theorem 4.2. Let

$$V_2 := \{[[(\tau, 0)]] \in K : 0 \text{ is a vertex of } \tau\}.$$

V_2 is finite under the assumption that all tiles are polygons. Let $V_3 := \{[[(\tau, 0)]] : 0$ is in the interior of an edge η of τ, $\tau \in T$, $(T, T') \in \mathcal{BP}(\Phi)$, and there is $v \in \partial S(T, T')$ with v not parallel with $\eta\}$. V_3 is the set of points where the finitely many line segments in K meet an edge of a polygonal face of X_Φ transversely, so V_3 is finite. Let $V_4 := \{[(\tau, 0)] : 0 \in \mathring{\tau}$ and there are $(T_1, T_1'), (T_2, T_2') \in \mathcal{BP}(\Phi)$ with $\partial S(T_1, T_1') \neq \partial S(T_2, T_2')$ and $\tau \in T_1 \cap T_2\}$. V_4 contains the points where distinct line segments $[(\tau, \tau \cap l_1)]$ and $[(\tau, \tau \cap l_2)]$ cross in the interior of faces X_Φ—it is finite because there are only finitely many such line segments. Then $V := V_1 \cup V_2 \cup V_3 \cup V_4$ is finite and forward invariant under f. K is a finite simplicial complex and f stretches length on the edges of K by a factor of λ. To confirm that f is expanding Markov it is only necessary to verify that f is one-to-one on the components of $K \backslash (V \cup f^{-1}(V))$. But this is clear: if α is such a component, then $f(\alpha)$ is a line segment in K with no point of V in its interior. Hence $f(\alpha)$ lies entirely in a single face of X_Φ and cannot self intersect.

Now assume that Φ is self-similar Pisot. We will argue that if $(T, T') \in \mathcal{PBP}(\Phi)$ and $v \in \partial S(T, T')$, then the ray $\{tv : t \geq 0\}$ crosses the tiles of T in only finitely many ways. The result then follows as above. Let $g : \Omega_\Phi \to \hat{\mathbb{T}}_A$ be geometric realization onto the maximal equicontinuous factor and let $\hat{F}_A : \hat{\mathbb{T}}_A \to \hat{\mathbb{T}}_A$ be the hyperbolic automorphism with $g \circ \Phi = \hat{F}_A \circ g$. (Here A is the $2d \times 2d$ block diagonal matrix whose blocks are the companion matrix of the Pisot inflation factor λ, $d = \deg(\lambda)$, F_A is the associated hyperbolic toral endomorphism, and $\hat{F}_A$ is the shift on the inverse limit $\hat{\mathbb{T}}_A := \varprojlim F_A$.) The map g also semiconjugates translation in Ω_Φ with an $\mathbb{R}^2$ action on $\hat{\mathbb{T}}_A$ that consists of translation along the two-dimensional unstable (under $\hat{F}_A$) leaves in $\hat{\mathbb{T}}_A$: we will use the notation $g(T) - x := g(T - x)$ for this action. More explicitly in coordinates, the $\mathbb{R}^2$

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action on $\hat{\mathbb{T}}_A$ is given by

$$(z_0, \ldots, z_n, \ldots) - x = (z_0 - \tilde{x}, \ldots, z_n - \lambda^{-n}\tilde{x}, \ldots),$$

where $x \mapsto \tilde{x}$ is an isomorphism of $\mathbb{R}^2$ with the unstable space W_A^u of F_A, which, abusing notation slightly, we will identify with the unstable space of A. We will also use the key fact that g is boundedly finite-to-one [**BKe**].

Given $(T, T') \in \mathcal{PBP}(\Phi)$ and $v \in \partial S(T, T')$, we claim there is $t_0 > 0$ and $\bar{T} \in \Omega_\Phi$, with $\bar{T}$ Φ-periodic, so that $T - t_0 v \in W^s(\bar{T})$. Indeed, if there is $t_1 > 0$ so that $T - t_1 v \sim_s T' - t_1 v$, then let $t_0 := \inf\{a : T - tv \sim_s T' - tv, \text{ for all } t \in [a, t_1]\}$. Then $t_0 > 0$ and $T - t_0 v \nsim_s T' - t_0 v$. Let $n \in \mathbb{N}$ be large and $k \in \mathbb{N}$ be such that $(B_0[\Phi^{n+k}(T - t_0 v)], B_0[\Phi^{n+k}(T' - t_0 v)])$ is the same as $(B_0[\Phi^n(T - t_0 v)], B_0[\Phi^n(T' - t_0 v)])$, up to translation. Such translation must be parallel with v (as, for example, in the proof of Proposition 3.19) and then must be 0, since $B_0[\Phi^{n+k}(T - t_0 v)]$ and $B_0[\Phi^{n+k}(T' - t_0 v)]$, as well as $B_0[\Phi^n(T - t_0 v)]$ and $B_0[\Phi^n(T' - t_0 v)]$, are stably related on $\{tv : t \geq 0\}$ intersected with the supports of the appropriate patches. That is,

$$(B_0[\Phi^{n+k}(T - t_0 v)], B_0[\Phi^{n+k}(T' - t_0 v)]) = (B_0[\Phi^n(T - t_0 v)], B_0[\Phi^n(T' - t_0 v)])$$

and $T - t_0 v \in W^s(\bar{T})$, with

$$\bar{T} := \bigcup_{n \geq 0} \Phi^{\mathrm{nk}}(B_0[T - t_0 v])$$

Φ-periodic of period k. If, instead, $T - tv \nsim_s T' - tv$, for all $t \geq 0$, there are only finitely many patches of the form $B_0[T - tv]$ for $t \geq 0$, up to translation parallel to v (as argued in the non-collapsing case above). Let n be the Φ-period of T. There then are arbitrarily large t_1 and $s \in \mathbb{R}^+$, $m \in \mathbb{N}$ with $B_0[T - t_1 v] + sv \subset \Phi^{\mathrm{mn}}(B_0[T - t_1 v])$. Let $t_0 := t_1 + s/(\lambda^{\mathrm{mn}} - 1)$ and let $\bar{T} := \bigcup_{l \geq 0} \Phi^{\mathrm{lmn}}(B_0[T - t_0 v])$. Then $\bar{T}$ is Φ-periodic and $T - t_0 v \in W^s(\bar{T})$.

To simplify notation, assume that T is fixed by Φ. Let t_0 and $\bar{T}$ be as above with $\bar{T}$ periodic of period k under Φ. Let $L_v = L_v(T)$ denote the ray $L_v := \{g(T - tv) : t \geq 0\} = \{g(T) - tv : t \geq 0\}$ in $\hat{\mathbb{T}}_A$. Note that

$$g(T - \lambda^{\mathrm{nk}} t_0 v) = g(\Phi^{\mathrm{nk}}(T - t_0 v)) = \hat{F}_A^{\mathrm{nk}}(g(T - t_0 v)) \to g(\bar{T})$$

as $n \to \infty$ so that $g(\bar{T}) \in \mathrm{cl}(L_v)$ and L_v meets the ϵ-stable manifold

$$W_\epsilon^s(g(\bar{T})) := \{z \in \hat{\mathbb{T}}_A : d(\hat{F}_A^n(z), \hat{F}_A^n(g(\bar{T}))) \leq \epsilon \ \forall n \geq 0\}$$

for each $\epsilon > 0$. For each $n \in \mathbb{N}$, let $\pi_n : \hat{\mathbb{T}}_A \to \mathbb{T}^{2d} = \mathbb{R}^{2d}/\mathbb{Z}^{2d}$ be the projection of $\hat{\mathbb{T}}_A = \varprojlim F_A$ onto the nth coordinate space. We will prove now that $\pi_0(\mathrm{cl}(L_v))$ is a subtorus of $\mathbb{T}^{2d}$ of dimension at most d. (In fact the dimension of $\pi_0(\mathrm{cl}(L_v))$ must then be exactly d, but we won't need that here.)

To see that this is the case, first note that the stable (W_A^s) and unstable (W_A^u) spaces of $A : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, at 0, have bases consisting of vectors with entries in the field $\mathbb{Q}(\lambda)$. (W_A^u is spanned by a pair of eigenvectors for A with eigenvalue λ, while W_A^s is the space orthogonal to a space spanned by a pair of eigenvectors for A^T with eigenvalue λ.) Now, $g(T) - t_0 v, g(T) - \lambda^k t_0 v \in W^s(g(\bar{T}))$ means that the vector $w := (\lambda^k - 1)t_0 \tilde{v}$

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satisfies: (1) $w - p \in W_A^s$ for some $p \in \mathbb{Z}^{2d}$. (Here $\tilde{v}$ denotes the image of v under the previously introduced isomorphism of $\mathbb{R}^2$ with W_A^u.) Also, since $\tilde{v} \in W_A^u$: (2) $w \in W_A^u$. Conditions (1) and (2) can be expressed as a system of linear equations with coefficients in $\mathbb{Q}(\lambda)$; the unique solution $w = (w_1, \ldots, w_{2d})$ thus has all of its entries in $\mathbb{Q}(\lambda)$. The dimension of $\pi_0(\mathrm{cl}(L_v)) = \mathrm{cl}(\{tw + \mathbb{Z}^{2d} : t \in \mathbb{R}\})$ equals the dimension over $\mathbb{Q}$ of the rational span of $\{w_1, \ldots, w_{2d}\}$. Since the dimension of $\mathbb{Q}(\lambda)$ over $\mathbb{Q}$ is d, the dimension of $\pi_0(\mathrm{cl}(L_v))$ is at most d. The same argument clearly works for $\pi_n(\mathrm{cl}(L_v))$, for all $n \geq 0$, so $\mathrm{cl}(L_v)$ is a d'-dimensional solenoid for some $d' \leq d$.

We are going to show that, for small $\epsilon > 0$, $W_\epsilon^s(g(\bar{T})) \cap \mathrm{cl}(L_v)$ is a $(d'-1)$-dimensional cross-section to the $\mathbb{R}$-action $x \mapsto x - tv$ on the solenoid $\mathrm{cl}(L_v)$, to which orbits must return with bounded gap. From this it will follow that there are t_i' with $t_{i+1}' - t_i'$ bounded and with $t_i' \to \pm\infty$ as $i \to \pm\infty$ so that $\{B_0[T - t_i'v]\}$ is finite. An immediate consequence is that $\{tv : t \in \mathbb{R}\}$ intersects the tiles of T in only finitely many ways.

There is $\delta = \delta(v) > 0$ so that if $x + \mathbb{Z}^{2d}$, $y + \mathbb{Z}^{2d} \in \pi_0(\mathrm{cl}(L_v))$ and $|x - y| < \delta$, then $sx + (1-s)y + \mathbb{Z}^{2d} \in \pi_0(\mathrm{cl}(L_v))$, for all $s \in \mathbb{R}$. We claim that for $\epsilon < \delta$, $B_{\epsilon/2}(\pi_0(g(\bar{T}))) \cap \pi_0(\mathrm{cl}(L_v)) \subset \pi_0(W_\epsilon^s(g(\bar{T}))$. If this were not the case, then, by the choice of δ, $W_{\epsilon/2}^u(\pi_0(g(\bar{T}))) \subset \pi_0(\mathrm{cl}(L_v))$ and then $W^s(\pi_0(g(\bar{T}))) \subset \pi_0(\mathrm{cl}(L_v))$. But $W^s(\pi_0(g(\bar{T})))$ is dense in $\mathbb{T}^{2d}$ and $\pi_0(\mathrm{cl}(L_v))$ is a proper closed subset of $\mathbb{T}^{2d}$. Clearly, this δ works for the corresponding statement in all coordinate projections π_n simultaneously.

Let $\Delta_n(\epsilon)$ denote the local ϵ-stable manifold at the point $\pi_n(g(\Phi^{-n}(\bar{T})))$ for the restriction of F_A to the torus $\mathbb{T} := \pi_0(\mathrm{cl}(L_v)) = \pi_n(\mathrm{cl}(L_v))$. Then $\Delta_n(\epsilon/2)$ is contained in $W_\epsilon^s(\pi_n(g(\Phi^{-n}(\bar{T}))))$ for $\epsilon < \delta$ with δ depending only on v (and not on T, $\bar{T}$ or n). We saw above that the local unstable manifold $W_{\epsilon/2}^u(\pi_n(g(\Phi^{-n}(\bar{T}))))$ of F_A meets $\mathbb{T}$ only along the arc $\pi_n(g(\Phi^{-n}(\bar{T}))) - t\tilde{v}$, $|t| < \epsilon/2$, and it follows that $\Delta_n(\epsilon)$ is a co-dimension one disk in $\mathbb{T}$ transverse to the $\mathbb{R}$-action in the direction of $\tilde{v}$. Hence, orbits of the $\mathbb{R}$-action on $\mathbb{T}$ return to $\Delta_n(\epsilon/2)$ with bounded gap. (We really have a different $\mathbb{R}$-action on $\mathbb{T} = \pi_n(\mathrm{cl}(L_v))$, for each n, namely $(y, t) \mapsto y - \lambda^{-n}t\tilde{v}$. We will take this into account.)

Now choose ϵ' small enough so that if $S, S' \in \Omega_\Phi$, $g^{-1}(\{g(S)\}) = \{S_1, \ldots, S_m\}$, and $g(S') \in W_{\epsilon'}^s(g(S))$ then $B_0[S'] = B_0[S_i]$, for some $i \in \{1, \ldots, m\}$. Let n be large enough so that if $z, z' \in \hat{\mathbb{T}}_A$ are such that $\pi_n(z) = \pi_n(z')$ then $d(z, z') < \epsilon'/2$, and let $\epsilon > 0$ be less that the δ above and also small enough so that if $d(\pi_n(z), \pi_n(z')) < \epsilon$ then $d(z, z') < \epsilon'$. There is then a, b, depending only on ϵ and v, and $\{t_i' : i \in \mathbb{Z}\}$ with $t_i' \to \pm\infty$ as $i \to \pm\infty$ and $0 < t_{i+1}' - t_i' \leq b$ for all $i \in \mathbb{Z}$, so that $\pi_n(g(T - t_i'v)) = \pi_n(g(T)) - t_i'\lambda^{-n}\tilde{v} \in \Delta_n(\epsilon/2)$ for all $i \in \mathbb{Z}$. Let $g^{-1}(\{g(\bar{T})\}) = \{\bar{T}_1, \ldots, \bar{T}_m\}$. We have that $B_0[T - t_i'v] \in \{B_0[\bar{T}_1], \ldots, B_0[\bar{T}_m]\}$ for all $i \in \mathbb{Z}$.

With b as above, let B be the maximum of the cardinalities of the collections $\{B_b[S'] : S' \in \Omega_\Phi \text{ and } B_0[S'] = B_0[S]\}$ over all $S \in \Omega_\Phi$. Finite local complexity ensures that $B < \infty$. Let M be the maximal cardinality of a fibre $g^{-1}(\{g(S)\})$ over all $S \in \Omega_\Phi$: $M < \infty$ by [**BKe**]. The number of distinct translation equivalence classes of pairs $(\tau, \tau \cap l)$ with $\tau \in T$, $l = \{tv : t \in \mathbb{R}\}$, and $l \cap \tau \neq \emptyset$, is bounded by MB and this bound depends only on v and not on T (provided, of course, that there is T' with $(T, T') \in \mathcal{PBP}(\Phi)$ and $v \in \partial S(T, T')$).

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We prove now that for a given $v \in \mathbb{S}^1$ there are only finitely many distinct sets $\mathrm{cl}\{T - tv : t \in \mathbb{R}\}$ where T varies over all members of periodic line pairs (T, T') with $v \in \partial S(T, T')$. Indeed, there is a finite collection of pairs of patches $\{(P_1, P_1'), \ldots, (P_n, P_n')\}$ so that if (T, T') is any line pair with $v \in \partial S(T, T')$ then $(B_0[T], B_0[T']) = (P_i - tv, P_i' - tv)$ for some $i \in \{1, \ldots, n\}$, $t \in \mathbb{R}$. Suppose that (T_1, T_1') and (T_2, T_2') are periodic line pairs with $v \in \partial S(T_i, T_i')$, $i = 1, 2$, and $(B_0[T_1], B_0[T_1']) = (B_0[T_2] - sv, B_0[T_2'] - sv)$ for some $s \in \mathbb{R}$. Let m be a common Φ-period for T_1 and T_2. Then $B_0[T_1] = B_0[T_2 - sv]$ implies that $\Phi^{\mathrm{km}}(T_2 - sv) = T_2 - \lambda^{\mathrm{km}} sv \to T_1$ as $k \to \infty$. Thus, $T_1 \in \mathrm{cl}\{T_2 - tv : t \in \mathbb{R}\}$. If $T_2 - t_i v \to T_1$, then $T_2 - (t_i + t)v \to T_1 - tv$. Thus, $\mathrm{cl}\{T_1 - tv : t \in \mathbb{R}\} \subset \mathrm{cl}\{T_2 - tv : t \in \mathbb{R}\}$. Interchanging 1 and 2 and replacing s by $-s$ in the foregoing shows that $\mathrm{cl}\{T_1 - tv : t \in \mathbb{R}\} = \mathrm{cl}\{T_2 - tv : t \in \mathbb{R}\}$.

As there are only finitely many sets $\mathrm{cl}(\{T - tv : t \in \mathbb{R}\})$ for (T, T'), a periodic line pair, and $v \in \partial S(T, T')$, there are only finitely many such v, and there are only finitely many corner and isolated pairs, $K := \pi(\mathcal{PBP}(\Phi))$ is a finite union of line segments (some may be degenerate, if there are isolated pairs) in X_Φ. The description of $\mathcal{PBP}(\Phi)$ as $\varprojlim(f : K \to K)$ proceeds as in the non-collapsing case. □

LEMMA 5.5. *Suppose that α_1 and α_3 are arcs on $\mathbb{S}^1$ with the property that $\alpha_1 \cup \alpha_3$ contains a closed semi-circle in its interior. If there are distinct tilings $T_1, T_2, T_3 \in \Omega_\Phi$ so that T_i and T_2 are uniformly asymptotic in directions α_i, $i = 1, 3$, then there is $w \in \mathbb{R}^2$ so that $T_2 - w$ is Φ-periodic.*

Proof. Let $T_i^n = \Phi^{-n}(T_i)$, $i = 1, 2, 3$. By Lemmas 3.5 and 3.11, there is an R so that $B_0[T_i^n - tv] = B_0[T_2^n - tv]$, for all $t \geq R$ and $v \in \alpha_i$, $i = 1, 3$. It follows that there are only finitely many triples of patches $(B_0[T_1^n], B_0[T_2^n], B_0[T_3^n])$, $n \geq 0$, up to translation. We may then choose a subsequence and a $k \in \mathbb{N}$ so that the triples $(B_0[T_1^{n_j}], B_0[T_2^{n_j}], B_0[T_3^{n_j}])$ and $(B_0[T_1^{n_j+k}], B_0[T_2^{n_j+k}], B_0[T_3^{n_j+k}])$ are all translationally equivalent and (without loss of generality) all n_i are divisible by k. By passing to a further subsequence, we may assume that $T_i^{n_j} \to \bar{T}_i \in \Omega_\Phi$ as $j \to \infty$ for $i = 1, 2, 3$. Just as in the proof of Proposition 3.21 (the '$\alpha < 0$' case) we may conclude that $\bar{T}_2$ is fixed by Φ^k and $\bar{T}_2 = T_2 - w$ for some w. □

Proof of Theorem 4.11. Let $\Omega = \Omega_\Phi$ and $\Omega' = \Omega_\Psi$. If Ω has an isolated branch pair or a corner branch pair, then by Lemma 3.11 such a pair is uniformly asymptotic in a set of directions containing a closed hemisphere in its interior, and we're done by Theorem 3.32.

Let $h_0 : \Omega_\Phi \to \Omega_\Psi$ be a homeomorphism for which there is a linear isomorphism L of $\mathbb{R}^2$ so that $h_0(T - x) = h_0(T) - Lx$ for all $T \in \Omega_\Phi$ and all $x \in \mathbb{R}^2$ [**K**, Theorem 1.1]. Suppose that (T_1, T_2) and (T_2, T_3) are line branch pairs for Φ with $\partial S(T_1, T_2) \neq \partial S(T_2, T_3)$. Then there is $w \in \mathbb{R}^2$ so that $T_2 - w$ is Φ-periodic by Lemma 5.5 (in fact, it's not hard to show $w = 0$ in this case). By the 'linearity' and uniform continuity of h_0, there are arcs α_i in $\mathbb{S}^1$, slightly smaller than the semi-circles $\{Lv/|Lv| : v \in S(T_i, T_2)\}$, $i = 1, 3$, so that $\alpha_1 \cup \alpha_3$ contains a closed semi-circle in its interior and so that $h_0(T_i)$ and $h_0(T_2)$ are uniformly asymptotic in directions α_i, $i = 1, 3$. By Lemma 5.5, there is $w' \in \mathbb{R}^2$ so that $h(T_2) - w'$ is Ψ-periodic. Let $h_1 : \Omega_\Phi \to \Omega_\Psi$ by $h_1(T) := h_0(T + w) - w'$. Then h_1 is a homeomorphism that takes a Φ-periodic orbit to a Ψ-periodic orbit and,

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by [**K**, Corollary 1.5], there is a 'linear' homeomorphism $h : \Omega_\Phi \to \Omega_\Psi$ that conjugates some powers of Φ and Ψ. It follows easily that $h(\mathcal{BL}(\Phi)) = \mathcal{BL}(\Psi)$.

Now suppose that Ω_Φ is Pisot. We are done if Ω_Φ has an isolated branch pair or a corner branch pair. Otherwise, there are non-collapsing line branch pairs (T_1, T_1') and (T_2, T_2') with $\partial S(T_1, T_1') \neq \partial S(T_2, T_2')$ (Theorem 3.26). Let $\partial S(T_i, T_i') = \{\pm v_i\}$, $i = 1, 2$, and let $g : \Omega_\Phi \to \hat{\mathbb{T}}_A$ be the map onto the maximal equicontinuous factor of the $\mathbb{R}^2$-action on Ω_Φ. Note that $g(T_i) = g(T_i')$, $i = 1, 2$, since T_i and T_i' are proximal. Then

$$\bigcup_{t\in\mathbb{R}} W^s(g(T_i - tv_i)) = \bigcup_{t\in\mathbb{R}} W^s(g(T_i' - tv_i)), \quad i = 1, 2,$$

are co-dimension one affine subspaces of $\hat{\mathbb{T}}_A$ that aren't parallel (since $\{\pm v_1\} \neq \{\pm v_2\}$). Thus, there are $t_1, t_2 \in \mathbb{R}$ with

$$g(T_1 - t_1 v_1) = g(T_1' - t_1 v_1) = g(T_2 - t_2 v_2) = g(T_2' - t_2 v_2).$$

It is proved in [**BKe**] that, up to translation, there are only finitely many sets of the form $\{B_0[T'] : g(T') = g(T)\}$ for $T \in \Omega_\Phi$. Thus, there are $n, k \in \mathbb{N}$ and $w \in \mathbb{R}^2$ so that $(B_0[\Phi^{n+k}(T_1 - t_1 v_1)], B_0[\Phi^{n+k}(T_1' - t_1 v_1)], B_0[\Phi^{n+k}(T_2 - t_2 v_2)], B_0[\Phi^{n+k}(T_2' - t_2 v_2)]) = (B_0[\Phi^n(T_1 - t_1 v_1)] - w,\ B_0[\Phi^n(T_1' - t_1 v_1)] - w,\ B_0[\Phi^n(T_2 - t_2 v_2)] - w, B_0[\Phi^n(T_2' - t_2 v_2) - w])$. Equality of the first two coordinates of these quadruples implies that w is parallel with v_1, while equality of the last two coordinates implies that w is parallel with v_2. Thus $w = 0$ and the branch pairs $(\bar{T}_1, \bar{T}_1')$ and $(\bar{T}_2, \bar{T}_2')$ defined by

$$\bar{T}_i := \bigcup_{m\geq 0} \Phi^{mk}(B_0[\Phi^n(T_i - t_i v_i)]), \quad \bar{T}_i' := \bigcup_{m\geq 0} \Phi^{mk}(B_0[\Phi^n(T_i' - t_i v_i)])$$

are Φ-periodic and $\partial S(\bar{T}_i, \bar{T}_i') = \{\pm v_i\}$, $i = 1, 2$. Moreover, $g(\bar{T}_1) = g(\bar{T}_1') = g(\bar{T}_2) = g(\bar{T}_2')$ since g is continuous, $g \circ \hat{F}_A = \hat{F}_A \circ g$, and

$$\bar{T}_i = \lim_{m\to\infty} \Phi^{n+mk}(T_i - t_i v_i), \quad \bar{T}_i' = \lim_{m\to\infty} \Phi^{n+mk}(T_i' - t_i v_i), \quad i = 1, 2.$$

Note that $\bar{T}_i \neq \bar{T}_i'$, $i = 1, 2$, by the non-collapsing condition satisfied by (T_i, T_i'), $i = 1, 2$.

Now Ω_Ψ must also be Pisot. (For example, the linear nature of h_0 guarantees that if ϕ is an eigenfunction of the $\mathbb{R}^2$-action on Ω_Φ, then $\phi \circ h_0^{-1}$ is an eigenfunction of the $\mathbb{R}^2$-action on Ω_Ψ, and these $\mathbb{R}^2$-actions have non-trivial eigenfunctions if and only if the inflations are Pisot, [**So3**].) Also, the linearity of h_0 insures that h_0 preserves proximality. In particular, $g(S_1) = g(S_1') = g(S_2) = g(S_2')$, where g now denotes the map onto the maximal equicontinuous factor of the $\mathbb{R}^2$ action on Ω_Ψ and $S_i := h_0(\bar{T}_i)$, $S_i' := h_0(\bar{T}_i')$, $i = 1, 2$. Let $u_1 \in S(\bar{T}_1, \bar{T}_1')$ satisfy $\langle u_1, v_1\rangle = 0$. Then for each $\delta > 0$ there is $R = R(\delta)$ so that if $t > R$ then $d(\bar{T}_1 - sv_1 - tu_1, \bar{T}_1' - sv_1 - tu_1) < \delta$, for all $s \in \mathbb{R}$ (Lemma 3.11). Let $\epsilon > 0$ be small enough so that if $S, S' \in \Omega_\Psi$ and $w \in \mathbb{R}^2$ are such that $d(S - tw, S' - tw) < \epsilon$, for all $t > 0$ and $\lim_{t\to\infty} d(S - tw, S' - tw) = 0$, then $B_0[S - tw] = B_0[S' - tw]$, for all $t > 0$ (see the proof of Lemma 3.4). Now if $\delta > 0$ is chosen so that $d(T, T') < \delta \Rightarrow d(h_0(T), h_0(T')) < \epsilon$, then, for $t > R(\delta)$, $B_0[S_1 - sLv_1 - tLu_1] = B_0[S_1' - sLv_1 - tLu_1]$, for all $s \in \mathbb{R}$. Similarly for S_2, S_2'. Thus, there is $w \in \mathbb{R}^2$ so that $S_i - w$ and $S_i' - w$ agree on one of the components of $\mathbb{R}^2 \backslash \{tLv_i : t \in \mathbb{R}\}$, $i = 1, 2$. For $n \in \mathbb{N} \cup \{0\}$, let $S_{n,i} := \Psi^{-n}(S_i - w)$ and $S_{n,i}' := \Psi^{-n}(S_i - w)$, $i = 1, 2$. Using the fact that there are only finitely

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many quadruples $(S_{n,1}, S'_{n,1}, S_{n,2}, S'_{n,2})$, up to translation (for any n the entries in the quadruple have the same image under g), and using that Lv_1 is not parallel with Lv_2 and $S_1 \neq S'_1$, $S_2 \neq S'_2$, we see, exactly as in the proof of Proposition 3.21, that there is $w' \in \mathbb{R}^2$ so that $S_{0,i} - w'$ and $S'_{0,i} - w'$, $i = 1, 2$, are Ψ-periodic. Let $h_1 : \Omega_\Phi \to \Omega_\Psi$ be given by $h_1(T) := h_0(T) - (w + w')$. Then h_1 is a homeomorphism that takes a Φ-periodic orbit to a Ψ-periodic orbit. Corollary 1.5 of [**K**] supplies the desired $h : \Omega_\Phi \to \Omega_\Psi$, as above. □

6. *Problems*

Consider a one-dimensional substitution tiling space Ω_Φ with set of branch points $\{T_1, \ldots, T_n\}$. Given $T \in \Omega_\Phi$, let $a < b$ be such that $B_0[T - a] = B_0[T_i]$ and $B_0[T - b] = B_0[T_j]$ for some $i, j \in \{1, \ldots, n\}$, and so that $B_0[T - t] \notin \{B_0[T_k] : k = 1, \ldots, n\}$ for $a < t < b$. For such a, b we define a new tile $([a, b], [P])$, labeled by the translational equivalence class of the patch P consisting of all tiles in T whose support meets $[a, b]$. There are just finitely many translational equivalence classes of such new tiles, T is tiled by them, and Φ induces a substitution Φ_B on these new tiles (since the T_i are Φ-periodic). It is not hard to see that Ω_Φ and Ω_{Φ_B} are homeomorphic and, in fact, Φ and Φ_B are topologically conjugate. It is proved in [**BD**] that (for $d = 1$) Ω_Φ and Ω_Ψ are homeomorphic if and only if the substitutions Φ_B and Ψ_B, or Φ_B and the reverse substitution $\bar{\Psi}_B$, are weakly equivalent. In going from Φ to Φ_B, the individual tiles are endowed with the topological information of the branching structure. In a sense, Φ_B is a maximally uncollared version of Φ.

In higher dimensions, one would like to similarly replace the original tile set with new tiles whose boundaries are determined by the branch locus. Consider, for example, the chair substitution Φ_C (Example 4.7) in which the tiles are unit squares with a diagonal arrow. If T is a member of a branch pair in Ω_{Φ_C}, then the origin is either at the corner of a tile in T or is on the arrowed diagonal of a tile of T. If we cut the tiles along these diagonals and appropriately glue the resulting triangles along horizontal and vertical edges, we obtain a new tile set consisting of tilted $\sqrt{2} \times \sqrt{2}$ squares, labeled by the arrows along their edges, and a substitution Φ_{CB} on these tiles. Cutting and reassembling tiles provides a homeomorphism from Ω_{Φ_C} to $\Omega_{\Phi_{CB}}$ that conjugates the substitutions and $\mathbb{R}^2$-actions. The branch set of the (uncollared) Anderson–Putnam complex X_{CB} for Φ_{CB} is exactly the image of the branch locus of $\Omega_{\Phi_{CB}}$ under the natural projection. If Ω_Ψ is a tiling space homeomorphic with Ω_{Φ_C}, then there is a homeomorphism of these spaces that conjugates some powers of Φ_C and Ψ and takes branch locus to branch locus (Theorem 4.11). The branch locus of Ω_Ψ can be used to cut up the tiles, which, when reassembled into parallelograms, support a new substitution Ψ_B with Anderson–Putnam complex Y_B. The branch locus preserving homeomorphism of Φ_{CB} with Ψ_B should then induce a combinatorial relation (ideally, shift equivalence of some powers) between the substitution induced maps on X_{CB} and Y_B, as happens for one-dimensional tilings.

An immediate problem in trying to carry out the above more generally, is that there may be no branching along hyperspaces; in particular, the branch locus may consist entirely of isolated points, as in the half-hex (Example 4.6 of §4) and the Φ of Example 3.33. We know of no such examples that are not just higher dimensional solenoids, or co-dimension one foliated tori, with finitely many leaves split into asymptotic k-tuples (as in the way

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the half-hex is obtained from a two-dimensional solenoid by splitting a single leaf into 3 asymptotic leaves).

Problem 6.1. Suppose that Ω_Φ is a d-dimensional ($d \geq 2$), FLC, non-periodic, primitive, self-similar substitution tiling space for which $\mathcal{BL}(\Phi)$ consists entirely of isolated points. Set $T - x \sim T' - x$ if $T = T'$ or $(T, T') \in \mathcal{BP}(\Phi)$. Is $\Omega_\Phi / \sim$ an algebraic solenoid or torus?

Problem 6.2. Suppose that Ω_Φ is a two-dimensional, FLC, non-periodic, primitive, self-similar substitution tiling space for which there is a non-isolated branch pair. Can the branch lines be used to define a new tile set, and substitution Φ_B, as in the one-dimensional case? Is there a topological classification of Ω_Φ in terms of the combinatorics of Φ_B?

Problem 6.3. Suppose that Ω_Φ is a d-dimensional, FLC, non-periodic, primitive, self-similar substitution tiling space, the branch locus of which does not consist entirely of isolated points. Is there a substitution Ψ on a set of polyhedral tiles so that Φ and Ψ are topologically conjugate?

Problem 6.4. Suppose that Ω_Φ is a d-dimensional, FLC, non-periodic, primitive, self-similar substitution tiling space.

(a) Are there just finitely many branch pairs (T, T') with $S(T, T')$ containing a closed hemisphere? (In Proposition 3.19 the closed hemisphere is fixed beforehand.)
(b) Is $\mathcal{BL}(\Phi)$ compact? (See Lemma 3.31 and Proposition 4.5.)
(c) Are there periodic branch pairs $(T_1, T_1'), (T_2, T_2') \in \mathcal{PBP}(\Phi)$ and $u_1, u_2 \in \mathbb{S}^{d-1}$, u_1 not parallel with u_2, so that $H_{u_i} \subset S(T_i, T_i')$, $i = 1, 2$? (see Theorem 3.26.)
(d) Is $\mathcal{BL}(\Phi)$ a topological invariant? (See Theorems 3.32 and 4.11.)

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