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# Adding and Reversing Arcs in Semicomplete Digraphs

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Let $T$ be a semicomplete digraph on $n$ vertices. Let $a_k(T)$ denote the minimum number of arcs whose addition to $T$ results in a $k$-connected semicomplete digraph and let $r_k(T)$ denote the minimum number of arcs whose reversal in $T$ results in a $k$-connected semicomplete digraph. We prove that if $n \geqslant 3k-1$, then $a_k(T) = r_k(T)$. We also show that this bound on $n$ is best possible.

## 1. Introduction

A directed graph $D = (V, A)$ on at least $k+1$ vertices is *$k$-connected* if $D - X$ is strongly connected for every subset $X \subset V$ such that $|X| \leqslant k-1$. The connectivity $\kappa(D)$ is defined to be the largest $k$ for which $D$ is $k$-connected. It is a trivial observation that if a digraph without two-cycles on $n$ vertices is $k$-connected, then $n \geqslant 2k+1$.

Consider the following problems:

(1) Given a digraph $D = (V, A)$ on at least $k+1$ vertices for some natural number $k$, find a minimum set $F$ of new arcs such that the digraph $D' = (V, A \cup F)$ is $k$-connected. Let $a_k(D) = |F|$.

(2) Given a digraph $D = (V, A)$ on at least $2k+1$ vertices for some natural number $k$, find a minimum set $F \subset A$ of arcs in $D$ such that the digraph $D'$ obtained from $D$ by reversing every arc in $F$ is $k$-connected. Let $r_k(D) = |F|$.

In [3] problem (1) is solved completely in the sense that a min-max formula for $a_k(D)$ is obtained and it is shown that an optimal augmenting set can be found in polynomial time. Concerning problem (2) the situation is completely open for general digraphs. Even the question as to whether $r_k(D)$ is finite is open. (For some related conjectures see [4].)

We first point out a couple of easy observations we will use later. The first one is that if $F$ is a minimal augmenting set of arcs for problem (1), then no arcs in $F$ are parallel and no arc in $F$ is parallel to an arc in $A$. Secondly, for problem (2) we remark that no optimal reversing set of arcs contains an arc which is in a directed two-cycle in $D$.

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Note that if $F$ is a set of arcs in a digraph $D$ whose reversal makes $D$ $k$-connected, then the set $F'$ consisting of all arcs in $F$ reversed is an augmenting set of arcs whose addition to $D$ results in a $k$-connected digraph. Thus we always have that $a_k(D) \leqslant r_k(D)$.

It is not difficult to show that for each $k \geqslant 1$ there are digraphs for which each of the numbers $a_k(D)$ and $r_k(D)$ is finite and $a_k(D) < r_k(D)$. For example, one can take the class of digraphs $\{D_k\}_{k\geqslant 1}$. Here, for each $k \geqslant 1$, $D_k$ is obtained from a $k$-connected tournament $T_k$ on vertices $v_1, v_2, \ldots, v_n$, where $n \geqslant 2k+1$ by adding two new vertices $v_0$ and $v_{n+1}$ and the arcs $\{v_iv_0 : 1 \leqslant i \leqslant k-1\} \cup \{v_0v_j : k \leqslant j \leqslant n\} \cup \{v_{n+1}v_i : n-k+2 \leqslant i \leqslant n\} \cup \{v_jv_{n+1} : 1 \leqslant j \leqslant n-k+1\}$. It is not difficult to see that $D_k$ is $(k-1)$-connected and that $a_k(D) = 1$ and $r_k(D) = 2$. Observe that $D_k$ has only one pair of non-adjacent vertices, namely $v_0, v_{n+1}$.

In this paper we show the existence of a class of digraphs where $a_k(D) = r_k(D)$, provided that the number of vertices of $D$ is sufficiently high. That is, we show that if $D$ is any semicomplete digraph on at least $3k-1$ vertices for some $k \geqslant 2$, then $a_k(D) = r_k(D)$. (A directed graph $D = (V, A)$ is *semicomplete* if $uv \in A$ or $vu \in A$ (or both) holds for any pair of distinct vertices $u, v \in V$. A *tournament* is a semicomplete digraph without two-cycles.)

In Section 3 we show a family of digraphs proving that this lower bound for the number $n$ of vertices is best possible. Our proof easily implies a polynomial algorithm for finding an optimal set of arcs to be reversed. Note that the above equality for $k = 1$ (and $n \geqslant 3$) is trivial, using the fact that there exists a directed Hamiltonian path in any semicomplete digraph.

## 2. Definitions and preliminaries

Let $G = (V, A)$ be a directed graph. A vertex $u$ is an *in-neighbour* (*out-neighbour*) of $X \subset V$ if $u \in V - X$ and $ux$ ($xu$) is an arc for some $x \in X$. The set of in-neighbours and out-neighbours of $X \subset V$ is denoted by $N^-(X)$ and $N^+(X)$, respectively. Observe that $G$ is $k$-connected if and only if, for any subset $X \subset V$ with $|V - X| \geqslant k$, $|N^+(X)| \geqslant k$ and $|N^-(X)| \geqslant k$ hold. We will use several times the well-known – and easy to check – submodular property of the function $|N^+|$ (and $|N^-|$), which is the following:

$$|N^+(X)| + |N^+(Y)| \geqslant |N^+(X \cap Y)| + |N^+(X \cup Y)| \tag{2.1}$$

for any two subsets $X, Y \subset V$.

In a digraph $G = (V, A)$ with connectivity $\kappa(G)$ we call a subset $X \subset V$ with $|V - X| \geqslant \kappa(G) + 1$ *out-tight* or *in-tight*, if $|N^+(X)| = \kappa(G)$ or $|N^-(X)| = \kappa(G)$, respectively. It is easy to see that if $X$ is – say – out-tight in $G$, then $V - X - N^+(X)$ is in-tight.

After reversing an arc $e$ (or a set $E$ of arcs) of $G$, the new graph is denoted by $G \leftrightarrow e$ ($G \leftrightarrow E$). For an arc $e = uv$ (or a set $E$ of arcs) the oppositely directed arc $vu$ (or the set of the oppositely directed arcs of $E$) is denoted by $\overleftarrow{e}$ ($\overleftarrow{E}$).

An arc $e = xy$ is *critical* in a graph $G = (V, A)$ with connectivity $k$ if $G - e$ is not $k$-connected. This is equivalent to the property that there exist two non-empty sets $X, Y \subset V$ with $X \cap Y = \emptyset$, $|V - (X \cup Y)| = k - 1$ and the unique arc from $X$ to $Y$ in $G$ is $e$.

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The following two lemmas hold for any $k$-connected digraph $G$.

**Lemma 2.1.** *Let $G = (V, A)$ be a $k$-connected digraph and $e$ a critical arc in $G$. Then there exists a unique minimal out-tight set $K$ and a unique minimal in-tight set $B$ in $G - e$. There is no arc from $K$ to $B$ in $G - e$ and, in addition, $G - e \cup f$ is $k$-connected for any arc $f = uv$ with $u \in K$ and $v \in B$.*

**Proof.** Suppose that there exist two different minimal out-tight sets $K_1$ and $K_2$ in $G - e$. Since we can make this graph $k$-connected by adding the arc $e = xy$, $K_1$ and $K_2$ cannot be disjoint and $y \in (V - K_1 - N^+(K_1)) \cap (V - K_2 - N^+(K_2))$. This implies that in $G - e$ we have $N^-(y) \subseteq V - K_1 \cup K_2$. Hence $|V - (K_1 \cup K_2)| \geqslant k$ and $N^+(K_1 \cup K_2) \geqslant k - 1$, since $G - e$ is $(k-1)$-connected. Thus, using the submodularity of the function $|N^+|$ in $G - e$ and the $(k-1)$-connectivity of $G - e$,

$$k - 1 + k - 1 = |N^+(K_1)| + |N^+(K_2)| \geqslant |N^+(K_1 \cap K_2)| + |N^+(K_1 \cup K_2)| \geqslant k - 1 + k - 1.$$

This gives $|N^+(K_1 \cap K_2)| = k - 1$, contradicting the minimality of $K_1$. The uniqueness of $B$ follows similarly.

To see the second statement, observe that for any out-tight set $L$ and the unique minimal out-tight set $K$ we have $K \subseteq L$ and $B \subseteq (V - L - N^+(L))$. (In particular, $K \cap B = \emptyset$.) □

**Lemma 2.2.** *Let $G = (V, A)$ be a $k$-connected digraph and $e$ a critical arc in $G$. The minimal in-tight and out-tight sets in $G - e$ are denoted by $B$ and $K$, respectively. Let $f = uv$ be an arc with $u \in B$ and $v \in K$ for which $G - f$ is $k$-connected. Then $(G - e) \leftrightarrow f = (G - e - f) \cup \overleftarrow{f}$ is $k$-connected.*

**Proof.** By Lemma 2.1 we know that $B$ and $K$ are the unique minimal in-tight and out-tight sets in $G - e$. There exist a unique minimal out-tight set $K'$ and a unique minimal in-tight set $B'$ in $G - f - e$ as well, since $G - f$ is $k$-connected.

By Lemma 2.1 it is enough to prove that the unique minimal in-tight and out-tight sets are the same in $G - e$ and in $G - f - e$. It is obvious that $K$ is out-tight and $B$ is in-tight in $G - e - f$ as well, and hence we have $K' \subseteq K$ and $B' \subseteq B$.

If $K'$ is a proper subset of $K$, we get that $K'$ is not out-tight in $G - e$ but out-tight in $G - f - e$. This would imply $u \in K' \subseteq K$, contradicting the facts that $u \in B$ and $B \cap K = \emptyset$. □

## 3. The main theorem

Now we are ready to state and prove the main result.

**Theorem 3.1.** *Let $G = (V, A)$ be a semicomplete digraph with $|V| \geqslant 3k - 1$ for some $k \geqslant 2$. Then $a_k(G) = r_k(G)$.*

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**Proof.** As we mentioned, $a_k(G) \leqslant r_k(G)$ for an arbitrary digraph. Also, note that for any semicomplete digraph on at least $2k+1$ vertices $r_k(G)$ is finite, and in our case this condition holds.

To see the other direction, let $G = (V, A)$ be a semicomplete digraph with $n := |V| \geqslant 3k-1$. We shall prove that there exists a set $H$ of arcs which is an optimal augmenting set and for which $G \leftrightarrow \overleftarrow{H}$ is $k$-connected. (Note that not every optimal augmenting set will do; see the remarks at the end of Section 3.)

Let $F$ be an optimal augmenting set – with respect to $k$-connectivity – for which $r(F) := \max\{|J| : J \subseteq F; (G \leftrightarrow \overleftarrow{J}) \cup (F-J)$ is $k$-connected$\}$ is maximal. If $r(F) = |F|$, we are done. Let $H \subset F$ be a set of new arcs with $|H| = r(F)$ such that $(G \leftrightarrow \overleftarrow{H}) \cup (F-H)$ is $k$-connected. We can assume that there exists an arc $e = xy \in F - H$. Let $C := (G \leftrightarrow \overleftarrow{H}) \cup (F-H)$ and let $M := F - H$. By the optimality of $F$, every arc $e \in M$ is critical – with respect to $k$-connectivity – in $C$ and, by the maximality of $H$, $\overleftarrow{e}$ is also critical for every arc $e \in M$.

Let us fix an arc $e = xy \in M$. Now $C - e$ is $(k-1)$-connected but not $k$-connected. (Hence in $C - e$ a tight set has $k-1$ in- or out-neighbours.) The unique minimal in-tight and out-tight sets in $C - e$ are denoted by $B$ and $K$, respectively. Let $D := C - e$.

By Lemma 2.2 and the choice of $F$ we can assume that all the arcs from $B$ to $K$ in $C$ are critical. Since $G$ (and hence $C$) is semicomplete, $C$ contains every arc $g = vu$ with $v \in B$ and $u \in K$. Thus, since the arcs $bx, yl$ $(b \in B, l \in K)$ are critical, it follows from Lemma 2.1 that for any vertex $b \in B$ there exists a unique minimal out-tight set $K_b$ in $C - bx$ and, similarly, for any vertex $l \in K$ there exists a unique minimal in-tight set $B_l$ in $C - yl$.

It follows from the definition of $K_b$, and the fact that every vertex of $B$ is an in-neighbour of $x$, that for every $b \in B$ we have $K_b \cap B = \{b\}$. Similarly, for every vertex $l \in K$ we have $B_l \cap K = \{l\}$.

**Claim 3.2.** *If $|B| \geqslant k+1$, then $K_b \cap K_{b'} = \emptyset$ for any two distinct vertices $b, b' \in B$. If $|K| \geqslant k+1$, then $B_l \cap B_{l'} = \emptyset$ for any two distinct vertices $l, l' \in K$.*

**Proof.** By symmetry, it is enough to prove the first assertion. Suppose that $K_b$ and $K_{b'}$ have a non-empty intersection $X$. In $C - x$ we have

$$k-1+k-1 = |N^+(K_b)| + |N^+(K_{b'})| \geqslant |N^+(K_b \cup K_{b'})| + |N^+(X)| \geqslant k-1+k-1,$$

since $C - x$ is $(k-1)$-connected and $|V - (K_b \cup K_{b'})| \geqslant k-1$, because $|B| \geqslant k+1$. Thus $|N_C^+(X)| = |N_{C-x}^+(X)| = k-1$, contradicting the $k$-connectivity of $C$. □

**Claim 3.3.** *Let $K'_y$ ($B'_x$) be an out-tight (in-tight) set in $C - yx$. If $K \cap K'_y \neq \emptyset$, then $|V - (K \cup K'_y)| \leqslant k-1$ and, similarly, if $B \cap B'_x \neq \emptyset$, then $|V - (B \cup B'_x)| \leqslant k-1$. Furthermore, $K \cap K_y = \emptyset$ or $B \cap B_x = \emptyset$.*

**Proof.** Assume that $K \cap K'_y \neq \emptyset$ and $|V - (K \cup K'_y)| \geqslant k$. Thus, using the $k$-connectivity of $C$ and the fact that $y \in K \cup K'_y$, we get $|N_C^+(K \cup K'_y)| = |N_D^+(K \cup K'_y)| \geqslant k$. Furthermore,

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$x \notin K \cap K'_y$, thus $|N^+_C(K \cap K'_y)| = |N^+_D(K \cap K'_y)| \geqslant k$ must hold. Hence in $D$ we get

$$k - 1 + k = |N^+(K)| + |N^+(K'_y)| \geqslant |N^+(K \cap K'_y)| + |N^+(K \cup K'_y)| \geqslant k + k,$$

a contradiction. The proof for $B'_x$ is similar.

Suppose now that $K \cap K_y \neq \emptyset$ and $B \cap B_x \neq \emptyset$. (Here $K_y$ and $B_x$ are the unique minimal tight sets in $C - yx$). By the definition of $K, B, K_y, B_x$, there is no edge in $C$ with tail in $K \cup K_y$ and head in $B \cap B_x$, thus the $k$-connectivity of $C$ implies $|V - (K \cup K_y)| \geqslant k + 1$, which was just proved to be impossible. □

**Case I.** $|B| \leqslant k$ and $|K| \leqslant k$.

By Claim 3.3 we can assume – without loss of generality – that $K \cap K_y = \emptyset$. Let $B' = V - K_y - N^+_{C-yx}(K_y)$. Since $K_y$ is out-tight in $C - yx$ it follows that $B'$ is in-tight in $C - yx$ and that $|V - (K_y \cup B')| = k - 1$. Note also that since every vertex of $B$ has an arc to every vertex of $K$ and there is no arc from $K_y$ to $B'$ in $C - yx$, we must have $B' \cap K = \{x\}$ and $K_y \cap B = \{y\}$.

Since $C$ is semicomplete, and there is no arc from $K_y$ to $B'$ except $yx$, we get that every vertex $v$ in $B' - B$ is an in-neighbour of $B$ (by the existence of the arc $vy$); hence, since $B$ is in-tight, $|B' - B| \leqslant k$.

Let $W := (V - (K \cup B)) \cap K_y$. We claim that if $|B' - B| = k$ then $W = \emptyset$. Indeed, if $W \neq \emptyset$, then $W$ has at most $k - 1$ out-neighbours in $C$, because it has no out-neighbours in $B'$ and if there was an arc from $W$ to $K_y \cap B$, then $B$ would have at least $k$ in-neighbours in $D$ (each of the $k - 1$ vertices in $B' - B - x$ are in-neighbours of $B$ in $D$), contradicting the fact $|N^-_D(B)| = k - 1$. Thus, since $C$ is $k$-connected, we must have $W = \emptyset$ (which implies $K_y = \{y\}$).

First assume that $B \cap B' \neq \emptyset$. Now, counting each vertex in $C$, and applying Claim 3.3 to $B'$, we get

$$|V| = |B| + |B' - B| + |V - (B \cup B')| \leqslant k + k + k - 1 = 3k - 1.$$

We claim that at least one of the inequalities must be strict inequality, from which $|V| \leqslant 3k - 2$ will follow. Suppose, to the contrary, that equality holds above. This implies, in particular, that $|B' - B| = k$. Since this was shown to imply $W = \emptyset$, we can count $|V|$ in a different way, namely

$$|V| \leqslant |B' - B| + |B| + |V - (B' \cup K_y)| \leqslant k + k + k - 1,$$

and if $B \cap (V - (B' \cup K_y)) \neq \emptyset$, then strict inequality holds, since we counted at least one vertex twice. Thus we may assume that $B \cap (V - (B' \cup K_y)) = \emptyset$.

Now we see that in $C$ there is no arc from $V - (B' \cup K_y)$ to $B \cap B'$, since $B$ is in-tight and already has $k$ in-neighbours in $B' - B$. This implies that every in-neighbour of $B \cap B'$ must be in $B' - B - x$. Thus $B' - B - x$, a set of size at most $k - 1$, separates $x$ from $B \cap B'$ in $C$, a contradiction. Hence strict inequality must hold above and $|V| \leqslant 3k - 2$.

Now we can assume that $B \cap B' = \emptyset$ holds. As we observed, $|B'| = |B' - B| \leqslant k$, and by a similar argument we get that $|K_y| = |K_y - K| \leqslant k$. Thus we can count as follows:

$$|V| = |B'| + |K_y| + |V - (K_y \cup B')| \leqslant k + k + k - 1 \leqslant 3k - 1.$$

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However, we saw that $W \neq \emptyset$ implies $|B'| = |B' - B| \leqslant k-1$ and hence $|V| \leqslant 3k-2$. If $W = \emptyset$, then $K_y = \{y\}$ and we can count as follows:

$$|V| \leqslant |B'| + |K_y| + |V - (K_y \cup B')| \leqslant k + 1 + k - 1 = 2k \leqslant 3k - 2.$$

This contradiction proves that Case I cannot occur.

**Case II.** $|B| \geqslant k+1$ or $|K| \geqslant k+1$.
By symmetry we can assume that $|B| \geqslant k+1$ holds. Let $T := N_D^-(B)$. By the definition of $B$, $|T| = k-1$.

First, observe that for every $b \in B$ the minimality of $K_b$ implies that the subgraph induced by $K_b$ is strongly connected if $|K_b| \geqslant 2$.

Recall that by the remark just before Claim 3.2 we have $K_b \cap B = \{b\}$ for every $b \in B$. By Claim 3.2 the sets $K_b$ $(b \in B)$ are pairwise disjoint. Let $\mathscr{K} := \{K_b : b \in B\}$ be the family of these out-tight sets. Clearly $|\mathscr{K}| = |B|$. Let $\mathscr{M} := \{K' : K' \in \mathscr{K}, K' \cap (V - B - T) \neq \emptyset\}$. Let $\mathscr{N} := \mathscr{K} - \mathscr{M}$. Let $m := |\mathscr{M}|$. For any set $K' \in \mathscr{M}$ we have $K' \cap T \neq \emptyset$, otherwise the digraph induced by $K'$ is not strongly connected. Thus $m \leqslant k-1$. Let $r := |V-B-T|$, and $s := |B|$. Let $Z := \{v \in B : K_v \in \mathscr{N}\}$. We have $|Z| = s - m$, and since $C$ is semicomplete, $Z$ also induces a semicomplete digraph and

$$\max\{\delta_Z(z) : z \in Z\} \geqslant (s - m - 1)/2. \tag{3.1}$$

(Here $\delta_Z$ denotes the out-degree in the subgraph induced by $Z$.) Moreover, there is an arc from any $z \in Z$ to any vertex in $V - B - T$, and any out-neighbour of $z$ in $B$ or $V - T - B$ is clearly an out-neighbour of $K_z$ as well. This implies

$$(s - m - 1)/2 + r \leqslant k, \tag{3.2}$$

and hence

$$|V| = r + k - 1 + s \leqslant r + k - 1 + 2(k - r) + m + 1 = 3k - r + m. \tag{3.3}$$

The members of $\mathscr{M}$ are pairwise disjoint and none of them can contain $x$, thus

$$m \leqslant r - 1, \tag{3.4}$$

and

$$|V| \leqslant 3k - 1 \tag{3.5}$$

follows.

We claim that at least one of the inequalities (3.1)–(3.4) is strict, which will imply that (3.5) is also strict, *i.e.* $|V| \leqslant 3k-2$, a contradiction.

If equality holds in (3.5), then, in particular, equality holds in (3.1) and (3.2), which means that there is no two-cycle in $Z$ and the average out-degree in $Z$ equals the maximum out-degree. This gives that $Z$ induces a regular tournament (a digraph $D = (V', A')$ is *regular* if there exists a natural number $t$ such that the in-degree and the out-degree equal $t$ for every vertex $v \in V'$), and any vertex $z \in Z$ has precisely $k$ out-neighbours in $Z \cup (V - B - T)$. This implies that there is no arc from $Z$ to $B - Z$. Since $B$ was a minimal in-tight set in $D$, $B$ is strongly connected, hence $B - Z = \emptyset$, which gives $m = 0$; hence $r = 1$ and $s = 2k-1$, if (3.4) holds with equality. In particular, $B$ induces a $(k-1)$-regular

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tournament in $C$. We can also conclude that $K_b \cap T = \emptyset$ for all $b \in B$. For suppose $t \in T \cap K_b$ for some $b \in B$. Let $b' \in B - b$ be an in-neighbour of $b$ in the $(k-1)$-regular tournament induced by $B$. Now, either $b't$ is an arc, contradicting that $K_{b'}$ has precisely $k$ out-neighbours in $C$ (recall that $x$ is also an out-neighbour of $b$ and that $K_b \cap K_{b'} = \emptyset$), or $tb'$ is an arc, contradicting that $K_b$ has precisely $k$ out-neighbours in $C$.

Now we can conclude that no vertex $b \in B$ has an out-neighbour in $T$ and hence the only in-neighbour of $T$ in $C$ is $x$, a contradiction, since $C$ is at least 2-connected (clearly $T \neq \emptyset$, since $k \geqslant 2$).

This final contradiction in Case II proves that there exists an arc $f$ with tail in $B$ and head in $K$ which is not critical in $C$. Thus, deleting $e$ and $f$ and adding $\overleftarrow{f}$ would yield a $k$-connected digraph by Lemmas 2.1 and 2.2, which contradicts the choice of $F$. □

We remark that our proof is constructive in the sense that starting from an optimal augmentation – which can be obtained in polynomial time by [3] – we can easily get in polynomial time an optimal set to be reversed.

One may argue that perhaps if we restrict ourselves to only adding arcs between adjacent vertices, then we could have $a_k(D) = r_k(D)$ for any digraph $D$, provided both numbers are finite and the number of vertices in $D$ is large enough. This is not true, however, as can be seen from the following example.

Let $T_1$ and $T_2$ be disjoint 2-connected digraphs, let $u \in V(T_1)$, $v \in V(T_2)$ be fixed vertices and let $D$ be the digraph obtained from $T_1$ and $T_2$ by adding new vertices $x, y, z$ and the following arcs $\{r \to y | r \in V(T_1)\} \cup \{y \to s | s \in V(T_2)\} \cup \{s \to r | s \in V(T_2), r \in V(T_1)\} \cup \{r \to x | r \in V(T_1) - u\} \cup \{s \to z | s \in V(T_2) - v\} \cup \{x \to u, z \to v, z \to x\}$. It is not difficult to see that $a_2(D) = 1$ and that any arc which makes $D$ 2-connected has tail $x$ and head in $T_2 \cup z$. On the other hand, it is also easy to see that $r_2(D) \geqslant 2$. This example can be modified to work for any $k \geqslant 1$.

If we add edges to $D$ without increasing the number of out-neighbours of $x$ and of $z$, we can construct a semicomplete digraph $D'$ of any given size for which $xz$ is an optimal augmentation but reversing $xz$ does not make $D'$ 2-connected. This – and similar constructions for higher connectivity – show that we cannot reverse along an arbitrary optimal augmenting set for $k \geqslant 2$.

It is also important to note that the problem of determining the optimal number of arcs to be reversed to make an arbitrary digraph $k$-arc-connected was shown to be polynomially solvable in [1].

## 4. The example

In this section we show that for any $k \geqslant 3$ there do exist semicomplete digraphs $G$ for which $n = 3k - 2$ but $a_k(G) < r_k(G)$.

Let $T'$ be a $(k-1)$-connected $(k-1)$-regular tournament on $2k-1$ vertices $\{v_0, ..., v_{2k-2}\}$ for some $k \geqslant 3$. Let $N = \{v_1, ..., v_{k-1}\}$ denote the set of in-neighbours of $v_0$ in $T'$. Replace the vertex $v_0$ by a complete digraph $D$ on $k-1$ vertices $\{v_{2k-1}, ..., v_{3k-3}\}$, and then add $k-1$ independent arcs $\{v_i v_{2k-2+i}\}$, $i = 1, ..., k-1$, from $N$ to $D$, and add all the arcs of form $xy$, $x \in D, y \in T' - v_0$. Then add a new vertex $w = v_{3k-2}$ to this modified graph and

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the arcs $xw$, $x \in D \cup (T' - v_0)$, and $wy$, $y \in D$. Finally, add one more arc $e = wz$ where $z$ is an arbitrary vertex in $T' - v_0$. Let us denote the new graph of $3k-2$ vertices by $T$.

**Claim 4.1.** *$T$ is $k$-connected.*

**Proof.** It is easy to check that replacing $v_0$ in $T'$ by $D$ does not destroy the $(k-1)$-connectivity, hence $T - w$ is $(k-1)$-connected. The additon of a vertex $w$ with at least $k-1$ out- and in-neighbours also preserves $(k-1)$-connectivity. Thus $T$ is $(k-1)$-connected.

Suppose that $T$ is not $k$-connected. Then there exists a separating set $S$ with size $k-1$ in $T$. This yields that there exist two non-empty subsets $A, B \subset V$ such that $A \cup B = V - S$, $A \cap B = \emptyset$, and there is no arc from $A$ to $B$ in $T - S$. We have $w \notin S$, otherwise $S - w$ would be a separating set of size $k-2$ in $T - w$. Since there is an arc from any vertex of $T$ to $w$, $w \in A$ holds. This implies $N^+(w) \subseteq A \cup S$. Since there is an arc from any vertex of $D$ to any other vertex, $D \cap A = \emptyset$, thus $D = S$. Furthermore, $|D| = k-1$, and $|N^+(w)| = k$ implies that $z \in A$. However, $T - D - w$ is strongly connected (at least $(k-2)$-connected). But now $B$ is not reachable from $z$ in $T - D - w$, a contradiction. □

Since $w$ has out-degree $k$ in $T$, $\kappa(T - e) = k-1$, and $a_k(T - e) = 1$. However, it is easy to check that the semicomplete digraph $T - e$ cannot be made $k$-connected by reversing only one arc. We need to reverse some arc with head $w$, but every vertex $z$ of $T' - v_0$ has just $k$ out-neighbours in $T - e$, so reversing the arc $zw$ does not make the digraph $k$-connected.

Note that some arcs can be deleted from $T - e$ without changing the connectivity and the values $a_k$ and $r_k$. But not every two-cycle can be eliminated. This suggests that for tournaments with at least $2k+1$ vertices perhaps $a_k = r_k$ always holds.

Finally, we make the easy observation that if a semicomplete digraph $D$ has at least $2k+1$ vertices then $r_k(D)$ is finite and, in fact, bounded by a function in $k$. To see this it suffices to use the following two simple observations.

(a) If $D$ is a $k$-connected digraph and $D'$ is obtained from $D$ by adding a new vertex $x$ and arcs from $x$ to a set $X$ of $k$ distinct vertices of $D$ and arcs from a set $Y$ of $k$ distinct vertices of $D$ to $x$, then $D'$ is also $k$-connected.

(b) If $T$ is a semicomplete digraph on at least $4k-1$ vertices, then $T$ contains a vertex with in-degree and out-degree at least $k$.

By observation (a) and (b), for every semicomplete digraph $T$, $r_k(T) \leqslant r_k(T')$ for some subdigraph $T'$ of $T$ with $|V(T')| \leqslant 4k-2$. This provides a simple upper bound for $r_k(T)$ (and hence for $a_k(T)$) as a function of $k$. It follows from [2, Theorem 4.2] and the fact that $T$ is semicomplete that to increase the connectivity of a semicomplete digraph from $k$ to $k+1$ one needs to add at most $2k+1$ new arcs. This leads to a better bound on $a_k(T)$, and by Theorem 3.1 for $r_k(T)$. We leave the details to the interested reader.

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