Artzt parabolas of a triangle

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The following article is about beauty in mathematics. It shows how simplicity can occur in apparently complex constructions. It explores the remarkable properties of parabolas which are tangent to two sides of a triangle where the tangent points are two vertices. The other side of the triangle is a chord of the corresponding parabola. These are known as the Artzt parabolas of the triangle since they were first described by Artzt in 1884 [1]. They are often described as being parabolas 'inscribed in a triangle' as in Bullard's papers [2, 3] and Eddy and Fritsch [4] (see Note 1).

This article is also about how new tools for geometric drawing can enable the discovery of such properties in a way which is well within the grasp of students. Such explorations are a means to learn about geometry for students who are inspired to do so (see Note 2). The advent of dynamic geometry packages such as Geometer's Sketchpad, Cabri, Cinderella and GeoGebra allow easy drawing and manipulation of curves, such as conic sections, in a way mathematicians in the past could only have dreamed about. Results can be found without a mass of calculation. Moreover, the results are visual and so much more accessible. By moving key points in a drawing, observation shows whether or not a theorem is true by noticing how incidences remain (see Note 2). They also show that, in what is known as advanced euclidean geometry, while not being in the forefront of mathematical research, there are still many results to be found. Indeed, with the internet to communicate such geometrical ideas, this geometry is flourishing as strongly as its heyday at the end of the nineteenth century. This is evidenced by the *Advanced Plane Geometry* e-mail group [5] which can post ten or more results in a day; and by the electronic journal *Forum Geometricorum* [6]. The drawings shown here and discussion are based on the free program GeoGebra [7].

The curve-stitched parabola

The parabola shown in Figure 1 will be familiar to everyone as the basis of curve stitching by forming it as an envelope of lines which are tangent to the curve. It is the usual way it is taught and is a special case by having the starting lines at right angles and equal. In this case, lines AB and AC have been divided into eight equal parts and the points are joined so that B is joined to A and then the point closest to B is joined to the point on AC closest to A. The construction continues by joining points moving from B to A and A to C, so finally A is joined to C. This means that the sum of the distances measured from A on AB and AC is a constant. Both AB and AC are divided into an even number of divisions so that the line marked in bold is the centre tangent which is the tangent at the vertex. This joins the midpoint of AB to the midpoint of AC.

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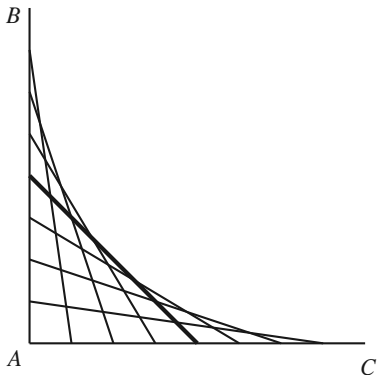


FIGURE 1

Very little is written about the geometry of this construction apart from saying that it is a parabola. A proof that the envelope is a parabola is given in Dorrie [8] as problem 51 using a theorem of Apollonius; a shorter proof using calculus is given as an example in the section on envelopes in [9, p. 76]. The construction applies to any pair of lines, not necessarily the two at right angles in Figure 1. It is just necessary that they be divided into the same number of equal parts. This particular case has special properties due to its symmetry.

One definition of a parabola is that it is the locus of a point which moves so that its distance from a given point (the focus) is always equal to its distance from a given straight line (the directrix). So the first obvious question to ask is "Where is the focus of the parabola?". The focus can be found using Lambert's theorem [10]; if you have a triangle formed by three tangents to a parabola, the circumcircle of the triangle goes through the focus. There are many such triangles in the tangent construction, so drawing the circumcircles such as the ones in Figure 2 intersect at the focus F of the parabola defined as the envelope of the lines. In this special case, the line joining *B* and *C* is a focal chord. The tangents at the end of a focal chord intersect at right angles on the directrix. In this special case, line *BC* (not shown) is a focal chord so it is possible to draw the directrix as a line through *A* which is parallel to *BC*. For a general non-symmetrical case the construction described below for Figure 3 needs to be used.

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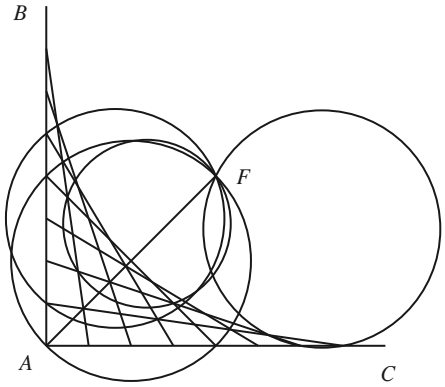


FIGURE 2

In Figure 2, the circle through A is the circumcircle of the triangle formed by the central tangent and AB and AC. Joining A to F gives the axis of the parabola and intersects the central tangent at the vertex at the centre of the line joining the midpoints of AB and AC.

Moving to the general case, Figure 3 shows an asymmetrical case with AB and AC each divided into eight equal parts. The line DE is parallel to the chord BC because the points D and E are midpoints of the sides. The point N is the midpoint of BC and line AN is a median of the triangle intersecting DE at G which is the tangent point. Note that this time the focus no longer lies on the median AN. The dashed line JK is the directrix which has been constructed using a theorem of Steiner whereby the orthocentres of triangles formed by three tangents to a parabola lie on the directrix (see Note 3). It is accidental that JK appears to intersect AB where the third tangent does.

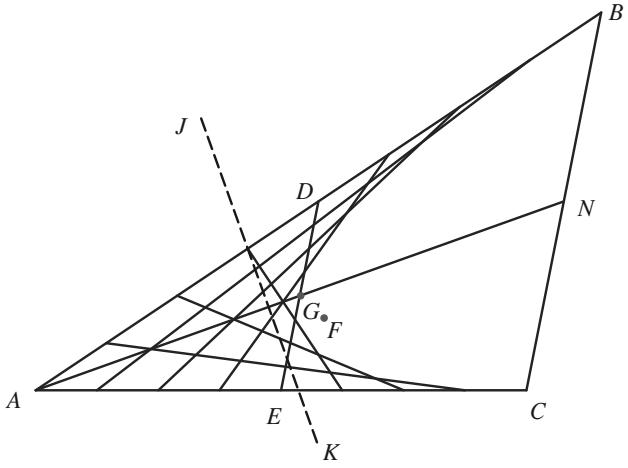


FIGURE 3

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In what follows, this is the basis of finding more points on the curve when we come to draw it. So far we know a point on the central tangent (its midpoint) and points B and C. In order to construct the curve we need five points. We can then use the tool in GeoGebra to construct a conic through the five points.

The construction of the parabola as a curve rather than an envelope

Figure 4 shows how to construct the five points to determine the parabola tangent to two sides which has the tangent points at the ends of the third side. The triangle is ABC. First construct points D and E as the midpoints of AB and AC. The point we require is G, the midpoint of DE. Consider the two tangents DB and DG as two sides of another triangle. Construct Q as the midpoint of DB and P as the midpoint of DG and then T as the midpoint of PQ. Then T is another point and similarly W is the fifth one on the other side. So with the five points B, T, G, W and C the parabola is defined and can be drawn using the five point conic option in GeoGebra. There is no need to draw the segments as in Figure 4. GeoGebra has a midpoint tool and so you just click on the appropriate pair of points to get the midpoint until you have enough points to define the parabola.

There are three such parabolas, tangent to each pair of vertices. Before looking at the remarkable properties of this configuration, we will determine the area of this segment of the parabola in a triangle and for this we need to use a result of Archimedes.

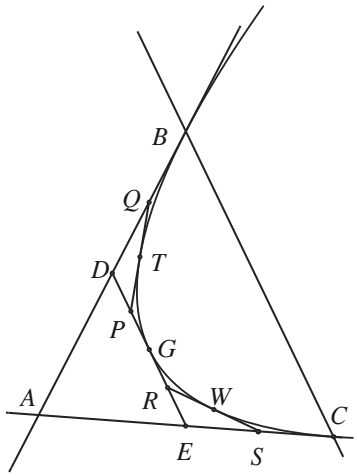


FIGURE 4

Archimedes' quadrature of the parabola

The famous Greek mathematician [11] determined the area of a segment of a parabola as a ratio of the area of the triangle as shown in Figure 5. He drew line BC to form the segment of the parabola and then drew a tangent

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parallel to *BC* at *G*. This defines triangle *BCG*. He then proved that the area of the segment of the parabola is 4/3 of the area of this triangle.

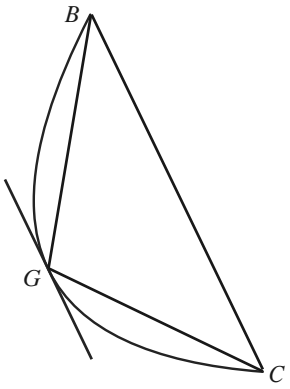


FIGURE 5

Figure 6 is now a combination of Figures 4 and 5. Because *D* and *E* are midpoints of *AB* and *AC*, *GN* is half *AN*. The area of a triangle is half the base times the perpendicular height and the heights of *ABC* and *GBC* are proportional to *GN* and *AN*. So the area of triangle *BGC* is half the area of *ABC*. From Archimedes' result that the area of the parabola is 4/3 of the area of triangle *BGC*, it follows that the area of the parabola is 2/3 that of the triangle *ABC*. The segment of a parabola tangent at two vertices in triangle *ANC* as in Figure 6 is often called an *Archimedean triangle*.

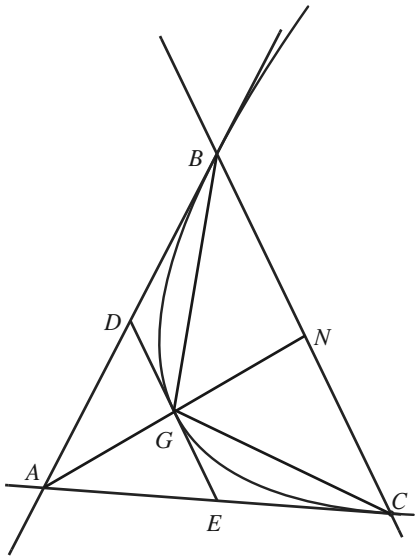


FIGURE 6

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The Artzt parabolas

The paper [1] by the mathematician August Artzt was the first to discuss the three parabolas tangent to two sides of a triangle at the vertices of the third side. I have been unable to obtain to find his paper of 1884 [1]; Artzt's title does not suggest the remarkable results that I describe here. French geometers were very active at the end of the nineteenth century, so it is possible to determine some of the contents of the paper in their references. The *Journal de Mathématiques Spéciales* was published by Gaston de Longchamps for candidates of the Écoles Polytechnique from 1881 and contains much of the foundations of advanced Euclidean geometry. In 1891 Emile Varigié [12] wrote a survey of the geometry of the triangle at 1890 of which item 9 (p. 59) is 'Les Paraboles de Artzt'. He says they form two groups which are confocal in pairs and they have their foci at the vertices of the second triangle of Brocard (see below). The first group are what are usually known as the Artzt parabolas although they should be termed the Artzt parabolas of the first type. They are tangent to two sides of the triangle at the extremities of the third. See below for more on the second type with an example in Figure 19. This article only deals in detail with the Artzt parabolas of the first type.

Brocard is, of course, famous for his discovery of the Brocard points of a triangle. He also wrote extensively on both types of Artzt parabolas. His paper on the nine point hyperbola and the six Artzt parabolas [13] is more complicated than his paper originally in the *Mémoires de L'Académie de Montpellier* which is just devoted to the Artzt parabolas [14].

Varigié summarises a paper by Gaston de Longchamps, [15] in which he gives the proof of areas described below. He also describes an ellipse which is discussed in the section on ellipses later in this article. The title of Artzt's paper mentions the Brocard circle so one might expect Brocard would explore Artzt's work. His paper [14] is not as simple as the one by de Longchamps and only briefly mentions Artzt and deals more with the Artzt parabolas of the second type. Bullard wrote two papers in 1935 and 1937 [2, 3] (without mentioning Artzt) which are the best source for the many remarkable properties of these parabolas. Further properties were also given by Thébault in 1957 [16] but other references are hard to find.

Figure 7 shows the configuration for the three parabolas of the first type, but to find more properties we need to draw the medians as shown in Figure 8.

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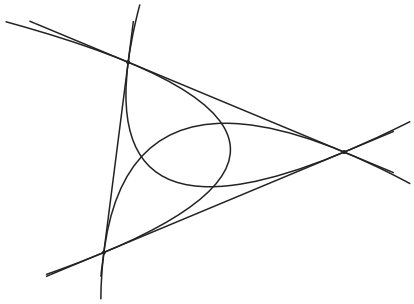


FIGURE 7

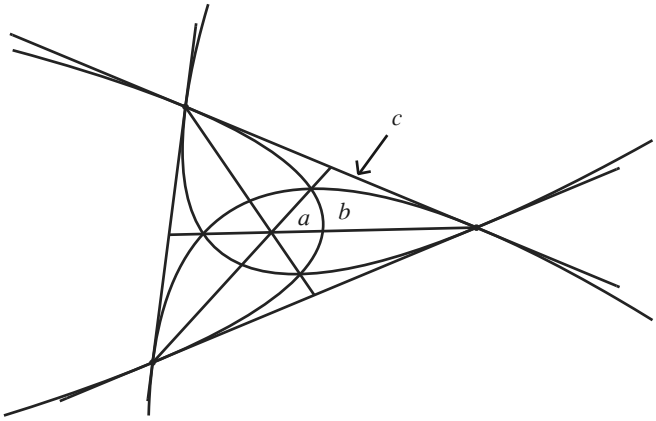


FIGURE 8

The first property is concerned with areas. As shown in Figure 8, there are three types of region. There are six Type *a* around the centroid where the medians intersect. There are another six Type *b* on the sides of the triangle with another six Type *c* either side of the medians. What is remarkable that all regions of the same type have the same area no matter what shape triangle you start from. The twelve *a* and *c* all have the same area which is 5/162 of the area of the triangle. Type *b* all have an area of 17/162 of the area of the triangle. So the curved area made up of the six *a* divisions around the intersection of the medians has an area of 5/27 of the area of the triangle. GeoGebra cannot measure such curved areas; Bullard [1] and Honsberger [17] have the calculation. The paper by de Longchamps [15] combines areas; for example the six areas marked *a* form a curved triangle with area 5/27 of the triangle. Honsberger [17] does not mention Artzt either since he gets his ideas from Bullard [2].

As well as interesting properties of area, the medians are also divided in simple ways. It is well known that the point where the medians intersect (the centroid) is 1/3 the way along the median from the triangle side. Following the segments from the vertex of the triangle, the parabola bisects the median

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as described above in the construction. Since the centroid is 2/3 from the vertex, the next segment is 1/6 of the median length. The next segment is 2/9 the median length so the final one close to the side of the triangle is 1/9. To measure them in GeoGebra, first draw a line as a segment for the median and then for the segment of the median required. The Algebra view in GeoGebra gives the absolute length against the name of the segment (usually just a letter). In the input box at the base of the screen type a name for the ratio followed by an equals sign and the division of the median length named by the segment length name. The result appears in the number section of the Algebra view. It is more convenient to do this than using the Measure tool. Also, by dividing the segment into the median it is easier to see a rational division.

There are other constant ratios of areas that are not in the literature which are easily found with GeoGebra. Figure 9 shows two triangles drawn with vertices at intersections. They were drawn using GeoGebra's polygon tool. Like the segments, their measurements (the area this time) show up in the Algebra view with names *poly1*, *poly2*, etc, as you draw them. Draw the complete triangle as the first polygon. Find the ratios of area to the whole triangle using the same technique as with the segments. You then find that the darker grey one has an area which is 1/9 and the lighter grey one 1/16.

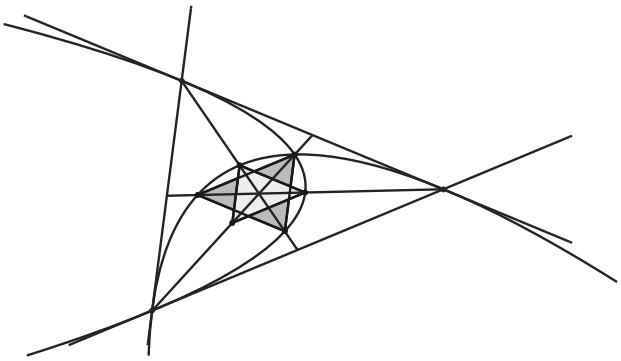


FIGURE 9

This process can be repeated with triangles on the intersections of the medians with the previous triangle as in Figure 10. The areas for the next two in this figure are 1/36 and 1/64. It is curious that the denominators are squares of 3, 4, 6 and 8. Continuing the construction and calculating why this should be would make a nice project.

Another remarkable situation arises if the tangents to the parabolas are drawn where they cross the medians closest to the sides of the triangle. These are shown as dashed lines in Figure 11. They are parallel in pairs to the medians. They divide the sides of the triangle into three equal parts. They also form a hexagon which has an area 2/9 the area of the triangle.

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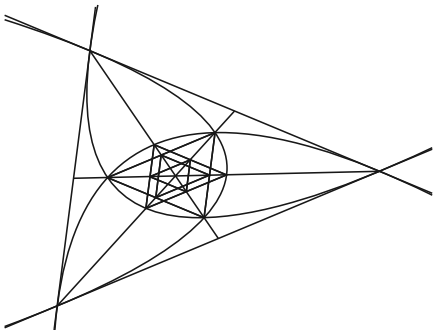


FIGURE 10

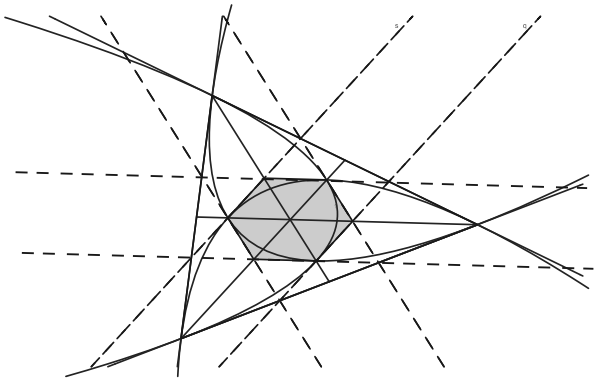


FIGURE 11

The property of these tangents trisecting the sides of the triangle gives another way to construct each parabola as a five point conic. Two points are vertices of the triangle. We know that one point is the intersection of a median and the medial line. The remaining points are the intersection of the other two medians and the tangent found by joining points which are at the trisection points of the sides. Trisection is easy using the standard Euclidean construction.

Figure 12 shows this construction. The medians are AX, BY and CZ. The vertices of the triangle B and C are the tangent points and M is the third point from the construction above as midpoint of ZY. Point T is at $\frac{1}{3}AB$ from B. Point S is at $\frac{1}{3}AC$ from A. Point R is at $\frac{1}{3}BC$ from C. Point U is found as the intersection of ST and median CZ. Point N is found at the intersection of SR and median BY. The five point conic then passes through B, C, U, M and N.

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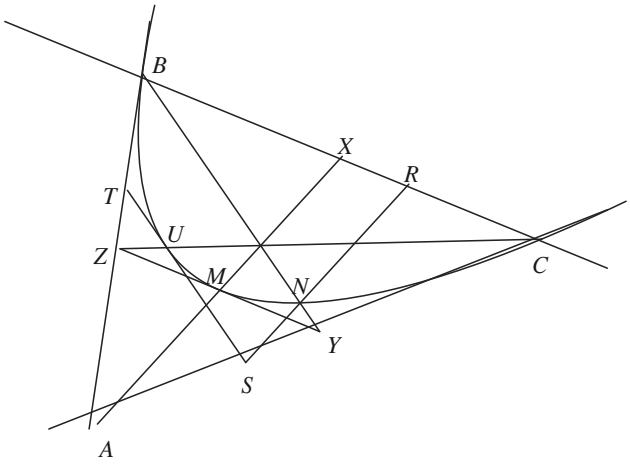


FIGURE 12

Circles

If circles are drawn defined by the three points where the parabolas meet at the medians (Figure 13), then the ratio of their radii is interesting. There are three circles: the circumcircle (C_1), one defined by the intersections of two parabolas (C_2) and one defined by the intersections of the parabolas with a median (C_3). The ratio of $C_1 : C_3$ is 4 : 1, that of $C_2 : C_3$ is 4 : 3 and that of $C_1 : C_2$ is 3 : 1. Also, these three centres lie on a line which goes through the centroid. This line, which also goes through the orthocentre and nine-point centre, is the Euler line of the triangle. The centre of the smaller of the three circles (Q) and the one through where the parabola intersect the medians once is the midpoint of the nine-point centre and the circumcentre.

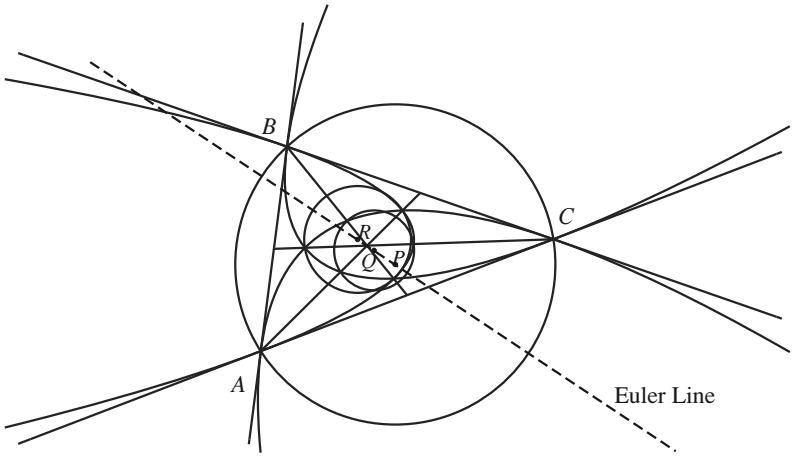


FIGURE 13

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The nine-point circle also has another part to play [2]; the directrix and the median from the non-tangent vertex intersect on the nine-point circle as does the altitude from this vertex. The diagram is hard to draw clearly since the intersections are close to one another. The result is much more obvious if you use GeoGebra and move the vertices of the triangle.

Another curious appearance of the nine-point circle comes about by applying a Miquel circles construction with the focus of a parabola as the Miquel point. See Note 4 for background on Miquel's theorems. In Figure 14, F, D and E are the midpoints of the sides of the triangle. The bold circle is thus the nine-point circle. Point M is the focus of the parabola tangent at A and C. Draw the circle through B, D and F which, by Lambert's law, will go through the focus M because BC, BA and FD are tangents to the parabola. Construct circles through C, D and M and A, F and M. By Miquel's corollary to his first theorem, these latter two circles will intersect at a point on AC, Q. Using GeoGebra, this is easily seen to be the second point where the nine-point circle (shown in bold) intersects AC, that is the foot of the altitude from B.

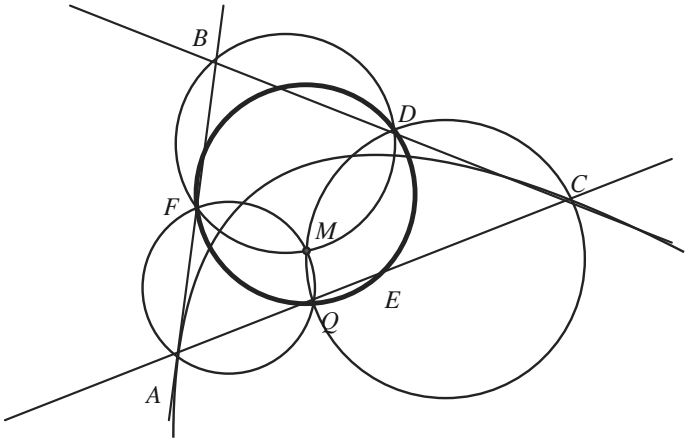


FIGURE 14

Ellipses

The French mathematician de Longchamps, apart from determining the areas in his paper [15] on the Artzt parabolas as mentioned above, says that Artzt constructed the ellipse shown in Figure 15. He constructed three extra points by reflecting in the centroid the points on the medians where two parabolas intersect. Constructing a five point conic shows that it goes through all six points to give the ellipse. It is easy to see that this construction applies to any three lines through a point with three points on each line.

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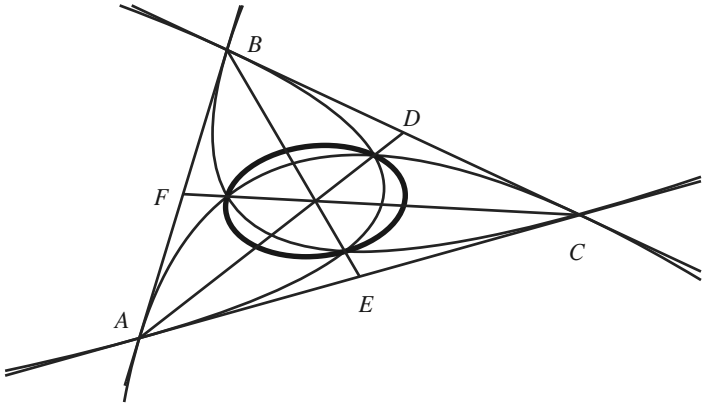


FIGURE 15

In his paper, de Longchamps says that the ellipse is remarkable, but does not provide any other property than the area. He calculates this and obtains the relation

$$\alpha\beta = \frac{8\Delta}{27\sqrt{3}},$$

where $\alpha\beta$ is the product of the semi-axes of the ellipse and Δ is the area of the triangle. Multiplying both sides of the equation by π makes the left side the area of the ellipse. Calculating the ratio of the triangle and ellipse areas gives the result that the triangle is 1.860735 times the area of the ellipse. However, when this ratio is measured using GeoGebra the result is 3.721470041. So de Longchamps is out by a factor of two.

The de Longchamps ellipse in fact belongs with a pair of ellipses known as the Steiner circumscribed ellipse and inellipse. The inscribed Steiner ellipse is tangent to the triangle at the midpoints of the sides. The circumscribed ellipse has the least area of any ellipse circumscribed about the triangle. They have the same eccentricity as the de Longchamps ellipse and have the centroid of the triangle as centre. They can be constructed by reflecting three points in the centroid to give six points of the ellipse, as in the de Longchamps method. In the case of the circumscribed ellipse, the points are the vertices of the triangle and for the inscribed ellipse, the midpoints of the sides. The de Longchamps ellipse is one third and the inscribed ellipse one half of the circumscribed ellipse.

GeoGebra allows exploration, for example to ask the question as to whether the points where two parabolae intersect plus the three points where they intersect singly on a median also lie on an ellipse. In making the construction only five points are required, so the choice of the five from six is important. It is natural to choose the three double intersection points first and then two of the others. The resultant ellipse does not go through the third single intersection point but in another remarkable result, the ellipse goes through a vertex of the triangle. The case in Figure 16 shows the single

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points on medians AD and FC with the vertex being B. This is at the end of the median BE which is where the parabola tangent to A and C intersects BE and is not part of the ellipse.

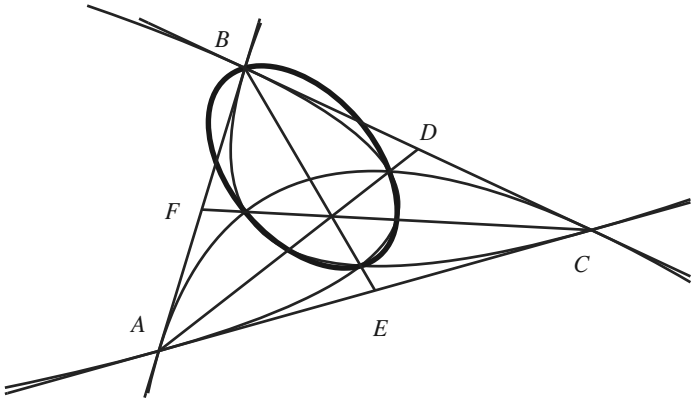


FIGURE 16

There are three such ellipses as shown in Figure 15. There is a tool in GeoGebra which allows measurement of the areas of an ellipse by clicking on it. What surprised me when I did this was that, despite different shaped ellipses, they all have the same area! The area is not the same as the Artzt ellipse of Figure 13. By moving a vertex and calculating the ratio of the ellipses of Figure 17 to the Artzt ellipse, it is seen to be constant at 1.7888534. The triangle is 2.0803649987 times the area of these three ellipses. I have not been able to simplify these values to an expression like the one for Artzt's ellipse.

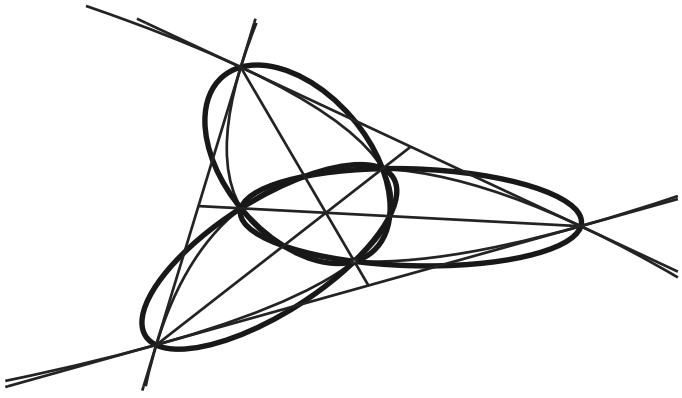


FIGURE 17

Another instance of ellipses is the result of isogonal transformation of the parabolas. This produces three ellipses which are tangent to two sides of the triangle at the vertices of the third side as shown in Figure 18. Parabolas

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do not usually give a conic as the isogonal conjugate curve; the parabolae of the second type, which do not go through the vertices, give a cubic. This begs the question (which the reader might like to follow up) as to whether this equivalent to the Artzt parabolas for ellipses has interesting properties.

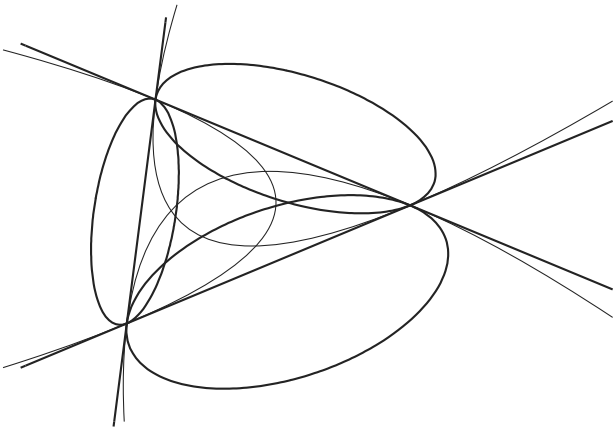


FIGURE 18

Foci and the second type parabolas

The papers by Bullard [2, 3] and Thébault [16] together with the French nineteenth century ones contain other properties of which the following are worth pointing out. The second type of parabolas seems to be mentioned sparsely in the literature. Emile Varigié [12] says the second type contains as tangents the bisectors of the angles and the corresponding medians. This is incorrect as construction does not give a parabola confocal with the type one parabola. Figure 19 shows parabolas, one of each type, which have the same focus. The median from B is the directrix of the type two parabola. Brocard [13] gives the correct details as construction verifies; the bisectors of the angle are two tangents, but the other two are the perpendiculars at the midpoints of the sides adjacent to the angle. Point M is the common focus.

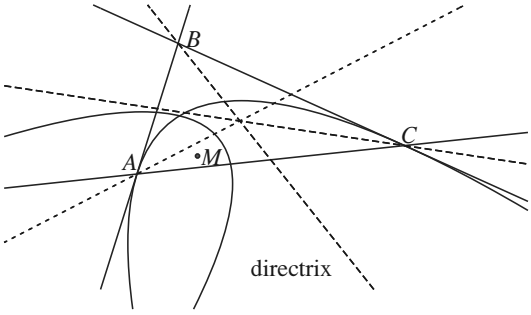


FIGURE 19

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The title of Artzt's paper mentions the Brocard circle. This is the circle whose diameter is the join of the symmedian point (K) and the circumcentre of the triangle. The three foci not only sit on this circle but they do so where the symmedian lines intersect this circle as shown in Figure 20. These points are the vertices of the second Brocard triangle. Bullard [2] also says that a perpendicular through a vertex of Brocard's second triangle (a focus), drawn to the corresponding third side of the original triangle, passes through the intersection of the nine-point circle and the corresponding median.

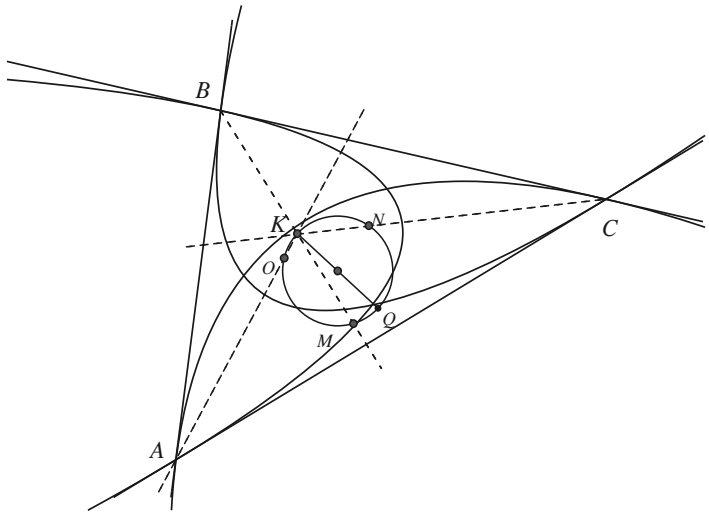


FIGURE 20

Another result from Bullard [2] is that the circle through a focus of a parabola and the two vertices at which the parabola is tangent goes through the circumcentre.

Taking it further

The above results are only part of the story. Many more properties are given in the references. The use of GeoGebra also poses a question about proof and whether a result demonstrated by constructions is as valid educationally as conventional proofs; see Note 2. I doubt that the new results such as the ellipses would have been discovered without the GeoGebra method. There are many instances in the above which could provide problems for solution as conventional proofs. So to finish, Figure 20 shows another result which would seem to be an obvious case for such a problem.

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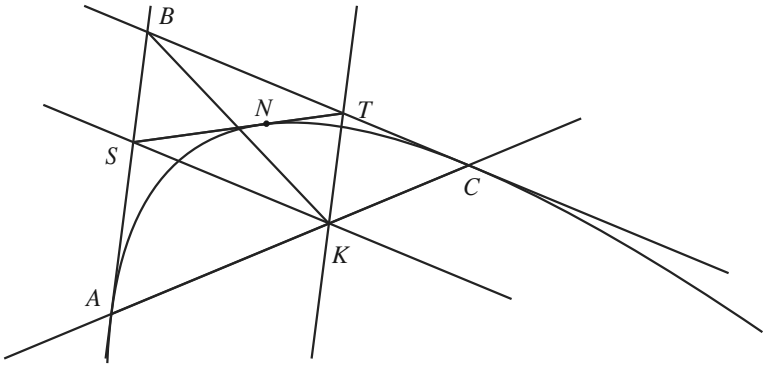


FIGURE 21

In GeoGebra, construct the Artzt parabola. Draw a point K anywhere on side AC of the Archimedean triangle. Draw KT parallel to side AB and KS parallel to side BC. Join S and T (with a segment line here). This shows ST is a tangent to the parabola. Move K along AC and see the tangent move along the curve. Next draw segments SN and NT on the tangent, and AK and KC on side AC. Using the techniques of calculating their ratios described above, the ratios SN/NT equals AK/KC. Now find the ratios SA/SB and BT/TC and you will find this is also equal to AK/KC. The line BK is not the median unless K is midway. This is a Cevian line. It does not intersect the tangent ST at N unless K is midway between A and C. However, if the intersection point is V, KV/VB also equals AK/KC. A conventional proof of these properties I leave to the reader.

As a means of finding results quickly, with additional benefits of giving an accurate visual feedback, the GeoGebra method is unsurpassed. I can only stress that it ought to be employed more as a research tool. I would not have found many of the results shown here by using conventional means. The educational benefits of GeoGebra should also be obvious since the ability of pupils does not have to be as great as requiring logical deduction nor the level of background knowledge. But above all, the beauty of mathematics shines out. Learning by exploring and doing is much better than following a teacher.

Notes

1. Conics inscribed in a triangle are those which touch all three sides of the triangle, for example the incircle, so that the triangle being referred to 'surrounds' the conic and is the region of the triangle in projective geometry which corresponds to Euclid's triangle. In the case of the triangle there is only one such circle. If the conic touches the three sides externally then the term is escribed; in this case the segments of the lines in projective geometry are one finite and two infinite. The literature abounds with the term 'inscribed parabola'. This is particularly true of *The Mathematical Gazette*;

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for example see [18]. It is only possible for a parabola to be inscribed if a triangle has one side which is the line at infinity (the ideal line) since all parabolae have this line as tangent. Moreover, for two lines plus the line at infinity there are many inscribed parabolas.

Notable instances of references to the inscribed parabola are Cremona [19] and Hatton [20], both of whom were distinguished mathematicians. Cremona has the index reference 'Parabola, inscribed in a triangle, its directrix passes through the orthocentre'. The reference (p. 273) uses the same phrase, which arises because it follows from the discussion of a quadrilateral circumscribing a conic in general. One of the lines is taken to infinity and the remaining three from a triangle whose orthocentre is on the directrix. So he is really then getting an escribed parabola. Hatton [20, p. 222] uses the same wording as Cremona. Whereas Cremona and Hatton are nineteenth century books, Lawrence's paper [18] in *The Mathematical Gazette* is 1930 and again has as its starting point two general conics.

2. The use of computer packages as a way of exploring mathematics needs to be encouraged more in education. Borwein [21] has written on the subject of 'Experimental Mathematics' and created a journal devoted to it.

It presents a problem to older teachers who have not experienced geometry other than by logical proof with Euclid and beyond. There will also be many who will only accept conventional proof and one can only agree with them in absolute terms. In the past there have been 'proofs' of geometrical fallacies such as proving two unequal lines are equal or there are two perpendiculars to a point from a line. Such proofs fall apart when drawn correctly in dynamic geometry packages with the ability to adjust the entities and see that an incorrect 'back of the envelope' drawing does not hold water. Observation of a drawing is not enough. It is by moving key entities and noticing how incidences remain will avoid the fallacies produced by accidental occurrences. Those who use such software know when they have discovered a result and may be sufficiently motivated to go on to produce a formal conventional proof. They may wish to leave it to others to take the result to that stage as happens with some of the results here. It raises the question as to whether the beauty is the result or whether it is in the proof or if both are aspects of the beauty of the discovery. I realise that older teachers who have not experienced geometry other than by logical proof with Euclid and beyond may be unhappy with the approach of finding results using the computer in this way but, as I have shown, it is unlikely that many of the results would not have been found using conventional methods.

3. The theorem that the orthocentres of triangles formed by three tangents to a parabola lie on the directrix dates back to the 1820s in two papers by Jacob Steiner. It is theorem 29 in [22] and on page 59 in [23]. The proof relies on Brianchon's theorem and is given in [19].

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4. August Miquel wrote a paper in 1838 [24] which includes a number of theorems on circles associated with triangles. One result was described in a recent paper [25]. What the author calls Miquel's theorem is in fact a corollary to his first theorem and is often named as Miquel's theorem. His first theorem is about three circles with a common point which Michael de Villers extends to four circles. Miquel's name is also associated with another theorem on parabolas. In his 1838 paper he gave a second theorem that if you have four lines and you construct the circumcircles of the four triangles this creates, they all intersect in one point. He published a further theorem in a later paper [26] in 1845 where he took five lines and constructed this point for four lines taken from this set of five. The five points lie on a circle. Now because four lines define a parabola, the intersection points are the foci of the five parabolas. William Kingdom Clifford [27] realised that these were the parabolas foci and published a paper which said that the foci of the five parabolae from five lines taken four at a time lie on a circle. This has become known as the Miquel-Clifford theorem.

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My thanks to the anonymous referee for obvious corrections, but also for extra ideas. These illustrate that there are yet more properties to be discovered and how fascinating and a remarkable subject the Artzt parabolas are especially if explored with dynamic geometry.

References

The subject is buried in the literature and more information can be obtained by searching the internet for ‘Artzt parabola’ and ‘parabola inscribed in a triangle’.

Klaoudatosa's recent paper [28] was found that way. Some of these references may require a university subscription to JSTOR. Also the Access Research internet reference available in public libraries is another resource. The *Journal de Mathématiques Spéciales* is available on the internet archive and is most easily accessed from http://onlinebooks.upenn/webbin/serial?id=jmathelemfr.

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