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Modulated and subsequential ergodic theorems in Hilbert and Banach spaces

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Abstract. Let $\{a_k\}_{k\geq 0}$ be a sequence of complex numbers. We obtain the necessary and sufficient conditions for the convergence of $n^{-1}\sum_{k=0}^{n} a_k T^k x$ for every contraction T on a Hilbert space H and every $x \in H$. It is shown that a natural strengthening of the conditions does not yield convergence for all weakly almost periodic operators in Banach spaces, and the relations between the conditions are exhibited. For a strictly increasing sequence of positive integers $\{k_j\}$, we study the problem of when $n^{-1}\sum_{j=1}^{n} T^{k_j} x$ converges to a T-fixed point for every weakly almost periodic T or for every contraction in a Hilbert space and not for every weakly almost periodic operator.

1. *Introduction*

Generalizing the result obtained by Blum and Hanson [**7**] for the isometry T induced in L_2 by a probability-preserving transformation, Akcoglu and Sucheston [**2**], Jones and Kuftinec [**15**] and Lin [**17**] (independently) proved the following.

THEOREM 1.1. *Let T be a contraction of a Hilbert space H and $x \in H$. Then the following are equivalent:*

(i) $T^n x \to 0$ *weakly;*

(ii) for every strictly increasing sequence of positive integers $\{k_j\}$ we have

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=1}^{n} T^{k_j} x \right\| = 0.$$

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Berend and Bergelson [**4**] generalized this by characterizing those bounded sequences $\{x_k\}$ in a Hilbert space satisfying $\|n^{-1}\sum_{j=1}^{n} x_{k_j}\| \to 0$ for every strictly increasing sequence $\{k_j\}$.

For a mean ergodic contraction T on a Banach space X we put $E(T)x = \lim_{n\to\infty} n^{-1}\sum_{k=1}^{n} T^k x$. We say that *the subsequential ergodic theorem holds for T along $\{k_j\}$* (or that $\{k_j\}$ *averages* T) *if* $\lim_{n\to\infty} n^{-1}\sum_{j=1}^{n} T^{k_j}x = E(T)x$ for every $x \in X$.

It follows from the work of Blum and Eisenberg [**6**] (see also [**21**]) that $\{k_j\}$ *averages every unitary operator U in a Hilbert space if and only if* $\lim_{n\to\infty} n^{-1}\sum_{j=1}^{n} \lambda^{k_j} = 0$ *for every* $\lambda \neq 1$ *with* $|\lambda| = 1$.

If $\{k_j\}$ is a sequence of positive density (i.e. n/k_n converges to a positive limit), the convergence of $\left\{n^{-1}\sum_{j=1}^{n} T^{k_j}x\right\}$ is equivalent to the convergence of $\left\{(1/k_n)\sum_{i=1}^{k_n} a_i T^i x\right\}$, when we define $a_i = 1$ if $i \in \{k_j\}$ and $a_i = 0$ otherwise. Following Tempelman [**27**], for a given sequence $\{a_j\}$ of (complex) numbers and a given contraction T on a (complex) Banach space X, we call the convergence of $\left\{n^{-1}\sum_{j=1}^{n} a_j T^j x\right\}$ for every $x \in X$ a *modulated ergodic theorem* for T. We are interested in sequences $\{a_k\}$ which yield a modulated ergodic theorem for every Hilbert space contraction. For additional references, including those dealing with a.e. convergence when T is a Dunford–Schwartz operator (in particular, induced by a measure-preserving transformation), we refer the reader to [**18**] and [**10**].

A general approach for obtaining some results for both modulated and subsequential ergodic theorems is to look at matrices $[\alpha_{n,k}]$, with $k \geq 0$ and $n > 0$, such that $\sum_{k=0}^{\infty} |\alpha_{n,k}| < \infty$ for every n, and study the convergence (in the strong or weak operator topology) of the sequence $A_n(T) := \sum_{k=0}^{\infty} \alpha_{n,k} T^k$ (e.g. [**18**]). For an invertible two-sided power-bounded T, we can also consider matrices with $\alpha_{n,k}$ defined for all $k \in \mathbb{Z}$. When all $\alpha_{n,k}$ are non-negative, with $\sum_k \alpha_{n,k} = 1$ for every n, each $A_n(T)$ is a weighted average of the powers of T. The matrix $[\alpha_{n,k}]$ *averages* a mean ergodic contraction T if $A_n(T)x \to E(T)x$ for every vector x.

2. *Modulated ergodic theorems*

In this section we find the precise necessary and sufficient conditions on a sequence $\{a_k\}$ which yield the modulated ergodic theorem for all contractions in complex Hilbert spaces. It turns out, as in the mean ergodic theorem, that weak convergence in the modulated ergodic theorem will already imply the strong one. An example shows that the conditions are strictly weaker than those obtained in [**18**] for the modulated ergodic theorem for all almost periodic operators on Banach spaces.

THEOREM 2.1. *Let $[\alpha_{n,k}]_{n>0,\,k\in\mathbb{Z}}$ be a matrix of complex numbers, such that $\sum_{k=-\infty}^{\infty} |\alpha_{n,k}| < \infty$ for every n. Then the following are equivalent:*

(i) $$\sup_{n\geq 1}\sup_{|\lambda|=1}\left|\sum_{k=-\infty}^{\infty} \alpha_{n,k}\lambda^k\right| < \infty;$$

(ii) *for every unitary operator U on a complex Hilbert space H we have*

$$\sup_{n\geq 1}\left\|\sum_{k=-\infty}^{\infty} \alpha_{n,k}U^k\right\| < \infty.$$

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Proof. Denote by Γ the unit circle $\{\lambda : |\lambda| = 1\}$ and define $f_n(\lambda) = \sum_{k=-\infty}^{\infty} \alpha_{n,k}\lambda^k$.

Assume (i), i.e. $\sup_n \|f_n\|_\infty = K < \infty$. Let U be unitary on H and let $x \in H$. The spectral theorem for U, with $\mu_x(\cdot) = \langle E(\cdot)x, x\rangle$, and Lebesgue's Theorem (since $\left|\sum_{|k|\le N} \alpha_{n,k}\lambda^k\right| \le \sum_{k=-\infty}^{\infty} |\alpha_{n,k}| < \infty$ for $|\lambda| = 1$) yield that for every positive n we have

$$\left\| \sum_{k=-\infty}^{\infty} \alpha_{n,k} U^k x \right\|^2 = \int_\Gamma |f_n(\lambda)|^2 \, d\mu_x(\lambda) \le \|f_n\|_\infty^2 \|x\|^2 \le K^2 \|x\|^2.$$

Hence $\sup_n \left\| \sum_{k=-\infty}^{\infty} \alpha_{n,k} U^k \right\| \le K$.

Let (ii) be satisfied, and assume (i) does not hold, so $\sup_n \||f_n|^2\|_\infty = \infty$. Let $\mathbf{M}$ be the space of finite complex-valued measures on Γ with the total variation norm, and define functionals φ_n on $\mathbf{M}$ by $\varphi_n(\mu) = \int_\Gamma |f_n|^2 \, d\mu$. Then $\|\varphi_n\| = \||f_n|^2\|_\infty$ and, by our assumption, $\sup_n \|\varphi_n\| = \infty$. By the Banach–Steinhaus Theorem, there exists a probability measure μ on Γ with $\sup_n |\varphi_n(\mu)| = \infty$. Let $H = L_2(\Gamma, \mu)$, and let U be the multiplication by λ, i.e. $Ug(\lambda) = \lambda g(\lambda)$ for $g \in H$. Take for x the function $x(\lambda) = 1$. Then $\langle U^k x, x\rangle = \int_\Gamma \lambda^k \, d\mu$ for every integer k, so $\mu_x = \mu$. Hence by the spectral theorem

$$\sup_{n\ge 1} \left\| \sum_{k=-\infty}^{\infty} \alpha_{n,k} U^k x \right\| = \sup_{n\ge 1} \int_\Gamma |f_n(\lambda)|^2 \, d\mu(\lambda) = \sup_{n\ge 1} |\varphi_n(\mu)| = \infty$$

which contradicts (ii). □

Remark. For a bounded measurable (in particular, continuous) function $f(\lambda)$ defined on Γ and a unitary operator U on H, the operator $f(U)$ is defined by $f(U)x = \int_\Gamma f(\lambda) E(d\lambda)x$ (in the weak operator topology) [**20**, §109]. The referee has observed that, in fact, our proof shows that, for a sequence $\{f_n\}$ of bounded measurable functions on Γ, the condition $\sup_n \|f_n\|_\infty < \infty$ *is necessary and sufficient for* $\sup_n \|f_n(U)\| < \infty$ *for every unitary operator U on a Hilbert space.* We will not use this general form here.

THEOREM 2.2. *Let $[\alpha_{n,k}]_{n>0,\, k\in\mathbb{Z}}$ be a matrix of complex numbers, such that $\sum_{k=-\infty}^{\infty} |\alpha_{n,k}| < \infty$ for every n. Then the following are equivalent.*

(i) *The matrix $[\alpha_{n,k}]$ satisfies the following two conditions:*

(A) *for every $\lambda \in \Gamma$,*

$$\lim_{n\to\infty} \sum_{k=-\infty}^{\infty} \alpha_{n,k}\lambda^k$$

exists; and

(B) $$\sup_{n\ge 1} \sup_{|\lambda|=1} \left| \sum_{k=-\infty}^{\infty} \alpha_{n,k}\lambda^k \right| < \infty.$$

(ii) *For every unitary operator U on a Hilbert space H and every $x \in H$ the sequence $\left\{\sum_{k=-\infty}^{\infty} \alpha_{n,k} U^k x\right\}$ converges in norm.*

(iii) *For every unitary operator U on a Hilbert space H and every $x \in H$ the sequence $\left\{\sum_{k=-\infty}^{\infty} \alpha_{n,k} U^k x\right\}$ converges weakly.*

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Proof. Clearly (ii) implies (iii). By applying (iii) to multiplication by λ on a one-dimensional space we get (A), and (iii) implies condition (B) by the Banach–Steinhaus theorem and the previous theorem; thus, (iii) implies (i).

Now (i) implies (ii) by the spectral theorem and Lebesgue's bounded convergence theorem—the proof is similar to that of [**18**, Theorem 2.1] (see also [**21**]). □

Remarks. (1) The limit function

$$c(\lambda) := \lim_{n\to\infty} \sum_{k=-\infty}^{\infty} \alpha_{n,k}\overline{\lambda}^k$$

may be non-zero on an uncountable set [**21**, Proposition 33].

(2) In [**18**], (ii) was proved with (B) replaced by the stronger condition $\sup_n \sum_k |\alpha_{n,k}| < \infty$.

COROLLARY 2.3. *Let $\{a_k\}_{k\geq 0}$ be a sequence of complex numbers. Then the following are equivalent.*

(i) *The sequence $\{a_k\}$ satisfies the following two conditions:*

(a) *for every $\lambda \in \Gamma$ there exists $c(\lambda)$ such that*

$$\frac{1}{n}\sum_{k=0}^{n-1} a_k\overline{\lambda}^k \underset{n\to\infty}{\longrightarrow} c(\lambda);$$

(b)
$$\sup_{n\geq 1}\sup_{|\lambda|=1}\left|\frac{1}{n}\sum_{k=0}^{n-1} a_k\lambda^k\right| < \infty.$$

(ii) *For every unitary operator U on a Hilbert space H and every $x \in H$ the sequence $\left\{n^{-1}\sum_{k=0}^{n} a_kU^kx\right\}$ converges in norm.*

(iii) *For every contraction T on a Hilbert space H and every $x \in H$ the sequence $\left\{n^{-1}\sum_{k=0}^{n} a_kT^kx\right\}$ converges in norm.*

(iv) *For every unitary operator U on a Hilbert space H and every $x \in H$ the sequence $\left\{n^{-1}\sum_{k=0}^{n} a_kU^kx\right\}$ converges weakly.*

When (i) holds, $c(\lambda) \neq 0$ for at most countably many $\lambda \in \Gamma$, and the limit in (iii) has the form

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n} a_kT^kx = \sum_{\lambda\in\Gamma} c(\overline{\lambda})E(\lambda, T)x$$

where $E(\lambda, T)$ is the orthogonal projection on the eigenspace of T corresponding to the eigenvalue λ, and the series converges strongly.

Proof. The equivalence of (i), (ii) and (iv) follows from the previous theorem, when $\alpha_{n,k} = a_k/n$ for $n > 0$ and $0 \leq k \leq n-1$, and 0 for $k < 0$ or $k \geq n$. Clearly (iii) implies (ii), and the unitary dilation theorem yields that (ii) implies (iii).

For the series representation of the limit, see [**18**]. □

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Remarks. (1) The referee has pointed out that the proof of (iii) can be obtained without using the unitary dilation theorem; instead, one can use the spectral theorem for contractions, as given in [**16**, Proposition 2.3.1]. Similarly, in Theorem 2.2 one can obtain (ii) with the unitary U replaced by a contraction T and U^n replaced by T_n, where $T_n = T^n$ for $n \geq 0$ and $T_n = T^{*|n|}$ for $n < 0$, without using the unitary dilation theorem.

(2) In [**18**] the convergence in (ii) and (iii) was proved when (b) is replaced by the obviously stronger condition

$$\text{(c)} \qquad \sup_n \frac{1}{n} \sum_{k=0}^{n-1} |a_k| < \infty.$$

We would like to show that there are no other implications between the three properties (a), (b) and (c). The first example shows that condition (a) cannot be omitted from (i) (so the boundedness given by Theorem 2.1 is not sufficient for the convergence proved in Theorem 2.2). The second example shows that property (c) is not necessary for the results of Theorem 2.2. The third example shows that (a) does not imply (b).

Example 2.4. A sequence $\{a_k\}$ satisfying (c) but not (a).

Let $(N_m)_{m=1}^{\infty}$ be an increasing sequence of integers with $N_1 = 0$ and satisfying $N_{m+1}/N_m \geq q > 1$ for $m > 1$. Put

$$a_n = \begin{cases} 1, & N_m \leq n < N_{m+1},\ m = 1, 3, 5, \ldots, \\ 0, & N_m \leq n < N_{m+1},\ m = 2, 4, 6, \ldots . \end{cases}$$

Clearly, property (a) is violated already for $\lambda = 1$, whereas property (c) is obviously satisfied.

Remarks. (1) Guy Cohen (private communication) has shown that for $\{a_n\}$ bounded (as in the previous example), $n^{-1} \sum_{k=0}^{n-1} a_k \lambda^k \to 0$ a.e. (with respect to Lebesgue's measure).

(2) Put $a_{2^j} = 2^j$, and $a_k = 0$ for k not a power of 2. Then (c) holds, but for any $\lambda \in \Gamma$ we have $\limsup_n |a_n \lambda^n / n| = 1$, so $n^{-1} \sum_{k=0}^{n-1} a_k \lambda^k$ does not converge. Similarly, for every isometry T and $0 \neq x \in H$ the sequence $\left\{n^{-1} \sum_{k=0}^{n-1} a_k T^k x\right\}$ does not converge in norm.

Example 2.5. A sequence $\{a_k\}$ satisfying (a) and (b), but not (c).

Let $\{\varepsilon_n\}$ be the Rudin–Shapiro sequence [**23**]: $\varepsilon_n = \pm 1$, and for each positive N we have

$$\left| \sum_{k=0}^{N-1} \varepsilon_k \lambda^k \right| \leq C\sqrt{N}, \quad |\lambda| = 1. \tag{2.1}$$

The best known upper bound for the constant C is $\sqrt{\frac{3}{5}}(2 + \sqrt{2})$ [**25**] (which improves that of [**23**]; see also [**3**]) and is close to the lower bound of $\sqrt{6}$ [**8**].

We now define $a_n = \varepsilon_n b_n$, with $b_n > 0$ increasing to ∞ and $b_n = o(\sqrt{n})$ (e.g. $b_n = n^{\alpha}$ with $0 < \alpha < \frac{1}{2}$ or $b_n = \sqrt{n}/\log n$ for large n). Since $\lim_n |a_n| = \infty$, property (c) is violated.

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Summation by parts, (2.1) and the monotonicity of b_n readily give (see [**30**, pp. 3–4])

$$\left|\sum_{n=0}^{N-1} a_n\lambda^n\right| = \left|\sum_{n=0}^{N-1} b_n\varepsilon_n\lambda^n\right| \le 2b_N \max_{1\le k\le N}\left|\sum_{n=0}^{k-1}\varepsilon_n\lambda^n\right| \le 2b_N C\sqrt{N}. \tag{2.2}$$

Since $b_N/\sqrt{N}\to 0$, this proves that properties (a) and (b) are satisfied.

Remarks. (1) For a given complex sequence $\{a_k\}$, put $f_n(\lambda) = n^{-1}\sum_{k=0}^{n-1} a_k\lambda^k$. Then, with respect to Lebesgue's measure on the circle, $\|f_n\|_2 \le \|f_n\|_\infty$, so (b) implies the existence of K such that $\sum_{k=0}^{n-1}|a_k|^2 < K^2n^2$ for every n. Thus, $|a_n| \le 2K\sqrt{n}$ for $|a_k|$ increasing, since

$$n|a_n|^2 \le \sum_{k=n}^{2n-1}|a_k|^2 \le 4K^2n^2.$$

Note that the sequence $\{\varepsilon_n\sqrt{n}\}$ satisfies (b), by putting $b_n = \sqrt{n}$ in (2.2).

(2) Condition (a) implies that $a_n/n\to 0$. Define $a_n = 0 = a_0$ for n not a power of 2, and $a_{2^k} = 2^k/k$ for $k\ge 1$. Since $\sum_{j=1}^{J} 2^j/j \le 2^{J+2}/J$, we have that $N^{-1}\sum_{n=1}^{N}|a_n|\to 0$, so $\{a_n\}$ satisfies (a) and (c), and hence also (b), with $a_n = o(n)$.

Example 2.6. A sequence $\{a_k\}$ satisfying (a) but not (b).

Let $(N_m)_{m=1}^{\infty}$ be a strictly increasing sequence of integers, starting with $N_1 = 0$ and $N_2 = 1$, such that for some fixed $\varepsilon > 0$, we have

$$(2+\varepsilon)N_m \le N_{m+1} \le \varepsilon^{-1}N_m, \quad m = 2, 3, \ldots . \tag{2.3}$$

Put $\zeta_m = e^{2\pi i/m}$, $m = 1, 2, \ldots$ and set

$$a_n = m\zeta_m^n, \quad N_m \le n < N_{m+1}, \quad m = 1, 2, \ldots .$$

Then,

$$\sum_{n=0}^{N_m-1} a_n\lambda^n = \sum_{j=1}^{m-1}\sum_{n=N_j}^{N_{j+1}-1} a_n\lambda^n = \sum_{j=1}^{m-1} j\sum_{n=N_j}^{N_{j+1}-1}(\zeta_j\lambda)^n. \tag{2.4}$$

Taking $\lambda_m = \zeta_m^{-1}$ and using (2.4), we obtain

$$\left|\sum_{n=0}^{N_{m+1}-1} a_n\lambda_m^n\right| \ge m(N_{m+1}-N_m) - \left|\sum_{n=0}^{N_m-1} a_n\lambda_m^n\right| \ge (N_{m+1}-N_m)m - N_m(m-1).$$

Using (2.3), we then have, for $\lambda_m = \zeta_m^{-1}$,

$$\left|\frac{1}{N_{m+1}}\sum_{n=0}^{N_{m+1}-1} a_n\lambda_m^n\right| > \frac{N_m}{N_{m+1}}\varepsilon m \ge \varepsilon^2 m \underset{m\to\infty}{\longrightarrow} \infty,$$

so that property (b) does not hold.

Now we claim that property (a) holds with $c(\lambda) = 0$ for every λ. In fact, suppose first $\lambda = 1$. Since (2.3) implies that N_m grows exponentially fast in m, and $|1-\zeta_j| > 1/j$ for $j > 1$, we have

$$\left|\sum_{n=0}^{N_m-1} a_n\lambda^n\right| = \left|1 + \sum_{j=2}^{m-1} j\zeta_j^{N_j}\frac{1-\zeta_j^{N_{j+1}-N_j}}{1-\zeta_j}\right| \le \sum_{j=1}^{m-1} 2j^2 = o(N_m).$$

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For $\lambda \neq 1$ on the unit circle, the numbers $|1-\zeta_j\lambda|$ are bounded away from 0 for sufficiently large j. When λ is not one of the ζ_j, we have for $N_m < N \leq N_{m+1}$:

$$\begin{aligned}\left|\sum_{n=0}^{N-1} a_n\lambda^n\right| &= \left|\sum_{j=1}^{m-1} j(\zeta_j\lambda)^{N_j}\frac{1-(\zeta_j\lambda)^{N_{j+1}-N_j}}{1-\zeta_j\lambda} + m(\zeta_m\lambda)^{N_m}\frac{1-(\zeta_m\lambda)^{N-N_m}}{1-\zeta_m\lambda}\right| \\ &\leq \frac{1}{\inf_{j\geq 1}|1-\zeta_j\lambda|}(m^2+2m) \\ &= o(N_m) = o(N).\end{aligned}$$

When $\lambda = \zeta_k$ for some $k > 1$, we obtain for $N_m < N \leq N_{m+1}$ the estimate

$$\left|\sum_{n=0}^{N-1} a_n\lambda^n\right| \leq C(\lambda) + \frac{1}{\inf_{j>k}|1-\zeta_j\lambda|}(m^2+2m) = o(N_m) = o(N).$$

It follows that property (a) is satisfied.

Recall that an operator T on a Banach space X is *(weakly) almost periodic* if for every $x \in X$ the sequence $\{T^k x : k \geq 0\}$ is conditionally (weakly) compact. A weakly almost periodic operator is necessarily power-bounded (and is a contraction under an equivalent norm).

It was shown in [**18**] that satisfying (a) and (c) is necessary and sufficient for the norm convergence of the sequence $\{n^{-1}\sum_{k=0}^{n} a_k T^k x\}$ for every T almost periodic on a Banach space X and $x \in X$. We now show that these two conditions are not sufficient for the modulated ergodic theorem to hold for every *weakly* almost periodic operator.

Example 2.7. A weakly almost periodic operator which is not modulated by a sequence satisfying (a) and (c).

Let $\{a_k\}$ be defined by $a_{j^2} = j$, and $a_k = 0$ if k is not a square. It was shown in [**18**] that this sequence satisfies (a), and we easily compute that $n^{-1}\sum_{k=1}^{n} a_k \to \frac{1}{2}$.

Recall that a (real or complex) bounded function $g(k)$ defined on $\mathbb{Z}$ is *(weakly) almost periodic* if the set of translates $\{g_j(k) = g(k+j) : j \in \mathbb{Z}\}$ is conditionally (weakly) compact in $\ell_\infty(\mathbb{Z})$. Let X be the space $WAP(\mathbb{Z})$ of weakly almost periodic functions on $\mathbb{Z}$, and let T be the unit translation operator, i.e. $Tg(k) = g(k+1)$. By the definition, T is weakly almost periodic on X. We now define a function f on $\mathbb{Z}$; we put $f(k) = 0$ for $k \leq 0$ and for $k > 0$ not a square. On the set A of squares f will be defined by alternating blocks of ones and zeros. Put $n_i = 2^i$. We define $f(j^2) = 1$ for $1 \leq j \leq n_1$, $n_2 < j \leq n_3, \ldots, n_{2\ell} < j \leq n_{2\ell+1}, \ldots$, and $f(j^2) = 0$ for the other squares. For even i we have

$$\frac{1}{n_i^2}\sum_{j=1}^{n_i} jf(j^2) \leq \frac{1}{n_i^2}\sum_{j=1}^{n_{i-1}} j \leq \frac{3}{16}.$$

For odd i we have

$$\frac{1}{n_i^2}\sum_{j=1}^{n_i} jf(j^2) \geq \frac{1}{n_i^2}\sum_{j=n_{i-1}+1}^{n_i} j > \frac{3}{8}.$$

The set of squares is clearly a thin set (i.e. $(A+i)\cap A$ is finite for every $0 \neq i \in \mathbb{Z}$), so by [**24**, Lemma 3] (see [**13**, Lemma 3.2] for general groups) f is weakly almost periodic

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on $\mathbb{Z}$. But

$$\frac{1}{n^2}\sum_{k=1}^{n^2} a_k T^k f(0) = \frac{1}{n^2}\sum_{k=1}^{n^2} a_k f(k) = \frac{1}{n^2}\sum_{j=1}^{n} j f(j^2)$$

does not converge, by the construction, so $n^{-1}\sum_{k=1}^{n} a_k T^k f$ does not converge even weakly in X.

Remarks. (1) The function f in the example is weakly almost periodic, but not in the uniform closure of the Fourier–Stieltjes algebra.

(2) In [**18**, Theorem 1.2], (c) is replaced by a stronger condition which, together with (a), yields a modulated ergodic theorem for every weakly almost periodic operator. It is not known whether this stronger condition is also necessary. That condition alone, without (a), yields a modulated ergodic theorem for every weakly almost periodic operator with a continuous peripheral spectrum (i.e. without any unimodular eigenvalues). The example of $a_{2^j} = 2^j$ and $a_k = 0$ for k not a power of 2 shows that (c) is not sufficient for the modulated ergodic theorem for weakly almost periodic isometries with continuous spectrum.

3. *Subsequential ergodic theorems and averaging sequences*

In this section we consider weighted averages and subsequential ergodic theorems, and study the averaging of weakly almost periodic functions by sequences of probabilities on $\mathbb{Z}$.

THEOREM 3.1. *Let $[\alpha_{n,k}]_{k\geq 0}$ be a matrix of non-negative numbers, such that $\sum_{k=0}^{\infty}\alpha_{n,k} = 1$ for every n. If $\sum_{k=0}^{\infty}\alpha_{n,k}\overline{\lambda}^k \underset{n\to\infty}{\longrightarrow} 0$ for every $\lambda \neq 1$ with $|\lambda| = 1$, then for every almost periodic operator T on a Banach space X and every contraction T on a Hilbert space we have*

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\alpha_{n,k}T^k x = E(T)x, \quad x \in X. \tag{3.1}$$

Proof. For Hilbert space contractions this follows from Theorem 2.2.

Let S be the shift on ℓ_2 defined by $Se_j = e_{j+1}$ for $j \geq 1$. Since S has no fixed point, applying (3.1) to S with $x = e_1$ we obtain

$$\max_j \alpha_{n,j}^2 \leq \sum_{k=0}^{\infty}\alpha_{n,k}^2 = \left\|\sum_{k=0}^{\infty}\alpha_{n,k}e_{k+1}\right\|^2 = \left\|\sum_{k=0}^{\infty}\alpha_{n,k}S^k e_1\right\|^2 \underset{n\to\infty}{\longrightarrow} 0.$$

Hence the assumptions imply $\lim_{n\to\infty}\alpha_{n,k} = 0$ for every $k \geq 0$.

Now let T be almost periodic on X, and put $M = \sup_{k\geq 0}\|T^k\|$. Since T is clearly weakly almost periodic, we have the Jacobs–Deleeuw–Glicksberg decomposition $X = X_0 \oplus X_p$, where X_p is the closed linear manifold generated by the eigenvectors of unimodular eigenvalues, and X_0 is the subspace of flight vectors (i.e. all vectors x such that $T^{n_i}x \to 0$ weakly for some $\{n_i\}$) (see [**16**]). The desired convergence on X_p follows immediately from the assumptions. Let $x \in X_0$. Since T is almost periodic, we have $\|T^n x\| \to 0$. For $\varepsilon > 0$ fix K such that $\|T^k x\| < \varepsilon$ for $k > K$. Then

$$\left\|\sum_{k=0}^{\infty}\alpha_{n,k}T^k x\right\| \leq M\|x\|\sum_{k=0}^{K}\alpha_{n,k} + \varepsilon\sum_{k=K+1}^{\infty}\alpha_{n,k}.$$

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Hence $\limsup_n \left\| \sum_{k=0}^{\infty} \alpha_{n,k} T^k x \right\| \leq \varepsilon$, which proves the claimed convergence for $x \in X_0$. □

Remark. An invertible almost periodic operator need not yield an almost periodic representation of $\mathbb{Z}$, since its inverse need not be almost periodic (e.g. multiplication by $0 < r < 1$). This is why we may have X_0 non-trivial for such operators.

PROPOSITION 3.2. *Let $\{k_j\}$ be a strictly increasing sequence of positive integers satisfying $\lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} \lambda^{k_j} = 0$ for every $\lambda \neq 1$ with $|\lambda| = 1$.*

(i) *For every almost periodic operator T on a Banach space X and every contraction T on a Hilbert space we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} T^{k_j} x = E(T)x, \quad x \in X. \tag{3.2}$$

(ii) *If $\{k_j\}$ has positive lower density (i.e. $\sup_j (k_j/j) < \infty$), then (3.2) holds for every* weakly *almost periodic operator T on a Banach space X.*

(iii) *If $\{k_j\}$ is thin (i.e. $k_{j+1} - k_j \to \infty$), then there exists a weakly almost periodic operator T for which (3.2) does not hold.*

Proof. (i) follows from the previous theorem.

(ii) We start again from the decomposition $X = X_p \oplus X_0$. The convergence on X_p follows from (i), since the restriction of T to X_p is almost periodic. Since $\{k_j\}$ has positive lower density, we have convergence to 0 on X_0 by [**16**, Theorem 8.1.1].

(iii) Let $X = WAP(\mathbb{Z})$ and $Tg(j) = g(j+1)$. Take f to be the indicator function of the set $\{k_j\} \subset \mathbb{Z}$. Since $\{k_j\}$ is thin, $f \in WAP(\mathbb{Z})$ [**24**] and $M(f) = \lim_n n^{-1} \sum_{i=1}^{n} T^i f = 0$. Obviously, $n^{-1} \sum_{j=1}^{n} T^{k_j} f(0) = 1$ for every n. □

Remarks. (1) A random construction which yields many sequences of positive density satisfying the assumption of the proposition (so (ii) applies) is given by Blum and Cogburn [**5**].

(2) The sequence $\{k_j\}$ constructed in [**6**], which satisfies the assumptions of the proposition, is not Følner (i.e. the sequence of sets $A_n := \{k_1, \ldots, k_n\}$ is not Følner), though (ii) applies (so also (i) holds). However, the sequence $\{k_j\}$ defined as the union of the intervals of integers $\{(2n)! + 1, \ldots, (2n+1)!\}$ is Følner, so (3.2) holds for every mean ergodic contraction, although $\{k_j\}$ has 0 lower density and upper density 1. It is possible also to have $\{k_j\}$ Følner with density 0.

(3) Explicit examples of thin sequences $\{k_j\}$ satisfying the assumption of the proposition (so (iii) applies) were constructed by Veech [**29**] and Boshernitzan [**9**]. Random constructions are given in Lemma 1 of [**1**] and at the end of [**11**].

Definition. A sequence $\{\mu_n\}$ of probabilities on $\mathbb{Z}$ is called *averaging* if for every almost periodic function f on $\mathbb{Z}$ we have $\lim_n \mu_n(f) = M(f)$, where $M(f)$ is the unique invariant mean on $WAP(\mathbb{Z})$. (For a finite measure μ on $\mathbb{Z}$, $\mu(f) := \sum_{k\in\mathbb{Z}} f(k)\mu(k)$.)

The Blum–Eisenberg theorem shows that if $\{\mu_n\}$ is averaging, then $A_n(U)x := \sum_{k\in\mathbb{Z}} \mu_n(k) U^k x$ converges strongly to $E(U)x$ for every unitary operator U.

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Hence (by Bochner's Theorem) $\mu_n * f(j) := \sum_k f(j+k)\mu_n(k)$ converges uniformly to $M(f)$ for every f in the uniform closure of the Fourier–Stieltjes algebra $B(\mathbb{Z})$ (which is contained in $WAP(\mathbb{Z})$ [**12**]).

The examples to which Proposition 3.2(iii) applies yield averaging sequences (defined by $\mu_n = n^{-1}\sum_{j=1}^{n}\delta_{k_j}$) such that for some $f \in WAP(\mathbb{Z})$, the sequence $\mu_n * f$ does not converge even pointwise to $M(f)$. Note that the sequence of Example 2.7 (after the correct normalization) does not give such an example, as it is not averaging—the limits of the Fourier transform are not zero for λ with rational angle. However, since the sequence $\{k_j\}$ is thin, it does not provide an example of non-modulation as needed in Example 2.7.

Remark. The existence of averaging sequences $\{\mu_n\}$ such that $\mu_n * f$ does not converge for some $f \in WAP(\mathbb{Z})$ shows that Theorem 2.3 in [**19**] does not follow from the Theorem 2.2 there.

Suppose that $\{\mu_n\}$ is averaging, which means that $\mu_n * f \to M(f)$ pointwise for $f \in AP(\mathbb{Z})$ (since $\mu_n * f(j) = \mu_n * (\delta_j * f)(0) = \int \delta_j * f \, d\mu_n$); this discussion shows that, in fact, the convergence is uniform on $\mathbb{Z}$. In fact, the same is true for the convergence of nets $\{\mu_\alpha\}$ (by combining Theorem 1.2.5, Corollary 4.8.11 and Lemma 4.8.11 of [**28**]). However, we have the result from [**22**], that weak convergence to invariance of nets does not imply strong convergence (in any amenable locally compact group), although for sequences it does [**26**]. This of course raises the question for weakly almost periodic functions: *If $\{\mu_n\}$ is an averaging sequence of probabilities, and $\mu_n * f \to M(f)$ pointwise for every $f \in WAP(\mathbb{Z})$, is the convergence uniform for every $f \in WAP(\mathbb{Z})$?* The following proposition shows that the corresponding question for nets has a negative answer.

PROPOSITION 3.3. *There exists a net (μ_α) of probability measures on $\mathbb{Z}$ such that, for all functions $g \in WAP(\mathbb{Z})$, we have $\lim_{\alpha\to\infty}\mu_\alpha(g) = M(g)$ but for some function, $0 \le f \in WAP(\mathbb{Z})$ with $M(f) = 0$, and for some (j_α) in $\mathbb{Z}$, we always have $\mu_\alpha * f(-j_\alpha) = 1$.*

Proof. Let $\{m_k\}$ be an averaging thin sequence (e.g. [**9**]). For $n \ge 1$ set $\nu_n = n^{-1}\sum_{k=1}^{n}\delta_{m_k}$.

By the Jacobs–Deleeuw–Glicksberg decomposition, each function $g \in WAP(\mathbb{Z})$ can be written uniquely as $g = g_1 + g_0$ where $g_1 \in AP(\mathbb{Z})$ and $g_0 \in WAP(\mathbb{Z})$ with $M(|g_0|) = 0$ (e.g. [**12**, Corollaries 2.8 and 2.18]). For each finite set $F \subset WAP(\mathbb{Z})$, take this representation $h = h_1 + h_0$ for $h \in F$. Now, since $M(|h_0|) = 0$, also $M(|\delta_{m_k} * h_0|) = 0$ for each k. Hence, for any $n \ge 1$ and $\varepsilon > 0$, we can choose N large enough so that

$$\frac{1}{N}\sum_{j=1}^{N}\sum_{h\in F}\sum_{k=1}^{n}|h_0(j+m_k)| < \varepsilon.$$

Consequently, there exists some value of j, $1 \le j \le N$, such that $\sum_{h\in F}\sum_{k=1}^{n}|h_0(j+m_k)| < \varepsilon$. Hence, for each n there is an integer $j \in \mathbb{Z}$, such that $\nu_n * \delta_j(|h_0|) = \nu_n(|\delta_j * h_0|) < \varepsilon$ for all $h \in F$. We take a net index α given by triples (n, F, ε), where $n \ge 1$, F is a finite subset of $WAP(\mathbb{Z})$ and $\varepsilon > 0$; we give the net the usual ordering, and then put $\mu_\alpha = \nu_n * \delta_{j_\alpha}$ with $j_\alpha = j(n, F, \varepsilon)$ chosen as before. Then $\lim_{\alpha\to\infty}\mu_\alpha(h) = 0$ for every $h \in WAP(\mathbb{Z})$ with $M(|h|) = 0$. Yet, we still have $\lim_{\alpha\to\infty}\hat{\mu}_\alpha = 0$ pointwise on

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Γ since $(m_k)_{k=1}^{\infty}$ is averaging. Hence, $\lim_{\alpha\to\infty}\mu_\alpha(h) = M(h)$ for all $h \in AP(\mathbb{Z})$. Thus, $\lim_{\alpha\to\infty}\mu_\alpha(h) = M(h)$ for every $h \in WAP(\mathbb{Z})$.

We now take $f = 1_{\{m_k : k=1,2,\dots\}}$. As before, $f \in WAP(\mathbb{Z})$ since $\{m_k\}$ is thin, and $M(|f|) = 0$. However, for each $\alpha = (n, F, \varepsilon)$ we have $\mu_\alpha * f(-j_\alpha) = \nu_n(f) = 1$ for the particular j_α corresponding to α. □

4. *Modulated ergodic theorems for unitary representations of locally compact Abelian groups*

The results of §2 can be generalized to unitary representations of σ-compact locally compact Abelian (LCA) groups. Let G be a σ-compact LCA group with dual group Γ. Following [**19**], we call a sequence $\{\mu_n\}$ of probabilities on G *unitarily averaging* if for every $U(t)$ continuous unitary representation of G in a Hilbert space, with P the orthogonal projection on its common fixed points, we have $\int U(t)x\, d\mu_n(t) \to Px$ for every vector x. Blum and Eisenberg [**6**] proved that $\{\mu_n\}$ *is unitarily averaging if and only if* $\widehat{\mu}_n(\gamma) \to 0$ *for every non-zero* $\gamma \in \Gamma$. Their proof uses the Naimark–Stone Spectral Theorem, which will also be used here.

PROPOSITION 4.1. *Let ν be a positive finite measure on Γ. Then there exists a continuous unitary representation U in $L_2(\Gamma, \nu)$ with spectral measure $E(\cdot)$, such that $\nu(\cdot) = \langle E(\cdot)h, h\rangle$ for some $h \in L_2(\Gamma, \nu)$.*

Proof. Define $U(t)g$ for $g \in L_2(\Gamma, \nu)$ by $U(t)g(\gamma) = \gamma(t)g(\gamma)$. Clearly $U(t)$ is unitary, and U is a representation. By the definition, we have

$$\|U(s)g - U(t)g\|_2^2 = \int |\gamma(s-t) - 1|^2 |g(\gamma)|^2\, d\nu(\gamma).$$

This and Lebesgue's Theorem yield the continuity of U when G is second countable. For continuity in the general case, see [**14**, Theorem 33.7].

Let $E(\cdot)$ be the spectral measure of U (by the Naimark–Stone Theorem); define $h(\gamma) \equiv 1$, and put $\nu_h(\cdot) = \langle E(\cdot)h, h\rangle$. For every $t \in G$ the spectral theorem yields

$$\widehat{\nu}(t) = \int_\Gamma \overline{\gamma(t)}\, d\nu(\gamma) = \langle U(t)^{-1}h, h\rangle = \int_\Gamma \overline{\gamma(t)}\, d\nu_h(\gamma) = \widehat{\nu}_h(t).$$

Hence $\nu = \nu_h$. □

THEOREM 4.2. *Let $\{\mu_n\}$ be a unitarily averaging sequence of probabilities on a σ-compact LCA group G. The following are equivalent for a function f measurable on G.*

(i) *For every continuous unitary representation U in a Hilbert space H and every $x \in H$, the sequence $\left\{\int_G f(t)U(t)x\, d\mu_n(t)\right\}$ converges strongly in H.*

(ii) *The function f satisfies the following two conditions:*

(a) $\lim_n \int_G f(t)\gamma(t)\, d\mu_n(t)$ *exists for every* $\gamma \in \Gamma$.

(b)
$$\sup_{n\geq 1}\sup_{\gamma\in\Gamma}\left|\int_G f(t)\gamma(t)\, d\mu_n(t)\right| < \infty.$$

The proof is an adaptation of the proofs of the theorems in §2, using the Naimark–Stone spectral theorem and the previous proposition.

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Remark. For Abelian groups, the previous theorem improves Theorem 2.3 of [**19**] (where (i) is proved when, instead of (b), f satisfies $\sup_n \int_G |f|\,d\mu_n(t) < \infty$ and can be approximated by functions f_k with $\sup_n \int_G |f_k|^{1+\varepsilon}\,d\mu_n(t) < \infty$).

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