Mathematical Structures in Computer Science (2025), **35**, e40, pp. 1–24
doi:10.1017/S0960129525100388

CAMBRIDGE UNIVERSITY PRESS

PAPER

Congruence relations on domains*

Mengjie Jin[1] and Qingguo Li[2]

[1]School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang, Henan, 471023, China and
[2]School of Mathematics, Hunan University, Changsha, Hunan, 410082, China
Correspoding author: Qingguo Li; Email: liqingguoli@aliyun.com

(Received 17 April 2024; revised 9 October 2025; accepted 3 December 2025)

Abstract

Domains exhibit a variety of different aspects, some are order theoretical, some are topological, some belong to topological algebra. In this paper, we introduce two kinds of congruence relations on domains: I-congruence relation and II-congruence relation on domains. We obtain that there is a bijection from the set of all kernel operators of domain P preserving directed sups onto the set of all I-congruence relations on P which exclude $P \times P$. There is also a bijection from the set of all closure operators of domain P preserving directed sups onto the set of all II-congruence relations on P which exclude $P \times P$. Furthermore, between two domains, we propose a new homomorphism called I-homomorphism and II-homomorphism, respectively. We conclude that the kernels of I-homomorphisms and II-homomorphisms between domains are I-congruence relations and II-congruence relations on domains, respectively. Therefore, we obtain the I-homomorphism and I-isomorphism theorems, as well as II-homomorphism and II-isomorphism theorems for domains. Besides, we give a positive answer to an open problem on homomorphisms and quotients of continuous semilattices posed by G. Gierz, et al.

Keywords: Domain; continuous semilattice; I-congruence relation on domains; I-homomorphism on domains; II-congruence relation on domains; II-homomorphism on domains

1. Introduction

Congruence relations play an important role in algebraic structures, such as groups, rings, and modules. One of the well-known fundamental facts about algebraic structures is that the surjective homomorphisms between them are completely characterized as the quotients of their domains modulo kernel congruences. From this, there are typical fundamental homomorphism and isomorphism theorems for algebraic structures. These are also true for certain ordered algebraic structures.

Domain theory was first introduced by Dana Scott as a foundation for program semantics in the 1970s. In Gierz et al. (2003), the authors proposed a certain kind of congruence relation on continuous lattices, bounded complete domains corresponding to kernel operators which preserve directed sups, respectively, and they raised an open problem: Does the same result hold for continuous semilattices (see the second problem on page 92 of Gierz et al. (2003))? As far as we know, this problem remains unresolved. Also, in Jung et al. (1992), the authors studied congruence relations on algebraic distributive bounded complete dcpos, namely distributive Scott domains. Furthermore, in Fiech (1996), Mahmoudi et al. (2017), and Mahmoudi et al. (2019), we

*This work is supported by the National Natural Science Foundation of China (Nos. 12231007, 12501638, 12571498) and by Natural Science Foundation of Henan Province (No. 252300420896).

© The Author(s), 2025. Published by Cambridge University Press.

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press

can see some results about congruence relations on dcpos. Recently, we obtained some kind of congruence relation on L-domains corresponding to strong homomorphisms among L-domains in Jin and Li (2022). There are no more results about quotients of domains and related structures according to the earlier literature.

In this paper, we introduce two kinds of congruence relations on domains called I-congruence relation and II-congruence relation on domains. Then, we conclude that there is a bijection between kernel operators preserving directed sups and I-congruence relations on domains which do not contain $P \times P$ on domain P. There is also a bijection between closure operators preserving directed sups and II-congruence relations on domains which do not contain $P \times P$ on domain P. Meanwhile, we describe and characterize the unique order on the quotient set.

In contrast to I-congruence relations and II-congruence relations on domains, we propose a new homomorphism between domains called I-homomorphism and II-homomorphism, respectively. The relationship of I-homomorphisms and II-homomorphisms and their corresponding congruence relations will be studied. Finally, we get the I-homomorphism and I-isomorphism theorems, as well as II-homomorphism and II-isomorphism theorems for domains.

Evidently, the images of I-homomorphism and II-homomorphism from a domain, an L-domain, a bounded complete domain, a continuous semilattice, and a continuous lattice also have the same structure. Thus, we get the I-congruence and II-congruence relations on domains, L-domains, bounded complete domains, continuous semilattices, continuous lattice, respectively. In particular, we show that the I-congruence relation on L-domains is a generalization of the congruence relation on L-domains in Jin and Li (2022). Furthermore, the I-congruence relation on continuous lattices is precisely the congruence relation on continuous lattices defined in Gierz et al. (2003). At last, we take examples to reveal that the I-congruence and II-congruence relations on continuous lattices are different.

In addition, applying the above results to continuous semilattices, we obtain a positive answer to the open problem on homomorphisms and quotients of continuous semilattices mentioned above.

2. Preliminaries

To start with, we recall some standard definitions and notations of order theory and domain theory to be used in this paper, for more details refer to Chajda and Snášel (1998), Chajda et al. (2009), Davey and Priestley (2002), Gierz et al. (2003), Hofmann and Mislove (1977), Goubault-Larrecq (2013), Kolibiar (1987), and Shum et al. (2008).

A *binary relation* R on a set X is any subset of $X \times X$. We write $x\,R\,y$ instead of $(x, y) \in R$. R is *reflexive* if $x\,R\,x$ for every $x \in X$. R is *transitive* whenever $x\,R\,y$ and $y\,R\,z$, then $x\,R\,z$. R is *antisymmetric* whenever $x\,R\,y$ and $y\,R\,x$, then $x = y$. A reflexive, transitive, and antisymmetric relation is called a *partial order* or an *order* for short and is usually written as $\leq$. A set with an order is called a *partially ordered set* or a *poset*. A binary relation R is *symmetric* whenever $x\,R\,y$, then $y\,R\,x$. An *equivalence relation* on X is a binary relation on X which is reflexive, symmetric, and transitive. An *equivalence class* or a *block* $[x]_R$ of $x \in X$ is the set of all $y \in X$ such that $x\,R\,y$. The *quotient set* X/R is the set of all equivalence classes.

Next, we define a binary relation $\leq_R$ on the quotient set. In this paper, we will prove $\leq_R$ is an order on the quotient set if R is a I-congruence relation and II-congruence relation on domains (see Lemmas 3.5 and 5.7, respectively).

Definition 2.1. *Let P be a poset and R an equivalence relation on P. Define a binary relation $\leq_R$ on P/R as follows:*

$$[x]_R \leq_R [y]_R \text{ if there exist } x_0 \in [x]_R \text{ and } y_0 \in [y]_R \text{ such that } x_0 \leq y_0.$$

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press

Let P be a poset and $A \subseteq P$. We say that x is an *upper bound* of A and y is a *lower bound* of A in P if $a \leq x$ for all $a \in A$ and $y \leq a$ for all $a \in A$, respectively. A has a *supremum* in P if A has the least upper bound in P, and it is written as $\vee A$ or $\sup A$. Dually, A has an *infimum* in P if A has the greatest lower bound in P, and we write it as $\wedge A$ or $\inf A$. P is called a *sup semilattice* if any elements $x, y \in P$ have a least upper bound $x \vee y$. We call P is a *complete lattice* if every subset B of P has a least upper bound $\sup B$ and a greatest lower bound $\inf B$. A subset D of P is *directed* provided that it is nonempty and every finite subset of D has an upper bound in D. The poset P is called a *dcpo* if every directed subset D in P has a supremum. A subset Q of P is called a *subdcpo* whenever D is a directed subset of Q, $\sup_P D$ belongs to Q. P is called a *complete semilattice* if every nonempty subset has an infimum and every directed subset has a supremum. Let $x, y \in P$, we say that x *is way below* y, in symbols $x \ll y$, if and only if for all directed subsets $D \subseteq P$ for which $\sup D$ exists, the relation $y \leq \sup D$ always implies the existence of an element $d \in D$ such that $x \leq d$.

In the following, we introduce various types of domains.

Definition 2.2. (Gierz et al. (2003), Definition I-1.6). *The poset P is called continuous if the set $\{y \in P \mid y \ll x\}$ is directed and $x = \sup\{y \in P \mid y \ll x\}$ for all $x \in P$.*

(1) *A dcpo which is continuous as a poset will be called a domain.*
(2) *A domain which is a complete lattice is called a continuous lattice.*
(3) *A domain which is also a semilattice is called a continuous semilattice.*
(4) *A complete semilattice which is a domain as a poset is called a complete continuous semilattice or alternatively a bounded complete domain.*
(5) *A domain in which every principal ideal $\downarrow x$ is a complete lattice (in its induced order) is called an L-domain.*

Lemma 2.3. (Gierz et al. (2003), Exercise I-1.38). *If P is a dcpo, then the following conditions are equivalent:*

1. *$\downarrow x$ is a continuous lattice for each $x \in P$;*
2. *P is a domain in which every nonempty set that is bounded above has a greatest lower bound;*
3. *P is an L-domain.*

This characterization of L-domains will be essential in Section 7 when we study I-congruence relation on L-domains.

Now, we introduce a general statement about posets.

Proposition 2.4. *Let P and Q be posets and $f : P \to Q$ a map. If f is surjective, then we have two groups of equivalent statements:*

(1) *For any $A \subseteq P$, f satisfies the equation: $\downarrow f(\cap_{a\in A} \downarrow a) = \cap_{a\in A} \downarrow f(a)$;*
(2) *For any $A \subseteq P$, f satisfies the equation: $f(\cap_{a\in A} \downarrow a) = \cap_{a\in A} \downarrow f(a)$.*
(1′) *For any $A \subseteq P$, f satisfies the equation: $\uparrow f(\cap_{a\in A} \uparrow a) = \cap_{a\in A} \uparrow f(a)$;*
(2′) *For any $A \subseteq P$, f satisfies the equation: $f(\cap_{a\in A} \uparrow a) = \cap_{a\in A} \uparrow f(a)$.*

Furthermore, f is order-preserving under any of the above four given equations.

Proof. We only prove the equivalence of (1) and (2). The proof of equivalence for (1′) and (2′) is similar.

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press

First, we show that f is order-preserving under any of the above four given equations. Here, we only prove that f is order-preserving when f satisfies the equation: $\downarrow f(\cap_{a\in A}\downarrow a) = \cap_{a\in A}\downarrow f(a)$. Let $x, y \in P$ and $x \leq y$. Then, $x \in \downarrow x \cap \downarrow y$. It implies that $f(x) \in \downarrow f(\downarrow x \cap \downarrow y) = \downarrow f(x) \cap \downarrow f(y)$. So, $f(x) \leq f(y)$.

(1) $\Rightarrow$ (2): For any $A \subseteq P$, we know that $f(\cap_{a\in A}\downarrow a) \subseteq \cap_{a\in A}\downarrow f(a)$. Conversely, let $x \in \cap_{a\in A}\downarrow f(a)$. Then, $x \in \downarrow f(\cap_{a\in A}\downarrow a)$ by (1). Thus, there is an element $p \in \cap_{a\in A}\downarrow a$ such that $x \leq f(p)$. Since f is surjective, there exists an element $p_x \in P$ such that $f(p_x) = x$. Note that $x \in \downarrow f(p_x) \cap f(p) = \downarrow f(\downarrow p_x \cap \downarrow p)$. It follows that there exists an element $y \in \downarrow p_x \cap \downarrow p$ such that $x \leq f(y)$. So we conclude that $y \in \cap_{a\in A}\downarrow a$ and $x = f(y)$. This implies $x = f(y) \in f(\cap_{a\in A}\downarrow a)$. Thus, $\cap_{a\in A}\downarrow f(a) \subseteq f(\cap_{a\in A}\downarrow a)$. Therefore, $f(\cap_{a\in A}\downarrow a) = \cap_{a\in A}\downarrow f(a)$.

(2) $\Rightarrow$ (1): For any $A \subseteq P$, we have $f(\cap_{a\in A}\downarrow a) \subseteq \downarrow f(\cap_{a\in A}\downarrow a)$. By (2), $f(\cap_{a\in A}\downarrow a) = \cap_{a\in A}\downarrow f(a) \subseteq \downarrow f(\cap_{a\in A}\downarrow a)$. Conversely, since f is order-preserving, $\downarrow f(\cap_{a\in A}\downarrow a) \subseteq \cap_{a\in A}\downarrow f(a)$. So, $\downarrow f(\cap_{a\in A}\downarrow a) = \cap_{a\in A}\downarrow f(a)$. □

Next, we introduce the definitions and properties of Scott-continuous map, projection operator, closure operator, and kernel operator. Particularly, closure operator and kernel operators provide powerful tools for constructing quotient structures of domains. Their importance will become evident throughout the paper.

Definition 2.5. (Goubault-Larrecq (2013)). *Let P, Q be posets and $f : P \to Q$. f is called Scott-continuous if f preserves sups of directed subsets, that is, f is order-preserving and $f(\sup D) = \sup f(D)$, for all directed subsets $D \subseteq P$ for which $\sup D$ exists.*

Definition 2.6. (Gierz et al. (2003), Definition O-3.8). *Let L be a poset.*

(1) *A projection operator (shortly projection) is an idempotent, monotone self-map $p : L \to L$.*

(2) *A closure operator is a projection c on L with $id_L \leq c$.*

(3) *A kernel operator is a projection k on L with $k \leq id_L$.*

Lemma 2.7. (Gierz et al. (2003), Corollary I-2.3). *Let M be the image of a projection $p : L \to L$ preserving sups of directed sets. If L is a domain, an L-domain, a bounded complete domain, a continuous semilattice, and a continuous lattice, the same is true for the image M.*

Lemma 2.8. (Gierz et al. (2003), Remark on Page 81). *For a closure operator c on a domain P preserving sups of directed sets, then for all $x, y \in P$, one has $x \ll y$ implies $c(x) \ll_{c(P)} c(y)$.*

Lemma 2.9. (Gierz et al. (2003), Lemma I-2.4). *A closure operator $c : L \to L$ on a dcpo L preserves sups of directed sets if and only if its image $c(L)$ is closed in L with respect to sups of directed sets.*

Lemma 2.10. *A kernel operator $k : L \to L$ on a dcpo L preserves sups of directed sets if and only if $(k \times k)^{-1}(\Delta) = \{(x, y) \in L \times L \mid k(x) = k(y)\}$ is a subdcpo of $L \times L$.*

Proof. Assume first that $(k \times k)^{-1}(\Delta)$ is a subdcpo of $L \times L$. Let D be a directed subset of L. As k is a kernel operator, it is order-preserving, $\sup_L k(D) \leq k(\sup_L D)$. Define $\mathcal{D} = \{(d, k(d) \mid d \in D)\}$. Then $\mathcal{D}$ is directed in $(k \times k)^{-1}(\Delta)$. Since $(k \times k)^{-1}(\Delta)$ is a subdcpo of $L \times L$, $\sup \mathcal{D} = (\sup_L D, \sup_L k(D)) \in (k \times k)^{-1}(\Delta)$. It means that $k(\sup_L D) = k(\sup_L k(D))$. Note that $k(\sup_L k(D)) \leq \sup_L k(D)$, which implies that $k(\sup_L D) \leq \sup_L k(D)$. So, $\sup_L k(D) = k(\sup_L D)$, that is, k preserves sups of directed sets. Conversely, assume that k preserves sups of directed sets. Let $\mathcal{D}$ be a directed subset in $(k \times k)^{-1}(\Delta)$. Set

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press

$$D_1 = \{d_1 \in L \mid \text{ there exists an element } d_2 \in L \text{ such that } (d_1, d_2) \in \mathcal{D}\},$$
$$D_2 = \{d_2 \in L \mid \text{ there exists an element } d_1 \in L \text{ such that } (d_1, d_2) \in \mathcal{D}\}.$$

Then, $\sup \mathcal{D} = (\sup D_1, \sup D_2)$. We claim that $(\sup D_1, \sup D_2) \in (k \times k)^{-1}(\Delta)$. It is easy to see that $k(D_1) = k(D_2)$. Now, as k preserves sups of directed sets,

$$k(\sup D_1) = \sup_{d_1 \in D_1} k(d_1) = \sup_{d_2 \in D_2} k(d_2) = k(\sup D_2),$$

which implies that $(\sup D_1, \sup D_2) \in (k \times k)^{-1}(\Delta)$. So, $(k \times k)^{-1}(\Delta)$ is a subdcpo of $L \times L$. □

Remark 2.11. *Let P be a dcpo and Q a poset. From the proof of Lemma 2.10, we could conclude that for a Scott-continuous map $f : P \to Q$ between P and Q, $ker(f) = \{(x, y) \in P \times P \mid f(x) = f(y)\}$ is a subdcpo of $P \times P$.*

We now introduce the definition and properties of Galois connection. Galois connection is one of the most efficient tools in dealing with domains and plays a central role in I-homomorphism and II-homomorphism on domains.

Definition 2.12. (Gierz et al. (2003), Definition O-3.1). *Let S and T be two posets. We shall say that a pair (g, d) of functions $g : S \to T$ and $d : T \to S$ is a Galois connection or an adjunction between S and T provided that*

(1) *both g and d are monotone, and*
(2) *the relations $g(s) \geq t$ and $s \geq d(t)$ are equivalent for all pairs of elements $(s, t) \in S \times T$.*

In an adjunction (g, d), the function g is called the *upper adjoint* and d the *lower adjoint*.

Theorem 2.13. (Gierz et al. (2003), Theorem O-3.6). *For every pair of order-preserving functions between posets, $g : S \to T$ and $d : T \to S$, the following conditions are equivalent:*

(1) *(g, d) an adjunction;*
(2) *$dg \leq id_S$ and $id_T \leq gd$.*
Moreover, these conditions imply
(3) *$d = dgd$ and $g = gdg$,*
(4) *gd and dg are idempotent.*

Proposition 2.14. (Gierz et al. (2003), Proposition O-3.7). *For an adjunction (g, d) between posets S and T, the following conditions are equivalent:*

(1) *d is surjective;*
(2) *$dg = id_S$.*

Likewise, the following statements are equivalent:

(1*) *g is surjective;*
(2*) *$gd = id_T$;*
(3*) *d is injective.*

Lemma 2.15. (Gierz et al. (2003), Theorem O-3.3). *Any upper adjoint preserves existing infs, any lower adjoint preserves existing sups.*

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press

Lemma 2.16. (Gierz et al. (2003), Theorem IV-1.4). *Let S be a poset, T a domain, and g the upper adjoint of $d : T \to S$. Then the following statements are equivalent:*

(1) *g is Scott-continuous;*
(2) *d preserves $\ll$, that is, if $t \ll t^*$ in T, then $d(t) \ll d(t^*)$.*

Let P, Q be posets and $f : P \to Q$. The *kernel of f* is denoted by $ker(f) = \{(x, y) \in P \times P | f(x) = f(y)\}$. In the following theorem, we investigate the relation between the kernels of kernel operators and closure operators, and the kernels of upper adjoints and lower adjoints. These are crucial for us to characterize the I-homomorphism theorem (see Theorem 4.3) and II-homomorphism theorem (see Theorem 6.9).

Theorem 2.17. *Let P be a poset and R an equivalence relation on P. Then we have two groups of equivalent statements:*

(1) *R is the kernel of a kernel operator $k : P \to P$;*
(2) *there is an adjunction (g, d) between P and some T with $k = d \circ g$ and $R = ker(g)$.*
(1′) *R is the kernel of a closure operator $c : P \to P$;*
(2′) *there is an adjunction (g, d) between some S and P with $c = g \circ d$ and $R = ker(d)$.*

Proof. We only prove the equivalence of (1) and (2).

From Proposition O-3.10 in Gierz et al. (2003), $k : P \to P$ is a kernel operator on P if and only if there is an adjunction (g, d) between P and some T where $k = d \circ g$. Next, we prove that $R = ker(g)$. For any $(x, y) \in R$, we have $k(x) = k(y)$. That is, $dg(x) = dg(y)$. Thus, $gdg(x) = gdg(y)$. By Theorem 2.13, $g(x) = g(y)$. So, $(x, y) \in ker(g)$. Conversely, if $(x, y) \in ker(g)$, then $g(x) = g(y)$; it implies that $dg(x) = dg(y)$. Thus, $k(x) = k(y)$. So, $(x, y) \in R$. Therefore, $R = ker(g)$. □

3. I-Congruence Relations on Domains

In this section, we investigate the I-congruence relation on domains. First, we propose the concept of I-congruence relation on domains. Then we obtain that there is a bijective correspondence between all kernel operators which preserve directed sups and I-congruence relations which do not contain $P \times P$ on a domain P. Furthermore, we describe and characterize the unique order on the quotient set. At last, we get a complete picture of the I-congruence relation on domains. In addition, we take some examples to illustrate that the conditions of I-congruence relation on domains are mutually independent and not a single one can be omitted.

Let R be an equivalence relation on a poset P. Then we call that R satisfies the *dual-link property* or the *co-link property* if for any $a, b, b_1 \in P$, the following condition holds: if $a \leq b$ and $b \, R \, b_1$, then there exists an element $a_1 \in P$ such that $a_1 \leq b_1$ and $a \, R \, a_1$.

Definition 3.1. (Mahmoudi et al. (2017)). *Let P be a dcpo and R an equivalence relation on P. We say that R is a dcpo congruence if P/R with a suitable order is a dcpo and the quotient map $q : P \to P/R$ is Scott-continuous.*

Definition 3.2. *An equivalence relation R on a domain P is called a I-congruence relation if either $R = P \times P$ or it satisfies:*

1. *R is a subdcpo of $P \times P$;*
2. *each block of R has the smallest element;*
3. *R satisfies the dual-link property.*

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press

Now, we can give a more complete picture of I-congruence relation on domains and how they relate to kernel operators.

Theorem 3.3. *Let P be a domain and $R \subseteq P \times P$ an equivalence relation. Then the following conditions are equivalent:*

(1) *R is a I-congruence relation on P;*

(2) *$R = P \times P$ or there is a kernel operator k on P preserving sups of directed sets such that $R = (k \times k)^{-1}(\Delta)$, where Δ is the diagonal of $P \times P$;*

(3) *P/R is a singleton set or P/R with a suitable order is a domain in such a way that the quotient map $q : P \to P/R$ is an upper adjoint preserving sups of directed sets.*

Proof. (1) $\Rightarrow$ (2): If $R = P \times P$, it naturally holds. Otherwise, for any $p \in P$, from Condition 2 of Definition 3.2, $\inf [p]_R$ exists and $\inf [p]_R \in [p]_R$. Define a map $k : P \to P$ by $k(p) = \inf [p]_R$, where $p \in P$. Then, k is well defined and $k(p) \leq p$. In fact, $k(p)$ is the smallest element in $[p]_R$. It implies $k(k(p)) = k(p)$, that is, k is idempotent. Next, for any $x, y \in P$, suppose that $x \leq y$. Then, $k(y)\, R\, y$. By Condition 3 of Definition 3.2, there exists an element $u \in P$ with $u \leq k(y)$ and $u\, R\, x$. Whence, $k(x) = k(u) \leq u \leq k(y)$. It shows that k is monotone. Thus, k is a kernel operator.

Indeed, if $(x, y) \in R$, then $k(x) = k(y)$, and vice versa; therefore, we see that $(k \times k)^{-1}(\Delta) = R$. From Condition 1 of Definition 3.2, $(k \times k)^{-1}(\Delta)$ is a subdcpo of $P \times P$. So, k preserves sups of directed sets by Lemma 2.10.

(2) $\Rightarrow$ (3): If $R = P \times P$, then P/R is a singleton set. Otherwise, assume that there is a kernel operator k on P preserving sups of directed sets such that $R = (k \times k)^{-1}(\Delta)$. Since the image of a kernel operator k on P preserving sups of directed sets is a subdcpo of P (see Gierz et al. (2003), Proposition O-3.12), R is a dcpo congruence (see Mahmoudi et al. (2019), Corollary 4.7) and P/R with the relation defined by $[x]_R \leq_{P/R} [y]_R$ if and only if $k(x) \leq k(y)$ is a dcpo and $q : P \to P/R$ preserves sups of directed sets (see Mahmoudi et al. (2019), Theorem 4.1). By the first isomorphism theorem for dcpos (see Mahmoudi et al. (2019), Corollary 5.6), $(P/R, \leq_{P/R}) \cong k(P)$ and since $k(P)$ is a domain by Lemma 2.7, we obtain that P/R is a domain as well. Applying Decomposition Theorem for dcpos (see Mahmoudi et al. (2019), Theorem 5.3), there is a Scott-continuous map $f : P/R \to P$ given by $f([p]_R) = k(p)$ for any $[p]_R \in P/R$, and $k = f \circ q$. Based on Proposition O-3.10 in Gierz et al. (2003), (q, f) is an adjunction. Therefore, q is an upper adjoint preserving sups of directed sets.

(3) $\Rightarrow$ (1): We need to check the conditions of I-congruence relation on domains in Definition 3.2. If P/R is a singleton set, then we can directly get that $R = P \times P$. Otherwise, since the quotient map q preserves sups of directed sets, we have R is a subdcpo of $P \times P$ by Remark 2.11.

Since q is an upper adjoint, there exists an order-preserving map $h : P/R \to P$ such that (q, h) is an adjunction. Let $p \in P$. For any element $x \in [p]_R$, we have that $q(x) = q(p)$, and it follows that $h \circ q(x) = h \circ q(p)$. Because (q, h) is an adjunction, $h \circ q(p) = h \circ q(x) \leq x$ and $q \circ h \circ q(p) = q(p)$ by Theorem 2.13. Thus, $h \circ q(p)$ is the smallest element in $[p]_R$.

Let $v, a, b \in P$. Assume that $a \leq b$ and $b\, R\, v$. Then, $q(a) \leq q(b) = q(v)$. So, $h \circ q(a) \leq h \circ q(v) \leq v$ and $h \circ q(a)\, R\, a$ by Theorem 2.13. Thus, R satisfies the dual-link property. □

From the above theorem, we immediately get that there is a bijective correspondence between the kernel operators of P which preserve directed sups and I-congruence relations which do not contain $P \times P$ on P.

Corollary 3.4. *Let P be a domain. Then the rule $k \mapsto (k \times k)^{-1}(\Delta)$, where Δ is the diagonal of $P \times P$, associates with a kernel operator, its kernel congruence, and induces a bijection from the set of all kernel operators of P preserving directed sups onto the set of all I-congruence relations on P excluding $P \times P$.*

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press

For a domain P and a I-congruence relation R on P, by Theorem 3.3, the quotient set P/R is a singleton set or P/R with a suitable order is a domain. In order to clarify the order on P/R, we have the following lemma.

Lemma 3.5. *If P is a domain and R is a I-congruence relation on P, then $\leq_R$ (see Definition 2.1) is an order on P/R. Furthermore, it is the unique order on P/R such that the quotient map q is an upper adjoint.*

Proof. Obviously, $\leq_R$ is reflexive.

Assume that $[x]_R \leq_R [y]_R$ and $[y]_R \leq_R [x]_R$. Then there exist x_1, $x_2 \in [x]_R$, y_1, and $y_2 \in [y]_R$ such that $x_1 \leq y_1$ and $y_2 \leq x_2$. It means that $x_1 \leq y_1$ and $y_1 \, R \, y_2$. Since R satisfies the dual-link property, then there exists an element $x_3 \in [x]_R$ such that $x_3 \leq y_2$. Thus, $x_3 \leq y_2 \leq x_2$. Since R is a I-congruence relation on P, there exists an order $\leq_{P/R}$ on P/R such that the quotient map $q : P \to P/R$ is an upper adjoint preserving sups of directed sets by Theorem 3.3. Thus, $[x_3]_R \leq_{P/R} [y_2]_R \leq_{P/R} [x_2]_R$. Because $[x_3]_R = [x_2]_R$, we have that $[x]_R = [x_3]_R = [y_2]_R = [y]_R$.

At last, for the transitivity of $\leq_R$, suppose $[x]_R \leq_R [y]_R$ and $[y]_R \leq_R [z]_R$. Then there exist $x_1 \in [x]_R$, y_1, $y_2 \in [y]_R$ and $z_2 \in [z]_R$ such that $x_1 \leq y_1$ and $y_2 \leq z_2$. Because R satisfies the dual-link property, there exists $x_2 \in [x]_R$ with $x_2 \leq y_2$. That is, $x_2 \leq y_2 \leq z_2$. Thus, $[x]_R \leq_R [z]_R$.

Therefore, $\leq_R$ is an order on P/R.

Suppose that $\leq_{P/R}$ is an arbitrary order on P/R such that q is an upper adjoint. Then there is an order-preserving map $f : P/R \to P$ such that (q, f) is an adjunction. For any $x, y \in P$, if $[x]_R \leq_{P/R} [y]_R$, then $f([x]_R) \leq f([y]_R)$, that is, $fq(x) \leq fq(y)$. Since $qfq(x) = q(x)$ and $qfq(y) = q(y)$, we have that $[x]_R \leq_R [y]_R$ by the definition of $\leq_R$. It follows that $\leq_{P/R} \subseteq \leq_R$. Conversely, assume that $[x]_R \leq_R [y]_R$. Then there exist $x_0 \in [x]_R$ and $y_0 \in [y]_R$ such that $x_0 \leq y_0$. Thus, $q(x) = q(x_0) \leq_{P/R} q(y_0) = q(y)$. It means that $[x]_R \leq_{P/R} [y]_R$. Thus, $\leq_R \subseteq \leq_{P/R}$. Therefore, $\leq_R = \leq_{P/R}$. □

For a domain P and a I-congruence relation R on P, we get that $\inf [p]_R$ exists and $\inf [p]_R \in [p]_R$ for any element $p \in P$ by Definition 3.2. So, we could define a new relation on P/R as follows:

$$[x]_R \leq_R^1 [y]_R \text{ if } \inf [x]_R \leq \inf [y]_R \text{ for all elements } x, y \in P.$$

Furthermore, we obtain the following proposition.

Proposition 3.6. *If R is a I-congruence relation on a domain P, then for all elements x, y in P, the following conditions are equivalent:*

(1) $[x]_R \leq_R [y]_R$;

(2) $[x]_R \leq_R^1 [y]_R$;

(3) $[x]_R \leq_R^2 [y]_R$ *which is defined by the condition that for any element $b \in [y]_R$, there exists an element $a_b \in [x]_R$ such that $a_b \leq b$.*

Proof. It is easy to prove that (2) ⇒ (1) and (3) ⇒ (1).

(1) ⇒ (2): Since $[x]_R \leq_R [y]_R$, there exist $x_0 \in [x]_R$ and $y_0 \in [y]_R$ such that $x_0 \leq y_0$. Note that $\inf [y]_R \, R \, y_0$ and $x_0 \leq y_0$, then there is an element $c \in [x]_R$ such that $c \leq \inf [y]_R$, which implies that $\inf [x]_R \leq c \leq \inf [y]_R$; hence, (2) holds.

(1) ⇒ (3): Since $[x]_R \leq_R [y]_R$, there exist $x_0 \in [x]_R$ and $y_0 \in [y]_R$ such that $x_0 \leq y_0$. For any element $b \in [y]_R$, because $b \, R \, y_0$ and $x_0 \leq y_0$, there exists an element $a_b \in [x]_R$ such that $a_b \leq b$. Thus, (3) holds. □

Theorem 3.7. *Let P be a domain and R an equivalence relation on P. Then the following are equivalent:*

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press



Figure 1. The Hasse diagrams in Example 3.9.

(1) *R is a I-congruence relation on P;*
(2) *P/R is a singleton set or $(P/R, \leq_R)$ is a domain such that the quotient map $q : P \to P/R$ is an upper adjoint preserving sups of directed sets;*
(3) *P/R is a singleton set or for any element $p \in P$, $[p]_R$ has the smallest element, and $(P/R, \leq_R^1)$ is a domain such that the quotient map $q : P \to P/R$ is an upper adjoint preserving sups of directed sets;*
(4) *P/R is a singleton set or $(P/R, \leq_R^2)$ is a domain such that the quotient map $q : P \to P/R$ is an upper adjoint preserving sups of directed sets.*

Proof. By Theorem 3.3, it is straightforward to check that (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (1), and (4) $\Rightarrow$ (1).

If R is a I-congruence relation on P, then $\leq_R$ is an order on P/R by Lemma 3.5. Moreover, from Proposition 3.6, $\leq_R = \leq_R^1 = \leq_R^2$. Hence, we only need to prove (1) $\Rightarrow$ (3). Again by Theorem 3.3, R is a I-congruence relation if and only if $R = P \times P$ or there is a kernel operator k on P preserving sups of directed sets such that $R = (k \times k)^{-1}(\Delta)$, where Δ is the diagonal of $P \times P$. Recall that the proof of (2) $\Rightarrow$ (3) of Theorem 3.3, the order on P/R is defined as follows:

$$[x]_R \leq_{P/R} [y]_R \text{ if and only if } k(x) \leq k(y) \text{ for all } x, y \in P.$$

Now it remains to prove that $\inf [p]_R = k(p)$ for any element $p \in P$. As $R = (k \times k)^{-1}(\Delta)$ and k is a kernel operator, we can see that $k(p) \in [p]_R$. Furthermore, let $x \in [p]_R$. It means that $k(p) = k(x) \leq x$. So, $k(p)$ is the smallest element in $[p]_R$. Whence, $\inf [p]_R = k(p)$. □

Remark 3.8. *All results about the I-congruence relation on domains do not rely on continuity; hence, these conclusions remain valid when the domain is replaced by a dcpo.*

We now illustrate by examples that Conditions 1, 2, and 3 given in Definition 3.2 are mutually independent.

Example 3.9. *Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ with the natural order and $P = \mathbb{N} \oplus \{\infty\}$. Then it is easy to verify that P is a domain. Consider an equivalence relation R on P consisting of exactly two blocks $\mathbb{N}$ and $\{\infty\}$. In particular, Conditions 2 and 3 of Definition 3.2 hold. However, Condition 1 does not hold.*

In fact, we could see that the quotient set $P/R = \{\mathbb{N}, [\infty]_R\}$. Then, $(P/R, \leq_R)$ is a domain and the quotient map $q : P \to (P/R, \leq_R)$ is not Scott-continuous. By Theorem 3.3, R is not a I-congruence relation on domain P. It implies that Condition 1 given in Definition 3.2 does not hold. The Hasse diagrams see Figure 1.

Example 3.10. *Suppose that the poset P is given in the following:*

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press



Figure 2. The Hasse diagrams in Example 3.10.



Figure 3. The Hasse diagrams in Example 3.11.

$P = \{a, b, c, d\}$ and the order $\leq_P$ is defined as follows: $a \leq b$, $a \leq c$, $b \leq d$, and *$c \leq d$.*

Let R be an equivalence relation consisting of exactly three blocks $\{a\}$, $\{b, c\}$, and $\{d\}$. Then the block $\{b, c\}$ does not have the smallest element. In particular, it is easy to check that R satisfies Condition 3 of Definition 3.2. Furthermore, we could see that the quotient set $P/R = \{[a]_R, [b]_R, [d]_R\}$. Then $(P/R, \leq_R)$ is a domain and the quotient map $q : P \to (P/R, \leq_R)$ is Scott-continuous. It means that R is a subdcpo of $P \times P$ by Remark 2.11. Hence, Condition 1 given in Definition 3.2 holds. The Hasse diagrams see Figure 2.

Example 3.11. *Take the domain P in Example 3.10. Let $R = \Delta_P \cup \{(c, d), (d, c)\}$. Then R is an equivalence relation on P and satisfies Condition 2 of Definition 3.2. Furthermore, we could see that the quotient set $P/R = \{[a]_R, [b]_R, [c]_R\}$. Then, $(P/R, \leq_R)$ is a three-element chain and the quotient map $q : P \to (P/R, \leq_R)$ is Scott-continuous. Hence, R satisfies Condition 1 given in Definition 3.2. We could know that $b \leq d$ and $c\, R\, d$, but b is the only element in the block $\{b\}$ and $b \not\leq c$. So, R does not satisfy the dual-link property. The Hasse diagrams see Figure 3.*

4. I-Homomorphisms on Domains

In this section, we first define the I-homomorphism on domains, and later on, we will show that the kernel of a I-homomorphism on domains is a I-congruence relation on domains.

Definition 4.1. *Let P and Q be posets. A Scott-continuous map $f : P \to Q$ is called a I-homomorphism if* $\mathrm{card}(f(P)) = 1$ *or it is an upper adjoint.*

Theorem 4.2. *Let M be the image of a I-homomorphism $f : P \to Q$. If P is a domain, an L-domain, a bounded complete domain, a continuous semilattice, and a continuous lattice, the same is true for the image M.*

Proof. If $\mathrm{card}(f(P)) = 1$, then M is a singleton set. It implies that M is a continuous lattice. Otherwise, f is an upper adjoint. Then there exists an order-preserving map $g : Q \to P$ such that (f, g) is an adjunction between Q and P. Since for any pair of adjunctions (f, g), the image of f

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press

is order-isomorphic with the image of gf which is a kernel operator and, of course, a projection map, by Lemma 2.7, this theorem trivially holds. □

Theorem 4.3. (I-Homomorphism theorem). *Let P and Q be domains, and let $f : P \to Q$ be a I-homomorphism on domains. Then $ker(f) = \{(x, y) \in P \times P \mid f(x) = f(y)\}$ is a I-congruence relation on P and $f(P) \cong P/ker(f)$.*

Proof. If $\text{card}(f(P)) = 1$, then $ker(f) = P \times P$, and it is a I-congruence relation on P and $f(P) \cong P/ker(f)$. Otherwise, f is an upper adjoint, based on Theorem 2.17, the kernel of an upper adjoint is the kernel of a kernel operator and by Theorem 3.3, $ker(f)$ is a I-congruence relation on P and $f(P) \cong P/ker(f)$, because the kernels of kernel operators are characterized as I-congruence relations on P. □

By Theorem 4.3, we get the following corollary.

Corollary 4.4. (I-Isomorphism theorem). *Let P and Q be domains and $f : P \to Q$ be a surjective I-homomorphism on domains. Then, $ker(f) = \{(x, y) \in P \times P \mid f(x) = f(y)\}$ is a I-congruence relation on P and $Q \cong P/ker(f)$.*

5. II-Congruence Relations on Domains

In this section, we investigate the II-congruence relation on domains. First, we propose the concept of II-congruence relation on domains. Additionally, some necessary conditions for an equivalence relation to be a II-congruence relation on domains are given. Then, we obtain that there is a bijective correspondence between the closure operators which preserve directed sups and II-congruence relations which do not contain $P \times P$ on domain P. Furthermore, we describe and characterize the unique order on the quotient set. At last, we get a complete picture of the II-congruence relation on domains. Furthermore, we take some examples to illustrate that the conditions of II-congruence relation on domains are mutually independent and not a single one can be omitted.

Let R be an equivalence relation on a poset P. Then we call that R satisfies the *link property* (see Blyth and Janowitz (1972) and Mahmoudi et al. (2017)) if for any $a, b, a_1 \in P$, the following condition holds: if $a \leq b$ and $a \, R \, a_1$, then there exists an element $b_1 \in P$ such that $a_1 \leq b_1$ and $b \, R \, b_1$.

Definition 5.1. *An equivalence relation R on a domain P is called a II-congruence relation if either $R = P \times P$ or it satisfies:*

1. *R is a subdcpo of $P \times P$;*
2. *for any $p \in P$, $[p]_R$ is a directed subset of P;*
3. *R satisfies the link property;*
4. *$\{\sup [p]_R \mid p \in P\}$ is a subdcpo of P.*

Proposition 5.2. *Let P be a domain and R an equivalence relation on P. If R satisfies Conditions 1 and 2 in Definition 5.1, then each block B of R is a subdcpo of P. Moreover, B has a top element.*

Proof. Let B be a block of R. If $D \subseteq B$ is directed, note that there is an element $p \in P$ such that $B = [p]_R$. Define $\mathcal{D} = \{(d, p) \mid d \in D\}$. Then $\mathcal{D}$ is directed in R. Since R is a subdcpo of $P \times P$, $\sup \mathcal{D} = (\sup D, p) \in R$. It implies that $\sup D \in [p]_R = B$. That is, B is a subdcpo of P. Furthermore,

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press

from Condition 2 of Definition 5.1, B is directed in P, we deduce that $\sup B$ exists, and hence, $\sup B \in B$. It shows that B has a top element. □

Now, we can give a more complete picture of II-congruence relation on domains and how they relate to closure operators.

Theorem 5.3. *Let P be a domain and $R \subseteq P \times P$ an equivalence relation. Then the following conditions are equivalent:*

(1) *R is a II-congruence relation on P;*

(2) *$R = P \times P$ or there is a closure operator c on P preserving sups of directed sets such that $R = (c \times c)^{-1}(\Delta)$, where Δ is the diagonal of $P \times P$;*

(3) *P/R is a singleton set or P/R with a suitable order is a domain in such a way that the quotient map $q : P \to P/R$ preserves sups of directed sets and the way-below relation $\ll$ and for any elements $x, y \in P$, q satisfies the following equation:*

$$q(\uparrow x \cap \uparrow y)) = \uparrow q(x) \cap \uparrow q(y).$$

Proof. (1) $\Rightarrow$ (2): If $R = P \times P$, it naturally holds. Otherwise, by Theorem 2.12 in M. Mahmoudi, H. Moghbeli & K. Pióro (2017), Lemma 2.9 and Proposition 5.2, (2) holds.

(2) $\Rightarrow$ (3): If $R = P \times P$, then P/R is a singleton set. Otherwise, assume that there is a closure operator c on P preserving sups of directed sets such that $R = (c \times c)^{-1}(\Delta)$. Since the image of a closure operator k on P preserving sups of directed sets is a subdcpo of P (see Lemma 2.9), R is a dcpo congruence (see Mahmoudi et al. (2019), Corollary 4.7) and P/R with the partial order $[x]_R \leq_{P/R} [y]_R$ if and only if $c(x) \leq c(y)$ is a dcpo and $q : P \to P/R$ preserves sups of directed sets (see Mahmoudi et al. (2019), Theorem 4.1). By the first isomorphism theorem for dcpos (see Mahmoudi et al. (2019), Corollary 5.6), $(P/R, \leq_{P/R}) \cong c(P)$, we obtain that P/R is a domain as well since $c(P)$ is a domain by Lemma 2.7. Applying Decomposition Theorem for dcpos (see Mahmoudi et al. (2019), Theorem 5.3), there is a Scott-continuous map $f : P/R \to P$ given by $f([p]_R) = c(p)$ for any $[p]_R \in P/R$, and $c = f \circ q$. For $x, y \in P$, suppose that $x \ll y$ in P. Let $\mathcal{D} \subseteq P/R$ be a directed subset and $[y]_R \leq_{P/R} \sup \mathcal{D}$. Define $D = \{c(d) \mid [d]_R \in \mathcal{D}\}$. Since f is Scott-continuous, we get that D is directed in $c(P)$ and

$$f(y) = c(y) \leq f(\sup \mathcal{D}) = \sup_{[d]_R \in \mathcal{D}} f([d]_R) = \sup_{[d]_R \in \mathcal{D}} c(d) = \sup D.$$

As $x \ll y$, we have that $c(x) \ll_{c(P)} c(y) \leq \sup D$ by Lemma 2.8. It implies that there exists an element $[d]_R \in \mathcal{D}$ such that $c(x) \leq c(d)$. That is, $[x]_R \leq_{P/R} [d]_R$. Thus, $[x]_R \ll [y]_R$. This indicates that q preserves the way-below relation $\ll$.

Finally, we show that for any $x, y \in P$, $q(\uparrow x \cap \uparrow y)) = \uparrow q(x) \cap \uparrow q(y)$. Since q is Scott-continuous, it is order-preserving, and we get that $q(\uparrow x \cap \uparrow y)) \subseteq \uparrow q(x) \cap \uparrow q(y)$. Conversely, assume that $[z]_R \in \uparrow q(x) \cap \uparrow q(y)$. Then $[x]_R \leq_{P/R} [z]_R$ and $[y]_R \leq_{P/R} [z]_R$. It means that $c(x) \leq c(z)$ and $c(y) \leq c(z)$. Since c is a closure operator, $x \leq c(x) \leq c(z)$, $y \leq c(y) \leq c(z)$, and $c(c(z)) = c(z)$. Therefore, $c(z) \in \uparrow x \cap \uparrow y$. By (2), $R = (c \times c)^{-1}(\Delta)$. So, we conclude that $c(z) \in [z]_R$. Hence, $q(c(z)) = [z]_R$. It follows that $\uparrow q(x) \cap \uparrow q(y) \subseteq q(\uparrow x \cap \uparrow y))$. Whence we have that $q(\uparrow x \cap \uparrow y)) = \uparrow q(x) \cap \uparrow q(y)$.

(3) $\Rightarrow$ (1): We need to check the conditions of Definition 5.1. If P/R is a singleton set, we can directly get that $R = P \times P$. Otherwise, first, since the quotient map q preserves sups of directed sets, we could show that R is a subdcpo of $P \times P$ by Remark 2.11.

Let $p \in P$. Then, $p \in [p]_R$. So, $[p]_R$ is nonempty. For any elements $x, y \in [p]_R$, we have that $q(x) = q(y) = q(p)$. It implies that $q(p) \in \uparrow q(x) \cap \uparrow q(y)$. As $q(\uparrow x \cap \uparrow y) = \uparrow q(x) \cap \uparrow q(y)$, there exists an element $z \in \uparrow x \cap \uparrow y$ such that $q(z) = q(p)$. Therefore, $[p]_R$ is directed.

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press

Let $u, a, b \in P$. Assume that $a \leq b$ and $u \, R \, a$. Then, $q(u) = q(a) \leq q(b)$. Note that $q(b) \in \uparrow q(u) \cap \uparrow q(b)$. Again since $q(\uparrow u \cap \uparrow b) = \uparrow q(u) \cap \uparrow q(b)$, there is an element $v \in \uparrow u \cap \uparrow b$ such that $q(v) = q(b)$, which means that $u \leq v$ and $b \, R \, v$.

From the above proof, we can see that for all $p \in P$, $\sup [p]_R$ exists. Finally, we need to prove that $\{\sup [p]_R \mid p \in P\}$ is a subdcpo of P. Let $D \subseteq \{\sup [p]_R \mid p \in P\}$ be a directed subset. It is also directed in P. So, $\sup D$ exists. We claim that $\sup D = \sup [\sup D]_R$. Clearly, $\sup D \in [\sup D]_R$. It is enough to show that $x \leq \sup D$ for all $x \in [\sup D]_R$. Suppose not, there is an element $x_0 \in [\sup D]_R$ such that $x_0 \not\leq \sup D$. Since P is a domain, there exists an element $y \ll x_0$ but $y \not\leq \sup D$. By (3), as q preserves the way-below relation $\ll$, we infer that $q(y) \ll q(x_0) = q(\sup D)$. Because q is Scott-continuous and D is directed,

$$q(y) \ll q(x_0) = q(\sup D) = \sup_{d \in D} q(d).$$

Thus, there exists $d \in D$ such that $q(y) \leq q(d)$. Note that $q(d) \in \uparrow q(y) \cap \uparrow q(d)) = q(\uparrow y \cap \uparrow d))$, it implies that there is an element $z \in \uparrow y \cap \uparrow d$ such that $q(z) = q(d)$. That is, $y \leq z$, $d \leq z$, and $d \, R \, z$. Since d is the largest element in $[d]_R$, $y \leq z = d$. Hence, $y \leq d \leq \sup D$, which is a contradiction. Whence $x \leq \sup D$ for all $x \in [\sup D]_R$. Therefore, $\sup D = \sup [\sup D]_R$. It follows that $\sup D \in \{\sup [p]_R \mid p \in P\}$. So, $\{\sup [p]_R \mid p \in P\}$ is a subdcpo of P. □

From the above theorem, we immediately get that there is a bijective correspondence between the closure operators of P preserving directed sups and II-congruence relations which do not contain $P \times P$ on P.

Corollary 5.4. *Let P be a domain. Then the rule $c \mapsto (c \times c)^{-1}(\Delta)$, where Δ is the diagonal of $P \times P$, associates with a closure operator its kernel congruence and induces a bijection from the set of all closure operators of P which preserve directed sups onto the set of all II-congruence relations on P excluding $P \times P$.*

Proposition 5.5. *Let P be a domain and R a II-congruence relation on P. Then each block B of R is a convex subset of P (i.e., for all $a, b, p \in P$, if $a, b \in B$ and $a \leq p \leq b$, then $p \in B$).*

Proof. Take elements $a, b, p \in P$ such that $a \leq p \leq b$ and $a \, R \, b$. By Theorem 5.3, there exists an order $\leq_{P/R}$ on the quotient set P/R such that the quotient map $q : P \to P/R$ is Scott-continuous, so it is also order-preserving. Then $q(a) = q(b)$ and $q(a) \leq q(p) \leq q(b)$. It implies that $q(a) = q(p)$. Thus, $a \, R \, p$. It also follows from the analogous result for posets (see Blyth and Janowitz (1972)), and dcpos (see Mahmoudi et al. (2017)), because R is, particularly, a poset congruence, respectively a dcpo congruence. □

From Proposition 5.2, Theorems 5.3 and 2.12 in Mahmoudi et al. (2017), we obtain the following corollary.

Corollary 5.6. *Let P be a domain and $R \neq P \times P$ an equivalence relation on P. Then the following conclusions are equivalent:*

(1) *R is a II-congruence relation on P;*
(2) (*a*) *each block of R has a top element and the set of all elements is a subdcpo of P,*
(*b*) *R satisfies the link property.*

For a domain P and a II-congruence relation R on P, from Theorem 5.3, the quotient set P/R is a singleton set or P/R with a suitable order is a domain. In order to clarify the order on P/R, we have the following lemma.

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press

Lemma 5.7. *If P is a domain and R is a II-congruence relation on P, then $\leq_R$ is an order on P/R. Furthermore, it is the unique order on P/R such that for any $x, y \in P$, the quotient map q satisfies the following equation:*

$$q(\uparrow x \cap \uparrow y)) = \uparrow q(x) \cap \uparrow q(y).$$

Proof. Obviously, $\leq_R$ is reflexive.

Assume that $[x]_R \leq_R [y]_R$ and $[y]_R \leq_R [x]_R$. Then there exist $x_1, x_2 \in [x]_R$, $y_1, y_2 \in [y]_R$ such that $x_1 \leq y_1$ and $y_2 \leq x_2$. It means that $x_1 \leq y_1$ and $x_1 \, R \, x_2$. Since R satisfies the link property, there exists an element $y_3 \in [y]_R$ such that $x_2 \leq y_3$. Thus $y_2 \leq x_2 \leq y_3$. By Proposition 5.5, $[y]_R$ is convex, then we have that $x_2 \in [y]_R$. As equivalence classes are mutually disjoint, we get that $[x]_R = [y]_R$. The antisymmetry of $\leq_R$ is proved.

At last, for the transitivity of $\leq_R$, suppose $[x]_R \leq_R [y]_R$ and $[y]_R \leq_R [z]_R$. Then there exist $x_1 \in [x]_R, y_1, y_2 \in [y]_R$, and $z_2 \in [z]_R$ such that $x_1 \leq y_1$ and $y_2 \leq z_2$. Because R satisfies the link property, there exists $z_1 \in [z]_R$ with $y_1 \leq z_1$. That is, $x_1 \leq y_1 \leq z_1$. Thus, $[x]_R \leq_R [z]_R$.

Therefore, $\leq_R$ is an order on P/R.

Suppose that $\leq_{P/R}$ is an arbitrary order on P/R such that for any $x, y \in P$, q satisfies the equation $q(\uparrow x \cap \uparrow y) = \uparrow q(x) \cap \uparrow q(y)$. For any $x, y \in P$, if $[x]_R \leq_{P/R} [y]_R$, then $[y]_R \in \uparrow q(x) \cap \uparrow q(y) = q(\uparrow x \cap \uparrow y)$. It implies that there exists $z \in \uparrow x \cap \uparrow y$ such that $z \in [y]_R$. Note that $x \leq z$. So $[x]_R \leq_R [y]_R$ by the definition of $\leq_R$. Thus, $\leq_{P/R} \subseteq \leq_R$. Conversely, assume that $[x]_R \leq_R [y]_R$. Then there exist $x_0 \in [x]_R$ and $y_0 \in [y]_R$ such that $x_0 \leq y_0$. Thus, $[y]_R \in q(\uparrow x_0 \cap \uparrow y_0) = \uparrow q(x_0) \cap \uparrow q(y_0)$. It follows that $[x]_R \leq_{P/R} [y]_R$. So $\leq_R \subseteq \leq_{P/R}$. Therefore, $\leq_R = \leq_{P/R}$. □

For a domain P and a II-congruence relation R on P, we get that for any element $p \in P$, $\sup [p]_R$ exists and $\sup [p]_R \in [p]_R$ by Proposition 5.2. So, we could define a new relation on P/R as follows:

$$[x]_R \leq'_R [y]_R \text{ if } \sup [x]_R \leq \sup [y]_R \text{ for all elements } x, y \in P.$$

Furthermore, we obtain the following proposition.

Proposition 5.8. *If R is a II-congruence relation on a domain P, then for all elements x, y in P, the following conditions are equivalent:*

(1) $[x]_R \leq_R [y]_R$;
(2) $[x]_R \leq'_R [y]_R$;
(3) $[x]_R \leq''_R [y]_R$ *which is defined by the condition that for any element $a \in [x]_R$, there exists an element $b_a \in [y]_R$ such that $a \leq b_a$.*

Proof. It is easy to prove that (2) ⇒ (3) and (3) ⇒ (1).

(1) ⇒ (2): Since $[x]_R \leq_R [y]_R$, there exist $x_0 \in [x]_R$ and $y_0 \in [y]_R$ such that $x_0 \leq y_0$. Note that $\sup [x]_R \, R \, x_0$ and $x_0 \leq y_0$, then there is an element $c \in [y]_R$ such that $\sup [x]_R \leq c$, which implies that $\sup [x]_R \leq c \leq \sup [y]_R$; hence, (2) holds. □

Theorem 5.9. *Let P be a domain and R an equivalence relation on P. Then the following are equivalent:*

(1) *R is a II-congruence relation on P;*
(2) *P/R is a singleton set or $(P/R, \leq_R)$ is a domain such that the quotient map $q : P \to P/R$ preserves sups of directed sets and the way-below relation $\ll$ and for any $x, y \in P$, q satisfies the following equation:*

$$q(\uparrow x \cap \uparrow y) = \uparrow q(x) \cap \uparrow q(y);$$

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press

(3) *P/R is a singleton set or for all element $p \in P$, $\sup [p]_R$ exists and $(P/R, \leq'_R)$ is a domain such that the quotient map $q : P \to P/R$ preserves sups of directed sets and the way-below relation $\ll$ and for any $x, y \in P$, q satisfies the following equation:*

$$q(\uparrow x \cap \uparrow y) = \uparrow q(x) \cap \uparrow q(y);$$

(4) *P/R is a singleton set or $(P/R, \leq''_R)$ is a domain such that the quotient map $q : P \to P/R$ preserves sups of directed sets and the way-below relation $\ll$ and for any $x, y \in P$, q satisfies the following equation:*

$$q(\uparrow x \cap \uparrow y) = \uparrow q(x) \cap \uparrow q(y).$$

Proof. By Theorem 5.3, it is straightforward to check that (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (1), and (4) $\Rightarrow$ (1).

If R is a II-congruence relation on P, then $\leq_R$ is an order on P/R by Lemma 5.7. Moreover, from Proposition 5.8, $\leq_R = \leq'_R = \leq''_R$. Hence, we only need to prove (1) $\Rightarrow$ (3). Again by Theorem 5.3, R is a II-congruence relation if and only if $R = P \times P$ or there is a closure operator c on P preserving sups of directed sets such that $R = (c \times c)^{-1}(\Delta)$, where Δ is the diagonal of $P \times P$. Recall that the proof of (2) $\Rightarrow$ (3) of Theorem 5.3, we define the order on P/R below:

$$[x]_R \leq_{P/R} [y]_R \text{ if and only if } c(x) \leq c(y) \text{ for all } x, y \in P.$$

Now it remains to prove that $\sup [p]_R = c(p)$ for any element $p \in P$. As $R = (c \times c)^{-1}(\Delta)$ and c is a closure operator, we can see that $c(p) \in [p]_R$. Furthermore, let $x \in [p]_R$. It follows that $x \leq c(x) = c(p)$. So, $c(p)$ is the largest element in $[p]_R$. Whence $\sup [p]_R = c(p)$. □

We now illustrate by examples that Conditions 1, 2, 3, and 4 given in Definition 5.1 are mutually independent.

Example 5.10. *Consider P and R as in Example 3.9. In particular, R satisfies Conditions 2, 3 and 4 of Definition 5.1, but it does not satisfy Condition 1 of Definition 5.1.*

In fact, $\sup [0]_R = \sup [\infty]_R = \infty$. So, $[0]_R \leq'_R [\infty]_R$ and $[\infty]_R \leq'_R [0]_R$, but $[0]_R \neq [\infty]_R$. Hence, we conclude that $\leq'_R$ is not an order on P/R. By Theorem 5.9, R is not a II-congruence relation on the domain P. It implies that Condition 1 given in Definition 5.1 does not hold.

Example 5.11. *Consider P and R as in Example 3.10. Then R is an equivalence relation on P and satisfies Conditions 3 and 4 of Definition 5.1. For the block $\{b, c\}$, it is not directed. So, R does not satisfy Condition 2 of Definition 5.1. Furthermore, we could see that the quotient set $P/R = \{[a]_R, [c]_R, [d]_R\}$. Then, $(P/R, \leq_R)$ is a domain and the quotient map $q : P \to (P/R, \leq_R)$ is Scott-continuous. It follows that R is a subdcpo of $P \times P$. Hence, Condition 1 given in Definition 5.1 holds.*

Example 5.12. *Take a domain P, as given in the following:*

$$P = \{a, b, c\} \text{ with the order } a \leq b,\ a \leq c.$$

Consider R is an equivalence relation composed of two blocks $\{a, b\}$ and $\{c\}$. Then, $(P/R, \leq_R)$ is a domain. It is easy to show that Conditions 1, 2, and 4 in Definition 5.1 hold. For Condition 3, we can see that $a \leq c$ and $a\, R\, b$, but c is the only element of the block $\{c\}$ and $b \not\leq c$. Hence, R does not hold the link property. The Hasse diagrams see Figure 4.

Let B be a nonempty subset of a set P. Recall that the equivalence relation on P with B being its only nontrivial block is called the *Rees equivalence relation* and denoted by R_B (see Mahmoudi et al. (2017)). That is, $p\, R_B\, q$ if and only if $p = q$ or $p, q \in B$. To simplify the notation, we write P/B and $[p]_B$ instead of P/R_B and $[p]_{R_B}$, respectively.

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press



Figure 4. The Hasse diagrams in Example 5.12.



Figure 5. The Hasse diagrams in Example 5.13.

Example 5.13. *Let* $\mathbb{N} = \{0, 1, 2, \cdots\}$ *and* $P = \mathbb{N} \cup \{\infty\} \cup \{a_n \mid n \in \mathbb{N}\} \cup \{\top\}$. *The order* $\leq$ *on* P *is defined as follows:*

- $n \leq \infty$ *for all* $n \in \mathbb{N}$;
- $0 \leq 1 \leq 2 \leq \cdots \leq n \leq \cdots$;
- $n \leq a_n$ *for all* $n \in \mathbb{N}$;
- $a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$;
- $a_n \leq \top$ *for all* $n \in \mathbb{N}$;
- $\infty \leq \top$.

Then, P is a domain. Consider R_B *is the Rees equivalence relation with the nontrivial block*

$$B = \{a_n \mid n \in \mathbb{N}\} \cup \{\infty, \top\}.$$

Now, we could see that Conditions 2 and 3 of Definition 5.1 hold. Moreover, $\sup [p]_B$ *exists for all* $p \in P$. *However,* $\{\sup [p]_B \mid p \in P\}$ *is not a subdcpo. So, Condition 4 given in Definition 5.1 does not hold. For Condition 1, we get that the quotient set* P/B *with the relation* $\leq_{R_B}$ *is a domain and the quotient map* $q : P \to (P/B, \leq_{R_B})$ *is Scott-continuous. It implies that* R_B *is a subdcpo of* $P \times P$. *Whence* R_B *satisfies Condition 1 of Definition 5.1. The Hasse diagrams see Figure 5.*

6. II-Homomorphisms on Domains

In this section, we define the concept of II-homomorphism on domains, for which analogs of the homomorphism and isomorphism theorems of universal algebras hold.

Definition 6.1. *Let* P *and* Q *be posets. A Scott-continuous map* $f : P \to Q$ *is called a II-homomorphism if* $\mathrm{card}(f(P)) = 1$ *or it preserves the way-below relation* $\ll$ *and for any* $x, y \in P$,

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press

f satisfies the following equation:

$$\uparrow f(\uparrow x \cap \uparrow y) = \uparrow f(x) \cap \uparrow f(y).$$

Proposition 6.2. *Let P and Q be posets and $f : P \to Q$ a map. If P is actually a sup semilattice, then the following are equivalent:*

(1) *for all elements $x, y \in P$, f satisfies the following equation:*

$$\uparrow f(\uparrow x \cap \uparrow y) = \uparrow f(x) \cap \uparrow f(y);$$

(2) *for all elements $x, y \in P$, we have that $f(x \vee y) = f(x) \vee f(y)$, that is, f preserves sups of finite sets.*

By Proposition 2.4, if f is surjective, then the above conditions are equivalent to the following equation:

$$f(\uparrow x \cap \uparrow y) = \uparrow f(x) \cap \uparrow f(y), \text{ for all } x, y \in P$$

Proof. (1) $\Rightarrow$ (2): Let $x, y \in P$ and $x \leq y$. Then, $y \in \uparrow x \cap \uparrow y$. Whence

$$f(y) \in \uparrow f(\uparrow x \cap \uparrow y) = \uparrow f(x) \cap \uparrow f(y).$$

It implies that $f(x) \leq f(y)$. Thus, f is order-preserving. The same argument gives that $f(x \vee y)$ is an upper bound of $f(x)$ and $f(y)$. Let z be another upper bound of $f(x)$ and $f(y)$. Then, $z \in \uparrow f(x) \cap \uparrow f(y) = \uparrow f(\uparrow x \cap \uparrow y)$. It means that there is an element $p \in \uparrow x \cap \uparrow y$ such that $f(p) \leq z$. Note that $x, y \leq p$. Since P is a sup semilattice, $x \vee y$ exists and $x \vee y \leq p$. We get that $f(x \vee y) \leq f(p) \leq z$. Hence, $f(x \vee y) = f(x) \vee f(y)$. Thus, f preserves sups of finite subsets.

(2) $\Rightarrow$ (1): Since f preserves sups of finite sets, then f is order-preserving. So, $\uparrow f(\uparrow x \cap \uparrow y) \subseteq \uparrow f(x) \cap \uparrow f(y)$. Conversely, let $z \in \uparrow f(x) \cap \uparrow f(y)$. Then, $f(x) \leq z$ and $f(y) \leq z$. It implies that $f(x \vee y) = f(x) \vee f(y) \leq z$. Note that $x \vee y \in \uparrow x \cap \uparrow y$. It follows that $f(x \vee y) \in f(\uparrow x \cap \uparrow y)$. Thus, $z \in \uparrow f(\uparrow x \cap \uparrow y)$. Hence, $\uparrow f(x) \cap \uparrow f(y) \subseteq \uparrow f(\uparrow x \cap \uparrow y)$. Therefore, $\uparrow f(x) \cap \uparrow f(y) = \uparrow f(\uparrow x \cap \uparrow y)$. □

Theorem 6.3. *Let P and Q be posets and $f : P \to Q$ a map. If P is a complete lattice, consider the following conditions:*

(1) *f is a II-homomorphism;*

(2) *f preserves sups of arbitrary subsets and the way-below relation $\ll$.*

Then, (2) $\Rightarrow$ (1). *Furthermore, if f is surjective,* (1) $\Rightarrow$ (2).

Proof. Since P is a complete lattice, the map f preserving sups of arbitrary subsets implies it is Scott-continuous and preserves sups of finite sets. Thus, (2) $\Rightarrow$ (1) by Proposition 6.2. When f is surjective, the whole proof will be complete if we prove the following.

(1) $\Rightarrow$ (2): If $\mathrm{card}(f(P)) = 1$, then it is easy to verify that f preserves sups of arbitrary subsets and the way-below relation $\ll$ by the assumption that f is surjective. Otherwise, since f is Scott-continuous and preserves sups of finite sets, we have that f preserves sups of nonempty subsets. It suffices to prove that f preserves sups of the empty subset. We know that $\sup \emptyset = \bot$. So, $f(\sup \emptyset) = f(\bot)$. Note that $f(\bot)$ is a lower bound of $f(P)$ and $f(\bot) \in f(P)$. Therefore, $f(\bot)$ is the smallest element in $f(P)$. As f is a surjective map, $f(P) = Q$. This implies that $f(\bot)$ is the smallest element in Q. Thus, $\sup f(\emptyset) = f(\bot)$. It follows that $f(\sup \emptyset) = \sup f(\emptyset)$. Whence we get that f preserves sups of arbitrary subsets. □

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press

Lemma 6.4. *If $f : P \to Q$ is an order-preserving map between posets which has an upper adjoint, then it is Scott-continuous and for all $x, y \in P$, f satisfies the following equation:*

$$\uparrow f(\uparrow x \cap \uparrow y) = \uparrow f(x) \cap \uparrow f(y).$$

Proof. Since f has an upper adjoint, there is an order-preserving map $g : Q \to P$ such that (g, f) is an adjunction. By Lemma 2.15, f preserves existing sups. So it is Scott-continuous. Thus, for all $x, y \in P$, $\uparrow f(\uparrow x \cap \uparrow y) \subseteq \uparrow f(x) \cap \uparrow f(y)$. Conversely , let $z \in \uparrow f(x) \cap \uparrow f(y)$. Then $f(x) \leq z$ and $f(y) \leq z$. So, $x \leq g(z)$ and $y \leq g(z)$. It follows that $g(z) \in \uparrow x \cap \uparrow y$. Hence, $fg(z) \in f(\uparrow x \cap \uparrow y)$. Note that $fg(z) \leq z$ by Theorem 2.13. Therefore, $z \in \uparrow f(\uparrow x \cap \uparrow y)$. This implies that $\uparrow f(x) \cap \uparrow f(y) \subseteq \uparrow f(\uparrow x \cap \uparrow y)$. Whence we get that for all $x, y \in P$, f satisfies the equation: $\uparrow f(\uparrow x \cap \uparrow y) = \uparrow f(x) \cap \uparrow f(y)$. □

Theorem 6.5. *Let P be a dcpo and Q a poset. If a map $f : P \to Q$ is surjective, then the following are equivalent:*

(1) *f has an upper adjoint;*

(2) *f is Scott-continuous and for all $x, y \in P$, it satisfies the following equation:*

$$f(\uparrow x \cap \uparrow y) = \uparrow f(x) \cap \uparrow f(y).$$

Proof. It remains to prove that (2) $\Rightarrow$ (1). For each $q \in Q$, as f is surjective, $f^{-1}(q) \neq \emptyset$. Let $d_1, d_2 \in f^{-1}(q)$. Then, $f(d_1) = f(d_2) = q$. It implies that $q \in \uparrow f(d_1) \cap \uparrow f(d_2)$. Since $f(\uparrow d_1 \cap \uparrow d_2) = \uparrow f(d_1) \cap \uparrow f(d_2)$, there exists $d_3 \in \uparrow d_1 \cap \uparrow d_2$ such that $f(d_3) = q$. So, $f^{-1}(q)$ is directed. Because P is a dcpo, $\sup f^{-1}(q)$ exists. Define a map $g : Q \to P$ by $q \mapsto \sup f^{-1}(q)$, where $q \in Q$.

Claim 1: g is order-preserving.

Let $q_1, q_2 \in Q$ and $q_1 \leq q_2$. Since f is surjective, there exist $p_1, p_2 \in P$ such that $f(p_1) = q_1$ and $f(p_2) = q_2$. For each element $x \in f^{-1}(q_1)$, we get that $f(x) = q_1$. So, $q_2 \in \uparrow f(x) \cap \uparrow f(p_2)$. As $f(\uparrow x \cap \uparrow p_2) = \uparrow f(x) \cap \uparrow f(p_2)$, there exists an element $y \in \uparrow x \cap \uparrow p_2$ such that $f(y) = q_2$. Note that $x \leq y$ and $y \in f^{-1}(q_2)$. It implies that $x \leq \sup f^{-1}(q_2)$. Thus, $\sup f^{-1}(q_1) \leq \sup f^{-1}(q_2)$. It follows that $g(q_1) \leq g(q_2)$. Hence, the map g is order-preserving.

Claim 2: (g, f) is an adjunction.

For all pairs of elements $(p, q) \in P \times Q$, first, let $f(p) \leq q$. Then there exists an element $x \in P$ such that $f(x) = q$. So, $q \in \uparrow f(p) \cap \uparrow f(x) = f(\uparrow p \cap \uparrow x)$. Thus, there is an element $y \in \uparrow p \cap \uparrow x$ such that $f(y) = q$. Hence, $p \leq y \leq \sup f^{-1}(q)$. So, $p \leq g(q)$. Second, let $p \leq g(q)$. Since f is Scott-continuous,

$$f(p) \leq f(g(q)) = f(\sup f^{-1}(q)) = \sup f(f^{-1}(q)) = q.$$

Thus, (g, f) is an adjunction. □

Proposition 6.6. *Let P be a dcpo and Q a poset. For a Scott-continuous map $f : P \to Q$ which satisfies the following equation:*

$$\uparrow f(\uparrow x \cap \uparrow y) = \uparrow f(x) \cap \uparrow f(y),$$

for all $x, y \in P$, consider its image $f(P)$ in Q with the induced order. Then the following properties hold.

(1) *If X is a subset of $f(P)$ and $f^{-1}(X)$ has a sup in P, then X has a sup in $f(P)$ and $\sup_{f(P)} X = f(\sup_P f^{-1}(X))$.*

(2) *If P is a semilattice, a lattice, a bounded complete dcpo, a complete lattice, respectively, then the same holds for $f(P)$.*

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press

Proof. (1) Let $X \subseteq f(P)$ and $\sup_P f^{-1}(X)$ exist. For $f : P \to f(P)$, we have that $f(\uparrow x \cap \uparrow y) = \uparrow_{f(P)} f(x) \cap \uparrow_{f(P)} f(y)$ for all $x, y \in P$. By Theorem 6.5, we deduce that f has an upper adjoint g. So, $f(\sup_P f^{-1}(X)) = \sup f(f^{-1}(X)) = \sup_{f(P)} X$ by Lemma 2.15.

(2) We only prove (2) for P which is a semilattice because the other cases are covered by (1). Suppose that P is a semilattice. Let $q_1, q_2 \in f(P)$. Then, $g(q_1), g(q_2) \in P$. Since P is a semilattice, $g(q_1) \wedge_P g(q_2)$ exists. We claim that $f(g(q_1) \wedge_P g(q_2)) = q_1 \wedge_{f(P)} q_2$. Note that $g(q_1) \wedge_P g(q_2) \leq g(q_1)$ and $g(q_1) \wedge_P g(q_2) \leq g(q_2)$, it implies that $f(g(q_1) \wedge_P g(q_2)) \leq q_1$ and $f(g(q_1) \wedge_P g(q_2)) \leq q_2$. Thus, we conclude that $f(g(q_1) \wedge_P g(q_2))$ is a lower bound of q_1 and q_2. Let $q \in f(P)$ be another lower bound of q_1 and q_2. Then there exists an element $p \in P$ such that $f(p) = q$. It means that $f(p) \leq q_1$ and $f(p) \leq q_2$. Whence $p \leq g(q_1)$ and $p \leq g(q_2)$. Thus, $p \leq g(q_1) \wedge_P g(q_2)$. So, $f(p) = q \leq f(g(q_1) \wedge_P g(q_2))$ by the monotonicity of f. Therefore, $f(g(q_1) \wedge_P g(q_2))$ is the largest lower bound of q_1 and q_2 in $f(P)$. It follows that $f(P)$ is a semilattice. □

By Lemma 2.16 and Theorem 6.5, we can get the following corollary directly.

Corollary 6.7. *Let P be a domain and Q a poset. For a map $f : P \to Q$, consider the following two conditions:*

(1) *f has an upper adjoint g and g is Scott-continuous.*
(2) *f preserves sups of directed subsets and the way-below relation $\ll$ and for all $x, y \in P$, satisfies the following equation:*

$$\uparrow f(\uparrow x \cap \uparrow y) = \uparrow f(x) \cap \uparrow f(y).$$

Then (1) $\Rightarrow$ (2), *and if the map $f : P \to Q$ is surjective, then the above two conditions are equivalent.*

Theorem 6.8. *Let M be the image of a II-homomorphism $f : P \to Q$. If P is a domain, an L-domain, a bounded complete domain, a continuous semilattice, and a continuous lattice, then the same is true for the image M.*

Proof. If $\mathrm{card}(M) = 1$, then the conclusions naturally hold. Otherwise, for $f : P \to f(P)$, there exists a Scott-continuous map $g : f(P) \to P$ which is an upper adjoint of f by Corollary 6.7. Since $f : P \to f(P)$ is surjective, we have that $f \circ g = id_{f(P)}$ by Proposition 2.14. Whence $f(P)$ is a Scott retract of P. Because P is a domain and the property of being a domain is preserved under Scott retracts, it follows that $f(P)$ is also a domain. As the various completeness properties are preserved under II-homomorphisms (see Proposition 6.6), we get the conclusions immediately. □

Theorem 6.9. (II-Homomorphism theorem). *Let P and Q be domains, and $f : P \to Q$ be a II-homomorphism. Then, $ker(f) = \{(x, y) \in P \times P \mid f(x) = f(y)\}$ is a II-congruence relation on P and $f(P) \cong P/ker(f)$.*

Proof. If $\mathrm{card}(f(P)) = 1$, then $ker(f) = P \times P$, it is a II-congruence relation on P. Otherwise, for $f : P \to f(P)$, we have that f is a surjective II-homomorphism. It implies that there exists a Scott-continuous map $g : f(P) \to P$ which is an upper adjoint of f by Corollary 6.7. Based on Theorem 2.17, the kernel of a lower adjoint is that of a closure operator and by Theorem 5.3 $ker(f)$ is a II-congruence relation on P and $f(P) \cong P/ker(f)$, because the kernels of closure operators are characterized as II-congruence relations on P. □

By Theorem 6.9, we get the following corollary.

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press

Corollary 6.10. (II-Isomorphism theorem). *Let P, Q be domains and $f : P \to Q$ be a surjective II-homomorphism. Then $ker(f) = \{(x, y) \in P \times P \mid f(x) = f(y)\}$ is a II-congruence relation on P and $Q \cong P/ker(f)$.*

7. Applications

This section is devoted to giving some applications of the results of domains to continuous semilattices, L-domains, bounded complete domains, and continuous lattices.

Evidently, the kernels of I-homomorphisms and II-homomorphisms between domains and I-congruence relations and II-congruence relations on domains are practically the same thing on account of the I-homomorphism theorem, respectively II-homomorphism theorem. By Theorems 4.2 and 6.8, the images of I-homomorphism and II-homomorphism from an L-domain, a bounded complete domain, a continuous semilattice, and a continuous lattice also have the same property. Thus, we can give the definition of I-congruence relation and II-congruence relation on L-domains, bounded complete domains, continuous semilattices, and continuous lattices, respectively. These conclusions are consistent with those on domains.

Applying the above results to continuous semilattices, we obtain a positive answer to an open problem on homomorphisms and quotients of continuous semilattices in Gierz et al. (2003) (the second problem on page 92 of Gierz et al. (2003)). Furthermore, for I-congruence relation on continuous semilattices, we have a more complete picture.

Theorem 7.1. *Let P be a continuous semilattice and $R \neq P \times P$ an equivalence relation on P. Then, the following conditions are equivalent:*

(1) *R is a I-congruence relation on P;*
(2) *(a) R is closed in $P \times P$ under the formation of finite infs and directed sups,*
(b) for $x \in P$, $[x]_R$ has the smallest element in P.

Proof. (1) $\Rightarrow$ (2): It remains to prove that R is closed in $P \times P$ under finite infs. Since R is a I-congruence relation, we conclude that the quotient set P/R with a suitable order is a continuous semilattice in such a way that the quotient map $q : P \to P/R$ is an upper adjoint preserving sups of directed sets by Theorem 3.3. From Lemma 2.15, q preserves finite infs. Let $(a, b), (c, d) \in R$. Then, $q(a) = q(b)$ and $q(c) = q(d)$. Note that $q(a) \wedge q(c) = q(a \wedge c)$ and $q(b) \wedge q(d) = q(b \wedge d)$. So, $q(a \wedge c) = q(b \wedge d)$. It follows that $(a \wedge c, b \wedge d) \in R$. Whence R is closed in $P \times P$ under finite infs.

(2) $\Rightarrow$ (1): We only need to prove that R satisfies the dual-link property. Let $a, b, b^* \in P$. If $a \leq b$ and $b\, R\, b^*$, then $(a, a), (b, b^*) \in R$. As R is closed in $P \times P$ under finite infs, $(a, a) \wedge (b, b^*) = (a, a \wedge b^*) \in R$. It implies that $a \wedge b^* \leq b^*$ and $a\, R\, a \wedge b^*$. Thus, R satisfies the dual-link property. □

Second, we consider the case of L-domains. For I-congruence relation on L-domains, we find that it is a generalization of the congruence relation on L-domains introduced in Jin and Li (2022).

Lemma 7.2. *Let P and Q be posets and $f : P \to Q$ a map. For arbitrary subset A of P, if f satisfies the following equation:*

$$\downarrow f(\cap_{a \in A} \downarrow a) = \cap_{a \in A} \downarrow f(a),$$

then f preserves existing infs.

Proof. By Proposition 2.4, f is order-preserving. Assume that A is an arbitrary subset in P for which $\inf A$ exists. As f is order-preserving, we have that $f(\inf A)$ is a lower bound of $f(A)$. Suppose

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press

q is another lower bound of $f(A)$ in Q. Then, $q \in \cap_{a\in A}\downarrow f(a)$. Since $\downarrow f(\cap_{a\in A}\downarrow a) = \cap_{a\in A}\downarrow f(a)$, $q \in \downarrow f(\cap_{a\in A}\downarrow a)$. Thus, there is an element $x \in \cap_{a\in A}\downarrow a$ such that $q \leq f(x)$. Note that $x \leq a$, for any $a \in A$, which implies that $x \leq \inf A$. So $q \leq f(x) \leq f(\inf A)$. Thus, $f(\inf A) = \inf f(A)$. □

Theorem 7.3. *Let P and Q be posets and $f : P \to Q$ a map. Consider the following two conditions:*

(1) *f is an upper adjoint;*

(2) *for arbitrary subset A of P, f satisfies the following equation:*

$$\downarrow f(\cap_{a\in A}\downarrow a) = \cap_{a\in A}\downarrow f(a).$$

Then, (1) ⇒ (2). *Furthermore, if P is an L-domain and f is a surjective map, then* (2) ⇒ (1).

Proof. (1) ⇒ (2): Let $A \subseteq P$. Since f is an upper adjoint, it is order-preserving, we have $\downarrow f(\cap_{a\in A}\downarrow a) \subseteq \cap_{a\in A}\downarrow f(a)$. Conversely, let $q \in \cap_{a\in A}\downarrow f(a)$. Then, $q \leq f(a)$, for any $a \in A$. Again since f is an upper adjoint, there exists an order-preserving map $g : Q \to P$ such that (f, g) is an adjunction. Thus, $g(q) \leq a$, for any $a \in A$. It follows that $g(q) \in \cap_{a\in A}\downarrow a$. So, $fg(q) \in f(\cap_{a\in A}\downarrow a)$. Because $q \leq fg(q)$, $q \in \downarrow f(\cap_{a\in A}\downarrow a)$. Hence, $\cap_{a\in A}\downarrow f(a) \subseteq \downarrow f(\cap_{a\in A}\downarrow a)$. Therefore, $\downarrow f(\cap_{a\in A}\downarrow a) = \cap_{a\in A}\downarrow f(a)$.

Assume that P is an L-domain and f is a surjective map. If for arbitrary subset A of P, f satisfies the equation: $\downarrow f(\cap_{a\in A}\downarrow a) = \cap_{a\in A}\downarrow f(a)$, then define a map $g : Q \to P$ by $q \mapsto \inf f^{-1}(q)$, where $q \in Q$.

Claim 1: For any $q \in Q$, $\inf f^{-1}(q)$ exists.

Since f is surjective, there exists an element $p \in P$ such that $f(p) = q$. Then, $\downarrow p \cap f^{-1}(q) \neq \varnothing$. As P is an L-domain, $\inf(\downarrow p \cap f^{-1}(q))$ exists by Lemma 2.3. For any $x \in f^{-1}(q)$, $f(x) = q = f(p)$. Thus, $f(x) \in \downarrow f(x) \cap \downarrow f(p) = \downarrow f(\downarrow x \cap \downarrow p)$. Hence, there is an element $p_x \in \downarrow x \cap \downarrow p$ such that $f(x) \leq f(p_x)$. Because $p_x \leq x$, we have $f(p_x) \leq f(x)$. So, $f(x) = f(p_x) = q$. That is, $p_x \in \downarrow p \cap f^{-1}(q)$. Thus, $\inf(\downarrow p \cap f^{-1}(q)) \leq p_x \leq x$. It follows that $\inf(\downarrow p \cap f^{-1}(q))$ is a lower bound of $f^{-1}(q)$. Let $y \in P$ be another lower bound of $f^{-1}(q)$. Then for any $z \in \downarrow p \cap f^{-1}(q)$, $y \leq z$. So, $z \leq \inf(\downarrow p \cap f^{-1}(q))$. Therefore, $\inf f^{-1}(q) = \inf(\downarrow p \cap f^{-1}(q))$.

Thus, g is well defined.

Claim 2: g is order-preserving.

Let $q_1, q_2 \in Q$ and $q_1 \leq q_2$. For any $a \in f^{-1}(q_2)$, $f(a) = q_2$. Since f is surjective, there exists an element $p_{q_1} \in P$ such that $f(p_{q_1}) = q_1$. Thus, $f(p_{q_1}) \in \downarrow f(p_{q_1}) \cap \downarrow f(a) = \downarrow f(\downarrow p_{q_1} \cap \downarrow a)$. It follows that there is an element $b \in \downarrow p_{q_1} \cap \downarrow a$ such that $f(p_{q_1}) \leq f(b)$. So $f(p_{q_1}) \leq f(b) \leq f(p_{q_1})$; hence, $f(p_{q_1}) = f(b) = q_1$. Note that $b \in f^{-1}(q_1)$ and $b \leq a$. Then, $\inf f^{-1}(q_1) \leq a$. So, $\inf f^{-1}(q_1) \leq \inf f^{-1}(q_2)$, that is, $g(q_1) \leq g(q_2)$.

Claim 3: (f, g) is an adjunction.

For all pairs of elements $(p, q) \in P \times Q$, first, let $g(q) \leq p$. Then $\inf f^{-1}(q) \leq p$. Thus, $f(\inf f^{-1}(q)) \leq f(p)$. By Lemma 7.2, f preserves existing infs. Hence,

$$f(\inf f^{-1}(q)) = \inf f(f^{-1}(q)) = q \leq f(p).$$

Second, let $q \leq f(p)$. Then $g(q) \leq gf(p)$. Since $g(f(p)) = \inf f^{-1}(f(p)) = \inf(\downarrow p \cap f^{-1}(f(p)) \leq p$, we have $g(q) \leq p$. Thus, (g, f) is an adjunction.

So f is an upper adjoint. Thus, (2) ⇒ (1). □

Theorem 7.4. *Let P be an L-domain and $R \neq P \times P$ an equivalence relation on P. Then the following conditions are equivalent:*

(1) *R is a I-congruence relation on P;*

(2) *(a) R is a subalgebra of $P \times P$ (i.e., R is closed in $P \times P$ under the formation of infima of nonempty sets bounded from above and directed sups);*
(b) for $x \in P$, $[x]_R$ has the smallest element in P;

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press

(3) *P/R with a suitable order is an L-domain in such a way that $q : P \to P/R$ preserves sups of directed sets and for arbitrary subset A of P, q satisfies the following equation:*

$$q(\cap_{a\in A} \downarrow a) = \cap_{a\in A} \downarrow q(a).$$

Proof. By Proposition 2.4 and Theorems 3.3, 7.3, we get that (1) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (2): We need to prove that R is closed in $P \times P$ under nonempty sets bounded above infs. Since (1) and (3) are equivalent and P is an L-domain, we have that q preserves nonempty sets bounded above infs by Lemma 3.5. Let $\mathcal{A}$ be an arbitrary nonempty sets bounded above in R. Set

$$A_1 = \{a_1 \in P \mid \text{there exists } a_2 \in P \text{ such that } (a_1, a_2) \in \mathcal{A}\},$$
$$A_2 = \{a_2 \in P \mid \text{there exists } a_1 \in P \text{ such that } (a_1, a_2) \in \mathcal{A}\}.$$

Then A_1 and A_2 are nonempty sets bounded above in P. Since P is an L-domain, $\inf A_1$ and $\inf A_2$ exist. As q preserves nonempty sets bounded above infs, we have

$$q(\inf A_1) = \inf q(A_1) = \inf q(A_2) = q(\inf A_2).$$

Thus, $(\inf A_1, \inf A_2) \in R$. So R is closed in $P \times P$ under the formation of infima of nonempty sets bounded from above.

(2) $\Rightarrow$ (1): It is enough to prove that R satisfies the dual-link property. Let $a, b, b^* \in P$. If $a \le b$ and $b\, R\, b^*$, then $(a, a), (b, \inf [b]_R) \in R$. Note that (b, b) is an upper bound of (a, a) and $(b, \inf [b]_R)$ in R. By Condition (a) of (2), $(a, a) \wedge (b, \inf [b]_R) = (a, a \wedge \inf [b]_R) \in R$. So, $\inf [a]_R \le a \wedge \inf [b]_R \le \inf [b]_R \le b^*$ and $a\, R \inf [a]_R$. Therefore, R satisfies the dual-link property. □

Let P be an L-domain and R an equivalence relation on P. If R is a congruence relation defined in Jin and Li (2022), recall Proposition 4.3 in Jin and Li (2022), we know that each block of R is a complete lattice. It follows that each block of R has the smallest element. So it is a I-congruence relation on P. Therefore, I-congruence relation on L-domains is a generalization of the congruence relation on L-domains introduced in Jin and Li (2022).

Now we consider the case of continuous lattices. For a continuous lattice P, there are various equivalent ways of looking at I-congruence relations and II-congruence relations on P, respectively.

In Gierz et al. (2003), it introduced a kind of congruence relation on continuous lattices, corresponding to kernel operators which preserve directed sups (see Theorem I-2.15 of Gierz et al. (2003)). Next, we will show that it is in fact the I-congruence relation on continuous lattices.

Lemma 7.5. (Gierz et al. (2003)). *Let $f : P \to Q$ be a map between posets of which P is a complete lattice. Then f preserves infs of arbitrary sets if and only if f is order-preserving and has a lower adjoint.*

Theorem 7.6. *Let P be a continuous lattice and R an equivalence relation on P. Then the following conditions are equivalent:*

(1) *R is a I-congruence relation on P;*

(2) *R is closed in $P \times P$ under the formation of arbitrary infs and directed sups;*

(3) *P/R with a suitable order is a continuous lattice in such a way that the quotient map $q : P \to P/R$ preserves infs of arbitrary sets and sups of directed sets.*

Proof. From Theorem I-2.15 in Gierz et al. (2003), (2) $\Leftrightarrow$ (3). For $R = P \times P$, since P is a continuous lattice, $P \times P$ is also a continuous lattice. So, $P \times P$ is closed under the formation of arbitrary infs and directed sups. By Theorem 3.3 and Lemma 7.5, (1) $\Leftrightarrow$ (3). □

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press



Figure 6. The Hasse diagrams in Example 7.8.

Theorem 7.7. *Let P be a continuous lattice and R an equivalence relation on P. Then the following conditions are equivalent:*

(1) *R is a II-congruence relation on P;*

(2) *(a) R is closed in $P \times P$ under the formation of arbitrary sups,*
(b) $\{\sup [p]_R \mid p \in P\}$ is a subdcpo of P;

(3) *P/R with a suitable order is a continuous lattice in such a way that the quotient map $q : P \to P/R$ preserves sups of arbitrary sets and the way-below relation $\ll$.*

Proof. By Theorems 6.3 and 5.3, (1) and (3) are equivalent.

(3) $\Rightarrow$ (2): We only need to prove Condition (a) of (2). Assume that there is an order $\leq_{P/R}$ on the quotient set P/R such that $q : P \to (P/R, \leq_{P/R})$ preserves sups of arbitrary sets and the way-below relation $\ll$. Let $\mathcal{A}$ be an arbitrary subset in R. Similar to the proof of (1) $\Rightarrow$ (2) in Theorem 7.4, we have $(\sup A_1, \sup A_2) \in R$. So, R is closed in $P \times P$ under the formation of arbitrary sups.

(2) $\Rightarrow$ (1): Since R is closed in $P \times P$ under arbitrary sups, we would get that R is closed in $P \times P$ under directed sups. Thus, R is a subdcpo of $P \times P$. Let $p \in P$ and $x, y \in [p]_R$. Then, $(x, y), (x, x) \in R$. It implies that $(x, y) \vee (x, x) = (x, x \vee y) \in R$. So, $x \vee y \in [p]_R$; hence, $[p]_R$ is directed in P. It remains to prove that R satisfies the link property. For $x, y, u \in P$, if $x \leq y$ and $x \, R \, u$, then $(x, u), (y, y) \in R$. So, $(x, u) \vee (y, y) = (y, u \vee y) \in R$, which implies that $u \leq u \vee y$ and $y \, R \, u \vee y$. Thus, R satisfies the link property. □

At last, we give examples to show that the I-congruence relation and II-congruence relation on continuous lattices mentioned above are different.

Example 7.8. *Consider P in Example 3.11. Then P is a continuous lattice. Let R be an equivalence relation comprising exactly two blocks $\{b, c, d\}$ and $\{a\}$. Then $P/R = \{\{b, c, d\}, \{a\}\}$ with the order $\leq_R$ is a two-element chain. Note that the block $\{b, c, d\}$ does not have the smallest element. So, R is not a I-congruence relation on continuous lattices. But it is straightforward to check that R is a II-congruence relation on continuous lattices. The Hasse diagrams see Figure 6.*

Example 7.9. *Take $\mathbb{N} = \{0, 1, 2, \cdots\}$. Let $P = \mathbb{N} \cup \{a, b\}$, where the order $\leq_P$ is defined as follows:*

- $0 \leq 1 \leq 2 \leq \cdots \leq n \leq \cdots$;
- $n \leq a$ *for all* $n \in \mathbb{N}$;
- $n \leq b$ *for all* $n \in \mathbb{N}$;
- $a \leq b$.

Obviously, P is a continuous lattice. Let $R = \Delta \cup \{(a, b), (b, a)\}$. Then R is an equivalence relation on P. Clearly, we can see that the quotient set P/R with the order $\leq_R$ is also a continuous lattice

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press



Figure 7. The Hasse diagrams in Example 7.9.

and the quotient map $q : P \to (P/R, \leq_R)$ preserves infs of arbitrary sets and directed sups. Then, R is a I-congruence relation on continuous lattices by Theorem 7.6. However, $\{\sup [p]_R \mid p \in P\} = \mathbb{N} \cup \{b\}$ is not a subdcpo of P. So, Condition 4 given in Definition 5.1 does not hold. Thus, R is not a II-congruence relation on continuous lattices. The Hasse diagrams see Figure 7.

Acknowledgments. We would like to thank the anonymous reviewers for their professional comments and valuable suggestions.

Competing interests. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

Blyth, T. S. and Janowitz, M. F. (1972). *Residuation Theory*, vol. 102 of International Series of Monographs in Pure and Applied Mathematics, Pergamon Press.

Chajda, I. and Snášel, V. (1998). Congruences in ordered sets. *Mathematica Bohemica* **123** 95–100.

Chajda, I., Snášel, V. and Jukl, M. (2009). Congruences in ordered sets and LU compatible equivalences, Acta Universitatis Palackianae Olomucensis. *Facultas Rerum Naturalium. Mathematica* **48** 153–156.

Davey, B. A. and Priestley, H. A. (2002). *Introduction to Lattices and Order*, Second edn. Cambridge University Press.

Fiech, A. (1996). Colimits in the category DCPO. *Mathmetical Structures in Computer Science* **6** 455–468.

Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M. and Scott, D. S. (2003). *Continuous lattices and domains*, vol. 93 of Encyclopedia of Mathematics and its Applications, Cambridge University Press.

Goubault-Larrecq, J. (2013). *Non-Hausdorff Topology and Domain Theory*, vol. 22 of New Mathematical Monographs, Cambridge University Press.

Hofmann, K. H. and Mislove, M. W. (1977). The lattice of kernel operators and topological algebra. *Mathematische Zeitschrift* **154**) 175–188.

Jin, M. and Li, Q. (2022). Quotients of L-domains. *Journal of Pure and Applied Algebra* **226** 106837.

Jung, A., Libkin, L. and Puhlmann, H. (1992). Decomposition of domains. In: *Proceedings of the Conference on Mathematical Foundations of Programming Semantics 91*, Lecture Notes in Computer Science, vol. 598, 235–258.

Kolibiar, M. (1987). Congruence relations and direct decomposition of ordered sets, Acta Universitatis Szegediensis. *Acta Scientiarum Mathematicarum* **51** 129–135.

Mahmoudi, M., Moghbeli, H. and Pióro, K. (2017). Natural congruences and isomorphism theorems for directed complete partially ordered sets. *Algebra Universalis* **77** (1) 79–99.

Mahmoudi, M., Moghbeli, H. and Pióro, K. (2019). Directed complete poset congruences. *Journal of Pure and Applied Algebra* **223** 4161–4170.

Shum, K. P., Zhu, P. and Kehayopulu, N. (2008). III-Homomorphisms and III-congruences on posets. *Discrete Mathematics* **308** 5006–5013.

Cite this article: Jin M and Li Q (2025). Congruence relations on domains. *Mathematical Structures in Computer Science* **35**(e40), 1–24. https://doi.org/10.1017/S0960129525100388

https://doi.org/10.1017/S0960129525100388 Published online by Cambridge University Press