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SPECIAL VALUES OF SHIFTED CONVOLUTION DIRICHLET SERIES

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For Jeff Hoffstein on his 61st birthday

Abstract. In a recent important paper, Hoffstein and Hulse [Multiple Dirichlet series and shifted convolutions, arXiv:1110.4868v2] generalized the notion of Rankin–Selberg convolution L-functions by defining *shifted convolution* L-functions. We investigate symmetrized versions of their functions, and we prove that the generating functions of certain special values are linear combinations of weakly holomorphic quasimodular forms and "mixed mock modular" forms.

§1. *Introduction and statement of results.* Suppose that $f_1(\tau) \in S_{k_1}(\Gamma_0(N))$ and $f_2(\tau) \in S_{k_2}(\Gamma_0(N))$ are cusp forms of even weights $k_1 \geqslant k_2$ with L-functions

$$L(f_i, s) = \sum_{n=1}^{\infty} \frac{a_i(n)}{n^s}.$$

In the case of equal weights, Rankin and Selberg [22, 24] independently introduced the so-called *Rankin–Selberg convolution*

$$L(f_1 \otimes f_2, s) := \sum_{n=1}^{\infty} \frac{a_1(n)\overline{a_2(n)}}{n^s},$$

works which are among the most important contributions to the modern theory of automorphic forms. Later, in 1965, Selberg [25] introduced *shifted convolution* L-functions, and these series have now played an important role in progress towards Ramanujan-type conjectures for Fourier coefficients and the Lindelöf hypothesis for automorphic L-functions inside the critical strip (for example, see [1, 10, 11, 17] and the references therein).

In a recent paper, Hoffstein and Hulse [14]† introduced the *shifted convolution series*

$$D(f_1, f_2, h; s) := \sum_{n=1}^{\infty} \frac{a_1(n+h)\overline{a_2(n)}}{n^s}. \tag{1.1}$$

When $k_1 = k_2$, they obtained the meromorphic continuation of these series and certain multiple Dirichlet series which are obtained by additionally summing in

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† Here we choose slightly different normalizations for Dirichlet series from those that appear in [14].

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h-aspect. Moreover, as an important application they obtained a Burgess-type bound for L-series associated to modular forms.

Here we study the arithmetic properties of these Dirichlet series, where we additionally allow the weights to be non-equal. For this, it is convenient to consider the *derived* shifted convolution series

$$D^{(\mu)}(f_1, f_2, h; s) := \sum_{n=1}^{\infty} \frac{a_1(n+h)\overline{a_2(n)}(n+h)^{\mu}}{n^s}, \tag{1.2}$$

where $h \geqslant 1$ and $\mu \geqslant 0$ are integers and $\mathrm{Re}(s) > (k_1 + k_2)/2 + \mu$. Of course, we have $D^{(0)}(f_1, f_2, h; s) = D(f_1, f_2, h; s)$.

For each $\nu \geqslant 0$ and each $h \geqslant 1$, we define (see §3.4) a *symmetrized shifted convolution* Dirichlet series $\widehat{D}^{(\nu)}(f_1, f_2, h; s)$ using the $D^{(\mu)}(f_1, f_2, h; s)$. We consider their special values at $s = k_1 - 1$ in h-aspect, which are well-defined as (conditionally) convergent series if $\nu \leqslant (k_1 - k_2)/2$. They are absolutely convergent if $\nu < (k_1 - k_2)/2$. In order to investigate these special values, we construct the q-series

$$\mathbb{L}^{(\nu)}(f_1, f_2; \tau) := \sum_{h=1}^{\infty} \widehat{D}^{(\nu)}(f_1, f_2, h; k_1 - 1) q^h, \tag{1.3}$$

where $q := \mathrm{e}^{2\pi \mathrm{i} \tau}$. In the special case where $\nu = 0$ and $k_1 = k_2$, we have

$$\widehat{D}(f_1, f_2, h; s) = D(f_1, f_2, h; s) - D(\overline{f_2}, \overline{f_1}, -h; s),$$

which then implies that

$$\mathbb{L}^{(0)}(f_1, f_2; \tau) = \sum_{h=1}^{\infty} \widehat{D}(f_1, f_2, h; k_1 - 1) q^h.$$

It is natural to ask for a characterization of these functions. For example, consider the case where $f_1 = f_2 = \Delta$, the unique normalized weight 12 cusp form on $\mathrm{SL}_2(\mathbb{Z})$. Then we have

$$\begin{aligned}\mathbb{L}^{(0)}(\Delta, \Delta; \tau) = &-33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3 \\ &+ 4827.434\ldots q^4 - \ldots.\end{aligned}$$

Using the usual Eisenstein series $E_{2k} = E_{2k}(\tau)$ and Klein's j-function, we let

$$\sum_{n=-1}^{\infty} r(n) q^n := -\Delta(j^2 - 1464 j - \alpha^2 + 1464\alpha),$$

where $\alpha = 106.10455\ldots$. By letting $\beta = 2.8402\ldots$, we find that

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$$-\frac{\Delta}{\beta}\left(\frac{65520}{691}+\frac{E_2}{\Delta}-\sum_{n\neq 0} r(n)n^{-11}q^n\right)$$
$$= -33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3$$
$$+ 4827.434\ldots q^4 - \ldots.$$

It turns out that this q-series indeed equals $\mathbb{L}^{(0)}(\Delta, \Delta; \tau)$, and the purpose of this paper is to explain such formulas.

The q-series $\mathbb{L}^{(\nu)}(f_1, f_2; \tau)$ arise from the *holomorphic projections* of completed *mock modular forms*. In 1980, Sturm [**26**] introduced the method of holomorphic projection and, in their celebrated work on Heegner points and derivatives of L-functions, Gross and Zagier [**12**] further developed this technique. In unpublished work, Zwegers suggested applying such holomorphic projections to the theory of mock modular forms. Recently, Imamoğlu *et al* [**15**] have obtained general theorems in this direction with applications to Ramanujan's mock theta functions in mind. This work, as well as the preprint [**19**] by the first author, are based on unpublished notes of Zagier. Here we continue this theme, and we show that the $\mathbb{L}^{(\nu)}(f_1, f_2; \tau)$ also arise in this way.

The results we obtain depend on the theory of *harmonic Maass forms*, certain non-holomorphic modular forms which explain Ramanujan's enigmatic *mock theta functions* (see [**21**, **28**, **29**] and the references therein). For any f_1, we denote by M_{f_1} a harmonic Maass form of weight $2-k_1$ whose *shadow* is f_1 (see §2 for the definition). Such forms always exist. Our main result gives a surprising relation between the νth *Rankin–Cohen bracket* (again, see §2 for definitions) of M_{f_1} and f_2 and the generating function $\mathbb{L}^{(\nu)}(f_1, f_2; \tau)$. These Rankin–Cohen brackets are (completed) *mixed mock modular forms* (see §2).

As a technical condition for our result, we need the notion of M_{f_1} being *good* for f_2. By this, we mean that the Rankin–Cohen bracket function $[M_{f_1}^+, f_2]_\nu$, where $M_{f_1}^+$ denotes the *holomorphic part* of M_{f_1}, is holomorphic on the upper half of the complex plane, and is bounded as one approaches all representatives for all cusps (see §3 for details). Our main result relates such Rankin–Cohen bracket functions to $\mathbb{L}^{(\nu)}(f_1, f_2; \tau)$ modulo $\widetilde{M}_{2\nu-k_1+k_2+2}(\Gamma_0(N))$. Here the space $\widetilde{M}_k(\Gamma_0(N))$ is given by

$$\widetilde{M}_k(\Gamma_0(N)) = \begin{cases} M_k(\Gamma_0(N)) & \text{if } k \geqslant 4, \\ \mathbb{C}E_2 \oplus M_2(\Gamma_0(N)) & \text{if } k = 2, \end{cases} \tag{1.4}$$

where $M_k(\Gamma_0(N))$ is the space of weight k holomorphic modular forms on $\Gamma_0(N)$. Special care is required for weight 2 because the Eisenstein series E_2 is not a holomorphic modular form; it is a *quasimodular form*. In general, we require the spaces $\widetilde{M}_k^!(\Gamma_0(N))$, which are the extension of $\widetilde{M}_k(\Gamma_0(N))$ by the weight k *weakly holomorphic modular forms* on $\Gamma_0(N)$. Weakly holomorphic modular forms are those meromorphic modular forms whose poles (if any) are supported at cusps. The space of such forms of weight k on $\Gamma_0(N)$ is denoted by $M_k^!(\Gamma_0(N))$.

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THEOREM 1.1. *Assume the notation above. If* $0 \leqslant \nu \leqslant (k_1 - k_2)/2$, *then*

$$\mathbb{L}^{(\nu)}(f_2, f_1; \tau) = -\frac{1}{(k_1 - 2)!} \cdot [M_{f_1}^+, f_2]_\nu(\tau) + F(\tau),$$

where $F \in \widetilde{M}^!_{2\nu+2-k_1+k_2}(\Gamma_0(N))$. *Moreover, if* M_{f_1} *is good for* f_2, *then* $F \in \widetilde{M}_{2\nu+2-k_1+k_2}(\Gamma_0(N))$.

Two remarks.

(1) One can derive Theorem 1.1 when $\nu = 0$ and M_{f_1} is good for f_2 from [**15**, Theorem 3.5].

(2) Given cusp forms f_1 and f_2, it is not generically true that there is a harmonic Maass form M_{f_1} which is good for f_2. To see this, consider the case where $f_1 = f_2 = f$, a normalized Hecke eigenform on $\mathrm{SL}_2(\mathbb{Z})$. For M_f to be good for f, it is necessary that M_f^+ has at most a simple pole at infinity. By the theory of Poincaré series (see §2.2), it is clear that most Hecke eigenforms f do not satisfy this condition.

The expression on the right-hand side of Theorem 1.1 is explicitly computable when $f_1 = f_2$ by making use of the theory of Poincaré series (see §2.2 for notation and definitions).

COROLLARY 1.2. *Suppose that* $k \geqslant 2$ *is even and* m *is a positive integer. If* $P(\tau) := P(m, k, N; \tau) \in S_k(\Gamma_0(N))$ *and* $Q(\tau) := Q(-m, k, N; \tau) \in H_{2-k}(\Gamma_0(N))$, *then*

$$\mathbb{L}^{(0)}(P, P; \tau) = \frac{1}{m^{k-1} \cdot (k-1)!} \cdot Q^+(\tau) P(\tau) + F(\tau),$$

where $F \in \widetilde{M}^!_2(\Gamma_0(N))$. *Moreover, if* $m = 1$, *then* $F \in \widetilde{M}_2(\Gamma_0(N))$.

Remark. In §2.2, we shall see that $Q(-m, k, N; \tau)$ has a pole of order m at $i\infty$, while $P(m, k, N; \tau)$ generally only has a simple zero there. This explains the special role of $m = 1$ in the corollary above.

Example 1. Here we consider the case where $f_1 = f_2 = \Delta$, the unique normalized cusp form of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$. Using the first 10^7 coefficients of Δ, one obtains the following numerical approximations for the first few shifted convolution values $\widehat{D}(\Delta, \Delta, h; 11)$:

h	1	2	3	4	5
$\widehat{D}(\Delta, \Delta, h; 11)$	$-33.383\ldots$	$266.439\ldots$	$-1519.218\ldots$	$4827.434\ldots$	$-5704.330\ldots$

We have $\Delta(\tau) = (1/\beta) P(1, 12, 1; \tau)$, where β can be described in terms of a Petersson norm, or as an infinite sum of Kloosterman sums weighted by J-Bessel functions as follows:

$$\beta := \frac{(4\pi)^{11}}{10!} \cdot \|P(1, 12, 1)\|^2 = 1 + 2\pi \sum_{c=1}^{\infty} \frac{K(1, 1, c)}{c} \cdot J_{11}(4\pi/c) = 2.8402\ldots.$$

Since $m = 1$ and there are no weight 2 holomorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$, Corollary 1.2 then implies that

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$$\begin{aligned}\mathbb{L}^{(0)}(\Delta, \Delta; \tau) &= \frac{Q^+(-1, 12, 1; \tau) \cdot \Delta(\tau)}{11! \cdot \beta} - \frac{E_2(\tau)}{\beta} \\ &= -33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3 \\ &\quad + 4827.434\ldots q^4 - \ldots.\end{aligned}$$

Since the expressions for the coefficients of $Q^+(-1, 12, 1; \tau)$ are rapidly convergent, this q-series identity provides an extremely efficient method for computing the values $\widehat{D}(\Delta, \Delta, h; 11)$.

Example 2. Here we consider the newform $f = f_1 = f_2 = \eta(3\tau)^8 \in S_4(\Gamma_0(9))$. This form has complex multiplication by $\mathbb{Q}(\sqrt{-3})$, and is a multiple of the Poincaré series $P(1, 4, 9; \tau)$ because this space of cusp forms is one dimensional. Here we show how complex multiplication implies some striking *rationality* properties for the corresponding special values $\widehat{D}(f, f, h; 3)$. Using a computer, one obtains the following numerical approximations for the first few shifted convolution values.

h	3	6	9	12	15
$\widehat{D}(f, f, h; 3)$	$-10.7466\ldots$	$12.7931\ldots$	$6.4671\ldots$	$-79.2777\ldots$	$64.2494\ldots$

We note that $\widehat{D}(f, f, h; 3) = 0$ whenever n is not a multiple of 3.

Below we define real numbers β, γ and δ which are approximately

$$\begin{aligned}\beta &:= \frac{(4\pi)^3}{2} \cdot \|P(1, 4, 9)\|^2 = 1.0468\ldots, \\ \gamma &= -0.0796\ldots, \qquad \delta = -0.8756\ldots.\end{aligned}$$

As we shall see, these real numbers arise naturally from the theory of Petersson inner products. In the table below, we consider the first few values of

$$T(f; h) := \beta \widehat{D}(f, f, h; 3) + 24\beta\gamma \sum_{d|h} d - 12\beta\delta \sum_{\substack{d|h \\ 3\nmid d}} d.$$

h	3	6	9	12
$T(f; h)$	$-8.250\ldots \sim -\frac{33}{4}$	$22.391\ldots \sim \frac{2799}{125}$	$-8.229\ldots \sim -\frac{32919}{4000}$	$-61.992\ldots \sim -\frac{8250771}{133100}$

It is indeed true that $T(f; h)$ is rational for all h. To see this, we note that $f = (1/\beta)P(1, 4, 9; \tau)$. By applying Corollary 1.2 with $m = 1$, we obtain a congruence between $Q^+(-1, 4, 9; \tau)f(\tau)/\beta$ and $\mathbb{L}^{(0)}(f, f; \tau)$ modulo $\widetilde{M}_2(\Gamma_0(9))$. In particular, we obtain the following identity:

$$\begin{aligned}\mathbb{L}^{(0)}(f, f; \tau) - \frac{Q^+(-1, 4, 9; \tau)f(\tau)}{\beta} &= \gamma\left(1 - 24\sum_{n=1}^{\infty} \sigma_1(3n)q^{3n}\right) \\ &\quad + \delta\left(1 + 12\sum_{n=1}^{\infty} \sum_{\substack{d|3n \\ 3\nmid d}} dq^{3n}\right).\end{aligned}$$

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The rationality of $T(f;h)$ follows from the deep fact that

$$Q^+(-1,4,9;\tau) = q^{-1} - \tfrac{1}{4}q^2 + \tfrac{49}{125}q^5 - \tfrac{3}{32}q^8 - \dots$$

has rational Fourier coefficients. Bruinier *et al* proved a general theorem (see [7, Theorem 1.3]) which establishes the algebraicity of mock modular forms whose shadows are CM forms.

Example 3. Here we consider the case of the weight 24 Poincaré series

$$\begin{aligned} f(\tau) &:= P(2,24,1;\tau) \\ &= 0.00001585\dots q + 2.45743060\dots q^2 - 114.85545780\dots q^3 \\ &\quad + 2845.49507680\dots q^4 - \dots. \end{aligned}$$

We have the following numerical approximations for the first few shifted convolution values $\widehat{D}(f,f,h;23)$:

h	1	2	3	4	5
$\widehat{D}(f,f,h;23)$	$-0.00000629\dots$	$-0.00092041\dots$	$0.033927\dots$	$-0.472079\dots$	$4.628028\dots$

We define

$$F := \frac{1}{2^{23}} \cdot \left(\gamma \cdot \frac{E_4^2 E_6}{\Delta} + \delta \cdot E_2 \right) = \frac{\gamma}{2^{23}} \cdot q^{-1} + \dots,$$

where $\gamma = -0.00001585\dots$ and $\delta = -2.45743\dots$. We have

$$F \in \widetilde{M}_2^!(\Gamma_0(1)) \backslash \widetilde{M}_2(\Gamma_0(1)).$$

It turns out that

$$\begin{aligned} \mathbb{L}^{(0)}(f,f;\tau) &= \frac{Q^+(-2,24,1;\tau)\cdot P(2,24,1;\tau)}{2^{23}\cdot 23!} + F \\ &= -0.00000629\dots q - 0.000920\dots q^2 + 0.0339\dots q^3 \\ &\quad - 0.4720\dots q^4 + 4.628\dots q^5 + \dots. \end{aligned}$$

This example illustrates Corollary 1.2 in a situation where there is no M_f which is good for f.

As mentioned above, Theorem 1.1 relies critically on the theory of holomorphic projections, harmonic Maass forms and the combinatorial properties of Rankin–Cohen brackets. In §2, we recall the essential properties of the theory of harmonic Maass forms, Maass–Poincaré series and Rankin–Cohen brackets. In §3.4, we define the relevant Dirichlet series and q-series $\mathbb{L}^{(\nu)}(f_1,f_2;\tau)$. To obtain Theorem 1.1, we must modify the existing theory of holomorphic projections to obtain a *regularized holomorphic projection*. This regularization is required for the general case of Theorem 1.1. We give this in §3 by suitably modifying earlier work of Gross and Zagier. We then conclude with the proofs of the main results of this paper.

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§2. *Nuts and bolts.*

2.1. *Harmonic Maass forms.* We recall some basic facts about harmonic Maass forms. These real-analytic modular forms were introduced by Bruinier and Funke in [5]. This theory explains the role of Ramanujan's enigmatic *mock theta functions* in the theory of automorphic forms (see [21, 28, 29]).

Throughout, we assume that $k \geqslant 2$ is even, and we let $\tau = x + \mathrm{i}y \in \mathbb{H}$, where $x, y \in \mathbb{R}$. We define the weight k *slash operator* acting on smooth functions $f : \mathbb{H} \to \mathbb{C}$ by

$$(f|_k\gamma)(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right),$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. The weight k *hyperbolic Laplacian* is defined by

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \mathrm{i}ky\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right).$$

Definition 2.1. A smooth function $f : \mathbb{H} \to \mathbb{C}$ is called a *harmonic weak Maass form*† of *weight* $2 - k$ on $\Gamma_0(N)$ if the following conditions hold:

(1) f is invariant under the action of $\Gamma_0(N)$, i.e. for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we have
$$(f|_{2-k}\gamma)(\tau) = f(\tau)$$
for all $\tau \in \mathbb{H}$;

(2) f lies in the kernel of the weight $2 - k$ hyperbolic Laplacian, i.e.
$$\Delta_{2-k} f \equiv 0;$$

(3) f grows at most linearly exponentially at the cusps of $\Gamma_0(N)$.

The $\mathbb{C}$-vector space of harmonic weak Maass forms of weight $2 - k$ on $\Gamma_0(N)$ is denoted by $H_{2-k}(\Gamma_0(N))$.

The following property of these functions is well known (for example, see [5, equations (3.2*a*) and (3.2*b*)]).

LEMMA 2.2. *Let $f \in H_{2-k}(\Gamma_0(N))$ be a harmonic Maass form. Then there is a canonical splitting*

$$f(\tau) = f^+(\tau) + \frac{(4\pi y)^{1-k}}{k-1}\overline{c_f^-(0)} + f^-(\tau), \tag{2.1}$$

where for some $m_0, n_0 \in \mathbb{Z}$ we have the Fourier expansions

$$f^+(\tau) := \sum_{n=m_0}^{\infty} c_f^+(n) q^n$$

† For convenience, we will usually use the term *harmonic Maass form* and omit the word "weak".

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and

$$f^-(\tau) := \sum_{\substack{n=n_0 \\ n\neq 0}}^{\infty} \overline{c_f^-(n)} n^{k-1} \Gamma(1-k; 4\pi n y) q^{-n},$$

where $\Gamma(\alpha; x)$ *denotes the usual incomplete Gamma function.*

The series f^+ in the above lemma is the *holomorphic part* of f, and it is known as a *mock modular form* when the *non-holomorphic part* $((4\pi y)^{1-k}/(k-1))\overline{c_f^-(0)} + f^-(\tau)$ does not vanish. Products of mock modular forms and usual modular forms, as well as linear combinations of such functions, are called *mixed mock modular forms*.

The explanation for the complex conjugation of the coefficients in the non-holomorphic part $((4\pi y)^{1-k}/(k-1))\overline{c_f^-(0)} + f^-(\tau)$ of f is given in the following proposition due to Bruinier and Funke (see [5, Proposition 3.2 and Theorem 3.7]).

PROPOSITION 2.3. *The operator*

$$\xi_{2-k} : H_{2-k}(\Gamma_0(N)) \to M_k^!(\Gamma_0(N)), \qquad f \mapsto \xi_{2-k} f := 2\mathrm{i} y^{2-k} \overline{\frac{\partial f}{\partial \overline{\tau}}}$$

is well-defined and surjective with kernel $M_{2-k}^!(\Gamma_0(N))$*, the space of weakly holomorphic modular forms of weight* $2-k$ *on* $\Gamma_0(N)$*. Moreover, we have*

$$(\xi_{2-k} f)(\tau) = -(4\pi)^{k-1} \sum_{n=n_0}^{\infty} c_f^-(n) q^n.$$

The function $-(4\pi)^{1-k}\xi_{2-k} f$ is called the *shadow* of the mock modular form† f^+. For our purpose, we only consider cases where the shadow of f^+ is a cusp form. Since the ξ-operator is surjective, the natural question arises of how to construct a harmonic Maass form with prescribed shadow. This can be done using Poincaré series, which we recall next.

2.2. *Poincaré series.* Here we recall the classical cuspidal Poincaré series and the harmonic Maass–Poincaré series. We then give their relationship under the ξ-operator described above.

A general Poincaré series of weight k for $\Gamma_0(N)$ is given by

$$\mathbb{P}(m, k, N, \varphi_m; \tau) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} (\varphi_m^* |_k \gamma)(\tau),$$

where m is an integer, $\Gamma_\infty := \left\{ \pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z} \right\}$ is the subgroup of translations in $\Gamma_0(N)$ and $\varphi_m^*(\tau) := \varphi_m(y)\exp(2\pi \mathrm{i} m x)$ for a function $\varphi_m : \mathbb{R}_{>0} \to \mathbb{C}$ which is $O(y^A)$ as $y \to 0$ for some $A \in \mathbb{R}$. We distinguish two special cases ($m > 0$)

$$P(m, k, N; \tau) := \mathbb{P}(m, k, N, \mathrm{e}^{-my}; \tau), \tag{2.2}$$

$$Q(-m, k, N; \tau) := \mathbb{P}(-m, 2-k, N, \mathcal{M}_{1-k/2}(-4\pi m y); \tau), \tag{2.3}$$

† We shall also call this the shadow of the harmonic Maass form f.

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where $\mathcal{M}_s(y)$ is defined in terms of the M-Whittaker function. We often refer to $Q(-m, k, N; \tau)$ as a *Maass–Poincaré series*.

Next we give the Fourier expansions of the Poincaré series $P(m, k, N; \tau)$ and $Q(-m, k, N; \tau)$. In order to do so, let us define the *Kloosterman sums* by

$$K(m, n, c) := \sum_{v(c)^\times} e\left(\frac{m\overline{v} + nv}{c}\right), \tag{2.4}$$

where $e(\alpha) := \mathrm{e}^{2\pi \mathrm{i}\alpha}$. The sum over v runs through the primitive residue classes modulo c, and $\overline{v}$ denotes the multiplicative inverse of v modulo c.

The Fourier expansions of the cuspidal Poincaré series (for example, see [16]) are described in terms of infinite sums of Kloosterman sums weighted by J-Bessel functions.

LEMMA 2.4. *If $k \geqslant 2$ is even and $m, N \in \mathbb{N}$, then the Poincaré series $P(m, k, N; \tau)$ is in $S_k(\Gamma_0(N))$ with a Fourier expansion*

$$P(m, k, N; \tau) =: q^m + \sum_{n=1}^{\infty} a(m, k, N; n) q^n, \tag{2.5}$$

with

$$a(m, k, N; n) = 2\pi(-1)^{k/2}\left(\frac{n}{m}\right)^{(k-1)/2} \cdot \sum_{\substack{c>0 \\ c\equiv 0 \pmod N}} \frac{K(m, n, c)}{c} \cdot J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Now we recall one family of Maass–Poincaré series which have appeared in earlier works (for example, see [3, 4, 9, 13, 20]).

LEMMA 2.5. *If $k \geqslant 2$ is even and $m, N \geqslant 1$, then $Q(-m, k, N; \tau)$ is in $H_{2-k}(\Gamma_0(N))$. Moreover, if I_{k-1} is the usual I-Bessel function, then*

$$Q(-m, k, N; \tau) = (1-k)(\Gamma(k-1, 4\pi my) - \Gamma(k-1))q^{-m} + \sum_{n\in\mathbb{Z}} c_m(n, y) q^n.$$

(1) *If $n < 0$, then*

$$c_m(n, y) = 2\pi \mathrm{i}^k (1-k)\Gamma(k-1, 4\pi|n|y)\left|\frac{n}{m}\right|^{(1-k)/2} \times \sum_{\substack{c>0 \\ c\equiv 0 \pmod N}} \frac{K(-m, n, c)}{c} \cdot J_{k-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right).$$

(2) *If $n > 0$, then*

$$c_m(n, y) = -2\pi \mathrm{i}^k \Gamma(k)\left(\frac{n}{m}\right)^{(1-k)/2} \times \sum_{\substack{c>0 \\ c\equiv 0 \pmod N}} \frac{K(-m, n, c)}{c} \cdot I_{k-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right).$$

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(3) *If $n = 0$, then*

$$c_m(0, y) = -(2\pi \mathrm{i})^k m^{k-1} \sum_{\substack{c>0 \\ c\equiv 0 \pmod N}} \frac{K(-m, 0, c)}{c^k}.$$

These two families of Poincaré series are closely related (for example, see [3, Theorem 1.1]).

LEMMA 2.6. *If $k \geqslant 2$ is even and $m, N \geqslant 1$, then*

$$\xi_{2-k}(Q(-m, k, N; \tau)) = (4\pi)^{k-1} m^{k-1}(k-1) \cdot P(m, k, N; \tau) \in S_k(\Gamma_0(N)).$$

Proof. The claim follows easily from the explicit expansions in Lemmas 2.4 and 2.5 and the definition of ξ_{2-k}. □

We conclude this section with a well-known result about the behavior of the Maass–Poincaré series at the cusps (see [6, Proposition 3.1]).

LEMMA 2.7. *For $k \geqslant 2$ and $m, N \geqslant 1$, the harmonic Maass form $Q(-m, k, N; \tau)$ grows like $\exp(2\pi m y)$ as τ approaches the cusp $\mathrm{i}\infty$ and grows moderately approaching any other cusp of $\Gamma_0(N)$. The cuspidal Poincaré series $P(m, k, N; \tau)$ decays like $\exp(-2\pi y)$ as τ approaches the cusp $\mathrm{i}\infty$ and (linearly) exponentially approaching any other cusp.*

2.3. *Rankin–Cohen brackets.* Rankin–Cohen brackets are a generalization of the usual product of modular forms. They are bilinear differential operators which map modular forms to modular forms. They were introduced independently by Rankin in [23] and Cohen in [8]. A very nice discussion of them may be found in [27].

Definition 2.8. Let $f, g : \mathbb{H} \to \mathbb{C}$ be smooth functions on the upper half-plane and $k, \ell \in \mathbb{R}$ be some real numbers, the *weights* of f and g. Then, for a non-negative integer ν, we define the νth *Rankin–Cohen bracket* of f and g by

$$[f, g]_\nu := \frac{1}{(2\pi \mathrm{i})^\nu} \sum_{\mu=0}^{\nu} (-1)^\mu \binom{k+\nu-1}{\nu-\mu} \binom{\ell+\nu-1}{\mu} \frac{\partial^\mu f}{\partial \tau^\mu} \frac{\partial^{\nu-\mu} g}{\partial \tau}.$$

The following result (see [8, Theorem 7.1]) gives the connection of this definition to modular forms.

THEOREM 2.9 (Cohen). *Let f, g be as in Definition 2.8 with integral weights k, ℓ. Then the νth Rankin–Cohen bracket commutes with the slash operator, i.e.*

$$[(f|_k\gamma), (g|_\ell\gamma)]_\nu = ([f, g]_\nu)|_{k+\ell+2\nu}\gamma$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{R})$. In particular, if f and g are modular of their respective weights, then $[f, g]_\nu$ is modular of weight $2\nu + k + \ell$.

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Remark. The 0th Rankin–Cohen bracket of f and g is just their usual product; the first bracket defines a Lie bracket on the graded algebra of real-analytic modular forms, giving it the structure of a so-called *Poisson algebra*.

Remark. The νth Rankin–Cohen bracket is symmetric in f and g if ν is even, otherwise it is antisymmetric.

Since the Rankin–Cohen bracket provides a "product" on the algebra of modular forms, it is natural to consider functions of the form $[f^+, g]_\nu$ for $f \in H_{2-k}(\Gamma_0(N))$ and $g \in M_\ell(\Gamma_0(N))$ to be *mixed mock modular forms* as well.

An interesting observation, which does not seem to appear in the literature, is the following property of Rankin–Cohen brackets of harmonic Maass forms and holomorphic modular forms.

LEMMA 2.10. *Let $f \in H_{2-k}(\Gamma_0(N))$ and $g \in M_\ell(\Gamma_0(N))$ for even, positive integers k, ℓ. Then, for $\nu > k-2$, the Rankin–Cohen bracket $[f, g]_\nu$ is a weakly holomorphic modular form of weight $2\nu - k + \ell + 2$.*

Proof. Consider the operators

$$R_k := 2\mathrm{i}\frac{\partial}{\partial \tau} + ky^{-1}, \tag{2.6}$$

$$L_k := -2\mathrm{i}y^2\frac{\partial}{\partial \overline{\tau}}, \tag{2.7}$$

known as the *Maass raising operator* and the *Maass lowering operator*, respectively. These operators map modular forms of weight k to modular forms of weight $k+2$ (respectively $k-2$).

An explicit description of the iterated raising operator $R_k^n := R_{k+2(n-1)} \circ \ldots R_{k+2} \circ R_k$ for $n > 0$, $R_k^0 := \mathrm{id}$ is given by (see [**18**, equation (4.15)])

$$R_k^n = \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(k+n)}{\Gamma(k+n-m)} y^{-m} (2\mathrm{i})^{n-m} \frac{\partial^{n-m}}{\partial \tau^{n-m}},$$

which follows easily by induction on n.

With this, one obtains by straightforward calculation that if $f \in H_{2-k}(\Gamma_0(N))$ and $g \in M_\ell(\Gamma_0(N))$, then we have

$$L_{2\nu-k+\ell+2}([f, g]_\nu) = \frac{1}{(4\pi)^\nu} \binom{k-2}{\nu} L_{2-k}(f) R_\ell^\nu(g).$$

But, for $k \in \mathbb{N}$, this is 0 whenever $\nu > k-2$; thus, $[f, g]_\nu$ must be holomorphic. □

§3. *Holomorphic projection and the proof of Theorem 1.1.* Here we prove Theorem 1.1. We begin by recalling the principle of *holomorphic projection.*

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3.1. *Holomorphic projection.* The basic idea behind holomorphic projection is the following. Suppose that you have a weight k real-analytic modular form $\tilde{f}$ on $\Gamma_0(N)$ with moderate growth at the cusps. Then this function defines a linear functional on the space of weight k holomorphic cusp forms by $g \mapsto \langle g, \tilde{f}\rangle$, where $\langle \cdot, \cdot \rangle$ denotes the Petersson scalar product. But this functional must be given by $\langle \cdot, f\rangle$ for some holomorphic cusp form f of the same weight. This f is essentially the holomorphic projection of f. In [26], Sturm introduced this notion, which was used and further developed later, for example in the seminal work of Gross and Zagier on Heegner points and derivatives of L-functions [12]. The technique was also used in order to obtain recurrence relations among Fourier coefficients of mock modular forms; see e.g. [15, 19].

Here we briefly recall this framework for holomorphic projections.

Definition 3.1. Let $f : \mathbb{H} \to \mathbb{C}$ be a (not necessarily holomorphic) modular form of weight k on $\Gamma_0(N)$ with a Fourier expansion

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_f(n, y) q^n,$$

where again $y = \mathrm{Im}(\tau)$. For a cusp κ_j, $j = 1, \dots, M$, and $\kappa_1 := \mathrm{i}\infty$ of $\Gamma_0(N)$, fix $\gamma_j \in \mathrm{SL}_2(\mathbb{Z})$ with $\gamma_j \kappa_j = \mathrm{i}\infty$. Suppose that for some $\varepsilon > 0$ we have:

(1) $f(\gamma_j^{-1} w)(\frac{d}{dw}\tau)^{k/2} = c_0^{(j)} + O(\mathrm{Im}(w)^{-\varepsilon})$ as $w \to \mathrm{i}\infty$ for all $j = 1, \dots, M$ and $w = \gamma_j \tau$;

(2) $a_f(n, y) = O(y^{2-k})$ as $y \to 0$ for all $n > 0$.

Then we define the *holomorphic projection* of f by

$$(\pi_{\mathrm{hol}} f)(\tau) := (\pi_{\mathrm{hol}}^k f)(\tau) := c_0 + \sum_{n=1}^{\infty} c(n) q^n,$$

with $c_0 = c_0^{(1)}$ and

$$c(n) = \frac{(4\pi n)^{k-1}}{(k-2)!} \int_0^{\infty} a_f(n, y) \mathrm{e}^{-4\pi n y} y^{k-2}\, dy \tag{3.1}$$

for $n > 0$.

This operator has several nice properties, some of which are summarized in the following proposition.

PROPOSITION 3.2. *Let f be as in Definition 3.1. Then the following are true.*

(1) *If f is holomorphic, then $\pi_{\mathrm{hol}} f = f$.*

(2) *The function $\pi_{\mathrm{hol}} f$ lies in the space $\widetilde{M}_k(\Gamma_0(N))$.*

Proof. Claim (1) of this proposition is an easy and elementary calculation, see e.g. [15, Proposition 3.2], while the first proof of (2) using Poincaré series (and "Hecke's trick" for the case of weight 2) was given in [12, Propositions 5.1 and 6.2]. A different proof which uses spectral methods and the language of vector-valued modular forms was given in [15, Theorem 3.3]. □

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Proposition 3.2, when combined with the previous results on Rankin–Cohen brackets and harmonic Maass forms, is sufficient for proving Theorem 1.1 when M_{f_1} is good for f_2. As mentioned in the introduction, this situation is quite rare. Therefore, we must extend this notion to accommodate the most general cases. To this end, we introduce *regularized holomorphic projections*.

3.2. *Regularized holomorphic projections.* The classical holomorphic projection is constructed to respect the Petersson inner product on the space of cusp forms. We construct a regularized holomorphic projection by the same recipe. This projection shall respect the regularized Petersson inner product introduced by Borcherds in [**2**].

Definition 3.3. Let $f : \mathbb{H} \to \mathbb{C}$ be a real-analytic modular form of weight $k \geqslant 2$ with a Fourier expansion

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_f(n, y) q^n.$$

Suppose further that for each cusp κ of $\Gamma_0(N)$ there exists a polynomial $H_\kappa(X) \in \mathbb{C}[X]$, such that we have

$$(f|_k \gamma_\kappa^{-1})(\tau) - H_\kappa(q^{-1}) = O(y^{-\varepsilon})$$

for some $\varepsilon > 0$ and further suppose that $a_f(n, y) = O(y^{2-k})$ as $y \to 0$ for all $n > 0$. Then we define the *regularized holomorphic projection* of f by

$$(\pi_{\mathrm{hol}}^{\mathrm{reg}} f) = H_{i\infty}(q^{-1}) + \sum_{n=1}^{\infty} c(n) q^n,$$

where

$$c(n) = \lim_{s \to 0} \frac{(4\pi n)^{k-1}}{(k-2)!} \int_0^\infty a_f(n, y) \mathrm{e}^{-4\pi n y} y^{k-2-s} \, dy. \tag{3.2}$$

This operator enjoys a natural analog of Proposition 3.2.

PROPOSITION 3.4. *Let f be a function as in Definition 3.3.*

(1) *If f is holomorphic on $\mathbb{H}$, then we have $\pi_{\mathrm{hol}}^{\mathrm{reg}} f = f$.*
(2) *The function $\pi_{\mathrm{hol}}^{\mathrm{reg}} f$ lies in the space $\widetilde{M}_k^!(\Gamma_0(N))$.*
(3) *If f satisfies Definition 3.1, then $\pi_{\mathrm{hol}}^{\mathrm{reg}}(f) = \pi_{\mathrm{hol}}(f)$.*

Proof. Claims (1) and (3) are clear. The proof of (2) is a modification of the proofs of Propositions 5.1 and 6.2 in [**12**], which concern the classical holomorphic projection. Here we indicate how to modify these proofs to this setting.

We begin by recalling the *regularized Petersson inner product* defined by Borcherds [**2**]. For $T > 0$, we denote by $\mathcal{F}_T(\mathrm{SL}_2(\mathbb{Z}))$ the truncated version of the usual fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$, i.e.

$$\mathcal{F}_T(\mathrm{SL}_2(\mathbb{Z})) := \{\tau \in \mathbb{H} : |\tau| \geqslant 1, |\mathrm{Re}(\tau)| \leqslant \tfrac{1}{2}, \mathrm{Im}(\tau) \leqslant T\}.$$

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For a finite index subgroup $\Gamma \leqslant \mathrm{SL}_2(\mathbb{Z})$, we define a truncated fundamental domain of Γ as

$$\mathcal{F}_T(\Gamma) := \bigcup_{\gamma \in \mathcal{V}} \gamma \mathcal{F}_T(\mathrm{SL}_2(\mathbb{Z})),$$

where $\mathcal{V}$ is a fixed set of representatives of $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$. The regularized inner product of $g \in M_k(\Gamma)$ and h a weight k real-analytic modular form on Γ ($k \geqslant 2$ even) with at most linearly exponential growth at the cusps is given as the constant term in the Laurent expansion of the meromorphic continuation of the expression

$$\langle g, h\rangle^{\mathrm{reg}} := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]} \lim_{T\to\infty} \int_{\mathcal{F}_T(\Gamma)} g(\tau)\overline{h(\tau)} y^{k-2s} \frac{dx\,dy}{y^2}.$$

It follows from [7, Corollary 4.2] that the weakly holomorphic Poincaré series from §2.2 together with the Eisenstein series (including the quasimodular Eisenstein series E_2 in case that $k = 2$) are orthogonal to cusp forms with respect to $\langle\cdot,\cdot\rangle^{\mathrm{reg}}$. By subtracting a suitable linear combination of these functions, one can replace f by a function that satisfies the growth condition in [12, Proposition 5.1]. The proof then follows *mutatis mutandis* as in [12].

We consider this product for $\Gamma = \Gamma_0(N)$, where g is the regularized Poincaré series

$$P_{m,s}(\tau) := \sum_{\gamma \in \Gamma_\infty\backslash\Gamma_0(N)} (y^s \mathrm{e}^{2\pi i m\tau})|_k\gamma,$$

where $m \geqslant 1$ and $\Gamma_\infty := \{\pm\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right), n \in \mathbb{Z}\}$ denotes the stabilizer of the cusp $i\infty$ in $\Gamma_0(N)$.

Note that this regularization is necessary to ensure convergence in the case of weight 2; if the weight is at least 4, these converge to the cuspidal Poincaré series in §2.2 as $s \to 0$. Since the domain of integration, the truncated fundamental domain, is compact and all functions in the integrand are continuous, it is legitimate to replace $P_{m,s}$ by its definition as a series and interchange summation and integration. This yields $\left(\gamma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\right)$

$$\begin{aligned}
\langle g, h\rangle^{\mathrm{reg}} &= \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]} \cdot \lim_{T\to\infty} \sum_{\gamma \in \Gamma_\infty\backslash\Gamma_0(N)} \int_{\mathcal{F}_T(\Gamma_0(N))} \mathrm{e}^{2\pi i m(a\tau+b)/(c\tau+d)} \\
&\quad \times \overline{h\left(\frac{a\tau+b}{c\tau+d}\right)} \frac{y^{k-s}}{|c\tau+d|^{2(k-s)}} \frac{dx\,dy}{y^2} \\
&= \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]} \cdot \lim_{T\to\infty} \int_{\mathcal{F}_T(\Gamma_\infty)} \mathrm{e}^{2\pi i m\tau} \overline{h(\tau)} y^{k-s} \frac{dx\,dy}{y^2} \\
&= \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]} \cdot \int_{\Gamma_\infty\backslash\mathbb{H}} \mathrm{e}^{2\pi i m\tau} \overline{h(\tau)} y^{k-s} \frac{dx\,dy}{y^2}.
\end{aligned}$$

From here, the proof literally is the same as in [12], so we refer the reader to there. □

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3.3. *Holomorphic projections of completed mixed mock modular forms.* For the proof of our main result, we need to know the action of the holomorphic projection operator on Rankin–Cohen brackets of harmonic Maass forms and cusp forms. The following result (see [19, Theorem 3.6]) gives an explicit formula for the Fourier coefficients of such a holomorphic projection. The case of ordinary products, i.e. $\nu = 0$, is already contained in [15, Theorem 3.5]. Before the statement of the proposition, let us define the polynomial

$$G_{a,b}(X,Y) := \sum_{j=0}^{a-2}(-1)^j\binom{a+b-3}{a-2-j}\binom{j+b-2}{j}X^{a-2-j}Y^j \in \mathbb{C}[X,Y] \tag{3.3}$$

for integers $a \geqslant 2$ and $b \geqslant 0$, which is homogeneous of degree $a-2$.

PROPOSITION 3.5. *Let $f_1 \in S_{k_1}(\Gamma_0(N))$ and $f_2 \in S_{k_2}(\Gamma_0(N))$ be cusp forms of even weights $k_1 \geqslant k_2$ as in the introduction and let $M_{f_1} \in H_{2-k_1}(\Gamma_0(N))$ be a harmonic Maass form with shadow f_1. Then we have for $0 \leqslant \nu \leqslant (k_1-k_2)/2$ that*

$$\begin{aligned}\pi_{\mathrm{hol}}^{\mathrm{reg}}&([M_{f_1},f_2]_\nu)(\tau)\\ &= [M_{f_1}^+,f_2]_\nu(\tau) - (k_1-2)!\sum_{h=1}^{\infty}q^h\Bigg[\sum_{n=1}^{\infty}a_2(n+h)\overline{a_1(n)}\sum_{\mu=0}^{\nu}\binom{\nu-k_1+1}{\nu-\mu}\\ &\quad\times\binom{\nu+k_2-1}{\mu}\big((n+h)^{-\nu-k_2+1}G_{2\nu-k_1+k_2+2,k_1-\mu}(n+h,n)\\ &\quad - n^{\mu-k_1+1}(n+h)^{\nu-\mu}\big)\Bigg].\end{aligned} \tag{3.4}$$

Proof. We write

$$M_{f_1}^-(\tau) = \sum_{n=1}^{\infty}n^{1-k_1}\overline{a_1(n)}\Gamma^*(k_1-1,4\pi ny)\overline{q}^{-n}$$

with $\Gamma^*(\alpha;x) := \mathrm{e}^x\Gamma(\alpha;x)$. This representation makes it easy to determine the μth derivative of $M_{f_1}^-$ as

$$\begin{aligned}&\frac{1}{(2\pi\mathrm{i})^\mu}\left(\frac{\partial^\mu}{\partial\tau^\mu}M_{f_1}^-\right)(\tau)\\ &\quad= (-1)^\mu\frac{\Gamma(k_1-1)}{\Gamma(k_1-\mu-1)}\sum_{n=1}^{\infty}n^{\mu-k_1+1}\overline{a_1(n)}\Gamma(k_1-\mu-1;4\pi ny)q^{-n}.\end{aligned}$$

Using this, we find that

$$[M_{f_1}^-,f_2]_\nu(\tau) = \sum_{h\in\mathbb{Z}}b(h,y)q^h,$$

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where

$$b(h,y)=\sum_{n=1}^{\infty}\sum_{\mu=0}^{\nu}\binom{1-k_1+\nu}{\nu-\mu}\binom{k_2+\nu-1}{\mu}\frac{\Gamma(k_1-1)}{\Gamma(k_1-\mu-1)}\times n^{1-k_1+\mu}\overline{a_1(n)}\Gamma(k_1-\mu-1;4\pi ny)a_2(n+h)(n+h)^{\nu-\mu}.$$

Thus, the holomorphic projection of this becomes

$$\pi_{\mathrm{hol}}^{\mathrm{reg}}([f^-,g]_\nu)=\sum_{h=1}^{\infty}b(h)q^h,$$

with

$$b(h)=\frac{(4\pi h)^{2\nu-k_1+k_2+1}}{(2\nu-k_1+k_2)!}\sum_{n=1}^{\infty}\sum_{\mu=0}^{\nu}\binom{1-k_1+\nu}{\nu-\mu}\binom{k_2+\nu-1}{\mu}\times\frac{\Gamma(k_1-1)}{\Gamma(k_1-\mu-1)}n^{1-k_1+\mu}\overline{a_1(n)}a_2(n+h)(n+h)^{\nu-\mu}\times\int_0^{\infty}\Gamma(k_1-\mu-1;4\pi ny)\mathrm{e}^{-4\pi hy}y^{2\nu-k_1+k_2}\,dy.$$

For the evaluation of the integral, we may interchange the outer integration and the implicit one in the definition of the incomplete Gamma function, which after several substitutions of variables and simplification steps, which are carried out in detail in the proof of Lemma 3.7 in [**19**], yields the claim. We point out that none of these steps actually changes the order of summation, so also the case of conditional convergence ($\nu=(k_1-k_2)/2$) works fine. □

Remark. In the case $k_1=k_2=k$ and $\nu=0$, equation (3.4) simplifies to

$$\pi_{\mathrm{hol}}^{\mathrm{reg}}(M_{f_1}\cdot f_2)(\tau)=M_{f_1}^{+}(\tau)\cdot f_2(\tau)-(k-2)!\times\sum_{h=1}^{\infty}\left[\sum_{n=1}^{\infty}a_2(n+h)\overline{a_1(n)}\left(\frac{1}{(n+h)^{k-1}}-\frac{1}{n^{k-1}}\right)\right]q^h.$$

3.4. *The Dirichlet series.* Here we define the general Dirichlet series in Theorem 1.1. We recall from the introduction the *derived* shifted convolution series

$$D^{(\mu)}(f_1,f_2,h;s):=\sum_{n=1}^{\infty}\frac{a_1(n+h)\overline{a_2(n)}(n+h)^{\mu}}{n^s}.$$

Obviously, we have $D(f_1,f_2,h;s)=D^{(0)}(f_1,f_2,h;s)$. Due to the absence of the symmetry in the f_i, it is natural to consider special values of the symmetrized Dirichlet series

$$\widehat{D}(f_1,f_2,h;s):=D(f_1,f_2,h;s)-\delta_{k_1,k_2}D(\overline{f_2},\overline{f_1},-h;s),\qquad(3.5)$$

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where we set $a_j(n) := 0$ for $n \leqslant 0$ and δ_{ij} denotes the usual Kronecker δ. As we shall see, one important consequence of symmetrizing these functions is the convergence of the special values that we consider here. Note that $\widehat{D}(f_1, f_2, h; s)$ is, as a Dirichlet series, in fact conditionally convergent at $s = k_1 - 1$. This can be seen immediately from the estimate

$$\sum_{m=1}^{M} a_1(m+h)\overline{a_2(m)} \ll_\varepsilon M^{(k_1+k_2)/2-\delta}$$

for $h \leqslant M^{(4/3)-\varepsilon}$ ($\varepsilon > 0$) and some $\delta > 0$; see [1, Corollary 1.4].

Remark. We note that Blomer chose a different normalization and only considered shifted convolution sums for a single cusp form (i.e. $a_1(n) = a_2(n)$ for all n). However, an inspection of the proof of Theorem 1.3 in [1] shows that all his arguments carry over directly to our case.

For integers $\nu \geqslant 0$, we consider the generating function in h-aspect of the symmetrized shifted convolution special values of the Dirichlet series, convergent for $\mathrm{Re}(s) > k_1$,

$$\begin{aligned}\widehat{D}^{(\nu)}(f_1, f_2, h; s) := &\sum_{\mu=0}^{\nu} \alpha(\nu, k_1, k_2, \mu) D^{(\nu-\mu)}(f_1, f_2, h; s-\mu) \\ &- \sum_{j=0}^{w} \beta(\nu, k_1, k_2, j) D^{(j)}(\overline{f_2}, \overline{f_1}, -h; s+j-\nu), \quad (3.6)\end{aligned}$$

where $w := 2\nu - k_1 + k_2$. The $\alpha(\nu, k_1, k_2, \mu)$ and $\beta(\nu, k_1, k_2, j)$ are integers defined by

$$\alpha(\nu, k_1, k_2, \mu) := \binom{\nu - k_1 + 1}{\nu - \mu}\binom{\nu + k_2 - 1}{\mu} \quad (3.7)$$

and

$$\begin{aligned}\beta(\nu, k_1, k_2, j) := &\sum_{\mu=0}^{\nu} \binom{\nu - k_1 + 1}{\nu - \mu}\binom{\nu + k_2 - 1}{\mu} \\ &\times \binom{2\nu - \mu + k_2 - 1}{2\nu - k_1 + k_2 - j}\binom{j + k_1 - \mu - 2}{j}. \quad (3.8)\end{aligned}$$

Note that again, viewed as a Dirichlet series, $\widehat{D}^{(\nu)}$ is conditionally convergent at $s = k_1 - 1$.

Of course, we have $\widehat{D}(f_1, f_2, h; s) = \widehat{D}^{(0)}(f_1, f_2, h; s)$. The generating function we study is

$$\mathbb{L}^{(\nu)}(f_1, f_2; \tau) := \sum_{h=1}^{\infty} \widehat{D}^{(\nu)}(f_1, f_2, h; k_1 - 1) q^h. \quad (3.9)$$

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In the special case when $\nu = 0$, we have $\mathbb{L}^{(0)}(f_1, f_2; \tau) = \mathbb{L}(f_1, f_2; \tau)$.

3.5. *Proofs of Theorem 1.1 and Corollary 1.2.* We can now prove Theorem 1.1.

Proof of Theorem 1.1. If one inserts the definition of $G_{a,b}$ in (3.3) into (3.4), one gets

$$\begin{aligned}
&\pi_{\mathrm{hol}}^{\mathrm{reg}}([M_{f_1}, f_2]_\nu)(\tau) \\
&\quad = [M_{f_1}^+, f_2]_\nu(\tau) - (k_1 - 2)! \sum_{h=1}^{\infty} q^h \Bigg\{ \sum_{n=1}^{\infty} a_2(n+h)\overline{a_1(n)} \Bigg[\sum_{\mu=0}^{\nu} \binom{\nu - k_1 + 1}{\nu - \mu} \\
&\qquad \times \binom{\nu + k_2 - 1}{\mu} \Bigg(\sum_{j=0}^{w} \binom{2\nu - \mu + k_2 - 1}{2\nu - k_1 + k_2 - j} \binom{j + k_1 - \mu - 2}{j} \\
&\qquad \times \frac{n^j}{(n+h)^{k_1 + j - \nu - 1}} \Bigg) - \binom{\nu - k_1 + 1}{\nu - \mu} \binom{\nu + k_2 - 1}{\mu} \frac{(n+h)^{\nu - \mu}}{n^{k_1 - \mu - 1}} \Bigg] \Bigg\},
\end{aligned}$$

where, as above, $w := 2\nu - k_1 + k_2$. By definition, this is

$$[M_{f_1}^+, f_2]_\nu(\tau) + (k_1 - 2)! \cdot \mathbb{L}^{(\nu)}(f_2, f_1; \tau),$$

where $\alpha(\nu, k_1, k_2, \mu)$ and $\beta(\nu, k_1, k_2, j)$ are defined by (3.7) and (3.8).

By Proposition 3.4, this function lies in the space $\widetilde{M}^!_{2\nu - k_1 + k_2 + 2}(\Gamma_0(N))$, which yields Theorem 1.1 in the general case. When M_{f_1} is good for f_2, Proposition 3.2 implies that it lies in the space $\widetilde{M}_{2\nu - k_1 + k_2 + 2}(\Gamma_0(N))$. □

Proof of Corollary 1.2. The formulas stated in the corollary are exactly the same as in Theorem 1.1; keeping in mind that by Lemma 2.6 the shadow of Q is precisely $(1 - k) \cdot P$, we only have to make sure that Q is good for P. But this follows immediately from Lemma 2.7 since P decays exponentially in every cusp of $\Gamma_0(N)$ and Q^+ has exactly one simple pole at infinity and grows moderately in the other cusps. □

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