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Physical measures for certain partially hyperbolic attractors on 3-manifolds

RICARDO T. BORTOLOTTI

Departamento de Matemática – UFPE, Recife, PE, Brazil
(e-mail: ricardo@dmat.ufpe.br)

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Abstract. In this work, we analyze ergodic properties of certain partially hyperbolic attractors whose central direction has a neutral behavior; the main feature is a condition of transversality between the projections of unstable leaves, projecting through the stable foliation. We prove that partial hyperbolic attractors satisfying this condition of transversality, neutrality in the central direction and regularity of the stable foliation admit a finite number of physical measures, coinciding with the ergodic u-Gibbs States, whose union of the basins has full Lebesgue measure. Moreover, we describe the construction of robustly non-hyperbolic attractors satisfying these properties.

1. *Introduction*

In this work, we study ergodic properties for some attractors 'beyond uniform hyperbolicity'. Our interest is in the existence and finiteness of physical measures for partially hyperbolic systems. The analysis of this kind of questions started with Sinai, Ruelle and Bowen in the case of hyperbolic systems [**9**, **10**, **24**, **26**] and was later extended for partially hyperbolic systems [**1**, **8**, **11**, **28**, **30**], Henon-like families [**4–6**], Lorenz-like attractors [**21**, **25**, **29**] and many other types of systems [**2**, **18**, **31**]. In most of the previous mentioned results, it is shown that the dynamical system admits a finite number of physical measures whose basins cover a full Lebesgue measure subset.

In [**20**], Palis conjectured that every dynamical system can be approximated by another dynamical system having finitely many physical measures and that the union of their basins has total Lebesgue measure. It is also possible to consider this conjecture for the open set of partially hyperbolic systems. In the works of Bonatti and Viana [**8**] for 'mostly contracting' and Alves–Bonatti–Viana [**1**] for 'mostly expanding' central direction, the existence and finiteness of physical measures for partially hyperbolic systems with some non-uniform contraction or expansion in the central direction is proved. However, the case where the central direction is neutral has not been well studied.

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Also related to the Palis conjecture, the existence of physical measures for generic partially hyperbolic systems is not known. Therefore, it would be interesting to obtain results of genericity for generic partially hyperbolic systems.

Another relevant work is [**28**], where a generic approach is made in the case of surface endomorphism. Actually, in that paper, Tsujii proved that a generic partial hyperbolic endomorphism admits finitely many physical measures and the union of their basins has full Lebesgue measure.

The present work is a first step to extend the analysis of Tsujii for the case of diffeomorphisms. More precisely, we consider partially hyperbolic attractors with central Lyapunov exponent close to zero and with a geometrical property of transversality between the projections of unstable leaves, projecting through the stable foliation, which implies the non-joint-integrability of $E^{ss} \oplus E^{uu}$ and proves the existence of physical measures for this set of attractors.

We summarize our results in the following theorem.

THEOREM. *Given a diffeomorphism $f_0 : M \to M$ of class C^r, $r \geq 2$, a three-dimensional manifold M and a partially hyperbolic attractor Λ_0 for f_0, if we assume that the attractor is dynamically coherent and the following conditions hold:*

(C1) *transversality between unstable leaves via the stable foliation;*

(C2) *the central direction is neutral;*

(C3′) *the stable foliation $\mathcal{F}^{ss}$ is Lipschitz, then f_0 admits finite physical measures and the union of their basins has total Lebesgue measure in the basin of attraction of the attractor.*

Moreover, if we assume that the attractor is robustly dynamically coherent and that

(C3) *$\mathcal{F}^{ss}_f$ varies continuously with f in the C^1-topology, then there exists an open set $\mathcal{U}$ containing f_0 such that the same result holds for every $f \in \mathcal{U}$.*

For precise definitions and statements of our results, see §2.

In the above theorem, the geometrical condition of transversality (C1) allows us to prove that u-Gibbs states of diffeomorphisms are sent by local stable projections into absolutely continuous measures. This step contains the technical part of our work and allows us to prove that the ergodic u-Gibbs states are the physical measures.

In the last part of this work, we construct examples of partially hyperbolic attractors that are robustly non-hyperbolic and satisfy conditions (C1)–(C3).

We emphasize that our results include situations where the central Lyapunov exponent is null, and therefore we can't use Pesin's theory.

It is natural to expect a weaker version of the property of transversality that is generic among partially hyperbolic systems. Before proving such a generic property, it is necessary to weaken the regularity assumptions for the stable foliation.

1.1. *Strategy of the proof.* The heart of the proof is an inequality similar to the Doeblin–Fortet inequality (also known as the Lasota–Yorke inequality). We will work with a norm for finite measures defined in center-unstable manifolds that is similar to the L^2 norm for the densities of measures when they are absolutely continuous. Our aim is to see that every

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u-Gibbs state has certain regularity after projecting locally by the stable foliation, which will imply that the ergodic u-Gibbs states are the physical measures.

The transversality condition plays an important role because the density of the u-Gibbs state is good *a priori* only in the unstable direction. Then, if the local stable projection of unstable discs gives many directions, we will obtain some mass in many directions, which will imply that the projected measure into center-unstable leaves is absolutely continuous.

1.2. *Structure of the work.* This work is organized as follows. In §2, we give the basic definitions and precise statements of our results. In §3, we define the boxes and norms that shall be used to state the main inequality and the technical lemmas. In §4, we prove estimates that will be used to prove the main inequality. In §5, we use the main inequality to show that every ergodic u-Gibbs state has better regularity after projecting via local stable projections into center-unstable leaves. We also show the existence and finiteness of physical measures and that the union of their basins has full Lebesgue measure. Finally, in §6, we describe the construction of non-hyperbolic attractors that have central direction close to neutral and satisfy the transversality condition. We also check that they are robustly transitive and robustly non-hyperbolic.

2. *Definitions and statements*

In this section, we give some definitions and state our main results.

2.1. *Prerequisites.* Let M be a compact Riemannian manifold with a respective normalized Lebesgue measure m, and let $f : M \to M$ be a differentiable function of class C^r with $r \geq 1$.

Definition 2.1. Given an f-invariant measure μ, we define the *basin of* μ as the set $B(\mu)$ of points x such that for every continuous function $\phi : M \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) = \int_M \phi \, d\mu.$$

We call an f-invariant measure μ a *physical measure* if the Lebesgue measure of $B(\mu)$ is positive.

Definition 2.2. Given an invariant set Λ for f, we call Λ an *attractor* if there exists an open set $U \subset M$ such that $\overline{f(U)} \subset U$ and $\Lambda = \bigcap_{n\geq 0} f^n(U)$. The set $B(\Lambda) = \bigcup_{j\geq 0} f^{-j}(U)$ is called the *basin of attraction* of Λ. This is the set of points whose orbit accumulates on Λ.

Definition 2.3. We say that an attractor Λ is *partially hyperbolic* for f if, for every $x \in \Lambda$, there exist constants $\lambda_{ss}^+ < \lambda_c^- < \lambda_c^+ < \lambda_{uu}^-$, $C > 1$ and an Df-invariant splitting $T_x M = E_x^{ss} \oplus E_x^c \oplus E_x^{uu}$ such that $\lambda_{ss}^+ < 1$, $\lambda_{uu}^- > 1$ and

$$\begin{aligned} \|Df^n v\| &< C(\lambda_{ss}^+)^n \|v\|, \quad v \in E_x^{ss} - \{0\}, \\ C^{-1}(\lambda_c^-)^n \|v\| < \|Df^n v\| &< C(\lambda_c^+)^n \|v\|, \quad v \in E_x^c - \{0\}, \\ C^{-1}(\lambda_{uu}^-)^n \|v\| &< \|Df^n v\|, \quad v \in E_x^{uu} - \{0\}. \end{aligned}$$

The subbundle E^{ss} is called stable direction and E^{uu} is called unstable direction. The distributions E^{ss} and E^{uu} integrate uniquely into invariant manifolds.

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THEOREM 2.1. *The stable and unstable subbundles E^{ss}_x and E^{uu}_y integrate uniquely into laminations $\mathcal{F}^{ss}_x$ and $\mathcal{F}^{uu}_y$ for every $x \in B(\Lambda)$ and $y \in \Lambda$.*

A proof of Theorem 2.1 can be found in [**3**] and [**15**]. The existence of $\mathcal{F}^{ss}$ can be guaranteed for every x in a full neighbourhoud of the attractor (see [**3**, Theorem 4.2]), but the unstable manifolds are only well defined for points in the attractor.

Remark 2.1. If Λ is a partially hyperbolic attractor, then $W^{uu}_x \subset \Lambda$ for every $x \in \Lambda$, so $\Lambda = \bigcup_{x\in\Lambda} W^{uu}(x)$.

THEOREM 2.2. *If the map f is of class C^r and satisfies the following bunching condition for a dominated splitting $E_1 \oplus E_2$ and $k \geq 1$:*

$$\sup_{x\in\Lambda} \|D_x f_{|E_1}\| \cdot \frac{\|D_x f_{|E_2}\|^k}{m(D_x f_{|E_2})} < 1, \tag{1}$$

then there exists an invariant foliation $\mathcal{F}_1$ tangent to E_1 of class C^l, where $l = \min\{k, r-1\}$. Moreover, the foliation $\mathcal{F}_1(g)$ tends to $\mathcal{F}_1(f)$ in the C^l topology when g tends to f in the C^r-topology.

For a proof of Theorem 2.2, see [**3**] and [**15**]. The C^l-continuity of the foliation follows from the use of the C^r-section theorem [**15**, Theorem 3.5] in the proof of [**3**, Theorem 4.12].

Invariant sets satisfying (1) for $k = 1$ are usually said to be normally dissipative (see [**23**]). In this case, the condition says that the rate of contraction along E_1 is smaller than the ratio between the minimum and maximum singular value of the restriction Df to E_2. If we take $k = 1$ and $r = 2$, from Theorem 2.2 we see that $\mathcal{F}^{ss}$ is of class C^1 and the stable foliation of any g C^2-close to f is close to the stable foliation of f in the C^1-topology, which means that $E^{ss}_x(g)$ is a C^1-subbundle and tends to $E^{ss}_x(f)$ when g tends to f.

Remark 2.2. When Λ is a transitive hyperbolic attractor (i.e., $E^c = 0$), the works of Sinai–Ruelle–Bowen guarantee that there exists a unique physical measure μ whose basin $B(\mu)$ has full Lebesgue measure in the basin of attraction of Λ.

Throughout the paper, the most important measures will be the so-called u-Gibbs states, which are measures whose disintegration along unstable leaves corresponds to absolutely continuous measures with respect to the induced Lebesgue measure in each unstable leaf.

Definition 2.4. Given $x \in \Lambda$, $r > 0$ and a C^1 disk Σ centered at x with dimension $\dim(E^{ss} \oplus E^c)$ and transversal to $\mathcal{F}^{uu}$, we define the *foliated box* $\Pi(x, \Sigma, r) := \bigcup_{z\in\Lambda\cap\Sigma} \gamma^{uu}_{(z,r)}$, where $\gamma^{uu}_{(z,r)}$ is the unstable neighborhood of z of radius r.

A *foliated chart* is an application $\Phi_{x,\Sigma,r} : \Pi(x, \Sigma, r) \to I^{uu}_r \times (\Sigma \cap \Lambda)$ that is a homeomorphism into the image and restricted to each γ^{uu} is a diffeomorphism into the horizontal.

Definition 2.5. Let μ be an invariant finite Borel measure. We say that μ is *absolutely continuous with respect to the Lebesgue measure on unstable leaves* and call it a *u-Gibbs state* if for every x, Σ and $r > 0$, the disintegration $\{\mu_z\}_{z\in\Sigma\cap\Lambda}$ of the measure $(\Phi_{x,\Sigma,r})_*\mu$ with respect to the partition of $I^{uu} \times (\Sigma \cap \Lambda)$ into horizontal lines is formed by absolutely

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continuous measures with respect to the induced Lebesgue measures $m_{\gamma^{uu}_{(z,r)}}$ for $\hat{\mu}$-almost every z.

It is well known that u-Gibbs states always exist. Actually, if D^{uu} is any compact disk contained in some unstable leaf, and $m_{D^{uu}}$ is the normalized Lebesgue measure induced on D^{uu} by the volume element associated to some Riemannian metric of M, we have the following proposition.

PROPOSITION 2.1. *Let f be a C^2 diffeomorphism, Λ a partially hyperbolic attractor and D^{uu} an unstable disk. If we consider the measures $\mu_n = (1/n)\sum_{i=0}^{n-1} f^j_*(m_{D^{uu}})$, then any accumulation point of μ_n is a u-Gibbs state.*

For more on u-Gibbs states one can check the book [**7**]. We will refer to these measures as 'u-Gibbs'. The main properties of u-Gibbs are given in the following result.

PROPOSITION 2.2. *Let $f : M \to M$ be a diffeomorphism of class C^r, $r \geq 2$ and Λ a partially hyperbolic attractor for f. Then the following hold.*

(1) *The densities of a u-Gibbs with respect to the induced Lebesgue measure along unstable plaques are positive and bounded from zero and infinity.*
(2) *The support of every u-Gibbs is W^{uu}-saturated. In particular, it is contained in the attractor Λ.*
(3) *The set of u-Gibbs is non-empty, weak-$*$ compact and convex.*
(4) *The ergodic components of a u-Gibbs are also u-Gibbs.*
(5) *Every physical measure supported in Λ is a u-Gibbs.*

Proof. See [**7**, §11]. □

Property (5) above says that the u-Gibbs are the appropriate candidates for the physical measures. In this work, we prove that under certain conditions, the ergodic u-Gibbs are, in fact, the physical measures.

PROPOSITION 2.3. *Let f be a C^2 diffeomorphism, Λ a partially hyperbolic attractor and $E \subset B(\Lambda)$ a measurable set with positive Lebesgue measure. If we consider m_E the restricted and normalized measure of m to E, then any point of accumulation of the measures $\mu_n = (1/n)\sum_{j=0}^{n-1} f^j_*(m_E)$ is a u-Gibbs.*

2.2. *The transversality condition.* Let M be a three-dimensional compact manifold and Λ a partially hyperbolic attractor for f with $\dim E^{ss} = \dim E^c = \dim E^{uu} = 1$.

Definition 2.6. We say that Λ is *dynamically coherent* if for every $x \in \Lambda$ there exist uniquely invariant manifolds W^{cu}_x and W^{cs}_x tangent to $E^c_x \oplus E^{uu}_x$ and $E^{ss}_x \oplus E^c_x$, respectively.

The invariant manifolds W^{cu}_x and W^{cs}_x will be called center-unstable and center-stable manifolds, respectively. When the attractor is dynamically coherent, we have the invariant center manifolds W^c given by $W^c_x = W^{cu}_x \cap W^{cs}_x$. When $z \notin \Lambda$ is in W^{cu}_x, we will denote $W^{cu}(z)$ by W^{cu}_x.

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Remark 2.3. It is not known whether dynamical coherence is an open property or not. However, if the distribution E^c is of class C^1 then it is open (see [**15**, Theorems 7.1 and 7.4]).

Definition 2.7. Given a dynamically coherent attractor Λ and two unstable curves of finite length γ_1 and γ_2, let W_i^{cu} be the center-unstable submanifold of radius R_0 around the curve γ_i, $i = 1, 2$. We define the *stable distance between these curves* by:

$$d^{ss}(\gamma_1, \gamma_2) = \begin{cases} \min\limits_{\gamma^{ss}} \left\{ l(\gamma^{ss}) \mid \gamma^{ss} \text{ is either a stable segment joining } \gamma_1 \text{ to } W_2^{cu} \right. \\ \qquad\qquad \left. \text{or a stable segment joining } \gamma_2 \text{ to } W_1^{cu} \right\} \\ \infty \quad \text{if } \gamma^{ss} \text{ does not exist as above.} \end{cases}$$

PROPOSITION 2.4. *There exists a finite covering $\{U_i\}_{i\in I}$ of Λ by open sets in M, and homeomorphisms $\psi_i : U_i \subset M \to I_i \times D_i \subset \mathbb{R}^3$, where $I_i \subset \mathbb{R}$ and D_i is a ball contained in $\mathbb{R}^2$, such that for every $z \in \Lambda \cap D_i$ there exists $a(z) \in I_i$ with $\psi_i(W^{cu}(z)) \subset \{a(z)\} \times D_i$ and $\psi_i|_{W^{cu}(z)}$ is a diffeomorphism into the image.*

Proof. The center-unstable manifolds form a lamination. So, for every point $x \in \Lambda$, considering a transversal section I_x, there exist a neighborhood U_x and a homeomorphism $\psi_x : U_x \to I_x \times D$ such that $\psi_x(W^{cu}(z)) \subset \{a(z)\} \times D$ for every $y \in \Lambda$ and every $z \in U_x \cap W_y^{cu}$, and $\psi_x|_{W^{cu}(z)}$ is a diffeomorphism into the image. By compactness of Λ, we can consider a finite sub-covering and the diffeomorphisms corresponding to this sub-covering. □

If the attractor is dynamically coherent, then, by compactness, there exists some constant R_0 such that every center-unstable manifold has an internal radius greater than R_0. Fix $K_1 \geq 1$ such that $K_1^{-1} \leq \|D(\psi_i|_{W^{cu}(z)})\| \leq K_1$ for every ψ_i as above and $R_1 > 0$ such that for every $x \in \Lambda$, the set $B^{cu}(x, R_1)$ is contained in some U_i.

Definition 2.8. We say that two continuous curves γ_1 and γ_2 of finite length contained in a subset of $\mathbb{R}^2$ are *θ-transversal in neighborhoods of radius r in $\mathbb{R}^2$* if for every $x_1 \in \gamma_1$, $x_2 \in \gamma_2$ such that $d(x_1, x_2) < r$, there exist cones C_1 and C_2 with vertices at the points x_1 and x_2 such that $\gamma_1 \cap B(x_1, r) \subset C_1$, $\gamma_2 \cap B(x_2, r) \subset C_2$ and $\angle(v_1, v_2) \geq \theta$ for every tangent vector v_1, v_2 at the points $x_1 \in \gamma_1, x_2 \in \gamma_2$ tangent to the cones C_1, C_2, respectively.

When these curves are differentiable we think of the cones above as having arbitrarily small width around the tangent direction to the curve. But if the curve is not differentiable, then the cone must contain every possible tangent direction to the curve.

Definition 2.9. We say that two continuous curves γ_1 and γ_2 of finite length contained in the same center-unstable manifold W^{cu} are *θ-transversal in neighborhoods of radius r* if $r < R_1 K_1^{-1}/2$ and for every $i \in I$, every connected component $\tilde{\gamma}_1$ of $\psi_i(\gamma_1 \cap U_i)$ and $\tilde{\gamma}_2$ of $\psi_i(\gamma_2 \cap U_i)$ are θ-transversal in neighborhoods of radius r in $\mathbb{R}^2$.

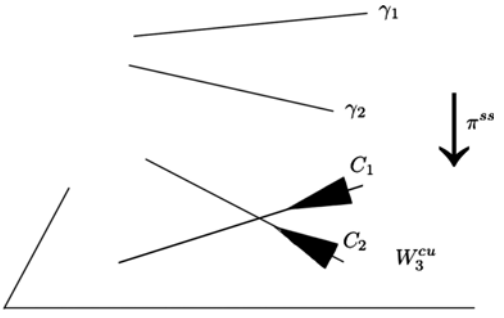
We will define a notion of transversality between two unstable curves via the stable foliation.

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Definition 2.10. (Condition (C1)) The *transversality condition (C1)* holds if there exist constants $\epsilon_0 > 0$, $L > 0$ and functions $\theta : (0, \epsilon_0) \to \mathbb{R}^+$ and $r : (0, \epsilon_0) \to \mathbb{R}^+$ such that the following is valid:

Given $\epsilon < \epsilon_0$, unstable curves γ_1 and γ_2 with length smaller than L and a center-unstable manifold W_3^{cu} with $d^{ss}(\gamma_i, W_3^{cu}) < \epsilon_0/2$, $i = 1, 2$, and $d^{ss}(\gamma_1, \gamma_2) > \epsilon$, let C be an open set with product structure $W^{ss} \times W^{cu}$, with d^{ss} diameter smaller than ϵ_0 and containing γ_1, γ_2, W_3^{cu}. Taking the stable projection $\pi^{ss} : C \to W_3^{cu}$, we ask that the curves $\pi^{ss}(\gamma_1)$ and $\pi^{ss}(\gamma_2)$ are $\theta(\epsilon)$-transversal in neighborhoods of radius $r(\epsilon)$.

Every time we refer to the transversality condition, it will be implicit that the manifold is three-dimensional and each subbundle is one-dimensional.



The transversality condition stated above is a quantitative way to say that $E^{ss} \oplus E^{uu}$ is non-integrable. It holds in several cases: for example, for contact Anosov flows [**13**, **14**, **17**], for algebraic automorphisms on Heisenberg nilmanifolds [**16**] and for the attractor that will be constructed in §6. As we will see in Proposition 5.4, this condition is open for attractors where the stable foliation varies continuously in the C^1 topology. It is important to say that a similar condition, called uniform non-integrability (UNI), plays a fundamental role in the works of Chernov–Dolgopyat–Liverani [**13**, **14**, **17**] for decay of correlation for contact Anosov flows.

2.3. *Main results.* In order to give the precise statements of our results, we will give two other main conditions: one of neutrality in the central direction and other of regularity of the stable foliation.

Definition 2.11. (Condition (C2)) Let Λ be a partially hyperbolic attractor and $\lambda_c^-, \lambda_c^+, \lambda_{uu}^-$ constants as in Definition 2.3. The attractor Λ has *central direction neutral* if $\lambda_c^- < 1 < \lambda_c^+$ and

$$\frac{\lambda_c^+}{(\lambda_c^-)^2 \cdot \lambda_{uu}^-} < 1.$$

This condition of neutrality in the central direction occurs when $Df_{|E^c}$ is close to an isometry. Note that if this condition is valid, then the center-unstable direction is volume-expanding.

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Definition 2.12. (Condition (C3)) Condition (C3) holds if the stable foliation $\mathcal{F}^{ss}_f$ is of class C^1 for every f in a neighborhood of f_0 and the map $f \to \mathcal{F}^{ss}_f$ is continuous at f_0 in the C^1-topology in the space of foliations defined in a fixed small neighbourhood of the attractor.

This condition guarantees that the stable foliation of the attractor Λ for f is close in the C^1-topology to the one of Λ_0 for f_0. It follows from Theorem 2.2 that this condition is valid when condition (1) is satisfied for $k = 1$.

We now give the main results of this paper.

THEOREM A. *Given a diffeomorphism $f : M \to M$ of class C^r, $r \geq 2$, a three-dimensional manifold M and a partially hyperbolic attractor Λ of f, if we assume that the attractor is dynamically coherent and that:*

(C1) *transversality between unstable leaves via the stable foliation;*
(C2) *the central direction is neutral;*
(C3′) *the stable foliation $\mathcal{F}^{ss}$ is Lipschitz, then f admits a finite number of physical measures supported in Λ, coinciding with the ergodic u-Gibbs, and the union of their basins has full measure in $B(\Lambda)$.*

Theorem A will be proved in §5. The technical tools for the proof of this result will be developed throughout §§3 and 4. As a consequence of Theorem A, we have the corollary below.

COROLLARY B. *Given a diffeomorphism $f_0 : M \to M$ of class C^r, $r \geq 2$, a three-dimensional manifold M and a partially hyperbolic attractor Λ_0 of f_0, if we assume that Λ_0 is robustly dynamically coherent and that:*

(C1) *transversality between unstable leaves via the stable foliation;*
(C2) *the central direction is neutral;*
(C3) *$f \to \mathcal{F}^{ss}_f$ is continuous at f_0 in the C^1-topology, then there exists an open set $\mathcal{U}$ containing f_0 such that every $f \in \mathcal{U}$ admits a finite number of physical measures supported in Λ, coinciding with the ergodic u-Gibbs, and the union of their basins has full measure in $B(\Lambda)$.*

In §6, we will describe the construction of an attractor with central direction neutral satisfying the conditions of transversality and of regularity of the stable foliation. Using this construction we will prove the following theorem.

THEOREM C. *There exist $f_0 : M^3 \to M^3$ and a partial hyperbolic attractor Λ_0 that is robustly non-hyperbolic and robustly dynamically coherent satisfying the conditions* (C1)–(C3). *Moreover, this attractor is robustly transitive.*

The proof of Theorem C corresponds to a construction considering initially a hyperbolic solenoidal attractor satisfying the transversality condition (C1) and deforming the dynamics in the central direction inside a neighborhood of a periodic point in a similar way to Mañé's example [**19**].

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3. *Toolbox*

For the technical part of Theorem A, we will make use of certain sets called boxes and a certain norm that will be defined in this section.

3.1. *The boxes.*

We will consider subsets of the manifold where the projection through the stable foliation into one fixed center-unstable manifold is well defined and such that every unstable curve that intersects these subsets must cross them.

Definition 3.1. Given a diffeomorphism $f : M \to M$ and a partially hyperbolic attractor Λ that is dynamically coherent, we call a quadruple $(C, W, \tilde{W}, \pi)$ a *box* if C is the image of an embedding $h : I^{uu} \times I^a \times I^b \to M$, where I^{uu}, I^a and I^b are intervals, such that the following hold.

(1) The function h_{z_0} given by $h_{z_0}(x, y) = h(x, y, z_0)$ is an embedding into a surface that coincides with a center-unstable manifold if its image intersects the attractor. The set W is the image of $h(I^{uu} \times I^a \times \{0\})$ and intersects Λ.

(2) If $h(x_0, y_0, z_0) \in \Lambda$, then $\gamma(t) = h(t, y_0, z_0)$ is an unstable curve.

(3) For every $x, y \in C$, every connected component of $W^{cu}_x \cap C$ and of $W^{ss}_y \cap C$ intersect at most once.

(4) The set $\tilde{W}$ is a connected center-unstable manifold with finite diameter containing W and such that $\tilde{W} \cap W^{ss}_y \neq \emptyset$ for every $y \in C$. The application $\pi : C \to \tilde{W}$ sends each $y \in C$ into the point $W^{ss}_{\text{loc}}(y) \cap \tilde{W}$.

The first condition above says that the set $C \cap \Lambda$ can be seen as a family of center-unstable manifolds. The second one states that if an unstable curve intersects C then it crosses C, and the third and fourth ones guarantee that the local stable projection $\pi^{ss} : C \to \tilde{W} \supset W$ are well defined.

Since every point in the attractor admits arbitrarily small boxes containing it, it is possible to consider a finite family of boxes $\{(C_i, W_i, \tilde{W}_i, \pi_i)\}$ such that the sets $\pi_i^{-1}(W_i)$ cover the attractor Λ, that is, $\Lambda \subset \bigcup_i \pi_i^{-1}(W_i)$.

3.2. *The norms.*

We will define a norm that estimates a kind of regularity of the projection of measures into a fixed center-unstable manifold. It will be in terms of this norm that we will describe a criterion of absolute continuity for the stable projection of measures. This norm will be used jointly with the boxes defined in the previous section.

Definition 3.2. Let $X \subset M$ be a submanifold of dimension 2, m_X the Lebesgue measure in X induced by the Riemannian metric, μ_1 and μ_2 finite measures defined in X and $r > 0$ fixed. We define the bilinear form $\langle \mu_1, \mu_2 \rangle_{X,r}$ by

$$\langle \mu_1, \mu_2 \rangle_{X,r} = \frac{1}{r^4} \int_X \mu_1(B(z, r)) \mu_2(B(z, r))\, dm_X(z).$$

The bilinear form $\langle \mu_1, \mu_2 \rangle_{X,r}$ induces the *norm*

$$\|\mu\|_{X,r} = \sqrt{\langle \mu, \mu \rangle_{X,r}}.$$

Let us now prove some facts for the norm $\| \cdot \|_{X,r}$.

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LEMMA 3.1. *Given two finite families of center-unstable manifolds* $\{W_i\}$ *and* $\{\tilde{W}_i\}$ *with bounded diameter,* $W_i \subset \tilde{W}_i$ *for every* i *and such that* $d(W_i, \partial\tilde{W}_i) > r_0$*, there exists a constant* $C_0 \geq 1$ *such that*

$$\|\nu\|_{\tilde{W}_i, r_2} \leq C_0 \|\nu\|_{\tilde{W}_i, r_1}$$

for every $0 < r_1 \leq r_2 \leq r_0 \leq 1$*, every* i *and every* ν *supported on* W_i*.*

Proof. Let $\tilde{C}_0$ be a constant independent of r_1, r_2 such that it is possible to cover every ball $B(0, r_2)$ in the plane $\mathbb{R}^2$ with $\lceil \tilde{C}_0 r_2/r_1 \rceil^2$ balls of radius r_1. If z_k are the center of these balls, then every ball $B(x, r_2)$ is covered by the balls $B(x + z_k, r_1)$. Since we have a finite number of center-unstable manifolds W_i^{cu} that are diffeomorphic images of subsets of $\mathbb{R}^2$ by $\phi_i = \psi_i|_{\psi_i^{-1}(W_i)}$, we may take a uniform Lipschitz constant $K > 0$ of every ϕ_i and ϕ_i^{-1} and get that the ball $B^{cu}(z, r_2)$ is contained in $\psi_i(B(\psi_i^{-1}(z), Kr_2))$, which is covered by $\lceil \tilde{C}_0(Kr_2/K^{-1}r_1)\rceil^2$ sets $\psi_i B(\psi_i^{-1}(z) + z_k, K^{-1}r_1)$, and each one of them is covered by a ball $B^{cu}(\psi_i(\psi_i^{-1}(z) + z_k), r_1)$. So there exists a constant C_0 independent of r_1, r_2, x and i such that it is possible to cover every $B^{cu}(x, r_2)$ in W^{cu} with $\lceil C_0(r_2/r_1)\rceil^2$ disks $B^{cu}(w_k, r_1)$.

We have the following estimate:

$$\begin{aligned}
\|\mu\|^2_{\tilde{W}_i, r_2} &= \frac{1}{r_2^4} \int_{W_i} \mu(B(z, r_2))^2 \, dm^{cu}(z) \\
&\leq \frac{1}{r_2^4} C_0^2 \int_{\tilde{W}_i} \left(\sum_k \mu(B(w_k, r_1))^2 \right) dz \\
&\leq \frac{1}{r_2^4} C_0^2 \left(\frac{r_2^2}{r_1^2} \right)^2 \sum_k \int_{\tilde{W}_i} \mu(B(w_k, r_1))^2 \, dw_k \\
&\leq C_0^2 \|\mu\|^2_{\tilde{W}_i, r_1}.
\end{aligned}$$

□

LEMMA 3.2. *Let* $\{\nu_n^1\}_n$ *and* ν^1 *be finite measures defined in a center-unstable manifold* W *such that* $\nu_n^1 \xrightarrow{*} \nu_\infty^1$ *and* $r > 0$ *fixed. Then,*

$$\lim_{n\to\infty} \|\nu_n^1\|_{W,r} = \|\nu^1\|_{W,r}.$$

Moreover, if we consider finite measures $\{\nu_n^2\}_n$ *and* ν^2 *also defined in* W *such that* $\nu_n^2 \xrightarrow{*} \nu_\infty^2$*, then*

$$\lim_{n\to\infty} \langle \nu_n^1, \nu_n^2 \rangle_{W,r} = \langle \nu^1, \nu^2 \rangle_{W,r}.$$

Proof. Note that $\nu^1(\partial B^{cu}(z, r)) = 0$ for m^{cu}-almost every z, because

$$\begin{aligned}
\int \nu^1(\partial B^{cu}(z, r) \, dm^{cu}(z) &= \iint_{d(z,w)=r} d\nu^1(w) dm^{cu}(z) \\
&= \int m^{cu}(\partial B^{cu}(w, r)) \, d\nu^1(w) \\
&= 0.
\end{aligned}$$

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If we take $J_r\nu(x) = \nu(B(x,r))/r^2$, then $J_r\nu_n^1 \to J_r\nu_\infty^1$, and since $\|J_r\nu_n^1\|_{L^2}$ is uniformly bounded, the lemma follows by the theorem of dominated convergence.

The proof that $\lim_{n\to\infty}\langle \nu_n^1, \nu_n^2\rangle_{W,r} = \langle \nu_\infty^1, \nu_\infty^2\rangle_{W,r}$ is analogous. □

Let us now consider a norm measuring the total mass of a measure.

Definition 3.3. For finite measures defined in the manifold M, we will consider the *norm* $|\cdot|$ as $|\mu| = \mu(M)$.

Definition 3.4. Given a finite family of applications $\pi_i : C_i \subset M \to \tilde{W}_i \supset W_i$, $i = 1, \ldots, s_0$ and a measure μ in M, we consider the families $\mathcal{W} = \{W_1, \ldots, W_{s_0}\}$ and $\tilde{\mathcal{W}} = \{\tilde{W}_1, \ldots, \tilde{W}_{s_0}\}$ and define

$$|||\mu|||_{\mathcal{W},r} := \max_{i=1,\ldots,s_0}\{\|(\pi_i)_*\mu\|_{W_i,r}\},$$
$$|||\mu|||_{\tilde{\mathcal{W}},r} := \max_{i=1,\ldots,s_0}\{\|(\pi_i)_*\mu\|_{\tilde{W}_i,r}\}.$$

Remark 3.1. Given a finite family of boxes $\{(C_i, W_i, \tilde{W}_i, \pi_i)\}$, $i = 1, \ldots, s_0$, with the property that $\{\pi_i^{-1}(W_i)\}$ covers Λ, for measures μ defined in Λ we have the following equivalence for $|||\mu|||_{\{W_i\},r}$ and $|||\mu|||_{\{\tilde{W}_i\},r}$:

$$|||\mu|||_{\{\mathcal{W}\},r} \le |||\mu|||_{\{\tilde{\mathcal{W}}\},r} \le s_0|||\mu|||_{\{\mathcal{W}\},r}.$$

When the boxes and the corresponding center-unstable manifolds are implicit, we will simply denote the norms by $\|\cdot\|_r$ and $|||\cdot|||_r$.

Given a center-unstable manifold W, we denote by m_W^{cu} the Lebesgue measure in this submanifold given by the Riemannian metric. When the set W is implicit, we will denote it by m^{cu}. Notice that we are considering dynamically coherent attractors with $\dim(E^c \oplus E^{uu}) = 2$. For absolute continuous measures defined in a center-unstable surface, the norm $\|\cdot\|_r$ is related to the L^2 norm of the density. This relation can be seen in the following criterion of absolute continuity for measures defined in a center-unstable manifold.

LEMMA 3.3. *Given a center-unstable manifold W, there exists a constant $I > 0$ such that every finite Borel measure ν defined in W satisfying*

$$\liminf_{r\to 0^+}\|\nu\|_{W,r} \le L,$$

for some $L > 0$, is absolutely continuous with respect to m^{cu} and

$$\left\|\frac{d\nu}{dm^{cu}}\right\|_{L^2(W,m^{cu})} \le IL.$$

Proof. First, we observe that there exist constants $C_1, C_2 > 0$ such that $C_1 \le m^{cu}(B(x,r))/r^2 \le C_2$, which is valid because the center-unstable manifolds form a lamination and the areas of balls with small radius depend only on the first derivative of the restriction of these charts to the horizontal planes.

Define $J_{r_n}\nu(x) = \nu(B_W^{cu}(x,r_n))/m^{cu}(B^{cu}(x,r_n))$. By hypothesis, we can consider a sequence $r_n \to 0^+$ such that $\|J_{r_n}\nu\|_{L^2(W,m^{cu})}$ is uniformly bounded by $C_1^{-1}L$. Taking a subsequence, we suppose that $J_{r_n}\nu \to \psi \in L^2$ weakly, so

$$\langle f, \psi\rangle_{L^2} = \lim_{n\to\infty}\int_W f \cdot J_{r_n}\nu \; dm^{cu} = \int_W f \, d\nu$$

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for every continuous function f. This implies that $\nu = \psi m_W^{cu}$, and that for $I = C_1^{-1}$,

$$\left\| \frac{d\nu}{dm^{cu}} \right\|_{L^2} = \lim_{n\to\infty} \|J_{r_n}\nu\|_{L^2} \le IL. \qquad \square$$

4. *The main inequality*

This section is dedicated to proving the main inequality of this work (Proposition 4.1 below), which will be used to estimate the norm $\|(\pi_i)_*\mu\|_{W_i,r}$ for small parameters r.

Given f and Λ satisfying the assumptions of Theorem A, consider a finite family of boxes $\{(C_i, W_i, \tilde{W}_i, \pi_i)\}$, $i = 1, \ldots, s_0$, such that $\Lambda \subset \bigcup_i \pi_i^{-1}(W_i)$ and the d^{ss}-diameter of each box is smaller than the ϵ_0 given in the transversality condition (C1), r_0 small such that $d^{cu}(\pi_i(C_i), \partial\tilde{W}_i) > r_0$ for every i, fixed constants a_1 and a_2 such that for every unstable curve γ^{uu} crossing C_i we have $a_1 \le l(\gamma^{uu}) \le a_2$ and an upper bound $L \ge 1$ for the Lipschitz constant of every π_i. In what follows, we will suppose that this family is fixed once for all.

The main inequality of this work is the following proposition.

PROPOSITION 4.1. (Main inequality) *There exist constants $B > 0$ and $\sigma > 1$ such that for every $n \in \mathbb{N}$, there exist $D_n > 0$, $r_n > 0$ and $c_n > 1$ such that for every ergodic u-Gibbs μ and every $r \le r_n$, we have*

$$|||f_*^n\mu|||_r^2 \le \frac{B}{\sigma^n}|||\mu|||_{c_n r}^2 + D_n|\mu|^2.$$

This type of inequality is often used in the study of the regularity of invariant measures in ergodic theory. There are two norms: one measuring the size and other measuring the regularity of the measure. These inequalities allow us to estimate the regularity of the fixed points. This kind of inequality is due to Doeblin–Fortet, and it is also known as the 'Lasota–Yorke type inequality'.

The proof of Proposition 4.1 is the main objective of this section.

4.1. *Approximating u-Gibbs inside the boxes.* To apply our arguments, it will be useful to approximate the restriction of a u-Gibbs to one box by a finite combination of measures supported on unstable curves that cross the box.

LEMMA 4.1. *There exist $C_1 > 0$ and $\alpha_1 \in (0, 1)$ such that the restriction of every ergodic u-Gibbs to a box C_i is equals to the limit $\lim_{n\to\infty}\mu_n$ with $\mu_n = \sum_{j=0}^{r_0}\rho_j^n m_{\gamma_j^n}$, where $\log\rho_j^n$ is (C_1, α_1)-Holder and γ_j^n is an unstable curve contained in C_i that crosses C_i.*

Proof. For every ergodic u-Gibbs μ, there exists some unstable curve γ^{uu} such that μ is written as a limit of $\mu_n := (1/n)\sum_{j=0}^{n-1} f_*^j(m_{|_{\gamma^{uu}}})$. Given C_1 and α_1, let $\mathcal{L}_{C_1,\alpha_1}$ be the closed space of measures whose logarithm of the density of the conditional measure to the unstable curves is of class (C_1, α_1)-Holder. One can easily see that there exist C_1, α_1 and $n_0 \in \mathbb{N}$ such that the space $\mathcal{L}_{C_1,\alpha_1}$ is invariant under f_*^n for every $n \in \mathbb{N}$.

Consider the measures $\nu_n := ((1/n)\sum_{j=0}^{n-1} f_*^j(m_{|_{\gamma^{uu}}}))_{|C_i} \to \mu_{|C_i}$. For every $j \in \mathbb{N}$, let $\tilde{\gamma}_j^{uu} \subset f^j(\gamma^{uu}) \cap C_i$ be the curve obtained by removing the connected components

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of $f^j(\gamma^{uu}) \cap C_i$ that do not cross the box C_i, and consider the measure $\tilde{\nu}_n := ((1/n)\sum_{j=0}^{n-1} f^j_*(m_{|_{f^{-j}\tilde{\gamma}^{uu}_j}}))_{|C_i}$. From the invariance of $\mathcal{L}_{C_1,\alpha_1}$, it follows that $\tilde{\nu}_n \in \mathcal{L}_{C_1,\alpha_1}$ for every $n \in \mathbb{N}$. Note that $\nu_n - \tilde{\nu}_n \overset{n\to\infty}{\to} 0$, because their difference is a measure supported in the connected components that were removed, but these connected components are at most $2n$, they have bounded length and the density of their conditional measure to these unstable curves converges to zero when $n \to \infty$. So $\lim_n \tilde{\nu}_n = \lim \nu_n = \mu_{|C_i}$, and thus the sequence $\{\tilde{\nu}_n\}_n$ satisfies the conditions that we want. □

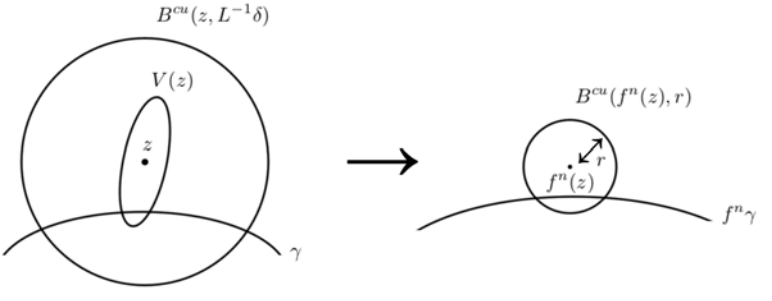
4.2. *Comparing sizes of cylinders.* Denote by $B^{cu}(x, r)$ the center-unstable ball of radius r centered at x. The next lemma estimates, for u-Gibbs measures, the measure of the sets $\pi_i^{-1} B^{cu}(x, r)$, which we will look at as cylinders.

Given $n \in \mathbb{N}$, let $\{R_{i,j,m}\}_{m=1,\dots,m_0(i,j,n)}$ be the connected components of $f^n(C_i) \cap C_j$, the sets $C_{i,j,m} := f^{-n} R_{i,j,m} \subset C_i$ and the maps $f_{i,j,m} := f^n_{|C_{i,j,m}}$. When the choices of C_i, C_j and n are implicit we will write f_m instead of $f_{i,j,m}$.

LEMMA 4.2. *There exists $B_1 > 0$ such that for every n, C_i and C_j there exists $\tilde{r}_n$ such that, taking $\delta = 10C^{-1}L^2(\lambda_c^-)^{-n}r$ and $\sigma_{n,r}(x) = (\lambda_c^-)^n \min_{w\in C(x)} \|Df^n_{|E^{uu}_w}\|$ with $C(x) = f_m^{-1}\pi_j^{-1} B^{cu}(\pi_j f^n(x), r)$, we have, for every ergodic u-Gibbs μ, every $x \in W_i$, every m and any $r < \tilde{r}_n$,*

$$\mu(f_m^{-1}\pi_j^{-1} B^{cu}(\pi_j f^n(x), r)) \le \left(\frac{r}{\delta}\right)^2 \frac{B_1}{\sigma_{n,r}(x)} \cdot \mu(\pi_i^{-1} B^{cu}(x, \delta)).$$

Proof. First, let us prove this lemma for measures $\mu = m_\gamma$, where γ is an unstable curve contained in C_i that crosses C_i. If $f^n(x) \notin C_j$ then the left-hand side of the inequality is zero, so we can suppose that $f^n(x) \in C_j$. Consider $z \in W^{ss}(x)$ such that $\gamma \subset W^{cu}(z)$, and define the set $V(z) = W^{cu}(z) \cap f_m^{-1}\pi_j^{-1} B^{cu}(\pi_j f^n(x), r)$. Let us also consider the intervals $I = \gamma^{-1}(V(z))$ and $J = \gamma^{-1}(B^{cu}(z, L^{-1}\delta))$. We note that $V(z) \subset B^{cu}(z, L^{-1}\delta) \subset W^{cu}(z) \cap \pi_i^{-1} B^{cu}(x, \delta)$.



We want to bound $m_\gamma(V(z))/m_\gamma(B^{cu}(z, L^{-1}\delta))$. For this purpose, we estimate $m_{\mathbb{R}}(I)/m_{\mathbb{R}}(J)$.

Fix $R_2 > 0$ such that for every $r < R_2$, every unstable curve intersects $B^{cu}(\cdot, Lr)$ with

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length at most $3Lr$. Taking $r < R_2$, we see that

$$\begin{aligned} 3Lr &\geq m_{f_m(\gamma)}(f_m\gamma \cap B^{cu}(f^n(z), Lr)) \\ &\geq m_{f_m(\gamma)}(f_m\gamma \cap \pi_j^{-1} B^{cu}(\pi_j f^n(z), r)) \\ &\geq m_\gamma(\gamma \cap f_m^{-1}\pi_i^{-1} B(x, r)) \cdot \min_{w\in\gamma} \|Df^n_{|E^{uu}_w}\| \\ &\geq m_{\mathbb{R}}(I) \cdot \min_{w\in C(z)} \|Df^n_{|E^{uu}_w}\|. \end{aligned}$$

Then, $m_{\mathbb{R}}(I) \leq 3Lr/\min_{w\in C(z)} \|Df^n_{|E^{uu}_w}\|$.

When $m_\gamma(f_m^{-1}\pi_j^{-1} B^{cu}(f^n(z), r)) = 0$, we have $m_{\mathbb{R}}(I) = 0$ and the inequality of the lemma holds in this case. When $m_\gamma(f_m^{-1}\pi^{-1} B^{cu}(f^n(z), r)) > 0$, the choice of δ $(= 10C^{-1}L^2(\lambda_c^-)^{-n}r)$ guarantees that

$$\begin{aligned} V(z) &\subset B^{cu}(z, \min_{w\in C(z)} \|Df^{-n}_{|E^c_w}\|^{-1} Lr) \\ &\subset B^{cu}(z, C^{-1}(\lambda_c^-)^{-n} Lr) \\ &\subset B^{cu}\left(z, \tfrac{1}{2}L^{-1}\delta\right). \end{aligned}$$

Take r small enough such that $a_1 \geq 10\delta$. Since $m_{\mathbb{R}}(\gamma) \geq a_1 \geq 10\delta$ and $\gamma \cap B^{cu}(z, \frac{1}{10}L^{-1}\delta) \neq \emptyset$, we have that the length of $\gamma \cap B^{cu}(z, \delta)$ is greater than $\frac{1}{2}L^{-1}\delta$, so $m_{\mathbb{R}}(J) \geq \frac{1}{2}L^{-1}\delta$. Then,

$$\frac{m_{\mathbb{R}}(I)}{m_{\mathbb{R}}(J)} \leq \frac{3Lr/\min \|Df^n_{|E^{uu}_w}\|}{1/2L^{-1}\delta} \leq \left(\frac{r}{\delta}\right)^2 \frac{60L^4C^{-1}}{(\lambda_c^-)^n \cdot \min_{w\in C(z)} \|Df^n_{|E^{uu}_w}\|}.$$

Thus, the lemma is valid in the case $\mu = m_\gamma$.

Now, taking $\mu = \rho m_\gamma$ with γ as before, $\log\rho$ of class (C_1, α_1)-Holder and ρ defined in an interval of length smaller than a_2, since $\int_I \rho(x)dx / \int_J \rho(x)dx \leq e^{2C_1 a_2^{\alpha_1}} (m_{\mathbb{R}}(I)/m_{\mathbb{R}}(J))$ for every $I \subset J \subset \mathbb{R}$, we have

$$\frac{(\rho m_\gamma)(f_m^{-1}\pi^{-1}B(x, r))}{(\rho m_\gamma)(\pi^{-1}B(\pi f_m^{-1}\pi^{-1}(x), \delta))} \leq \frac{e^{2C_1 a_2^{\alpha_1}} \cdot 60L^4C^{-1}}{\sigma_{n,r}} \cdot \left(\frac{r}{\delta}\right)^2.$$

Hence, Lemma 4.2 also holds for measures $\mu = \rho m_\gamma$, with $B_1 = e^{2C_1 a_2^{\alpha_1}} \cdot 60L^4C^{-1}$.

In the case where μ is a finite sum $\mu = \sum_{i=1}^{s_0} \rho_i m_{\gamma_i}$, with ρ_i and γ_i as before, the result follows by linearity, since both sides of the inequality are linear in μ.

Finally, if μ is an ergodic u-Gibbs, we proceed using Lemma 4.1. The measures μ_n given by Lemma 4.1 are finite sums of measures of the type ρm_γ, so the desired inequality holds for μ_n. If we take r smaller than the inverse of the curvature of every unstable curve, then $\partial B^{cu}(\cdot, r)$ intersects every unstable curve at most finitely; so $\mu(\partial B^{cu}(\cdot, r)) = 0$ for such r. Since μ is the limit of μ_n, we can take the limit in each term of the inequality, so what we desired also holds for μ. □

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4.3. *One inequality for the norm.*

LEMMA 4.3. *There are two constants $B_2 > 0$ and $\sigma_2 > 1$ such that for every n, C_i and C_j there exists a constant $R_n > 0$ such that, for $\delta = 10CL(\lambda_c^-)^{-n}r$, for every $r < R_n$ and any ergodic u-Gibbs μ,*

$$\sum_m \|(\pi_j)_*(f_m)_*\mu\|^2_{W_j,r} \le \frac{B_2}{\sigma_2^n}\|(\pi_i)_*\mu\|^2_{\tilde{W}_i,\delta}.$$

Proof. Applying Lemma 4.2 to the measure μ, we see that

$$\begin{aligned}\|(\pi_j)_*(f_m)_*\mu\|^2_{W_i,r} &= \frac{1}{r^4}\int_{W_j} \mu(f_m^{-1}\pi_j^{-1}B^{cu}(y,r))^2\,dm^{cu}(y)\\ &\le \frac{B_1^2}{\delta^4}\int_{W_j}\frac{\mu(\pi_i^{-1}B(\pi_i f_m^{-1}\pi_j^{-1}(y),\delta))^2}{(\lambda_c^-)^{2n}\cdot\min_{w\in C(y)}\|Df^n_{|E^{uu}_w}\|^2}\,dm^{cu}(y).\end{aligned}$$

Changing the variables $\tilde{y} = \pi_i \circ f_m^{-1} \circ \pi_j^{-1}(y)$ and noticing that $\pi_i \circ f_m^{-1} \circ \pi_j^{-1} : W_j \to \tilde{W}_i$ is an absolutely continuous homeomorphism into the image with Jacobian bounded by $L^2 \cdot \sup_{x\in W^{ss}(y)} |\det Df^n_{|E^{cu}_x}|$, we have

$$\begin{aligned}\|(\pi_j)_*(f_m)_*\mu\|^2_{W_i,r} &\le \frac{B_1^2}{\delta^4}\int_{\tilde{W}_i\cap R_{i,j,m}}\frac{\mu(\pi_i^{-1}B(\tilde{y},\delta))^2\cdot \mathrm{Jac}(\pi_i f_m \pi_j^{-1}(y))}{(\lambda_c^-)^{2n}\min_{w\in\pi^{-1}B(\tilde{y},L\delta)}\|Df^n_{|E^{uu}_w}\|^2}\,dm^{cu}(\tilde{y})\\ &\le \frac{B_1^2L^2}{\delta^4}\int_{\tilde{W}_i\cap R_{i,j,m}}\frac{\mu(\pi_i^{-1}B(\tilde{y},\delta))^2\cdot\sup_{y_1\in W^{ss}_{\tilde{x}}}|\det Df^n_{|E^{cu}_{y_1}}|}{(\lambda_c^-)^{2n}\min_{w\in\pi^{-1}B(\tilde{x},L\delta)}\|Df^n_{|E^{uu}_w}\|^2}\,dm^{cu}(\tilde{y}).\end{aligned}$$

To estimate the term inside the integral we use the central direction neutral condition. For that, we use the following claim, which will be proved later.

CLAIM 4.1. *There exists a constant $K_2 > 0$ such that for every $n \in \mathbb{N}$, there exists $\tilde{r}_n > 0$ such that for every $r < \tilde{r}_n$,*

$$\frac{\sup_{y_1\in W^{ss}_{\tilde{x}}}|\det Df^n_{|E^{cu}_{y_1}}|}{(\lambda_c^-)^{2n}\min_{w\in\pi^{-1}B^{cu}(\tilde{x},L\delta)}\|Df^n_{|E^{uu}_w}\|^2} \le K_2\left[\frac{(\lambda_c^+)}{(\lambda_c^-)^2(\lambda_u^-)}\right]^n.$$

Supposing that $r < \tilde{r}_n$ and considering $\sigma_2^{-1} = \lambda_c^+/(\lambda_c^-)^2(\lambda_u^-) < 1$,

$$\|(\pi_j)_*(f_m)_*\mu\|^2_{W_i,r} \le \frac{B_1^2L^2K_2}{\sigma_2^n}\int_{\tilde{W}_i\cap R_{i,j,m}}\frac{\mu(\pi_i^{-1}B^{cu}(\tilde{x},\delta))^2}{\delta^4}\,dm^{cu}(\tilde{x}) \qquad (2)$$

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Defining $B_2 = B_1^2 L^2 K_2$ and summing inequality (2) in m, we get

$$\begin{aligned}\sum_m \|(\pi_j)_*(f_m)_*\mu\|^2_{\tilde{W}_i,r} &\le \frac{B_2}{\sigma_2^n}\sum_m \int_{\tilde{W}_i\cap R_{i,j,m}} \frac{\mu(\pi_i^{-1}B^{cu}(\tilde{x},\,\delta))^2}{\delta^4}\,dm^{cu}(\tilde{x})\\ &= \frac{B_2}{\sigma_2^n}\int_{\tilde{W}_i}\frac{(\pi_i)_*\mu(B(\tilde{x},\,\delta))^2}{\delta^4}\,dm^{cu}(\tilde{x})\\ &= \frac{B_2}{\sigma_2^n}\|(\pi_i)_*\mu\|^2_{\tilde{W}_i,\delta}.\end{aligned}$$

□

Let us now prove Claim 4.1.

Proof of Claim 4.1. We first consider the case where $y_1 \in W^{ss}_{\tilde{x}}$ and $w = \tilde{x}$.

There exists $\tilde{K}_2 > 0$ such that $|\det Df^n_{|E^{cu}_{y_1}}|/|\det Df^n_{|E^{cu}_{y_2}}| \le \tilde{K}_2$ for every $y_2 \in W^{ss}_{y_1}$ and every $n \ge 0$. This is due to the following:

$$\begin{aligned}\log|\det Df^n_{|E^{cu}_{y_1}}| - \log|\det Df^n_{|E^{cu}_{y_2}}| &= \sum_{j=1}^{n}\log|\det Df_{|E^{cu}_{f^j(y_1)}}| - \log|\det Df_{|E^{cu}_{f^j(y_2)}}|\\ &\le \sum_{j\ge 0}\tilde{C}_2 d(f^j(y_1),\,f^j(y_2))^{\alpha_2}\\ &\le \sum_{j=0}^{+\infty}\tilde{C}_2 l((\lambda_{ss})^{\alpha_2})^j := \tilde{K}_2.\end{aligned}$$

Here, we have used that the function $x \to \log|\det Df_{|E^{cu}_x}|$ is $(\tilde{C}_2, \alpha_2)$-Holder and that the length of W^{ss} is uniformly bounded by l inside the boxes. We then have

$$\begin{aligned}\frac{|\det Df^n_{|E^{cu}_{y_1}}|}{(\lambda_c^-)^{2n}\cdot\|Df^n_{|E^{uu}_{\tilde{x}}}\|^2} &\le \tilde{K}_2\cdot\left[\frac{|\det Df^n_{|E^{cu}_{\tilde{x}}}|}{(\lambda_c^-)^{2n}\cdot\|Df^n_{|E^{uu}_{\tilde{x}}}\|^2}\right]\\ &= \frac{\tilde{K}_2\cdot\|Df^n_{|E^{c}_{\tilde{x}}}\|\cdot\|Df^n_{|E^{uu}_{\tilde{x}}}\|}{(\lambda_c^-)^{2n}\|Df^n_{|E^{uu}_{\tilde{x}}}\|^2}\\ &\le \frac{C^2\tilde{K}_2(\lambda_c^+)^n}{C^{-1}(\lambda_c^-)^{2n}(\lambda_u^-)^n}\\ &\le \frac{C^3\tilde{K}_2}{\sigma_2^n}.\end{aligned}$$

Now, if $y_1 \in W^{ss}_{\tilde{x}}$ and $w \in \pi_j^{-1}B^{cu}(\tilde{x},\,L\delta)$, by continuity, there exists $\tilde{r}_n = \tilde{r}_n(w_2)$ such that $\frac{1}{2} \le \|Df^n_{|E^{uu}_{w_1}}\|/\|Df^n_{|E^{uu}_{w_2}}\| \le 2$ for every $w_1 \in B^{cu}(w_2, r_n(w_2))$. By compactness, $\tilde{r}_n$ can always be taken to be uniform in w_2. Taking $\tilde{r}_n$ small enough, for every $r < \tilde{r}_n$, every $w \in B^{cu}(z, \tilde{r}_n)$ and any $z \in W^{ss}_{\tilde{x}} = W^{ss}_{y_1}$, we have

$$\frac{|\det Df^n_{|E^{cu}_{y_1}}|}{(\lambda_c^-)^{2n}\cdot\|Df^n_{|E^{uu}_{w}}\|^2} \le \frac{C^3\tilde{K}_2\cdot|\det Df^n_{|E^{cu}_{z}}|}{(\lambda_c^-)^{2n}\cdot 1/2\|Df^n_{|E^{uu}_{z}}\|^2} \le \frac{K_2}{\sigma_2^n}.$$

□

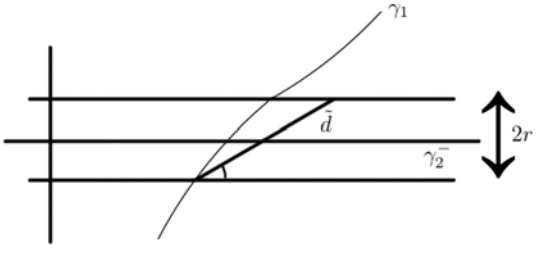
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4.4. *Inequalities using the transversality.*

LEMMA 4.4. *For every $\theta \in (0, \pi/2)$, there exist constants r_θ and C_θ for which the following is valid: given Lipschitz curves γ_1 and γ_2 of finite length contained in some center-unstable manifold contained in V that are θ-transversal in neighborhoods of radius r_θ, for every $r < r_\theta/4$ and every connected component E_r of $\{s, |d(\gamma_1(s), \gamma_2)| \leq r\}$, the length of $\gamma_1(E_r)$ is at most $C_\theta r$.*

Proof. We consider first the case where γ_1 and γ_2 are curves contained in $\mathbb{R}^2$.

If the curves γ_1 and γ_2 are contained in $\mathbb{R}^2$, for every point $\gamma_2(t)$ we consider cones $C_1(t)$ and $C_2(t)$ as in the definition of θ-transversality. By transversality in neighborhoods of radius r, the curves intersect at most once on neighborhoods of radius smaller than $r/2$. We also suppose that the extremal line of $C_2(t)$ closest to γ_1 is contained in the axis x and that $\gamma_1|_{\gamma_1^{-1}E_r}$ is a graph over it. We will consider $\gamma_1(|_{\gamma_1^{-1}E_r})$ as a graph over γ_2^-.



Notice that $\gamma_1(t)$ increases at least $C_\theta t$. Actually, the length of $\tilde{d}$ is at most $2r/\sin(\theta)$, and this geometric comparison passes to the following inequality:

$$\begin{aligned} l(\gamma_1(E_r)) &\leq m_{\mathbb{R}}(\{s, d(\gamma_1^+(s), \gamma_2^-) \leq r\}) \\ &\leq \frac{2}{\tan(\theta)} r = C_\theta r. \end{aligned}$$

Now, suppose that the curves are contained in the same center-unstable manifold. Let U_i be the sets given by Proposition 2.4 and the open set $V = \bigcup_i U_i$ that contains Λ, since the family of diffeomorphisms $\{\psi_i\}_{i\in I}$ given by Proposition 2.4 have derivative uniformly bounded. Taking $T = \sup_{i\in I} \|D\psi_i|_{W^{cu}_z}\|$, the length of $\gamma_1(E_r)$ is smaller than the length of $\psi_i(\gamma_1(E_r))$ times T. Let r_θ be such that $C_\theta r_\theta T < R_1/2$, then $\gamma_1(E_r)$ is contained in some U_{i_0}, $i_0 \in I$. The result follows by taking $r < \min\{R_1, r_\theta\}$ and noticing that in V the distances $d^{cu}(x, y)$ and $d(x, y)$ are equivalent and that $\{s, d(\gamma_1(s), \gamma_2) \leq r\} \subset \{s, d(\psi_{i_0}\gamma_1(s), \psi_{i_0}\gamma_2) \leq Tr\}$. □

The transversality between stable projections of unstable curves can be used to estimate the bilinear form of stable projections of u-Gibbs.

LEMMA 4.5. *Given n, C_i and C_j, there exist $\hat{r}_n > 0$ and $A_n > 0$ such that for every ergodic u-Gibbs μ, $r < \hat{r}_n$ and for every $m \neq m'$, the following holds:*

$$\langle (\pi_j)_*(f_m)_*\mu, (\pi_j)_*(f_{m'})_*\mu\rangle_{W_j,r} \leq A_n \cdot |(\pi_j)_*(f_m)_*\mu| \cdot |(\pi_j)_*(f_{m'})_*\mu|.$$

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Proof. By finiteness, there exist constants $d_1 > 0$ and $d_2 > 0$ such that for every pair of components $f^n(C_{i,j,m})$ and $f^n(C_{i,j,m'})$, $m \neq m'$, it is valid that either $d^{ss}(R_{i,j,m}, R_{i,j,m'}) > d_1$ or $d^{cu}(R_{i,j,m}, R_{i,j,m'}) > d_2$. Consider $\theta_n = \theta(d_1)$, $\hat{R}_n = r(d_1)$ given by the transversality condition and $r < \min\{\hat{R}_n/4, d_2/4\}$.

Lemma 4.5 will be proved first for measures supported on single unstable curves and after for approximations of μ by combination of measures supported in unstable curves.

Suppose $\mu_1 = \rho_1 m_{\gamma_1}$ and $\mu_2 = \rho_2 m_{\gamma_2}$, where $\log \rho_i$ is (C_1, α_1)-Holder and γ_i is an unstable curve contained in C_i that crosses C_i, $i = 1, 2$. Consider the Lipschitz curves $\tilde{\gamma}_m = \pi_j \circ f_m(\gamma_1)$ and $\tilde{\gamma}_{m'} = \pi_j \circ f_{m'}(\gamma_2)$, the measures $\nu_m = (\pi_j)_*(f_m)_*\mu_1$ and $\nu_{m'} = (\pi_j)_*(f_{m'})_*\mu_2$ supported on these curves, and consider the densities $\rho_m = d\nu_m/dm_{\tilde{\gamma}_m} = (\mathrm{Jac}\pi_j^{ss})^{-1}(\mathrm{Jac}\, f_m)^{-1}\rho_1$ and $\rho_{m'} = d\nu_{m'}/dm_{\tilde{\gamma}_{m'}} = (\mathrm{Jac}\pi_j^{ss})^{-1}(\mathrm{Jac}\, f_{m'})^{-1}\rho_2$. In this case, we will prove that $\langle \nu_m, \nu_{m'} \rangle_{W_j,r} \leq C_{\theta_n} |\nu_m||\nu_{m'}|$.

There exist constants C_3 and α_3 (depending on C_1, α_1, L) such that the densities ρ_m and $\rho_{m'}$ are (C_3, α_3)-Holder functions. Since the curves $\tilde{\gamma}_*$ have length uniformly bounded by $L^{-1}a_1$ and La_2, we have $\rho_*(\gamma_*(x)) \leq e^{C_3(La_2)^{\alpha_3}} \rho_*(\gamma_*(y))$, $* \in \{m, m'\}$. Let C_4 be a constant such that $m^{cu}(B^{cu}(z, r)) \leq C_4 r^2$ for every $z \in \bigcup C_i$.

Define the function $\mathbb{1}_r$ by $\mathbb{1}_r(z_1, z_2) = 1$ if $d^{cu}(z_1, z_2) < r$ and $\mathbb{1}_r(z_1, z_2) = 0$ otherwise.

If $d^{cu}(R_{i,j,m}, R_{i,j,m'}) < r$ then $\langle \nu_m, \nu_{m'} \rangle_{W_j,r}$ is zero and the inequality is valid. Else, $d^{ss}(R_{i,j,m}, R_{i,j,m'}) > d_1$, and so

$$
\begin{aligned}
\langle \nu_m, \nu_{m'} \rangle_{W_j,r} &= \frac{1}{r^4} \int_{W_j} \nu_m(B(z, r)) \nu_{m'}(B(z, r))\, dm^{cu}(z) \\
&= \frac{1}{r^4} \int_{W_j} \left(\int_{\tilde{\gamma}_m \cap B(z,r)} \rho_m(\tilde{\gamma}_m(s))\, dm_{\tilde{\gamma}_m}(s) \right) \\
&\quad \times \left(\int_{\tilde{\gamma}_{m'} \cap B(z,r)} \rho_{m'}(\tilde{\gamma}_{m'}(t))\, dm_{\tilde{\gamma}_{m'}}(t) \right) dm^{cu}(z) \\
&= \frac{1}{r^4} \int_{W_j} \int_{\tilde{\gamma}_m} \int_{\tilde{\gamma}_{m'}} (\rho_m(\tilde{\gamma}_m(s)) \rho_{m'}(\tilde{\gamma}_{m'}(t)) \mathbb{1}_r(\tilde{\gamma}_m(s), z) \mathbb{1}_r(\tilde{\gamma}_{m'}(t), z))\, ds\, dt\, dz \\
&\leq \frac{1}{r^4} \int_{\tilde{\gamma}_{m'}} \rho_{m'}(t) \left(\int_{\tilde{\gamma}_{m'}} \rho_m(s) \cdot \mathbb{1}_{2r}(\tilde{\gamma}_m(s), \tilde{\gamma}_{m'}(t)) \left(\int_{W_j} \mathbb{1}_r(\tilde{\gamma}_m(s), z)\, dz \right) ds \right) dt \\
&\leq \frac{C_4}{r^2} \int_{\tilde{\gamma}_m} \int_{\tilde{\gamma}_{m'}} (\rho_m(\tilde{\gamma}_m(s)) \rho_{m'}(\tilde{\gamma}_{m'}(t)) \mathbb{1}_{2r}(\tilde{\gamma}_m(s), \tilde{\gamma}_{m'}(t)))\, ds\, dt \\
&\leq \frac{C_4 e^{2C_3(La_2)^{\alpha_3}}}{r^2} \rho_m(\gamma_m(s_0)) \rho_{m'}(\gamma_{m'}(t_0)) \\
&\quad \times \left(\int_{\tilde{\gamma}_m} \mathbb{1}_{2r}(\tilde{\gamma}_m(s), \tilde{\gamma}_{m'})\, ds \right) \left(\int_{\tilde{\gamma}_{m'}} \mathbb{1}_{2r}(\tilde{\gamma}_m, \tilde{\gamma}_{m'}(t))\, dt \right).
\end{aligned}
$$

To estimate the last term in parentheses, we will use Lemma 4.4. Take $r < R_{\theta_n}$ small such that Lemma 4.4 holds, and consider the constant C_{θ_n} given by that lemma. Note also

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that for these projections of unstable curves, by transversality in neighborhoods with radius smaller than $\hat{R}_n$, there exists an integer $M_n \in \mathbb{N}$ such that each $\tilde{\gamma}_m$ intersects the neighborhood of radius r of $\tilde{\gamma}_{m'}$ in at most M_n connected components. Then,

$$\begin{aligned}\langle \nu_m, \nu_{m'} \rangle_{W_j,r} &\leq \frac{C_5}{r^2} \rho_m(\gamma_m(s_0)) \rho_{m'}(\gamma_{m'}(t_0)) \cdot (M_n C_{\theta_n} r)^2 \\ &= (C_5 (M_n C_{\theta_n})^2) \cdot \rho_m(\gamma_m(s_0)) \rho_{m'}(\gamma_{m'}(t_0)).\end{aligned}$$

In what follows, we will use that $\int_{\tilde{\gamma}_m} \rho_m(\gamma_m(s))\, ds \geq K_3 \rho_m(\gamma_m(s_0))$ for some constant $K_3 > 0$. This is due to a simple calculation:

$$\begin{aligned}\int_{\tilde{\gamma}_m} \rho_m(\gamma_m(s))\, ds &\geq \int_{\tilde{\gamma}_m} e^{-(C_2(La_2)^{\alpha_2})} \rho_m(\gamma_m(s_0))\, ds \\ &= e^{-(C_2(La_2)^{\alpha_2})} \rho_m(\gamma_m(s_0)) l(\tilde{\gamma}_m) \\ &\geq (L^{-1} a_1 e^{-(C_2(La_2)^{\alpha_2})}) \rho_m(\gamma_m(s_0)) \\ &= K_3 \rho_m(\gamma_m(s_0)).\end{aligned}$$

Analogously, it is valid that $\rho_{m'}(\gamma_{m'}(t_0)) \leq K_3^{-1} \int_{\tilde{\gamma}_{m'}} \rho_{m'}(\gamma_{m'}(t))\, dt$, so we have the estimate

$$\begin{aligned}\langle \nu_m, \nu_{m'} \rangle_{W_j,r} &\leq (C_5 (M_n C_{\theta_n})^2 (K_3)^2) \int_{\tilde{\gamma}_m} \rho_m(\gamma_m(s))\, ds \int_{\tilde{\gamma}_{m'}} \rho_{m'}(\gamma_{m'}(t))\, dt \\ &= A_n \cdot |\nu_m| \cdot |\nu_{m'}|.\end{aligned}$$

The second case to be considered is when the measure μ is a finite sum $\mu = \sum_{k=1}^{s_0} \rho_k m_{\gamma_k}$. In this case, the inequality holds by linearity because we also have θ_n-transversality of the stable projection of unstable curves. Actually, taking $\nu_m = (\pi_j)_*(f_m)_*\mu$ and $\nu_{m'} = (\pi_j)_*(f_{m'})_*\mu$, we see that

$$\begin{aligned}\langle \nu_m, \nu_{m'} \rangle_{W_j,r} &= \left\langle (\pi_j)_*(f_m)_* \left(\sum_k \rho_k m_{\gamma_k} \right), (\pi_j)_*(f_{m'})_* \left(\sum_{k'} \rho_{k'} m_{\gamma_{k'}} \right) \right\rangle_{W_j,r} \\ &= \sum_{k,k'} \langle (\pi_j)_*(f_m)_*(\rho_k m_{\gamma_k}), (\pi_j)_*(f_{m'})_*(\rho_{k'} m_{\gamma_{k'}}) \rangle_{W_j,r} \\ &\leq \sum_{k,k'} A_n |(\pi_j)_*(f_m)_*(\rho_k m_{\gamma_k})| |(\pi_j)_*(f_{m'})_*(\rho_{k'} m_{\gamma_{k'}})| \\ &= A_n |(\pi_j)_*(f_m)_*\mu| |(\pi_j)_*(f_{m'})_*\mu| \\ &= A_n |\nu_m| |\nu_{m'}|.\end{aligned}$$

Finally, let us suppose that μ is an ergodic u-Gibbs. In this case, we use Lemma 4.1 to approximate μ by probability measures $\sum_i \rho_i m_{\gamma_i}$ and apply Lemma 3.3 to approximate to the bilinear form on the left-hand side of the inequality. □

4.5. *Proof of the main inequality.* Let us give a localized version of the main inequality when two boxes C_i, C_j and an integer n are fixed.

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PROPOSITION 4.2. (Main inequality—localized version) *There exist* $\tilde{B} > 0$ *and* $\sigma > 1$ *such that for every* $n \in \mathbb{N}$ *and* C_i, C_j *fixed, there exist* $\tilde{D}_n > 0$, $r_n > 0$ *and* $c_n > 1$ *such that for every ergodic u-Gibbs* μ *and any* $r < r_n$, *it holds that*

$$\|(\pi_j)_*(f_*^n(\mu_{|C_i}))\|^2_{W_j,r} \le \frac{\tilde{B}}{\sigma^n}\|(\pi_i)_*\mu\|^2_{\tilde{W}_i,c_nr} + \tilde{D}_n|(\pi_i)_*\mu|^2.$$

Proof of the localized version. Take r_n small such that Lemmas 4.3 and 4.5 hold for every $r < r_n$. Then:

$$\begin{aligned}
\|(\pi_j)_*(f_*^n(\mu_{|C_i}))||^2_{W_j,r} &= \left\|(\pi_j)_*\sum_m (f_m)_*\mu\right\|^2_{W_j,r} \\
&= \sum_m \|(\pi_j)_*(f_m)_*\mu\|^2_{W_j,r} \\
&\quad + \sum_{m\neq m'} \langle(\pi_j)_*(f_m)_*\mu,\, (\pi_j)_*(f_{m'})_*\mu\rangle_{W_j,r} \\
&\le \frac{B_2}{\sigma^n}\|(\pi_i)_*\mu\|^2_{\tilde{W}_i,c_nr} + \sum_{m\neq m'} A_n|(\pi_j)_*(f_m)_*\mu||(\pi_j)_*(f_{m'})_*\mu| \\
&= \frac{\tilde{B}}{\sigma^n}\|(\pi_i)_*\mu\|^2_{\tilde{W}_i,c_nr} + \tilde{D}_n|(\pi_i)_*\mu|^2. \qquad \square
\end{aligned}$$

Proof of the main inequality. Note that $|||(f_{n;i,j,m})_*\mu|||_{\{\mathcal{W}\},r} = \|(\pi_j)_*\mu\|_{W_j,r}$, where $\mathcal{W} = \{W_1, \ldots, W_{s_0}\}$ is the finite family of boxes fixed in the beginning of this section, and note also that $f_*^n\mu(E) \le \sum_{i,j,m}(f_m)_*\mu(E)$ for every measurable set E. Then,

$$\begin{aligned}
|||f_*^n\mu|||^2_{\{\mathcal{W}\},r} &\le \sum_{i,j}\|(\pi_j)_*(f_*^n(\mu_{|C_i}))\|^2_{W_j,r} \\
&\le \sum_{i,j}\left[\frac{\tilde{B}}{\sigma^n}\|(\pi_i)_*\mu\|^2_{\tilde{W}_i,c_nr} + \tilde{D}_n|(\pi_i)_*\mu|^2\right] \\
&\le \sum_{i,j}\left[\frac{\tilde{B}}{\sigma^n}|||\mu|||^2_{\{\tilde{\mathcal{W}}\},c_nr} + \tilde{D}_n|\mu|^2\right] \\
&\le \frac{\tilde{B}s_0^2}{\sigma^n}|||\mu|||^2_{\{\mathcal{W}\},c_nr} + \tilde{D}_n s_0|\mu|^2 \\
&\le \frac{B}{\sigma^n}|||\mu|||^2_{\{\mathcal{W}\},c_nr} + D_n|\mu|^2. \qquad \square
\end{aligned}$$

5. *Physical measures*

Throughout this section we will assume that f satisfies the assumptions of Theorem A and the same boxes and norms considered in the lemmas in §4.

5.1. *Existence of physical measures.*

We will prove that every u-Gibbs projects by the stable projection into absolutely continuous measures in the center-unstable manifolds W_i, and that this fact implies a positive measure for the basin of these measures.

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PROPOSITION 5.1. *Every ergodic u-Gibbs projects into an absolutely continuous measure in $\tilde{W}_i$ by the applications π_i. Moreover, for every ergodic u-Gibbs μ, there exists a constant $L > 0$ such that $\|d((\pi_i)_*\mu)/dm^{cu}\|_{L^2} \leq L$.*

Proof. Given the ergodic u-Gibbs μ, consider B and σ as in the main inequality. Fix N such that $B/\sigma^N < 1$ and consider D_N, r_N and c_N as in the main inequality. Then, for $r < r_N$,

$$|||\mu|||_r^2 = |||f_*^n\mu|||_r^2 \leq \frac{B}{\sigma^N}|||\mu|||_{c_n r}^2 + D_N|\mu|^2.$$

For $r < r_N/c_N$, we have

$$|||\mu|||_{c_N^{-1}r}^2 \leq \frac{B}{\sigma^n}|||\mu|||_r^2 + D_N|\mu|^2.$$

Define the constant $K_0 := |||\mu|||_{r_N/2}^2$ and iterate j times the inequality

$$\begin{aligned}
|||\mu|||_{c_N^{-j}r_N/2}^2 &\leq \left(\frac{B}{\sigma^N}\right)|||\mu|||_{c_N^{-(j-1)}r_N/2}^2 + D_N|\mu|^2 \\
&\leq \cdots \\
&\leq \left(\frac{B}{\sigma^N}\right)^j |||\mu|||_{r_N/2}^2 + D_N\left(1 + \frac{B}{\sigma^N} + \cdots + \left(\frac{B}{\sigma^N}\right)^{j-1}\right)|\mu|^2 \\
&\leq 1 \cdot K_0 + D_N \cdot \frac{1}{1 - (B/\sigma^N)} =: \tilde{L}.
\end{aligned}$$

Hence, for every π_i we have

$$\liminf_{r\to 0^+}\|(\pi_i)_*\mu\|_{W_i,r} \leq \liminf_{j\to\infty}|||\mu|||_{c_N^{-j}r_N/2}^2 \leq \tilde{L} < +\infty,$$

and the criteria of absolute continuity for measures (Lemma 3.3) gives that

$$(\pi_i)_*\mu \ll m^{cu} \quad \text{and} \quad \left\|\frac{d((\pi_i)_*\mu)}{dm^{cu}}\right\|_{L^2} \leq L. \qquad \square$$

Since we have fixed the boxes $\{(C_i, W_i, \tilde{W}_i, \pi_i)\}$ that cover Λ, for every u-Gibbs μ there exists some i_0 such that $\mu(C_{i_0}) > 1/2s_0$.

PROPOSITION 5.2. *Let μ be an ergodic u-Gibbs for f, and let i be such that $\mu(C_i) > 1/2s_0$. If the measure $\nu = (\pi_i^{ss})_*\mu$ is absolutely continuous with respect to the Lebesgue measure m^{cu} and $\|d\nu/dm^{cu}\|_{L^2} \leq K$, then μ is a physical measure and there exists a constant $M > 0$ depending on K such that $m(B(\mu)) \geq M$.*

Proof. Since μ is ergodic, we have that $\mu(B(\mu)) = 1$. Note also that $\pi_i(B(\mu))$ is measurable ([**12**, Theorem 3.23] guarantees the measurability of this projection). Then,

$$\nu(\pi_i B(\mu)) = \mu((\pi_i)^{-1}\pi_i B(\mu)) \geq \mu(B(\mu) \cap C_i) > \frac{1}{2s_0}.$$

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By the absolute continuity of ν, it follows that $m^{cu}(\pi_i B(\mu)) > 0$. Actually, we have $m^{cu}(\pi_i B(\mu)) > (2s_0K)^{-1}$. In fact,

$$\begin{aligned}(2s_0)^{-1} &< \nu(\pi_i B(\mu))\\ &= \int \mathbb{1}_{\pi_i B(\mu)} \frac{d\nu}{dm^{cu}} dm^{cu}\\ &\leq \|\mathbb{1}_{\pi_i B(\mu)}\|_{L^2} \left\| \frac{d\nu}{dm^{cu}} \right\|_{L^2}\\ &\leq m^{cu}(\pi_i B(\mu))K.\end{aligned}$$

From the fact that $B(\mu)$ is $\mathcal{F}^{ss}$-saturated and $\mathcal{F}^{ss}$ is absolutely continuous, it follows that $m(B(\mu)) > 0$, so μ is a physical measure. Moreover, considering a constant that bounds from below the Jacobian of h^{ss}, we have that $m(B(\mu)) > (2s_0K)^{-1}\mathrm{Jac}(h^{ss})^{-1} =: M$. □

5.2. *Proof of Theorem A.* Proposition 5.2 guarantees that every ergodic u-Gibbs is a physical measure. To conclude the proof of Theorem A, it remains to prove that there exist at most finite physical measures (finiteness) and that the union of their basin has full Lebesgue measure in the basin of attraction (problem of the basins).

5.2.1. *Finiteness.* We will show that there exist at most finite ergodic physical measures for f. The finiteness of physical measures will be obtained as a consequence of the full Lebesgue measure of the union of the basins of the ergodic physical measures.

Suppose that there are infinitely many ergodic u-Gibbs μ_n. Taking a subsequence, we can suppose that $\mu_n \to \mu$, and we consider i_0 and a subsequence also denoted by μ_n such that $\mu_n(C_{i_0}) > 1/2s_0$ for every $n \in \mathbb{N}$. Proposition 5.2 implies that every $(\pi_{i_0})_*\mu_n$ is absolutely continuous with respect to m^{cu} and that $d(\pi_{i_0})_*\mu_n/dm^{cu}$ is uniformly bounded in L^2 by some constant $\hat{K} > 0$.

Considering $\hat{M}$ as in Proposition 5.2, there exist at most $1/\hat{M}$ ergodic physical measures with $\|d\nu/dm^{cu}\|_{L^2} \leq \hat{K}$. Actually, supposing that there are $\mu_1, \ldots, \mu_l$ ergodic physical measures for $l \geq (1/\hat{M}) + 1$, Proposition 2.1 implies that they are all ergodic u-Gibbs, and Proposition 5.2 implies that the Lebesgue measure of their basin is bounded below by $\hat{M} > 0$. Since their basins are disjoint, $1 \geq m(\bigcup_i B(\mu_i)) = \sum_i m(B(\mu_i)) \geq l \cdot \hat{C}_5 > 1$, which is a contradiction. Thus, there exist a finite number of ergodic physical measures.

5.2.2. *The problem of the basins.* We are now ready to show that the union of the basins of the physical measures has full Lebesgue measure in the whole basin of attraction.

Denote the ergodic u-Gibbs by $\mu_1, \ldots, \mu_l$. Consider the sets $X = \bigcup B(\mu_i)$, $E = B(\Lambda)\setminus \bigcup B(\mu_i)$, and assume by contradiction that $m(E) > 0$. Let m_E be the normalized measure, $\sigma_n = (1/n)\sum_{j=0}^{n-1} f_*^j(m_E)$, and let σ_∞ be an accumulation point of σ_n. By Proposition 2.2, σ_∞ is a u-Gibbs, so it projects by π_j into an absolutely continuous measure with $\|d(\pi_j)_*\sigma_\infty/dm^{cu}\|_{L^2} \leq K$ for every $j = 1, \ldots, s_0$. Consider $\{n_k\}$ the sequence such that $\sigma_{n_k} \to \sigma_\infty$, take j such that $(\pi_j)_*\sigma_\infty$ is non-zero and define the measures $\tilde{\sigma}_{n_k} = (\pi_j)_*\sigma_{n_k}$ and $\tilde{\sigma}_\infty = (\pi_j)_*\sigma_\infty$.

By the invariance of E and X, we have that $\sigma_{n_k}(X) = 0$ and $0 = \hat{\sigma}_{n_k}(\pi_j(X)) = \int_{\pi_j(X)} d\hat{\sigma}_{n_k}/dm^{cu} dm^{cu}$, which implies that $d\hat{\sigma}_{n_k}/dm^{cu}(x) = 0$ for m^{cu}-almost every

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$x \in \pi_j(X)$. Analogously, $\sigma_\infty(E) = 0$ implies that $d\hat{\sigma}_\infty/dm^{cu}(x) = 0$ for m^{cu}-almost every $x \in \pi_j(E)$. Putting it together, it holds that $d\hat{\sigma}_{n_k}/dm^{cu}(x) \cdot d\hat{\sigma}_\infty/dm^{cu}(x) = 0$ for m^{cu}-almost every $x \in \tilde{W}_j$. Hence,

$$\left\langle \frac{d\hat{\sigma}_{n_k}}{dm^{cu}}, \frac{d\hat{\sigma}_\infty}{dm^{cu}} \right\rangle_{L^2} = \int \frac{d\hat{\sigma}_{n_k}}{dm^{cu}}(x) \cdot \frac{d\hat{\sigma}_\infty}{dm^{cu}}(x)\, dm^{cu} = 0,$$

which is a contradiction with the fact that $\langle d\hat{\sigma}_{n_k}/dm^{cu}, d\hat{\sigma}_\infty/dm^{cu}\rangle_{L^2} \to \|\hat{\sigma}_\infty/dm^{cu}\|_{L^2} > 0$. □

5.3. *Proof of Corollary B.* We will see how Corollary B follows from Theorem A under the condition (C3) of regularity of the stable foliation.

In our setting, we have the trapping region U such that the attractor Λ_{f_0} is equal to $\bigcap_{n\geq 0} f_0^n(U)$, and the stable subbundle $E^{ss}_x(g)$ defined for every $x \in U$ tends to $E^{ss}_x(f_0)$ when f tends to f_0.

PROPOSITION 5.3. *Suppose that $\mathcal{F}^{ss}$ is of class C^1 and that there are constants a, ϵ_1, θ_2, L and r_2 such that for every unstable curve γ_1, γ_2 of length at most L and every center-unstable manifold W^{cu}_3 with $d^{ss}(\gamma_i, W^{cu}_3) \leq \epsilon_1$ and $d^{ss}(\gamma_1, \gamma_2) \in J = [a, Ia]$, where $I = \max_{x\in B(\Lambda)}\{\|Df|_{E^{ss}_x}\|^{-1}\}$, the curves $\pi^{ss}\gamma_1$ and $\pi^{ss}\gamma_2$ are θ_2-transversal in neighborhoods of radius r_2 in W^{cu}_3.*

Then, the transversality condition (C1) *holds.*

Proof. First, let us suppose that W^{cu}_3 contains the curve γ_2. Given the unstable curves γ_1 and γ_2 with $d^{ss} = d < \epsilon_0$, we consider $\epsilon_0 = \min\{\epsilon_1, a\}$ and the smallest integer $n \in \mathbb{N}$ such that $d(f_0^{-n}\gamma_1, f_0^{-n}\gamma_2) \in J = [a, Ia]$ (this is a fundamental domain for the stable distances of iterations of curves). By hypothesis, the projections of these curves into $f_0^{-n}W^{cu}_2$ are θ_2-transversal in neighborhoods of radius r_2. Iterating forward n times the projection of the curves, there exists a constant τ depending on the norm of Df_0 restricted to each subbundle such that the angle of the images of the tangent vectors to the projected curves is bounded by $\tau^n\theta_2$ in neighborhoods of radius $(\lambda_c^-)^n r_2$.

CLAIM 5.1. *Given two open sets U, $V \subset \mathbb{R}^2$ with bounded diameter and a diffeomorphism $h : U \to V$ of class C^1 with $\|h - \mathrm{id}\|_{C^1} \leq \frac{1}{2}$, for every $\theta < \pi/3$ and $r > 0$, there exist constants $\tilde{\theta}$ and $\tilde{r}$ such that for every pair of curves γ_1, γ_2 contained in U that are θ-transversal in neighborhoods of radius r, the curves $\tilde{\gamma}_1 = h(\gamma_1)$ and $\tilde{\gamma}_2 = h(\gamma_2)$ are $\tilde{\theta}$-transversal in neighborhoods of radius $\tilde{r}$.*

Proof of Claim 5.1. Let v_i be a unit vector tangent to γ_i at the point x_i and $\tilde{v}_i$ a unit vector tangent to $h(\gamma_i)$ at the point $h(x_i)$, for $i = 1, 2$. The unit vector tangent to the curve $\tilde{\gamma}_i$ at the point $h(x_i) \in \tilde{\gamma}_i$ is given by $\tilde{v}_i = \lambda_i \cdot dh_{x_i}(v_i)$, where $\lambda_i = \|dh_{x_i}(v_i)\|^{-1}$.

A geometric consideration gives that $\sin(\angle(v, w)/2) = \|v - w\|/2$ for every pair of unit vectors v and w. Let C be such that $C^{-1} \leq \sin x/x \leq C$ for every $x \in [-\pi/3, \pi/3]$. Then, $C^{-1}\|v - w\| \leq \angle(v, w) \leq C\|v - w\|$ whenever $\angle(v, w) < \pi/3$.

By continuity of the derivative, given r and θ, we take a constant $\tilde{r}_1$ such that $d(x, y) \leq \tilde{r}_1$ implies that $\|\mathrm{id} - dh_x^{-1}dh_y\| < C^{-1}\theta/20$. Also note that $\|h - \mathrm{id}\|_{C^1} \leq \frac{1}{2}$ implies that $\frac{1}{2} \leq \lambda_1, \lambda_2 \leq 2$ and that $\frac{1}{2} \leq \|(dh_x)^{-1}\|^{-1} \leq 2$.

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Now, supposing that $\angle(v_1 - v_2) \geq \theta$, that v_1 is closer to v_2 than to $-v_2$, and that $\hat{x}_1 = h(x_1)$, $\hat{x}_2 = h(x_2)$ satisfy $d(\hat{x}_1, \hat{x}_2) < \tilde{r} = \min\{r/2, \tilde{r}_1/2\}$, let us conclude that $\angle(\tilde{v}_1, \tilde{v}_2) \geq \tilde{\theta}$. In fact,

$$\begin{aligned}
\|\tilde{v}_1 - \tilde{v}_2\| &= \|\lambda_1 dh_{x_1} v_1 - \lambda_2 dh_{x_2} v_2\| \\
&\geq \lambda_1 \cdot \|dh_{x_1}^{-1}\|^{-1} \|v_1 - \lambda_1 \lambda_2^{-1} (dh_{x_1}^{-1} dh_{x_2}) v_2\| \\
&\geq \frac{1}{4}[\|v_1 - \lambda_1 \lambda_2^{-1} \cdot dh_x^{-1} dh_y(v_2))\|] \\
&\geq \frac{1}{4}[\|v_1 - \lambda_1 \lambda_2^{-1} v_2\| - \|\lambda_1 \lambda_2^{-1}(v_2 - dh_x^{-1} dh_y(v_2))\|] \\
&\geq \frac{1}{4}[\|v_1 - (v_1 \cdot v_2) v_2\| - \lambda_1 \lambda_2^{-1} \|v_2 - dh_{x_1}^{-1} dh_{x_2}(v_2)\|] \\
&\geq \frac{1}{4}[\|v_1 - v_2\| - \|v_2 - (v_1 \cdot v_2) v_2\| - 4\|\mathrm{id} - dh_{x_1}^{-1} dh_{x_2}\|] \\
&\geq \frac{1}{4}\left[\|v_1 - v_2\| - |1 - v_1 \cdot v_2| - 4\frac{C^{-1}\theta}{20}\right] \\
&\geq \frac{1}{4}\left[\|v_1 - v_2\| - \frac{\|v_1 - v_2\|^2}{2} - 4\frac{C^{-1}\theta}{20}\right] \\
&\geq \frac{1}{4}\|v_1 - v_2\|\left(1 - \frac{\|v_1 - v_2\|}{2}\right) - \frac{C^{-1}\theta}{20} \\
&\geq \frac{1}{4} \cdot C^{-1}\theta \cdot \frac{1}{4} - \frac{C^{-1}\theta}{20} = \frac{C^{-1}\theta}{80}.
\end{aligned}$$

Here, we have used the triangle inequality, that $\|v_1 - v_2\| \leq \sqrt{2} < \frac{3}{2}$, $\|x_1 - x_2\| \leq \tilde{r}_1$ and that $\|v - \lambda u\|$ is minimum for $\lambda = u \cdot v / \|u\|^2$. Taking $\tilde{\theta} := C^{-2}\theta/80$, it follows that the curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are $\tilde{\theta}$-transversal in neighborhoods of radius $\tilde{r}$. □

When γ_2 is not contained in W_3^{cu}, consider W_2^{cu} that contains γ_2, the same n as before, $\theta = \tau^n \theta_2$ and $r = (\lambda_c^-)^n r_2$. Take $\tilde{\epsilon}_0$ small such that it is possible to apply Claim 5.1 for the stable holonomy between center-unstable manifolds with diameter smaller than L whose d^{ss}-distance is smaller than $\tilde{\epsilon}_0$ (we take a local chart with product structure $W^{cu} \times W^{ss}$ to say that the holonomy is C^1 close to the vertical projection). Taking $\epsilon_0 < \tilde{\epsilon}_0$ and applying Claim 5.1, we obtain $\theta(\epsilon)$ and $r(\epsilon)$ such that the projections into W_3^{cu} are $\theta(\epsilon)$-transversal in neighborhoods of radius $r(\epsilon)$. □

Now we are able to prove the robustness of the transversality condition under the condition of regularity of the stable foliation, which implies that the stable subbundle $E_x^{ss}(f)$ converges to $E_x^{ss}(f_0)$ for every $x \in U$ when f tends to f_0.

PROPOSITION 5.4. *The transversality condition* (C1) *is an open property under condition* (C3).

Proof. As seen in Proposition 5.3, it is enough to check the transversality for projection of curves when $d^{ss} \in [a, Ia] = J$, where J is a fundamental domain for the size of iterates of stable segments.

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Since $x \to E^{uu}_x$ is continuous, we can consider a family of unstable cones with small width and fix a constant $\alpha > 0$ that bounds from below the angle of each pair of stable projections of unstable cones of this family when $d^{ss} \in J$. If a curve at x is contained in a cone C, then every curve at x that is C^1 close is also contained in the cone C. Thus, the family of cones and the limitation α of the projections can be taken as constant for every projection of f in a neighborhood of f_0 in the C^1 topology. □

Proof of Corollary B. Given f_0, there exists a neighborhood $\mathcal{U}$ of f_0 such that every $f \in \mathcal{U}$ satisfies the hypotheses of Theorem A. Actually, conditions (C2), (C3) and robust dynamical coherence are open in f, and it follows from Proposition 5.4 that the transversality condition is open under condition (C3).

6. *Attractors with transversality*

We will describe how to construct a family of non-hyperbolic attractors with central direction neutral and transversality between unstable leaves via the stable projection.

6.1. *The attractor F_0.* Consider $M = S^1 \times [-1, 1] \times [-1, 1]$ and $F_0 : M \to M$ of the type

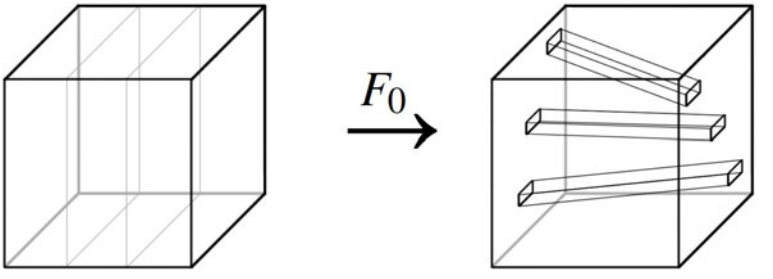
$$F_0(x, y, z) = (lx, \lambda_c y + g(x), \lambda_{ss} z + h(x)),$$

where $\lambda_{ss} < \lambda_c < 1$, $\lambda_c > 1/l$ and $\lambda_{uu} = l \in \mathbb{N}$.

In what follows, we will describe two examples of such attractors F_0 that satisfies the transversality condition.

6.1.1. *Example 1.* Let $l = \lambda_{uu} = 3$ and $i \in \{1, 2, 3\}$. We consider the rectangles $R_1 = [0, \frac{1}{3}) \times [-1, 1]^2$, $R_2 = [\frac{1}{3}, \frac{2}{3}) \times [-1, 1]^2$ and $R_3 = [\frac{2}{3}, 1) \times [-1, 1]^2$ and the functions g and h such that $g(x) = \alpha_i x + c_i$ and $h(x) = d_i$ if $x \in R_i$, where the constants c_i and d_i are parameters of translation guaranteeing that $F_0(M) \subset M$ and that F_0 is invertible.

The map F_0 is a linear model of a hyperbolic attractor. Restricted to each rectangle R_i it is a hyperbolic affine transformation inserting R_i into M with slope α_i in the y-direction into the xy-plane. For simplicity, we will consider a constant $\alpha \in (0, 1 - \lambda_c)$ and choose the parameters $\alpha_1 = \alpha$, $\alpha_2 = 0$, $\alpha_3 = -\alpha$ and $c_i = -\alpha_i$, $i = 1, 2, 3$. Then, the rectangles are inserted transversally with respect to the stable (vertical) foliation. This dynamics can be represented by the picture below.



The sets $D_i = \{i/3\} \times [-1, 1]^2$ are sent by F_i into $\{0\} \times [-1, 1]^2$. As F_0 is uniformly hyperbolic when restricted to the interior of the R_i, the unstable manifolds are well defined

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for every point in the attractor whose backward orbit never intersects any D_i, which corresponds to almost every point.

PROPOSITION 6.1. *The attractor Λ_0 for F_0 satisfies the transversality condition* (C1).

Proof. The calculation is done noticing that the stable projection π^{ss} of F_0 coincides with the vertical projection, the center-unstable direction at every point corresponds to the xy-plane, and E^{uu} is given for every $p \in \Lambda_0$ by $E^{uu}(p) = (1, \sum_j \lambda_{uu}^{-1}\alpha_j(p)(\lambda_c/\lambda_{uu})^j, 0)$, where $\alpha_j(p) = \alpha(F_0^{-j-1}(p)) \in \{\alpha_1, \alpha_2, \alpha_3\}$ depends on the itinerary of the x-coordinate of p by the expansion in S^1 by the factor $\lambda_{uu} = 3$. We obtain the expression for $E^{uu}_p = (1, \alpha^{uu}(p), 0)$, rewriting the condition $DF_0(p) \cdot E^{uu}_p = E^{uu}_{F_0(p)}$ as $(\lambda_{uu}, \alpha_i(p) + \alpha^{uu}(p)\lambda_c, 0) = \lambda_{uu}(1, \alpha^{uu}(F_0(p)), 0)$, so α^{uu} is a function that satisfies $\alpha^{uu}(F_0(p)) = (\lambda_{uu})^{-1}\alpha(p) + (\lambda_c/\lambda_{uu})\alpha^{uu}(p)$, where $\alpha(x, y, z) = \alpha_i x$. Iterating the last equation j times and considering $\rho := \lambda_c/\lambda_{uu} < \frac{1}{3}$, we get

$$\alpha^{uu}(F_0^j(p)) = (\lambda_{uu})^{-1}\alpha(F_0^{j-1}(p)) + \rho(\lambda_{uu})^{-1}\alpha(F_0^{j-2}(p)) + \cdots + \rho^j\alpha^{uu}(p).$$

So, for every $\tilde{p} \in \Lambda$, we have

$$\begin{aligned}\alpha^{uu}(\tilde{p}) &= (\lambda_{uu})^{-1}\alpha(F_0^{-1}(\tilde{p})) + \rho(\lambda_{uu})^{-1}\alpha(F_0^{-2}(\tilde{p})) + \cdots + \rho^j\alpha^{uu}(F_0^{-j}(\tilde{p}))\\ &= \sum_{j=0}^{j_0}(\lambda_{uu})^{-1}\rho^j\alpha(F_0^{-(j+1)}(\tilde{p})) + \rho^{j_0}\alpha^{uu}(F_0^{-j_0}(\tilde{p}))\\ &\to \sum_{j\geq 0}(\lambda_{uu})^{-1}\rho^j\alpha(F_0^{-(j+1)}(\tilde{p})).\end{aligned}$$

Consider multi-indexes $[k] = (k_0, \ldots, k_{t-1})$ of size t, where $k_j \in \{1, 2, 3\}$, and define the rectangle $R_{[k]}$ as the set of points q such that $F_0^j(q) \in R_{k_j}$, $0 \leq j \leq t-1$ and $f_{[k]}$ as the restriction of F_0^t to $R_{[k]}$. Since the d^{ss}-diameter of $R_{(k_0,\ldots,k_{t-1})}$ is equal to $(\lambda_{ss})^t$, if $d^{ss}(\gamma_1^{uu}, \gamma_2^{uu}) > \epsilon$ then these curves are contained in distinct $f_{[k]}(M)$, $f_{[k']}(M)$ for multi-indexes $[k] = (k_0, \ldots, k_{t_0-1})$ and $[k'] = (k'_0, \ldots, k'_{t_0-1})$ of size $t_0 \geq \log\epsilon/\log\lambda_{ss}$. For $p \in f_{[k]}(M)$ and $p' \in f_{[k']}(M)$, taking $j_0 = \inf\{j, k_j \neq k'_j\} \leq t_0$, we have

$$\begin{aligned}\alpha^{uu}(p) - \alpha^{uu}(p') &= \sum_{j\geq j_0}\frac{1}{\lambda_{uu}}\rho^j[\alpha(F^{-(j+1)}(p)) - \alpha(F^{-(j+1)}(p'))]\\ &= \frac{\rho^{j_0}}{\lambda_{uu}}[\alpha(F^{-(j_0+1)}(p)) - \alpha(F^{-(j+1)}(p'))] + \sum_{j>j_0}\frac{\rho^j}{\lambda_{uu}}(\alpha - \alpha')\\ &\geq \frac{1}{\lambda_{uu}}\left[\rho^{j_0}\alpha - 2\alpha\cdot\frac{\rho^{j_0+1}}{1-\rho}\right]\\ &\geq \frac{1}{\lambda_{uu}}\rho^{\log\epsilon/\log\lambda_{ss}}\alpha\left(\frac{1-3\rho}{1-\rho}\right) := C(\epsilon).\end{aligned}$$

Since $\pi_0^{ss}E^{uu}(p) = (1, \alpha^{uu}(p))$, it follows that $\angle((1, \alpha^{uu}(p)), (1, \alpha^{uu}(p')))$ is bounded from below, because the function α^{uu} is uniformly bounded and the difference $|\alpha^{uu}(p) - \alpha^{uu}(p')|$ is bounded from below by $C(\epsilon)$. Then, there exists a function $\theta : (0, 1) \to \mathbb{R}^+$ such that the angle between $\pi^{ss}(\gamma_1^{uu})$ and $\pi^{ss}(\gamma_2^{uu})$ is greater than $\theta(\epsilon)$ whenever $d^{ss}(\gamma_1^{uu}, \gamma_2^{uu}) > \epsilon$. □

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6.1.2. *Example 2.* Consider $F_0 : M \to M$ given by

$$F_0(x, y, z) = (2x, \lambda_c y + \sin(2\pi x), \lambda_{ss} z + h(x))$$

with $\lambda_{ss} < 0$, $5 < \lambda_c < 0, 51$, $l = \lambda_{uu} = 2$ and $h(x)$ equal to $\frac{1}{2}$ if $0 \le x \le \frac{1}{2}$ and equal to $-\frac{1}{2}$ if $\frac{1}{2} \le x \le 1$. This dynamics is an invertible version of [**27**, Example 1], considering M the quotient of $[0, 1] \times [-1, 1] \times [-1, 1]$ by the identification $(1, y, z) \sim (0, y, 1 - z)$.

PROPOSITION 6.2. *The attractor Λ_0 for F_0 satisfies the transversality condition* (C1).

Proof. Once again the stable projection π^{ss} is the projection into the z-coordinate, the center-unstable direction is tangent to the xy-plane, and E^{uu} is given for every $p \in \Lambda_0$ by $E^{uu}(p) = (1, \sum_j \lambda_{uu}^{-1} \alpha(F_0^{-j-1} p)(\lambda_c/\lambda_{uu})^j, 0)$, where $\alpha(x, y, z) = \partial/\partial x \sin(2\pi x) = 2\pi \cos(2\pi x)$.

From Proposition 5.3, it is sufficient to check the transversality for every pair of unstable curves whose stable distance is in the interval $J = [\frac{1}{4}, 1/4\lambda_{ss}]$. This is obtained considering $a = \frac{1}{4}$, $I = 1/\lambda_{ss} > 2$ and $\epsilon_1 = 1/4\lambda_{ss}$ in the statement of Proposition 5.3.

Given two unstable curves γ_1^{uu} and γ_2^{uu} with $d^{ss}(\gamma_1^{uu}, \gamma_2^{uu}) \in J$, $p = (x, y, z) \in \gamma_1^{uu}$ and $p' = (x', y', z') \in \gamma_2^{uu}$, the condition $d^{ss}(\gamma_1^{uu}, \gamma_2^{uu}) \in J$ implies that γ_1^{uu} and γ_2^{uu} are not contained in the same set $F_0(R_1)$ or $F_0(R_2)$.

Define $F_0^{-j}(p) = (x_j, y_j, z_j)$ and $F_0^{-j}(p') = (x'_j, y'_j, z'_j)$. When $x_1, x'_1 \in [0, \frac{2}{5}] \cup [\frac{3}{5}, 1]$, one can check that $|\cos(2\pi x_1) - \cos(2\pi x'_1)| \ge \cos(2\pi/5)$ and $|\cos(2\pi x_2) - \cos(2\pi x'_2)| \le \cos(\pi/5) + \cos(\pi/2 - \pi/5)$. This implies that

$$\begin{aligned} |\alpha^{uu}(p) - \alpha^{uu}(p')| &= \left| \sum_{j \ge 1} \frac{1}{\lambda_{uu}} \rho^j (\alpha(F^{-(j+1)}(p)) - \alpha(F^{-(j+1)}(p'))) \right| \\ &\ge \pi |\cos(2\pi x_1) - \cos(2\pi x'_1)| - \frac{\pi \lambda_c}{2} |\cos(2\pi x_2) - \cos(2\pi x'_2)| \\ &\quad - \left| \sum_{j>2} \frac{2\pi \rho^j}{\lambda_{uu}} (\alpha(x_j) - \alpha(x'_j)) \right| \end{aligned}$$

which is bounded from below by

$$K = 2\pi \left[\cos\left(\frac{2\pi}{5}\right) - \frac{\lambda}{4} \left(\cos(\frac{\pi}{5}) + \cos\left(\frac{\pi}{2} - \frac{\pi}{5}\right) \right) - \frac{\lambda^2}{4} \frac{2}{2 - \lambda} \right] > 0.$$

When $x_1, x'_1 \in [\frac{2}{5}, \frac{3}{5}]$, a lower bound for the distance of $\pi^{ss}(x_1)$ and $\pi^{ss}(x'_1)$ is obtained similarly using the relation $y = \sum_{i=1}^{\infty} \lambda_c^{i-1} g(F_0^{-i}(p))$, which is verified using the fact that $F_0^n(p) = \big(\lambda_{uu}^n x, \lambda_c^n y + \sum_{j=1}^n \lambda_c^{n-j} g(F_0^{j-1}(p)), \lambda_{ss}^n z + \sum_{j=1}^n \lambda_{ss}^{n-j} g(F_0^{j-1}(p))\big)$, writing $g(x, y, z) = g(x)$ and $h(x, y, z) = h(x)$, taking $p = F_0^n(F_0^{-n}(p))$ and sending $n \to \infty$. The condition $x_1, x'_1 \in [\frac{2}{5}, \frac{3}{5}]$ implies that the distance of the points x_2, x'_2 to the set $\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\}$ is smaller than $\frac{1}{40}$, implying

$$|\pi^{ss}(p) - \pi^{ss}(p')| \ge |y_1 - y'_1| \ge 2 \left[\sin\left(\frac{2\pi}{5}\right) - \lambda_c \sin\left(\frac{3\pi}{10}\right) - \frac{\lambda_c^2}{1 - \lambda_c} \right] =: K_2 > 0.$$

By continuity, we have that $|\pi^{ss}(p) - \pi^{ss}(p')| > K_2/2$ whenever $x_1, x'_1 \notin [\frac{2}{5} - \epsilon_1, \frac{3}{5} + \epsilon_1]$. So, for $r_2 = \min\{\epsilon_1/2, K_2/2\}$, we get θ_2-transversality in neighborhoods of radius r_2. □

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6.2. *The family of attractors $F_{\mu,n}$.* Let us give an example of non-hyperbolic attractors satisfying the transversality condition.

Let F_0 be as in any of the two examples before. We have $F_0^n(x, y, z) = (\tau^n(x), \lambda_c^n y + g_n(x), \lambda_{ss}^n z + h_n(x))$, where $\tau : S^1 \to S^1$ is the expansion in the circle by the factor λ_{uu}, $g_n(x) = \sum_{j=0}^{n-1} \lambda_c^{n-j-1} g(\tau^j(x))$ and $h_n(x) = \sum_{j=0}^{n-1} \lambda_c^{n-j-1} h(\tau^j(x))$.

Let q be a periodic of F_0, $\delta < 1/10\lambda_{uu}$, and consider an affine function $\psi_0 : B^{cu}(p, \delta) \subset M \to B(0, \delta) \subset \mathbb{R}^2$ that sends $q \to 0$ and $\{E_0^{uu}, E_0^c\} \to \{e_1, e_2\}$. Take a bump function $\psi_1 : B(0, \delta) \subset \mathbb{R}^2 \to [0, 1]$ of class C^∞, with $\psi_1(x) = 1$ if $\|x\| \le \delta/3$, $\psi_1(x) \in (0, 1)$ if $\frac{1}{3} \le \|x\| \le \frac{2}{3}$ and $\psi_1(x) = 0$ if $\|x\| \ge 2\delta/3$; this is done with $\|\psi_1\|_{C^1} \le 2(\delta^{-1}/3)$, $\|\psi_1\|_{C^0} \le 1$ and $\|\psi_0\|_{C^0} \le \delta$. Fixing some $\lambda_c^+ > 1$, define $\Phi_{\mu,n} : S^1 \times [-1, 1]^2 \to \mathbb{R}$ by $\Phi_{\mu,n}(x, y, z) = \mu\psi_1(\psi_0(x, y))[(\lambda_c^+ - \lambda_c^n)y]$ and define the family $F_{\mu,n} : M \to M$ by

$$F_{\mu,n}(x, y, z) = (\lambda_{uu}^n, \lambda_c^n y + g_n(x) + \Phi_{\mu,n}(x, y), \lambda_{ss}^n z + h_n(x)).$$

This family $F_{\mu,n}$ corresponds to a deformation of F_0^n changing the index of the periodic point p when passing through a pitchfork bifurcation. The deformation is done along the central direction, keeping the same central direction for every parameter μ. The attractor is the set $\Lambda_{\mu,n} = \bigcap_{j\ge 0} F_{\mu,n}^j(M)$. We will see that for an appropriate choice of n, it is possible to keep the unstable direction close to the original one in order that the transversality condition still holds for every parameter $\mu \in [0, 1]$.

PROPOSITION 6.3. *For every $\epsilon_1 > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that $d(E^{uu}_{F_{\mu,n}}(x), E^{uu}_{F_0^n}(x)) < \epsilon_1$ for every $x \in \Lambda_{\mu,n}$, every $\mu \in [0, 1]$ and every $n \ge n_0$.*

Proof. For parameters μ and n, the unstable direction $E_\mu^{uu}(p) = (1, \alpha_\mu^{uu}(p), 0)$ satisfies $DF_{\mu,n}(p) \cdot (1, \alpha^{uu}(p), 0) = \lambda_{uu}^n(1, \alpha^{uu}(F_{\mu,n}(p)), 0)$ for every $q \in \Lambda_\mu$, n, which can be rewritten as

$$\alpha^{uu}(F_{\mu,n}(p)) = \lambda_{uu}^{-n} g_n + \lambda_{uu}^{-n} \frac{\partial \Phi_{\mu,n}}{\partial x} + \lambda_{uu}^{-n} \left(\lambda_c^n + \frac{\partial \Phi_{\mu,n}}{\partial y}\right) \alpha^{uu}(p).$$

The function α^{uu} is obtained as the fixed point of the operator $T_{\mu,n}$, given by

$$\begin{aligned} T_{\mu,n}(\alpha)(p) = {} & \lambda_{uu}^{-n} \frac{dg_n}{dx}(F_{\mu,n}^{-1}(p)) + \lambda_{uu}^{-n} \frac{\partial \Phi_{\mu,n}}{\partial x}(F_{\mu,n}^{-1}(p)) \\ & + \lambda_{uu}^{-n} \left(\lambda_c^n + \frac{\partial \Phi_{\mu,n}}{\partial y}\right) \alpha(F_{\mu,n}^{-1}(p)). \end{aligned}$$

This fixed point $\alpha_{\mu,n}^{uu}$ is well defined because the operator $T_{\mu,n}$ is a contraction in the Banach space of continuous functions $\beta : \Lambda \to \mathbb{R}$ endowed with the norm of the supremum; in fact, the relation $\lambda_{\mu,n} = \lambda_{uu}^{-n}(\lambda_c^n + (\partial \Phi_{\mu,n}/\partial y)) \le \eta < 1$ guarantees that the operator is a contraction.

It remains to check that $\|\alpha_{\mu,n}^{uu} - \alpha_{0,n}^{uu}\| < \epsilon_1$, which is valid when the respective operators are close. Writing $T_{\mu,n}(\alpha) = A_{\mu,n} + \lambda_{\mu,n}\alpha \circ F_{\mu,n}^{-1}$, we have

$$\begin{aligned} d(\alpha_{\mu,n}^{uu}, \alpha_{0,n}^{uu}) &= d(T_{\mu,n}(\alpha_{\mu,n}^{uu}), T_{0,n}(\alpha_{0,n}^{uu})) \\ &\le d(T_{\mu,n}(\alpha_{\mu,n}^{uu}), T_{\mu,n}(\alpha_{0,n}^{uu})) + d(T_{\mu,n}(\alpha_{0,n}^{uu}), T_{0,n}(\alpha_{0,n}^{uu})) \\ &\le \|T_{\mu,n}\| \cdot d(\alpha_{\mu,n}^{uu}, \alpha_{0,n}^{uu}) + \|A_{\mu,n} - A_{0,n}\| + \|\lambda_{\mu,n}\alpha_{0,n}^{uu} - \lambda_{0,n}\alpha_{0,n}^{uu}\|. \end{aligned}$$

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Therefore,

$$d(\alpha^{uu}_{\mu,n}, \alpha^{uu}_{0,n}) \leq \frac{d(T_{\mu,n}, T_{0,n})}{1 - \|T_{\mu,n}\|} \leq \frac{\|A_{\mu,n} - A_{0,n}\| + (\lambda_{\mu,n} - \lambda_{0,n})\|\alpha_{0,n}\|}{1 - \eta}.$$

Since $\alpha_n(p)$ depends only on the x-coordinate of p, and since the x-coordinate of $F^{-1}_{\mu,n}(p)$ and of $F^{-1}_{0,n}(p)$ are the same, we have

$$(T_{\mu,n} - T_{0,n})(\beta)(p) = (\lambda_{uu})^{-n}\left[\frac{\partial \Phi_{\mu,n}}{\partial x}(F^{-1}_{\mu,n}(p))\right] + (\lambda_{uu})^{-n}\left[\left(\lambda_c^n + \frac{\partial \Phi_{\mu,n}}{\partial y}\right)\beta(F^{-1}_{\mu,n}(p)) - (\lambda_c^n)\beta(F^{-1}_{0,n}(p))\right].$$

Each line above goes to zero when $n \to \infty$, since $\lambda_c^n(\lambda_{uu})^{-n} \overset{n\to\infty}{\to} 0$ and $|\partial \Phi_{\mu,n}/\partial x|$, $|\partial \Phi_{\mu,n}/\partial y|$ are bounded. Then we take n large, D_1 and λ_1 small such that

$$\|\alpha^{uu}_{\mu,n} - \alpha^{uu}_{0,n}\| \leq \frac{D_1 + \lambda_1\|\alpha^{uu}\|}{1 - \eta} < \epsilon_1. \qquad \square$$

The non-hyperbolicity of $\Lambda_{\mu,n}$ will follow from the fact that the attractor admits hyperbolic periodic points of different indexes and that it is robustly transitive.

PROPOSITION 6.4. *There exists an integer n_1 such that the attractor $\Lambda_{n,\mu}$ of $F_{\mu,n}$ is robustly transitive for every $\mu \in [0, 1]$ and every $n \geq n_1$.*

Proof of Proposition 6.3. In our situation, the deformation is done inside a set D such that the dynamics outside D contracts vectors tangent to the central direction. Hence, we can make an argument similar to the one of Mañé [**8**, **19**] to prove the robust transversality.

Since the unstable foliation for F_0 is minimal in Λ_0, for every ϱ there exists a corresponding $L > 0$ such that the unstable foliation is $(L/2, \varrho/2)$-dense in center-stable manifolds, that is, every center-stable ball of radius $\varrho/2$ intersects every unstable curve γ^{uu} of length greater than $L/2$.

Applying Proposition 6.2 for an ϵ_1 sufficiently small, there exists n_1 such that the unstable foliation of $F_{\mu,n}$ is (L, ϱ)-dense in center-stable manifolds for every $n \geq n_1$ (the center-stable foliation is the same for $F_{\mu,n}$ and F_0). Let $\mathcal{U}_1$ be a neighborhood of $F_{\mu,n}$ such that the same (L, ϱ)-density holds for every f in this open set $\mathcal{U}_1$. In what follows, we assume that $\delta \leq 1/10\lambda_{uu} \leq \frac{1}{20}$.

CLAIM 6.1. *For every open set U intersecting Λ there exist a point $x \in U \cap \Lambda$ and an integer $N \in \mathbb{N}$ such that $f^{-n}(x) \notin D$ for every $n \geq N$.*

Proof of Claim 6.1. Take a negative iterate $\tilde{U}$ of U with central-stable diameter greater than 100δ, which exists since $Df_{|_{E^{cs}}}$ contracts area. Considering $R = 3\lambda_{ss}$, we note that every center-stable leaf of radius greater than R contains a center-stable ball B of radius $R/3$ such that $f^{-1}(B)$ does not intersect D.

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Let Γ_1 be a subset of $\tilde{U}$ such that $f^{-1}(\Gamma_1) \cap D = \emptyset$. Since the center-stable diameter of $f^{-1}(\Gamma_1)$ is greater than R, we can consider $\Gamma_2 \subset f^{-1}(\Gamma_1)$ such that $f^{-1}(\Gamma_2) \cap D = \emptyset$. Inductively, we consider a sequence $\Gamma_n \subset f^{-1}(\Gamma_{n-1})$ such that $f^{-1}(\Gamma_n) \cap D = \emptyset$. So, any $x \in \bigcap_n f^n(\Gamma_n)$ is a point that satisfies the conclusion of the claim. □

Given two open subsets U and V of Λ, we consider a point $x \in U \cap \Lambda$ given by Claim 6.1 and an unstable curve contained in V. Iterate the unstable curve until it has length greater than L and preiterate the center-stable leaf of x contained in U until it has internal radius greater than 2δ. Due to the (L, δ)-density we know that these sets have non-empty intersection, so $U \cap f^n(V) \neq \emptyset$ for some $n \geq 0$. □

6.3. *Proof of Theorem C.*

PROPOSITION 6.5. *There exists an integer $n_2 \in \mathbb{N}$ such that the transversality condition holds for the dynamics $F_{\mu,n}$, for every $\mu \in [0, 1]$ and every $n \geq n_2$.*

Proof. As seen in Proposition 6.1, the transversality holds for every $F_{0,n}$. Fix $a \in (0, \lambda_{ss}/10)$ and a fundamental stable domain $J = [a, \lambda_{ss}^{-1}a]$, and consider $\theta(a)$ a lower bound of the angle of projections of unstable curves with $d^{ss} \in J$ for F_0.

Since the stable projection is the vertical one, for each unstable curve we consider $\omega > 0$ small such that the center-unstable cone of width ω around the unstable direction is $\theta(a)/2$-transversal to each other center-unstable cone of width ω around another unstable direction with stable distance between them contained in the interval J; by compactness, this width ω can be taken to be uniform. Fixing this family of unstable cones of width ω containing the original $E^{uu}_{0,n}$, we have, by Proposition 6.2, that there exists $n_2 \in \mathbb{N}$ such that this family of cones contains the unstable direction of $F_{\mu,n}$ for every $\mu \in [0, 1]$ and every $n \geq n_2$. This implies that $\angle(\pi^{ss}(E^{uu}_{\mu,n}(x_1)), \pi^{ss}(E^{uu}_{\mu,n}(x_2))) > \theta(a)/2$ whenever $d^{ss}(x_1, x_2) > a$. By Proposition 5.1, this is enough to guarantee the transversality. □

Proof of Theorem C. Take F_0 as in any of the two examples above, and construct $F_{\mu,n}$ as before with $\lambda_c^+ > 1$ such that $(\lambda_c^+)^2/3\lambda_c < 1$. We have that $(\lambda_c^+)^2/(3\lambda_c)^n < 1$ (central direction is neutral for $F_{\mu,n}$), which guarantees Condition (C2). Taking $n_3 \in \mathbb{N}$ such that Proposition 6.3 is valid for every $F_{\mu,n}$, $\mu \in [0, 1]$, $n \geq n_3$, it follows that the transversality condition holds for every $F_{\mu,n}$. The C^1 regularity follows by choosing $\lambda_{ss} < 1$ with $3\lambda_{ss}/\lambda_c^+ < 1$ (this condition implies the C^1 regularity of the stable foliation and also the continuity of this foliation in the C^1-topology). The attractor is robustly dynamically coherent because the central direction for $F_{\mu,n}$ is of class C^1; actually, $E^c = y$-direction, $E^{cu} = xy$-plane, $E^{cs} = yz$-plane for every $F_{\mu,n}$, and [**15**, Theorems 7.1 and 7.4] guarantees that these laminations are structurally stable. Taking $N > n_2$ as given in Proposition 6.3, the map $F_{1,N}$ is robustly non-hyperbolic because it has periodic points of different indexes and it is robustly transitive. □

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