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Optimal path planning of manipulatory systems subjected to non-autonomous motion laws

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SUMMARY

A path planning method is presented based on non-autonomous dynamic modeling of open-loops in articulated systems. It is assumed that one part of the mechanical system is submitted to specified motions laws, while movements of the complementary part are free. Thus, motion optimization is related to free joint movements but it is achieved on the basis of the dynamic model of the whole mechanical system. This approach introduces a non-autonomous state equation of a special type in the sense that it can not only depend on the running time but also on the unknown travelling time. The cost function to be minimized involves the travelling time and the actuating inputs. Optimization is achieved by applying the Pontryagin Maximum Principle which yields a new optimality condition concerning the travelling time dependency of the stated problem. Two simulation examples are presented. The first one shows how the developed technique makes possible both the reducing and mastering the dynamic complexity of a four degrees of freedom-vertical manipulator. Set at four degrees of freedom, the second one deals with a redundant planar manipulator characterized by a mobile base that is submitted to a specified driving motion.

KEYWORDS: Robot dynamics; Path planning; Motion optimization.

1. INTRODUCTION

Industrial robot manipulators are usually used to perform repetitive tasks. In other respects, the complexity of their dynamic behaviour, especially when working at a fast rate, can result in very fast varying joint velocities and actuating inputs. The consequences can be repeated backlashes, joint high strains with overloaded actuators, link bending and torsion, making it difficult to efficiently control the global motion. Thus, the problem is to try to make compatible high performance requirements with both reliable robot functioning and safe payload transfer. The solution can be achieved by improving the dynamic performance of the controlled mechanical system, which means smoothing its motion while minimizing the travel time. Typically, we are confronted with an optimal transfer problem of which the statement must be based on accurate dynamic modeling of the multibody system.

This quite general problem, which has received a great deal of attention these last two decades, has been tackled in many ways. A basic way of dealing with motion optimization is found in the content of the cost function from which minimization must yield an optimal trajectory. At first, most authors favored searching for minimum-time solutions, either along specified paths,[1–7] or in free path planning.[8–10] This approach is not completely satisfactory in robotics because control inputs, which in fact are the actuator forces and torques, can be subjected to discontinuous variations at switching times. This may result in jerky motions and vibrations of the articulated system. A simple means in order to avoid this drawback calls for introducing a quadratic energy term, or better a quadratic control input functional, in the cost function.[11–18] Such a mixed and convex cost leads to a smoothing of the control inputs while minimizing the transfer time. Furthermore, besides the fact that the problem is mathematically well defined, this simple means makes it possible to develop robust algorithmic approaches to compute optimal solutions. This is the choice we have made for this study.

The implementation of an optimization technique is the second fundamental choice when dealing with optimal path planning. A few authors such as Shin and McKay,[4] Vukobratović and Kircanski,[19] Singh and Leu,[20] Heimann and Krüger[21] used dynamic programming approaches. Mathematical programing has been studied as well. Dissanayake[10] used this technique to solve a minimum-time trajectory problem, while Khoukhi and Hamam,[22] Wang et al.,[23] Wang and Hamam[24] developed on the same basis a general approach for solving complete constraint problems while considering a fixed total-motion time. Lastly, within the framework of the optimal control theory, the Pontryagin Maximum Principle constitutes a powerful means to deal with optimal transfer problems. It is a quite suitable method to account for hard-control variable constraints as well as mixed-control-phase constraints.[12,14] Control problems with bounded-state variables can also be efficiently dealt with by implementing a penalty method.[17] We must also mention that to solve state constrained problems Zefran and Kumar[16] have formulated the constrained optimal-control problem as an unconstrained problem of the calculus of variations. In the following, our choice is the Maximum Principle because, in addition to its general

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applicability, it offers insights into the structure of optimality conditions. Moreover, this method proved to be efficient when computed.

A prominent difficulty in dealing with optimal path planning of controlled articulated systems lies in the fact that the dynamic model of the mechanical system is too complex to handle since its size increases rapidly with the number N of degrees of freedom. This difficulty is drastically accentuated when expanding the optimality conditions of the underlying optimal transfer problem. Furthermore, kinematic redundancies appear with increasing N, resulting in a more complex dynamic behaviour of the mechanical system. The optimal planning problem can include a multiplicity of solutions for which computation may be untractable, most of them being unworkable. This difficulty can be overcome by adding *ad hoc* complementary constraints in order for the performance index to select a feasible optimal solution.

For this purpose, the planning method developed in this study continues and widens a previous approach presented by the authors[18] and is based on the introduction of non-autonomous phase constraints. These constraints specify movement laws imposed on certain joints or parts of the mechanical system. They can simultaneously or separately depend on independent configuration parameters, the running time and also the total transfer time which is an unknown of the stated problem. In this regard, such constraints are novel and yield a non-autonomous dynamic model with a new type dependency with respect to the travelling time. This approach is aimed at: (i) accounting for imposed driving motions, (ii) dealing with redundancies, (iii) reducing the dynamic model complexity of mechanical systems with a large number of degrees of freedom. Doing so, the optimization problem is easier to handle and allows one to account for specified motions of one part of the multibody system while optimizing the free movements of the complementary part.

Several numerical simulations are presented. First, motion optimization of a four degrees of freedom vertical manipulator has been examined. Three numerical examples concerning the same transfer task illustrate the optimal dynamic behaviour of the system according to the kind of complementary constraints introduced. Second, an example shows how to optimize the movement of a redundant planar manipulator when accounting for a specified driving motion of its mobile base.

2. SPECIFYING MOTION LAWS

The free dynamic behaviour of open-loop articulated systems used as serial robots can generate complex motions with large and rough variations of certain joint parameters, particularly at the end of the chain (see section 5, Figs. 1 and 2). Such dynamic effects are even more evident when increasing the number of degrees of freedom in the mechanical system. Hard motions to control can also be generated by driving movements exerted by a mobile base. In other respects, this difficulty of planning and controlling the movement of systems with a large number of active joints is coupled with the difficulty of mastering their dynamic modelization. The technique we propose in this study is intended to overcome this second difficulty while coping with the first one.

It is made up of introducing a limited set of independent configuration parameters, the others being submitted to a specified dependency with respect to the previous ones.

2.1 General formulation

We will consider a multibody system articulated as an open-loop kinematic chain with N degrees of freedom, and possibly including those of a mobile base. Selecting n free parameters written q_i, $i = 1, \ldots, n, n < N$, and brought together in the n-vector $q = (q_1, \ldots, q_n)^T$, we assume that the remaining parameters q_{n+j}, denoted φ_j in the following and put together in the m-vector $\varphi = (\varphi_1, \ldots, \varphi_m)^T$, $m = N - n$, are defined under the generic form of rheonomous constraints solved with respect to the free configuration parameters:

$$t \in [0, T], \quad \varphi_j(t) = \phi_j(q(t), t, T), \quad j = 1, \ldots, m \quad (1)$$

expression where t is the running time and T is the total motion time. It is assumed that T is unknown and must be optimized when solving the optimal control problem stated in section 4. The reason for which φ_i can explicitly depend on T is due to the fact that it must be equal to the values specified for φ_j at the ends of the interval $[0, T]$, especially at the final time T. Moreover, when doing so, the motion laws taken into account are automatically adapted to the optimal travel time corresponding to the best dynamic transfer the mechanical system is able to complete. Also, we must mention that ϕ_j does not necessarily depend explicitly on its three arguments in all examples (see section 5, (29), (32), (33), (34)).

In the next section, constraints such as (1) will result in formulating a reduced-order, non-autonomous dynamic problem of a new type in the field of optimal control theory owing to its explicit dependency on the unknown traveling time.

2.2 Applications

Here, we will present two kinds of applications.

The first one affects systems which are partly submitted to imposed motions. It is then necessary to integrate the corresponding movement laws into the dynamic model of the articulated system.

Our first example includes robotic manipulators mounted on a mobile base which have to go back and forth between two distant locations (see subsection 5.2) to complete a pick-and-place operation. If the motion of the base is controlled independently from the robot motions, it is interesting to link the first motion to the travelling time in order to optimize the latter, while accounting for both the specified motion of the base and

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the dynamic capacities of the robot arm. Describing the base movement by means of motion laws such as (1), but without explicit dependency with respect to q, i.e.:

$$\varphi_k(t) = \phi_k(t, T) \tag{2}$$

allows one to satisfy this double requirement (section 5, (33)).

Our second example under consideration is a manipulator that must respect special orientation specifications during the payload transfer in order, for instance, to avoid overturning the payload or losing some of its contents. In such cases, the movement of the gripper must be described with respect to either the reference horizontal plane or vertical direction. So, if the payload orientation must only vary between absolute limits, it is then possible to satisfy this requirement during the motion by submitting the wrist parameters to rheonomous constraints of general type (1).

The second kind of application comes from efforts to simplify the dynamic model of the mechanical system. This simplification can be obtained simply by introducing a limited set of independent configuration parameters. This approach is, for instance, suitable for the description of the dynamic behaviour of a robotic manipulator with an arm-wrist structure. In this case, during a pick-and-place transfer, it is for the most part the arm's dynamic capacities which are at stake. At the wrist level, actuators control relative motions of which amplitudes are in general limited, and which are related to links with smaller inertia. This situation allows one to limit dynamic optimization of the robot transfer to the movements of the arm while accounting for specified wrist motions defined by constraints of type (1) or (2). This approach is a simple means to efficiently manage the robot's dynamic behaviour, especially at the wrist level. Moreover it leads to drastic simplifications in solving the optimal control problem that results.

Finally, we must underline that the introduction of well-chosen specified joint movements may help as well in solving optimal dynamic planning of motions in redundant manipulatory systems.

In all above-mentioned cases, it is worth noting that in spite of the reduction in the number of independent configuration parameters, the reduced-order dynamic model that results, represents the dynamic behaviour of the whole mechanical system. However, we must not forget that the effects of dynamic optimization are centered on motions at selected joints.

3. DYNAMICS

With the application of the Pontryagin Maximum Principle (PMP) in mind, an appropriate formulation of the equations of motion is required. Indeed, these equations must be written in state space form that is to say as a first-order differential system solved according to the time derivatives of phase variables, this system constituting the vector state equation which describes the evolution of the controlled mechanical system. The Hamiltonian formulation naturally fulfills this requirement. Moreover, with this formulation, dynamic equations present symmetries which facilitate their processing when dealing with optimality conditions yielded by the PMP. It is this choice we present in this section.

3.1 Kinetic energy

Considering the free parameters and the dependent ones as defined in section 2-1, the kinetic energy takes the form:

$$K = \tfrac{1}{2}(\dot{q}^T, \dot{\varphi}^T)A(q, \varphi)\begin{pmatrix}\dot{q}\\ \dot{\varphi}\end{pmatrix} \tag{3}$$

where $\dot{q} \equiv dq/dt$, $\dot{\varphi} \equiv d\varphi/dt$.

In this expression, the inertia matrix A of the whole mechanical system can be split up into four submatrices adapted to the dimension of vectors $\dot{q}$ and $\dot{\varphi}$:

$$A(q, \varphi) = \begin{pmatrix} A_1(q, \varphi) & B(q, \varphi) \\ B^T(q, \varphi) & A_2(q, \varphi)\end{pmatrix}$$

where A_1, B and A_2 are respectively $n \times n$, $n \times m$ and $m \times m$ matrices.

Moreover, introducing the abbreviated notations:

$$\phi_{j,i} = \partial\phi_j/\partial q_i, \quad \phi_{j,t} = \partial\phi_j/\partial t$$

the general formula (1) implies:

$$\dot{\varphi}_j = \sum_{i=1}^{n} \phi_{j,i}\dot{q}_i + \phi_{j,t} \tag{4}$$

Thus, the quadratic expression with respect to $\dot{q}$ becomes:

$$K(q, \dot{q}, t, T) = \tfrac{1}{2}\dot{q}^T C\dot{q} + \dot{q}^T D\phi_{,t} + \tfrac{1}{2}\phi_{,t}^T E\phi_{,t}$$

with:

$$\phi_{,t} \equiv (\phi_{1,t}, \ldots, \phi_{m,t})^T$$

$$C \equiv A_1 + 2B\phi_{,q} + \phi_{,q}^T A_2 \phi_{,q} \quad (n \times n \text{ matrix})$$

$$\phi_{,q} \equiv (\phi_{j,i})_{j\le m, i\le n}$$

$$D \equiv B + \phi_{,q}^T A_2 \quad (n \times m \text{ matrix})$$

$$E \equiv A_2 \quad (m \times m \text{ matrix})$$

where A_1, A_2 and B are considered to be dependent on the arguments q, t and T which are shown through the arguments (q, φ) with $\varphi \equiv \phi(q, t, T)$. Therefore, the explicit dependence of K on t is conveyed by the vector function ϕ and its partial derivatives with respect to t and q.

3.2 Hamiltonian formulation

Let us define the Lagrangian of the mechanical system:

$$L(q, \dot{q}, t) = K(q, \dot{q}, t) - V(q) \tag{5}$$

where V stands for the potential of conservative force

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fields (e.g. the gravity field). Lagrange's equations of motion can be written as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i, \qquad i \le n \tag{6}$$

Here, the generalized force component Q_i is the sum:

$$Q_i = Q_i^d(q, \dot{q}) + Q_i^a$$

where Q_i^d (resp. Q_i^a) is the dissipative (resp. actuating) force/torque at joint i.

Then, introduce the conjugate variables:

$$p_i = \partial L/\partial \dot{q}_i, \quad p = (p_1, \ldots, p_n)^T \tag{7}$$

which, through (5) and (6), appear in the form of vector formula:

$$p = C\dot{q} + D\phi_{,t} \tag{8}$$

Also, let us show the derivative:

$$\frac{\partial L}{\partial q_i} = \tfrac{1}{2}\dot{q}^T C_{,i}\dot{q} + \dot{q}^T(D_{,i}\phi_{,t} + D\phi_{,ti}) + \tfrac{1}{2}\phi_{,t}^T E_{,i}\phi_{,t} + \phi_{,t}^T E\phi_{,ti} - V_{,i} \tag{9}$$

where the subscript "$_{,i}$" (resp. "$_{,t}$") denotes derivations with respect to q_i (resp. t).

Substituting (7) and (9) in (6), and accounting for (8) we obtain the Hamiltonian vector equations of motion as:

$$\dot{q} = C^{-1}(p - D\phi_{,t}) \tag{10a}$$

$$\dot{p} = \tfrac{1}{2}G^T C_{,q} G + G^T(D_{,q}\phi_{,t} + D\phi_{,tq}) + \tfrac{1}{2}\phi_{,t}^T E_{,q}\phi_{,t} + \phi_{,t}^T E\phi_{,tq} - V_{,q} + Q^d + Q^a \tag{10b}$$

where the subscript symbols "$_{,q}$" denotes the gradient derivation with respect to q; furthermore:

$$G = C^{-1}(p - D\phi_{,t})$$
$$Q^d = (Q_1^d, \ldots, Q_n^d)^T$$
$$Q^a = (Q_1^a, \ldots, Q_n^a)^T$$

We assume that the generalized dissipative components Q_i^d are reduced to quadratic damping laws such as:

$$Q_i^d(q, \dot{q}) = -a_i\dot{q}_i - b_i\dot{q}_i|\dot{q}_i|, \qquad i = 1, \ldots, n \tag{11}$$

where a_i and b_i are positive constants.

It is worth noting that the Hamiltonian dynamic model is computationally more efficient than the Lagrangian one when dealing with motion optimization by means of Pontryagin's Maximum Principle (cf. Bessonnet[25]). This remark is a further argument for using such a formulation.

3.3 Generic state space formulation

Define the $2n$-vector of state variables as:

$$x_i = q_i, \quad x_{n+i} = p_i, \quad x = (x_1, \ldots, x_{2n})^T, \tag{12}$$

and let the n-vector of control variables be the vector of actuating inputs:

$$u_i = Q_i^a, \quad u = (u_1, \ldots, u_n)^T \tag{13}$$

Next redefine the phase velocities (i.e. functions defining $\dot{q}_i$ and $\dot{p}_i$) by setting:

$$\left.\begin{aligned} G_i &= C_{ij}^{-1}(x_{n+j} - D_{jk}\phi_{k,t}), \quad G = (G_1, \ldots, G_n)^T \\ F_i &= G_i \end{aligned}\right\} \tag{14a}$$

$$\left.\begin{aligned} F_{n+i} &= \tfrac{1}{2}G^T C_{,i} G + G^T(D_{,i}\phi_{,t} + D\phi_{,ti}) \\ &\quad + \tfrac{1}{2}\phi_{,t}^T E_{,i}\phi_{,t} + \phi_{,t}^T E\phi_{,ti} - V_{,i} + Q_i^d \\ F &= (F_1, \ldots, F_{2n})^T \end{aligned}\right\} \tag{14b}$$

Then, the system of vector equations (10a,b) can be reformulated as the single $2n$-vector state equation:

$$t \in [0, T],$$
$$\dot{x}(t) = F(x(t), t, T) + I^* u(t), \tag{15}$$
$$I^* = [0_{n\times n}, I_{n\times n}]^T$$

where the explicit dependence on t and T is due to the presence of the vector function ϕ in (14a,b) which stands for the motion laws (1).

Equation (15) can also be written in the simplest form:

$$t \in [0, T], \quad \dot{x}(t) = f(x(t), u(t), t, T) \tag{16}$$

which gives prominence to its special non-autonomous feature. Indeed, it has the distinctive characteristic of depending explicitly on the final (or the total motion) time T which is an unknown of the optimal control problem stated in the next section. This is the main original aspect of this problem.

We assume that initial and final states of the mechanical system are known. Thus the state vector x of independent Hamiltonian phase variables must satisfy the end constraints:

$$x(0) = x^0, \quad x(T) = x^T \tag{17}$$

where x^0 and x^T are specified vector values. It is the same for the parameters φ_j submitted to constraints of type (1) and which must verify:

$$(\varphi(0), \dot{\varphi}(0)) = (\varphi^0, \dot{\varphi}^0), \quad (\varphi(t), \dot{\varphi}(T)) = (\varphi^T, \dot{\varphi}^T)$$

where values φ^0, $\dot{\varphi}^0$, φ^T, $\dot{\varphi}^t$ are given. Such end conditions are supposed to be directly involved in the explicit formulation of constraints (1) (see section 5, (29), (33), (35)).

4. OPTIMAL CONTROL PROBLEM

First, we define conditions of optimal transfer of the mechanical system, including control input and phase variable constraints, and involving a performance

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criterion to be minimized. Then, we apply the Pontryagin Maximum Principle to the optimal control problem which results in order to compute optimal solutions.

4.1 Constraining control inputs and phase variables

In order to acount for technological limitations of actuators, a set U of feasible controls is defined by fixed bounds $Q_i^{a\,\max}$ set on actuating force or torque (13) at free joint:

$$U = \{u \in \Re^n / |u_i| \le Q_i^{a\,\max},\ i \le n\} \tag{18}$$

In practice, each joint of the articulated system allows limited link movements. Furthermore, technological characteristics of reducers and transmissions impose limitations of joint velocities.

Thus, we set constant bounds to joint coordinates x_i and φ_i and their time derivatives (x_i, $\dot{x}_i$ as defined by (12) stand for q_i, $\dot{q}_i$):

$$x_i^{\min} \le x_i \le x_i^{\max}, \quad \dot{x}_i^{\min} \le \dot{x}_i \le \dot{x}_i^{\max}, \quad i \le n$$
$$\varphi_j^{\min} \le \varphi_j \le \varphi_j^{\max}, \quad \dot{\varphi}_j^{\min} \le \dot{\varphi}_j \le \dot{\varphi}_j^{\max}, \quad j \le m$$

Since $n + m = N$, these inequalities define $4N$ unilateral state constraints. The first $4n$ can be explicitly written as:

$$h_i(x(t)) \equiv x_i^{\min} - x_i(t) \le 0,$$
$$h_{n+i}(x(t)) \equiv x_i(t) - x_i^{\max} \le 0,$$
$$h_{2n+i}(x(t), t, T) \equiv \dot{x}_i^{\min} - F_i(x(t), t, T \le 0,$$
$$h_{3n+i}(x(t), t, T) \equiv F_i(x(t), t, T) - \dot{x}_i^{\max} \le 0,$$

for $i = 1, \ldots, n$, and where F_i represents the ith phase velocity (i.e. $\dot{x}_i$). Next, accounting for (1) and (4), become, with $j = 1, \ldots, m$:

$$h_{4n+j}(x(t), t, T) \equiv \varphi_j^{\min} - \phi_j(x(t), t, T) \le 0,$$

then, omitting the arguments x, t and T for brevity, we have:

$$h_{4n+m+j} \equiv \phi_j - \varphi_j^{\max} \le 0,$$
$$h_{4n+2m+j} \equiv \dot{\varphi}_j^{\min} - \phi_{j,x} F - \phi_{j,t} \le 0,$$
$$h_{4n+3m+j} \equiv \phi_{j,x} F + \phi_{j,t} - \dot{\varphi}_j^{\max} \le 0$$

Thus, we have defined $4N$ non-autonomous state constraints of the general type:

$$h_k(x(t), t, T) \le 0, \quad k \le 4N \tag{19}$$

And we also define the $4N$-vector function h as:

$$h = (h_1, \ldots, h_{4N})^T \tag{20}$$

4.2 Performance criterion

Similar to the earlier studies,[17,18] we now will introduce a mixed criterion which combines in the same function the total motion time and the quadratic control variables. Moreover, the state constraints are dealt with by means of an exterior penalty technique as developed in Bessonnet.[17] Thus, we are searching for optimal time T and control u which minimize the functional:

$$J(u, T) = \int_0^T \mathscr{L}(x(t), u(t), t, T)dt \tag{21}$$

where the integrand $\mathscr{L}$ is the sum of three terms:

$$\mathscr{L} = 1 - \mu + \frac{\mu}{2} u^T D_1 u + \frac{r}{2} (h^+)^T D_2 h^+,$$
$$\mu \in]0, 1[, r > 0 \tag{22}$$

D_1 and D_2 are diagonal weighting matrices:

$$D_1 = \text{diag}\,(\xi_1, \ldots, \xi_n), \quad D_2 = \text{diag}\,(\zeta_1, \ldots, \zeta_{4N})$$

The vector function h^+ is defined as:

$$h^+ = (h_1^+, \ldots, h_{4N}^+)^T \ldots, \quad h_k^+ = \tfrac{1}{2}(h_k + |h_k|), \tag{23}$$

where h_k^+ represents the positive values of h_k when the kth constraint is infringed.

The state constraints are satisfied if the penalty function in (22) vanishes. This situation occurs when increasing the multiplier r towards sufficiently high values. In theory, r must tend towards infinity in order for the constraints to be exactly satisfied. In practice, satisfactory solutions are obtained for finite values of r.

Note that, when the weight factor μ decreases, the minimization of the travel time is favored while the controls are increasingly saturated. Adding that the quadratic dependency of L on u yields continuous optimal control, it results in a smoother optimal motion, and computation of the solution is made easier.

4.3 Necessary optimality conditions

We have to find a state x, a control u and a travel time T that satisfy the state equation (16) together with the two-point boundary conditions (17) while minimizing the functional (21) on the set of feasible controls (18). The application of Pontryagin's Maximum Principle allows one to recast this optimal control problem in a more tractable formulation for numerical processing purposes.

For any $2n$-vector $w \in \Re^{2n}$, define the Hamiltonian:

$$H(x, u, w, t, T) \equiv w^T f(x, u, t, T) - \mathscr{L}(x, u, t, T)$$

Its explicit dependence with respect to T introduces a partial change in one of the usual conditions stated by the PMP. This new formulation can be formally obtained by varying the augmented criterion:

$$J^*(u, T) = \int_0^T \{w^T(t)\dot{x}(t) - H(x(t), u(t), w(t), t, T)\}dt,$$

using a classic calculus of variations technique (see for instance Lewis[26]). The first two optimality conditions provided by the PMP are unchanged: if (x, u, T) is an optimal control process, then there exists a vector function w satisfying the adjoint differential system:

$$\dot{w}^T = -\partial H / \partial x, \tag{24}$$

while the optimal control u satisfies the maximality condition:

$$\forall t \in [0, T],$$
$$H(x(t), u(t), w(t), t, T) = \max_{v \in U} H(x(t), v, w(t), t, T) \tag{25}$$

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Finally, at the final time T, the Hamiltonian must satisfy:

$$H(x(T), u(T), w(T), T, T) + \int_0^T (\partial H(x(t), u(t), w(t), t, T)/\partial T)dt = 0 \quad (26)$$

This last condition contains a new term due to the explicit dependency of H on T. The additional integral term makes it possible to adapt the optimal travel time T to the dynamic capacities of the mechanical system while accounting for the specified motion laws.

4.4 Solving the unconstrained control problem

Write the adjoint system (24) in the most explicit form:

$$\dot{w}(t) = -\left[\frac{\partial f}{\partial x}(x(t), u(t), w(t), t, T)\right]^T w(t) + r\left[\frac{\partial h}{\partial x}(x(t), t, T)\right]^T D_2 h^+(x(t), t, T) \quad (27)$$

Assuming that h as defined by (19, 20) is a smooth vector-function of its three arguments, the penalty term in (27) containing h^+ is obviously continuous and even lipschitzian when considered as a function of t through the function $t \to h(x(t), t, T)$. Thus, smoothness of the penalty term ensures accounting for phase constraints in a simple and efficient manner through the equation (27).

Now, we will sum up computation approaches presented in references.[12,14,25] The first step consists in extracting the optimal controls u_i from the maximality condition (25) which yields the expression of u_i as the explicit function of the costate w:

$$u_i(t) = \text{Sat}\,((u_i^{\max})^2 w_{n+i}(t)/\mu, u_i^{\max}), \quad i = 1, \ldots, n, \quad (28)$$

where we recall that sat is the saturation function defined here as: if $|x| < a$, then sat $(x, a) = x$, else sat $(x, a) = a$ (sign x).

When expression (28) is substituted for u in both state equation (16) and costate system (27), we have to solve a $4n$ order differential system with respect to the unknown vector functions $x(t)$ and $w(t)$. This system in which the function x must satisfy the two-point boundary conditions (17) has to be integrated on the interval $[0, T]$, with free end T submitted to condition (26). The latter condition which is of global type, since it depends on the values of x and w on the entire interval $[0, T]$, restricts the choice of existing algorithms for numerically solving the two-point boundary value problem which results. We have favored a shooting method implemented in references,[12,14,25] on the basis of the so-called transition matrix algorithm described in Bryson.[27] Adaptation of this technique to the specificity of (26) is straightforward.

The way the weight factor μ and the penalty multiplier r are varied defines the computation process. At first a guess solution is easily computed for $r = 0$ and $\mu = 0.9$. Next the shooting method is repeatedly used to compute optimal solutions for increasing values of r and decreasing values of μ. When r increases greater, the state constraints come closer to their bounds, while decreasing μ results in both reducing the travel time and saturating the actuator inputs.

5. NUMERICAL SIMULATIONS

The technique developed in the previous sections is applied to optimal motion planning of two quite different manipulatory systems. This new approach is illustrated by various specified motion laws showing the adaptability of the method and its ability to master the dynamic behaviour of the studied systems.

5.1 Four rotational-joint vertical manipulator (Figure 1)

In this section we will compare three different optimal transfers between identical initial and final positions of the robot. End locations of the arm are defined at the end of appendix 1. In order to make the comparison more significant, the weight factor μ has been decreased in each case until the same optimal travel time is reached.

At first, let's examine the optimal motion obtained without accounting for kinematic constraints of any type. The four joint parameters q_1 to q_4 are free. The eight-order Hamiltonian dynamic model which results is autonomous. The relative movement of the arm in its rotating plane is represented in Figure 2. Notice that the fourth link executes a full revolution (exactly 365°) and its speed of rotation (Figure 3) attains very high values which are incompatible with the technological limitations of commonly used reducers in robotics. The terminal link seems to react as a flail or as a whip. Such a motion is unrealistic. This result clearly reveals the necessity of introducing kinematic constraints in the formulation of optimal control problems.

In both constrained examples that follow, the joint parameters q_3, which describes the rotation of the third joint is submitted to a bound condition of type (19) in order to avoid too much bending, and the rotation of the fourth joint, denoted in this case φ_1, obeys a motion law of type (1) different in both cases. Thus, we assume that

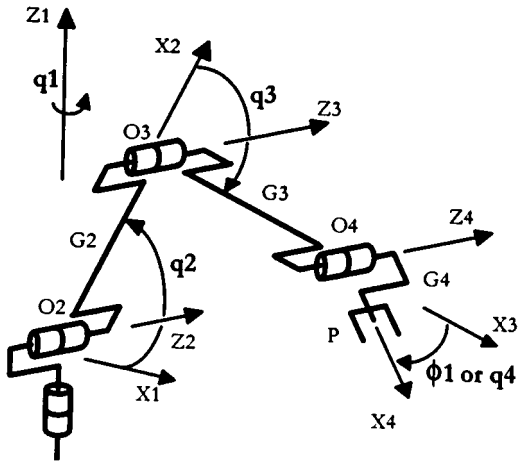


Fig. 1. Four rotational joint vertical robot.

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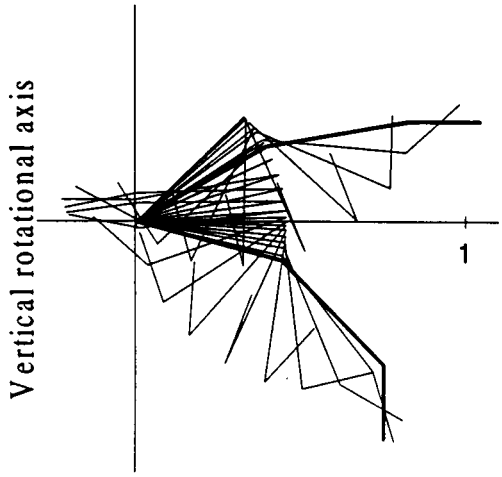


Fig. 2. Optimal motion in the rotating vertical plane.

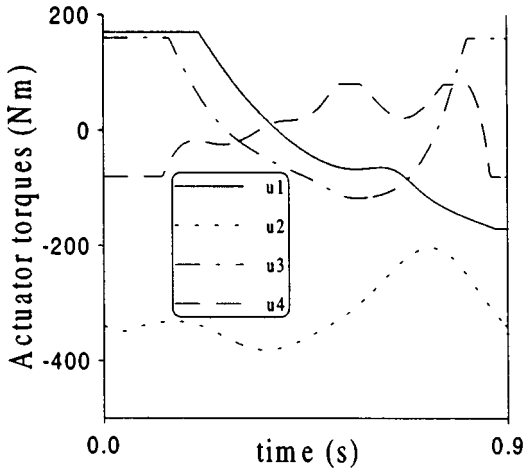


Fig. 4. Evolution of actuator torques.

the motion of the arm is free and has to be optimized, while the effector motion is specified. Consequently, the free configuration of the mechanical system is reduced to three independent parameters q_1, q_2 and q_3.

Example 1: Constraining the relative motion of the fourth link

We are searching for a motion law so that φ_1 can only vary between its initial and final values. Obviously it will be the case for q_1. Therefore, the simplest manner to specify φ_1 consists in linking its variations to those of q_1 by means of the linear function:

$$\phi_1(q) = \frac{\varphi_1^T - \varphi_1^0}{q_1^T - q_1^0}(q_1 - q_1^0) + \varphi_1^0 \qquad (29)$$

where $q = (q_1, q_2, q_3)^T$, $\varphi_1^0 = \varphi_1(0)$, $\varphi_1^T = \varphi_1(T)$.

In this way, we have defined an autonomous motion law in which the general dependency introduced in (1) is reduced to the free configuration q.

The optimal motion is easy to compute. Figure 5 shows that the relative motion of the arm is feasible. We notice that the variations of the joint velocity $\dot{\varphi}_1$ (Figure 6) together with the actuating torque u_4 (Figure 7) that drives the fourth link, are quite moderate.

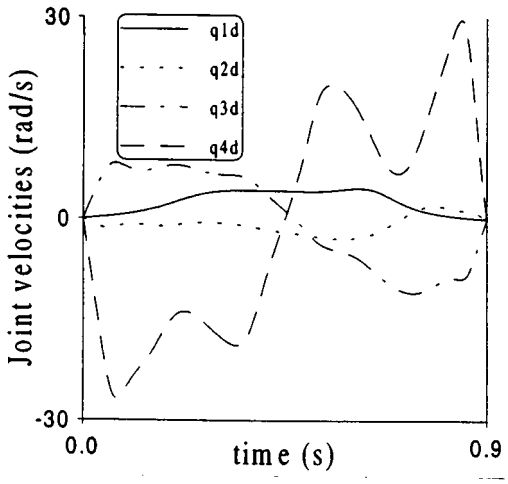


Fig. 3. Evolution of joint velocities.

However the absolute motion of the effector is not fully satisfactory because it moves inward at the beginning of the motion. For practical reasons it would be necessary to avoid this situation. This is done in the next example.

Example 2: Constraining the absolute rotation of the fourth link

Let us define a motion law φ_1 for q_4 so that the effector moves only outward with respect to the vertical axis of the robot.

Let α be the sum:

$$\alpha = q_2 + q_3 + \varphi_1$$

which represents the absolute angular position of the fourth link with respect to the vertical direction. Define α as an explicit function of the running time t under the form of the cubic spline:

$$t \in [0, T], \quad \alpha(t) = -\frac{\pi}{2}\left[1 - \left(3 - 2\frac{t}{T}\right)\frac{t^2}{T^2}\right] \qquad (30)$$

which increases from $-\frac{\pi}{2}$ to 0 and satisfies the end

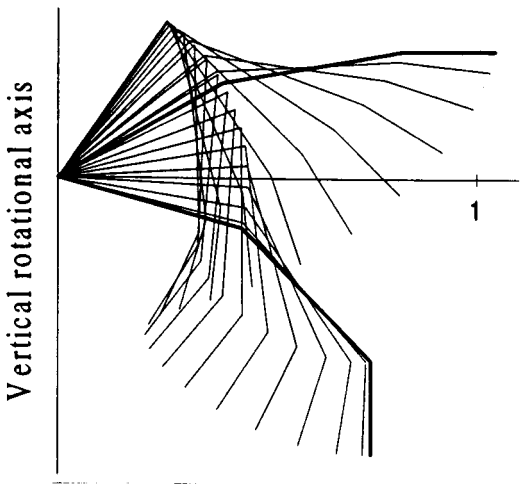


Fig. 5. Optimal motion in the rotating vertical plane.

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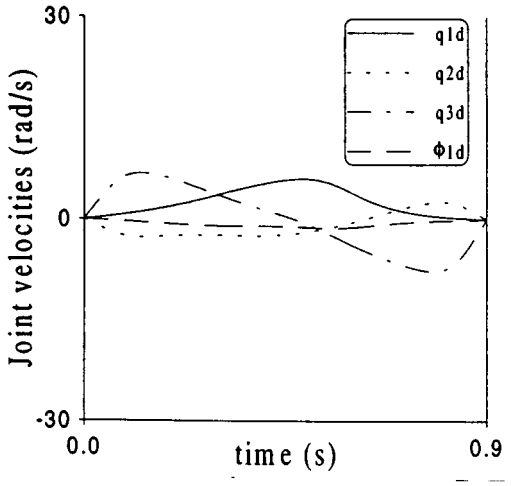


Fig. 6. Evolution of joint velocities.

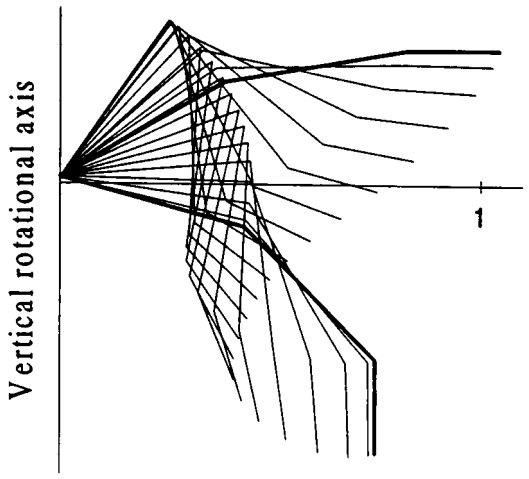


Fig. 8. Optimal motion in the rotating vertical plane.

velocity conditions $\dot{\alpha}(0) = \dot{\alpha}(T) = 0$. Thus, the motion law we choose for φ_1 is:

$$\varphi_1(t) = -q_2(t) - q_3(t) + \alpha(t) \tag{31}$$

or in term of ϕ_1 as defined in (1):

$$\phi_1(q(t), t, T) = -q_2(t) - q_3(t) - \frac{\pi}{2}\left[1 - \left(3 - 2\frac{t}{T}\right)\frac{t^2}{T^2}\right] \tag{32}$$

Observe that this motion law is non-autonomous and explicitly depends on the three arguments q, t and T.

The optimal motion of the arm in its vertical rotating plane is described in Figure 8 which clearly shows the outward motion of the fourth link. As in example 1, joint velocities (Figure 9) have very smooth variations. We can see (Figure 10) that the actuator torques are a little more saturated than in the previous case (Figure 7). Of course, the minimum saturation is obtained in Figure 4 when there is no kinematic constraint.

To conclude, underline the fact that in the absence of kinematic constraints the terminal link of the robot can be submitted to a rough motion which is easily mastered and changed into a smooth movement by means of the technique presented. Furthermore, the reduced-order dynamic model which results is easy to handle, and the optimal control problem is solved faster and the computation time is cut by at least a factor of five.

5.2 Planar redundant manipulator with mobile base

The planar manipulator schematized in Figure 11 is assumed to be driven by a mobile base in order to go back and forth between two distant work stations. We suppose that the motion of the base is controlled separately from the arm's, and must be specified so that the arm can move without surpassing its maximum actuating torques. Thus, this motion, although considered as independent, has to be linked in a certain manner to the arm's. This can be done by means of a constraint of type (2).

We assume that the motion parameter φ_1 of the base must satisfy the end conditions:

$$\varphi_1(0) = \varphi_1^0, \quad \varphi_1(T) = \varphi_1^T; \quad \dot{\varphi}_1(0) = \dot{\varphi}_1(T) = 0$$

Furthermore we want a smooth time derivative for φ_1. The simplest function that fulfils these requirements is a cubic spline of type (30):

$$\varphi_1(t) \equiv \phi_1(t, T) = (\varphi_1^T - \varphi_1^0)(3 - 2t/T)(t/T)^2 + \varphi_1^0 \tag{33}$$

We may recall that the explicit dependency with respect

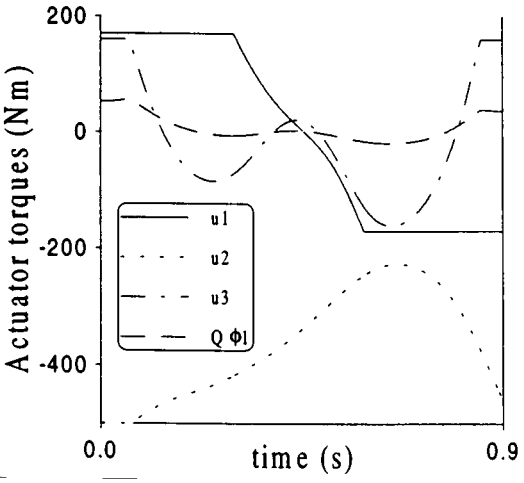


Fig. 7. Evolution of actuator torques.

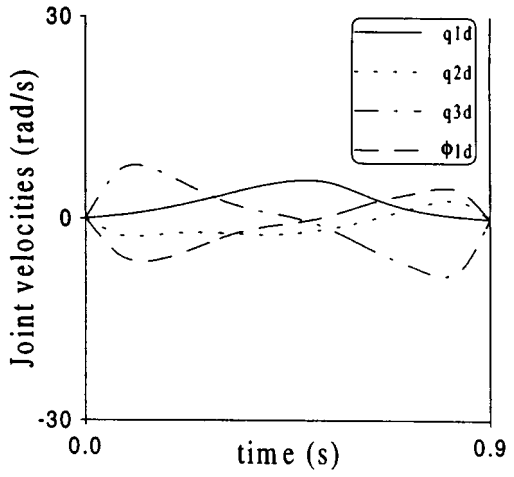


Fig. 9. Evolution of joint velocities.

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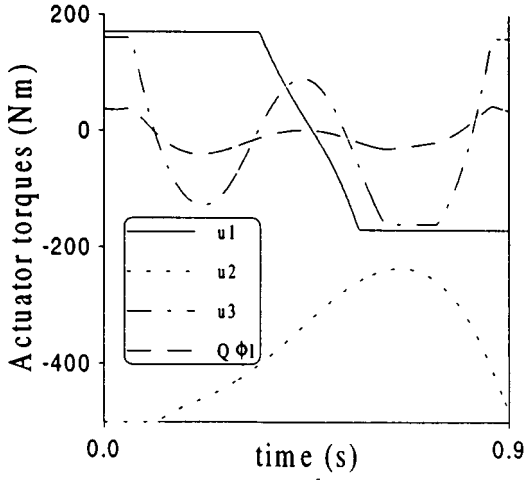


Fig. 10. Evolution of actuator torques.

to the travel time T makes it possible to compute an optimal value of T while accounting for the dynamic effects of the driving motion on the values of the optimal actuating torques which move the arm links.

Moreover, in order to easily control the movement of the terminal link of the arm, we submit its relative rotation to a motion law similar to the one defined in (29). But here, we have chosen a linear combination of the remaining independent parameters q_1 and q_2 so as to avoid too much variation for ϕ_2:

$$\phi_2(q) = \beta[\alpha_1(q_1 - q_1^0)\,\mathrm{sign}\,(q_1^T - q_1^0) + \alpha_2(q_2 - q_2^0)\,\mathrm{sign}\,(q_2^T - q_2^0)] + \varphi_2^0, \qquad (34)$$

expression where α_1 and α_2 are weight factors, and β is the constant coefficient:

$$\beta = (\varphi_2^T - \varphi_2^0)/[\alpha_1 |q_1^T - q_1^0| + \alpha_2 |q_2^T - q_2^0|], \qquad (35)$$

with $\varphi_2^0 = \varphi_2(0)$ and $\varphi_2^T = \varphi_2(T)$. In the same manner $q_i^0 = q_i(0)$, $q_i^T = q_i(T)$, $i = 1,2$.

Add that we could have chosen for φ_2 ($\varphi_2(t) \equiv \phi_2(q(t))$) a motion law of type (33) as well. The later choice simply illustrates the adaptability of the method. A lot of other formulations could be suitable for constraining the φ_i parameters.

Characteristics of the mechanical system and the end conditions of the motion we want to optimize are given in appendix 2. The free parameter q_2 is submitted to a bound condition of type (19). The optimal motion (Figure 12) is quite realistic. Time evolutions of joint velocities (Figure 13) and actuating inputs (Figure 14) have moderate values including those which are relative to fully constrained parameters φ_1 and φ_2.

5.3 Computational burden

The computation time may vary extensively according to (i) the number of independent motion parameters involved in the dynamic model, (ii) the amplitude of motion, (iii) the saturation level of hard constraints such as (18), (19) to which the control and state variables are submitted. Using a guess solution computed on a medium range of amplitude, with intermediate values for the factors μ and r, it can take a few minutes to a few tenths of minutes to achieve the computation with a VAX 4000 DEC computer. Owing to this computation time the technique presented is aimed at off-line planning.

6. CONCLUSION

The optimal motion planning method presented is aimed at mastering the dynamic behaviour of open-loop manipulatory systems. The technique developed consists in partitioning the set of joint parameters into a subset of free parameters and a complementary set of parameters submitted to a specified dependency with respect to free configuration parameters, running time and total transfer time. The non-autonomous reduced-order dynamic problem which results, leads to an optimal transfer problem of a new type owing to the explicit dependency of the state equation of the mechanical system with respect to the travelling time which is an unknown of the stated problem.

Two main situations have been dealt with, as follows:

(i) The base and/or the free end of the manipulator can be submitted to imposed motion laws in order to fulfill special transfer conditions. For instance, motions of the terminal link must satisfy absolute geometric constraints. Moreover, it may be necessary to account for dynamic coupling between the motion of the manipulator and its base when the latter is submitted to a motion which is controlled independently from the arm's.

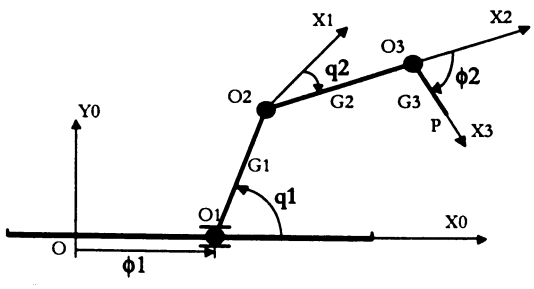


Fig. 11. Redundant planar manipulator with a mobile base.

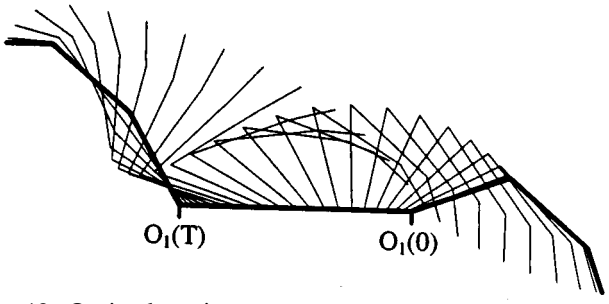


Fig. 12. Optimal motion.

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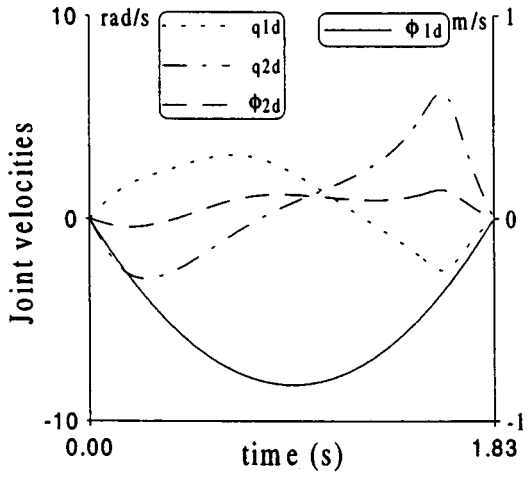


Fig. 13. Evolution of joint velocities.

(ii) Suitable joint motion laws have been introduced in order to have a better control of the dynamic behaviour of the whole mechanical system. Such specified motions can be particularly useful at the wrist level in order to avoid sharp reactions of the distal link. In other respects, this approach is quite suitable for dealing with redundancies of manipulators which have a high number of degrees of freedom.

In both above situations, the reduced-order dynamic model which results is much easier to handle, the numerical computation of optimal solutions is safer and the amount of computing time can be drastically reduced.

It is worth mentioning that since continuous phase constraints are accounted for (section 4, (19)), the method is compatible with obstacle avoidance as dealt with in Danes.[28,29] Finally, it is pointed out that for dealing with systems with six degrees of freedom or more it becomes necessary to develop a special software for symbolically generating all dynamics formulations and optimality conditions. This study is in progress and based on special iterative formulations developed in Bessonnet[25] which generate a complete Hamiltonian dynamic model of open-loop kinematic chains.

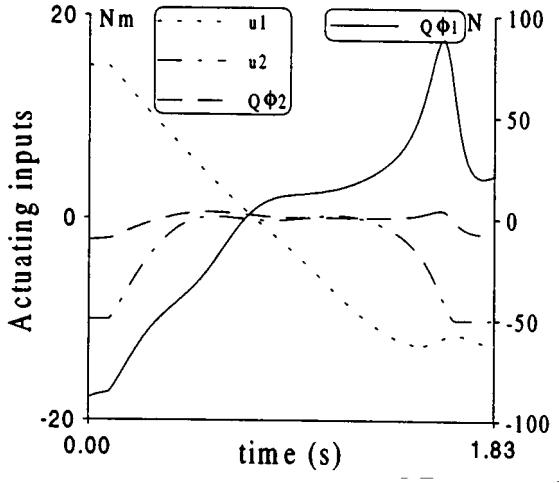


Fig. 14. Evolution of actuating inputs.

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APPENDIX 1:

Link lengths (m):	$O_2O_3 = 0.45 \quad O_3O_4 = 0.44 \quad O_4P = 0.22$
Centre of mass positions (m):	$O_2G_2 = 0.16 \quad O_3G_3 = 0.16 \quad O_4G_4 = 0.14$
Link masses (kg):	$m_2 = 77 \quad m_3 = 36 \quad m_4 = 16$
Link moment of inertia/Z_0 (kg m^2):	$I_{O_2Z_0} = 1.5$
	$I_{O_2X_2} = 0.75 \quad I_{O_2Y_2} = 6 \quad I_{O_2Z_2} = 7$
	$I_{O_3C_3} = 0.15 \quad I_{O_3Y_3} = 3 \quad I_{O_3Z_3} = 3$
	$I_{O_4X_4} = 0.1 \quad I_{O_4Y_4} = 0.45 \quad I_{O_4Z_4} = 0.45$
Damping coefficients (Nms):	$a_1 = 0.73 \quad a_2 = 0.73 \quad a_3 = 1.46 \quad a_4 = 1.46$
(Nms^2)	$b_1 = 0.029 \quad b_2 = 0.029 \quad b_3 = 0.029 \quad b_4 = 0.029$
Gravity acceleration (ms^{-2}):	$g = 9.81$
Bounds for actuator torques (Nm):	$Q_1^{a\,\max} = 170 \quad Q_2^{a\,\max} = 500 \quad Q_3^{a\,\max} = 160 \quad (Q_4^{a\,\max} = 80)$
Bound for phase parameters:	$q_3^{\max} = 135°$
Initial state:	$q_1^0 = 0° \quad q_2^0 = 15° \quad q_3^0 = 30° \quad q_4^0$ or $\varphi_1^0 = 45°$
Final state:	$q_1^T = 135° \quad Q_2^T = -30° \quad q_3^T = 20° \quad q_4^T$ or $\varphi_1^T = 10°$

Initial and final velocities are equal to zero.

APPENDIX 2:

Link lengths (m):	$O_1O_2 = 0.45 \quad O_2O_3 = 0.45 \quad O_3P = 0.2$
Center of mass positions (m):	$O_1G_1 = 0.22 \quad O_2G_2 = 0.22 \quad O_3G_3 = 0.1$
Link masses (kg):	$m_0 = 10 \quad m_1 = 7 \quad m_2 = 6 \quad m_3 = 3$
Link moment of inertia/Z_0 (kg m^2):	$I_{O_1Z_0} = 0.5 \quad I_{O_2Z_0} = 0.4 \quad I_{O_3Z_0} = 0.04$
Damping coefficients (Nms):	$a_0 = 0.23 \quad a_1 = 0.23 \quad a_2 = 0.23 \quad a_3 = 0.23$
(Nms^2)	$b_0 = 0.015 \quad b_1 = 0.015 \quad b_2 = 0.015 \quad b_2 = 0.015 \quad b_3 = 0.015$
Bounds for actuator torques (Nm):	$Q_1^{a\,\max} = 15 \quad Q_2^{a\,\max} = 10$
Bound for phase parameters:	$q_2^{\max} = 135°$
Initial state;	$\varphi_1^0 = 0.5\,\text{m} \quad q_1^0 = 20° \quad q_2^0 = -60° \quad \varphi_2^0 = -30°$
Final state:	$\varphi_1^T = -0.5\,\text{m} \quad q_1^T = 120° \quad q_2^T = 20° \quad \varphi_2^T = 40°$

Initial and final velocities are equal to zero.

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