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Abstract

In this paper, we prove the abundance conjecture for threefolds over a perfect field k of characteristic $p > 3$ in the case where the numerical dimension is equal to 2. More precisely, we show that, if (X, B) is a projective log canonical pair of dimension 3 over k such that $K_X + B$ is nef and $\nu(K_X + B) = 2$, then $K_X + B$ is semi-ample.

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1. Introduction

In birational geometry, the minimal model program (MMP) is a conjectural program to classify all algebraic varieties up to birational equivalence. This program predicts that, after finitely many birational transforms, every variety with mild singularities admits a birational model which is either a minimal model or a Mori fiber space. In characteristic 0, much progress has been made toward the MMP. For example, it has been fully established in dimensions ≤ 3. In higher dimensions, it is proved in [BCH+10] that the minimal models for varieties of general type exist. However, in positive characteristics, due to the failure of vanishing theorems, there has been some major progress in the MMP only after the work of Hacon and Xu (see [HX15]). They proved the existence of minimal models for terminal threefolds over an algebraically closed field k of characteristic $p > 5$. Then Cascini, Tanaka, and Xu proved that an arbitrary terminal threefold over k is birational to either a minimal model or a Mori fiber space [CTX15]. Based on this, Birkar and Waldron established the MMP for Kawamata log terminal (klt) threefolds over k (see [Bir16, BW17]). Moreover, there are some generalizations of it in various directions. For example, see [HNT17, Wal18] for its generalization to log canonical (lc) pairs, [GNT19, HW22a, HW22b]

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for its generalization to low characteristics, [DW22, Wal23] for its generalization to imperfect base fields, and [BMP+23, TY23] for its analog in mixed characteristics.

Now we can run MMP for lc pairs of dimension 3 over a perfect field of characteristic $p>3$ (see Theorem 2.20). Hence a central problem remaining is the following conjecture.

1.1 Abundance conjecture

Let (X,B) be a projective lc pair of dimension 3 over a perfect field k of characteristic $p>3$. If K_X+B is nef, then it is semi-ample.

In characteristic 0, the abundance conjecture for threefolds was proved in [Kaw92, KMM94]. The proof is dependent on the numerical dimension $\nu(K_X+B)$. In characteristic $p>3$, the conjecture has been proved in the cases of $\nu(K_X+B)=0$ and 3. When $\nu(K_X+B)=1$ and 2, it is open even in the terminal case. For details, please see [BW17, DW19, HW22b, Wal17, Wal18, Wit21, Xu24, Zha19, Zha20], for example. In mixed characteristic, the abundance conjecture for arithmetic threefolds whose closed points have residue characteristic >5 was proved in [BBS24]. In this paper, we prove the conjecture in the case of $\nu(K_X+B)=2$.

Theorem 1.1 (Theorem 5.8). *Let (X,B) be a projective lc pair of dimension 3 over a perfect field k of characteristic $p>3$. If K_X+B is nef and $\nu(K_X+B)=2$, then it is semi-ample.*

When k is $\mathbb{C}$, this result is proved in the terminal case in [Kaw92] (see [Kol92, Theorem 14.4.1] for a simplified proof). It is proved in the lc case in [KMM94]. The proofs rely on numerous results on the intersection theory for $\mathbb{Q}$-varieties together with positivity theorems, such as vanishing theorems and generic semi-positivity theorems. These results may fail in positive characteristics.

1.2 Sketch of the proof of Theorem 5.8

For simplicity, we assume that k is an algebraically closed field of characteristic $p>5$ since, in this case, klt singularities of dimension 3 are rational singularities by [ABL22, Corollary 1.3]. By a standard lemma (see Lemma 5.7), we can reduce the assertion to the case when (X,B) is an lc projective pair of dimension 3 and the following assertions hold:

(1) X is $\mathbb{Q}$-factorial, B is reduced, and $X\setminus B$ is terminal;
(2) $K_X+B\sim_{\mathbb{Q}} D$ for an effective $\mathbb{Q}$-divisor D with $\operatorname{Supp} D=B$;
(3) $K_X+(1-\varepsilon)B$ is nef for any sufficiently small $\varepsilon>0$;
(4) $\nu(K_X+B)=2$; and
(5) if C is a curve in X with $C\cdot(K_X+B)>0$, then (X,B) is purely log terminal (plt) at the generic point of C.

In this special case, we can mimic the proof in characteristic 0 (see [Kol92, Theorem 14.4.1]). Let $\rho: V\to X$ be a desingularization of X and m be a positive integer such that $L:=m(K_X+B)$ is Cartier. By the Riemann–Roch theorem, we can prove that

$$\chi(X,\mathcal{O}_X(nL))=\frac{n}{12}(K_V^2+c_2(V))\cdot\rho^*L+\chi(\mathcal{O}_V).$$

Here we use the assumption of the characteristic so that X has rational singularities. Then we meet two cases.

Case I: T_V is strongly (ρ^*L,ρ^*L)-semistable (see Definition 2.30).

Case II: T_V is not strongly (ρ^*L,ρ^*L)-semistable.

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In Case II, we use Langer's results on bend and break by 1-foliations (see Theorem 2.36) to prove that the nef dimension $n(K_X + B) \leq 2$. Then the assertion follows from Theorem 2.28. Case I is much more difficult since we do not have a good intersection theory for singular varieties in positive characteristic. Instead of the intersection theory for $\mathbb{Q}$-sheaves used in characteristic 0, we use the intersection theory developed by Langer [Lan25]. In this case, $\Omega_X^{[1]} := (\Omega_X)^{**}$ is also strongly (L, L)-semistable since ρ is a birational morphism. Then we can apply Langer's Bogomolov inequality for reflexive sheaves (see Theorem 4.8) to prove that

$$L \cdot (K_X^2 + c_2(\Omega_X^{[1]})) \geq 0,$$

where the second Chern class is defined in § 4. Then the most important step of the proof is to show the following key inequality:

$$\rho^* L \cdot (K_V^2 + c_2(V)) \geq L \cdot (K_X^2 + c_2(\Omega_X^{[1]})).$$

To do this, we prove some Bertini-type results in characteristic $p > 3$ (see § 3) and reduce the proof of the key inequality to a problem on surfaces (see Proposition 5.3). Then we need to compute relative Chern classes (see § 4 for definition) of desingularizations of some cyclic quotient surface singularities. We explain how to compute these classes in § 4.3. Therefore, we can prove the key inequality and modify the proof of [Kol92, Theorem 14.4.1] to prove that $\kappa(K_X + B) > 0$, and hence the assertion follows from Theorem 2.28.

1.3 Notation and conventions

We use the following notation and conventions.

- All schemes considered in this article are excellent, in particular Noetherian.
- A singular point of a scheme is a point that is not regular.
- A variety is an integral and separated scheme which is of finite type over a field k.
- We call (X, B) a pair if X is a normal variety and B is an effective $\mathbb{Q}$-divisor on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. For more notions in the theory of MMP, such as klt (dlt, lc, plt) pairs, flips, divisorial contractions, and so on, we refer to [KM98].
- Let X be a normal projective variety over a field k and D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. If $|mD| = \emptyset$ for all $m > 0$, we define the Kodaira dimension $\kappa(X, D) = -\infty$. Otherwise, we let $\Phi : X \dashrightarrow Z$ be the Iitaka map (we refer to [Laz04, 2.1.C]) of D and we define the Kodaira dimension $\kappa(X, D)$ to be the dimension of the image of Φ. Sometimes, we write $\kappa(D)$ for $\kappa(X, D)$. We denote $\kappa(X, K_X)$ by $\kappa(X)$. For a projective variety Y over a field k admitting a smooth model $\widetilde{Y}$, we define $\kappa(Y) := \kappa(\widetilde{Y})$.
- Let X be a normal projective variety of dimension n over a field k and D be a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then we can define

$$\nu(X, D) := \max\{k \in \mathbb{N} \mid D^k \cdot A^{n-k} > 0 \text{ for an ample divisor } A \text{ on } X\}.$$

  Sometimes, we write $\nu(D)$ for $\nu(X, D)$.
- Assume that $f : X \to Y$ is a morphism between normal schemes, $\mathcal{F}$ is a reflexive sheaf on X, and $\mathcal{E}$ is a reflexive sheaf on Y. Then we set

$$f^{[*]}\mathcal{E} := (f^*\mathcal{E})^{**}, \quad f_{[*]}\mathcal{F} := (f_*\mathcal{F})^{**}.$$

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2. Preliminaries

In this section we recall some basic results.

2.1 Nef reduction map

In this subsection, we recall the notion of a nef reduction map.

Definition 2.1. Let X be a normal projective variety defined over an uncountable field and let L be a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. We call a rational map $\varphi : X \dashrightarrow Z$ a *nef reduction map* of L if Z is a normal projective variety and there exist open dense subsets $U \subseteq X$ and $V \subseteq Z$ such that:

(1) $\varphi|_U : U \to V$ is proper and $\varphi_* \mathcal{O}_U = \mathcal{O}_V$;
(2) $L|_F \equiv 0$ for all fibers F of φ over V; and
(3) if $x \in X$ is a very general point and C is a curve passing through it, then $C \cdot L = 0$ if and only if C is contracted by φ.

It is proved that the nef reduction map exists over an uncountable algebraically closed field.

Theorem 2.2 [BCE+02, Theorem 2.1]. *A nef reduction map exists for normal projective varieties defined over an uncountable algebraically closed field. Furthermore, it is unique up to birational equivalence.*

We call $n(X, L) = \dim Z$ the *nef dimension* of L, where Z is the target of a nef reduction map $\varphi : X \dashrightarrow Z$. When the base field is countable and algebraically closed, we can define $n(X, L) := n(X_K, L_K)$ by [Wit21, Proposition 2.16], where K is an uncountable algebraically closed field that contains k, and X_K and L_K are the base changes of X and L to K. Then $\kappa(X, L) \leq n(X, L)$ is satisfied.

2.2 Surface singularities and semi-log canonical surfaces

In this subsection, we recall the notion of the dual graph of a surface singularity and collect some results on lc surface singularities and semi-log canonical (slc) surfaces. The main references for this subsection are [Kol13, Sat25]. First we recall some basic knowledge on the intersection theory for curves over a field, which is not necessarily perfect. Let C be an integral projective scheme over a field k with $\dim C = 1$ (not necessarily regular).

Definition 2.3. We define the *arithmetic genus* $g(C)$ of C by

$$g(C) := \frac{\dim_k(H^1(C, \mathcal{O}_C))}{\dim_k(H^0(C, \mathcal{O}_C))}.$$

For a 0-cycle $D = \sum_{i=1}^n a_i P_i$ on C, we define the *degree* of D over k by

$$\deg_{C/k}(D) := \sum_{i=1}^{n} a_i \dim_k(k(P_i)) \in \mathbb{Z},$$

where $k(P_i)$ are the residue fields of C at P_i. Note that any line bundle $L \in \mathrm{Pic}(C)$ is isomorphic to $\mathcal{O}_C(B)$ for some 0-cycle B on C. We define

$$\deg_{C/k}(L) := \deg_{C/k}(B).$$

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Since rationally equivalent cycles have the same degree, $\deg_{C/k}(L)$ is well-defined. The map $\deg_{C/k}$ gives the homomorphism

$$\deg_{C/k} : \mathrm{Pic}(C) \to \mathbb{Z}.$$

See [Ful98, § 1.4] for details.

Let X be a normal integral scheme with a dualizing complex $\omega_X^\bullet$. Then we can define resolutions of singularities (desingularizations for short), the canonical divisor K_X, and lc (klt, plt) pairs as usual (see [Sat25, § 2] for example).

DEFINITION 2.4. We say that (X, x) (or $(x \in X)$) is a *normal surface singularity* if X is a two-dimensional normal integral scheme with a dualizing complex $\omega_X^\bullet$ and x is a closed point of X.

Remark 2.5. The definition is more general than that in [Sat25, Definition 2.10], as it does not require that X be the spectrum of a local ring. For an effective $\mathbb{Q}$-Weil divisor B on X, we also say that $(x \in X, B)$ is a normal surface singularity.

Let (X, x) be a normal surface singularity, and let $f : Y \to X$ be a proper birational morphism from a normal integral scheme Y with the exceptional divisor $\mathrm{Exc}(f) = \sum_i E_i$ satisfying $f(\mathrm{Exc}(f)) = x$. Note that E_i are defined over the residue field $k(x)$ of X at x.

DEFINITION 2.6. For a Cartier divisor D on Y and an exceptional Weil divisor $Z = \sum_i b_i E_i$ on Y, we define the *intersection number* $D \cdot Z$ by

$$D \cdot Z := \sum_i b_i \deg_{E_i/k(x)}(\mathcal{O}_Y(D)|_{E_i}).$$

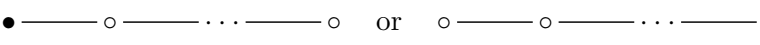
By [Sat25, Definition 2.11 and Remark 2.12], the *minimal resolution* of a normal surface singularity (X, x) can be defined, and it exists uniquely. Now we can give the definition of the dual graph as follows.

DEFINITION 2.7. Let (X, x) be a normal surface singularity, and $f : Y \to X$ be the minimal resolution with the exceptional divisor $\mathrm{Exc}(f) = \sum_{i=1}^n E_i$ satisfying $f(\mathrm{Exc}(f)) = x$. The *dual graph* of (X, x) is a graph whose set of vertices is $\{E_1, \dots, E_n\}$, the number of edges between E_i and E_j is $(E_i \cdot E_j) \in \mathbb{N}$, and the weight at E_i is

$$(\dim_{k(x)} H^0(E_i, \mathcal{O}_{E_i}), g(E_i), E_i^2) \in \mathbb{Z}^3.$$

Remark 2.8. We can define the *extended dual graph* of a normal surface singularity $(x \in X, B)$ similarly (see [Kol13, 2.26] for details). We often omit the weights when they are either unknown or not relevant. When we work with klt surface singularities over algebraically closed fields, the notion of dual graph can be greatly simplified since $\dim_{k(x)} H^0(E_i, \mathcal{O}_{E_i}) = 1$ and $g(E_i) = 0$ (see [KM98, Theorem 4.7] for example).

DEFINITION 2.9 [Kol13, Corollary 3.31]. Let S be a two-dimensional normal integral scheme with a dualizing complex, $s \in S$ a closed point, and B a reduced Weil divisor on S. Assume that $(s \in S, B)$ is lc and let (Γ, B) be its extended dual graph. We denote the residue field of S at s by $k(s)$. We say that $(s \in S, B)$ is a *cyclic quotient singularity* if (Γ, B) is



where the solid bullet stands for the strict transform of B and every circle stands for an exceptional curve over S. Moreover, every exceptional curve is a $\mathbb{P}^1_{k(s)}$, except in the case $B = 0$ with only one exceptional curve, which may instead be a $k(s)$-irreducible conic.

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We use the following result in Corollary 3.8 and § 5.

Proposition 2.10. *Let S be a two-dimensional normal integral scheme with a dualizing complex, $s \in S$ a closed point, and B a reduced Weil divisor on S. Assume that $(s \in S, B)$ is lc and let (Γ, B) be its extended dual graph. If $(s \in S, B)$ is plt and $s \in B \neq 0$, then (Γ, B) is*

$$\bullet \text{———} \circ \text{———} \cdots \text{———} \circ,$$

where the solid bullet stands for the strict transform of B, every circle stands for $\mathbb{P}^1_{k(s)}$, and $k(s)$ is the residue field of S at s. In particular, $(s \in S, B)$ is a cyclic quotient singularity.

Proof. By [Kol13, Corollary 3.31], the pair $(s \in S, B)$ must be one of the following singularities: (1), (2), or (3) in [Kol13, Corollary 3.31], or those in [Kol13, Examples 3.27 and 3.28]. The singularities in (2) and (3) of [Kol13, Corollary 3.31], as well as those in [Kol13, Example 3.27], require that $B = 0$. The singularities in [Kol13, Example 3.28] require that $(s \in S, B)$ is not plt. Hence we are in case (1) of [Kol13, Corollary 3.31]. By Definition 2.9, this means that $(s \in S, B)$ is a cyclic quotient singularity. Moreover, since $B \neq 0$, the assertion holds. □

We recall the following adjunction formula of lc surface singularities.

Theorem 2.11 [Kol13, Theorem 3.36]. *Let S be a two-dimensional normal integral scheme with a dualizing complex, and let $B \subset S$ be a one-dimensional reduced proper scheme over a field such that (S, B) is lc. Then*

$$(K_S + B) \cdot B = \deg \omega_B + \sum_{s \text{ cyclic, plt}} \left(1 - \frac{1}{\det(\Gamma_s)}\right) \deg[s] + \sum_{s \text{ dihedral}} \deg[s],$$

where $\det(\Gamma_s)$ are the determinants of the dual graphs Γ_s of cyclic quotient singularities (S, s).

Remark 2.12. We also call $\det(\Gamma_s)$ the index of (S, s). By [Kol13, Theorem 3.32] for a cyclic quotient singularity (S, s), if the residue field $k(s)$ of S at s is algebraically closed and S is defined over $k(s)$, then the completion of S at s is isomorphic to

$$\widehat{\mathbb{A}}^2_{k(s)}/\mu_m,$$

where μ_m is the group scheme $\operatorname{Spec}(k(s)[\mathbb{Z}_m])$. By [Kol13, Definition 3.33], we have $\det(\Gamma_s) = m$. Moreover, if m is not divisible by the characteristic of $k(s)$, then by [Kol13, 3.19], the action of $\mu_m \cong \mathbb{Z}_m$ on $\widehat{\mathbb{A}}^2_{k(s)}$ is linear and diagonalized.

Now we turn to the notion of slc surfaces.

Definition 2.13. Let X be a scheme. A coherent $\mathcal{O}_X$-module $\mathcal{F}$ is S_i, where i is a positive integer if it satisfies the condition

$$\operatorname{depth}_{\mathcal{O}_{X,x}} \mathcal{F}_x \geq \min\{i, \dim(\operatorname{Supp} \mathcal{F}_x)\}$$

for all $x \in X$. We say that X is S_i if $\mathcal{O}_X$ is S_i.

Proposition 2.14 [Wal18, Proposition 2.6]. *Let X be a reduced equidimensional quasi-projective scheme. Then the locus U where X is S_2 is an open subset with $\operatorname{codim}_X(X \setminus U) \geq 2$, and there exists a birational morphism called S_2-fication $g : X' \to X$ satisfying:*

(1) *X' is S_2 and reduced;*
(2) *g is finite and an isomorphism precisely above U; and*
(3) *the normalization $X^n \to X$ factorizes through g.*

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DEFINITION 2.15. A scheme (or ring) is called *nodal* if its codimension 1 local rings are regular or nodal. It is called *demi-normal* if it is S_2 and nodal.

We recall the following Riemann–Roch theorem for demi-normal surfaces.

THEOREM 2.16. *Let X be a demi-normal surface over an algebraically closed field. Let L be a Cartier divisor on X. Then we have*

$$\chi(\mathcal{O}_X(L)) = \tfrac{1}{2}L \cdot (L - K_X) + \chi(\mathcal{O}_X).$$

Proof. In characteristic 0, this is proved in [LR16, Theorem 3.1]. Assume that X is defined over an algebraically closed field of arbitrary characteristic. Let $\pi : \overline{X} \to X$ be the normalization of X, and let $D \subseteq X$ and $\overline{D} \subseteq \overline{X}$ be conductor subschemes. By the proof of [LR16, Theorem 3.1], it suffices to show that

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{\overline{X}}) + \chi(\mathcal{O}_D) - \chi(\mathcal{O}_{\overline{D}})$$

and

$$\chi(\mathcal{O}_X(L)) = \chi(\pi^*\mathcal{O}_X(L)) + \chi(\mathcal{O}_X(L)_{|_D}) - \chi(\pi^*\mathcal{O}_X(L)_{|_{\overline{D}}}).$$

We consider the exact sequences

$$0 \to \mathcal{O}_{\overline{X}}(-\overline{D}) \to \mathcal{O}_{\overline{X}} \to \mathcal{O}_{\overline{D}} \to 0$$

and

$$0 \to \mathcal{I}_D \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

By the definition of conductor subschemes [Kol13, 5.2], we have $\pi_*\mathcal{O}_{\overline{X}}(-\overline{D}) \cong \mathcal{I}_D$. It follows that

$$\chi(\mathcal{O}_X) - \chi(\mathcal{O}_D) = \chi(\mathcal{I}_D) = \chi(\pi_*\mathcal{O}_{\overline{X}}(-\overline{D})) = \chi(\mathcal{O}_{\overline{X}}) - \chi(\mathcal{O}_{\overline{D}}).$$

Similarly, we have

$$\chi(\mathcal{O}_X(L)) - \chi(\mathcal{O}_X(L)_{|_D}) = \chi(\pi^*\mathcal{O}_X(L)) - \chi(\pi^*\mathcal{O}_X(L)_{|_{\overline{D}}}).$$

Therefore, the assertion holds. □

DEFINITION 2.17 [Kol13, 5.10]. We say that (X, Δ) is a *semi-log canonical (slc)* pair if X is demi-normal, Δ is an effective $\mathbb{Q}$-divisor with no components along D, $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and the normalization $(\overline{X}, \overline{D} + \overline{\Delta})$ is an lc pair, where D and $\overline{D}$ are conductor subschemes, and $\overline{\Delta}$ is the birational transform of Δ.

The abundance for slc surfaces over any field of positive characteristic holds.

THEOREM 2.18 [Pos23, Theorem 1]. *Let (X, Δ) be a projective slc surface over a field of characteristic $p > 0$. If $K_X + \Delta$ is nef, then $K_X + \Delta$ is semi-ample.*

In characteristic 0, the reduced part of the boundary of an lc pair of dimension 3 is slc. It fails in positive characteristic, despite several partial positive results (see [Arv23, AP26, ABP24] for example). However, we have the following analogous result.

LEMMA 2.19. *Let (X, B) be an lc projective pair of dimension 3 over an algebraically closed field k of characteristic $p > 0$. Assume that B is reduced and let $g : B' \to B$ be its S_2-fication. Then the pair $(B', \mathrm{Diff}_{B'}(0))$ defined by the adjunction $K_{B'} + \mathrm{Diff}_{B'}(0) = (K_X + B)|_{B'}$ is an slc pair, where the existence of $\mathrm{Diff}_{B'}(0)$ is given by [Kol13, Definition 4.2].*

Proof. By the classification of lc surface singularities (see [KM98, 4.1] for example), we know B is nodal. Since g is an isomorphism outside some points by Proposition 2.14, we have B' is

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nodal. Therefore, B' is demi-normal. Then by [Kol92, Theorem 17.2] we know $(B', \mathrm{Diff}_{B'}(0))$ is slc. □

2.3 MMP for threefolds in positive characteristic

In this subsection, we recall the theory of MMP for projective lc pairs of dimension 3 over a perfect field of characteristic $p > 3$. Moreover, we define a partial MMP over algebraically closed fields of characteristic $p > 3$ (see Definition 2.23) and we will use this construction to study the abundance in § 5.

Theorem 2.20 ([HNT17, Theorem 1.1] and [HW22b]). *Let (X, B) be an lc pair of dimension 3 over a perfect field k of characteristic $p > 3$, and let $f : X \to Y$ be a projective surjective morphism to a quasi-projective variety. If $K_X + B$ is pseudo-effective (respectively not pseudo-effective) over Y, then we can run a $(K_X + B)$-MMP to get a log minimal model (respectively Mori fiber space) over Y.*

We recall the notion of MMP with scaling. Let (X, B) be a projective lc pair of dimension 3 over a perfect field k of characteristic $p > 3$ and $A \neq 0$ an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Suppose that there is $t_0 > 0$ such that $(X, B + t_0 A)$ is lc and $K_X + B + t_0 A$ is nef. We describe how to run a $(K_X + B)$-MMP with scaling of A as follows.

Let $\lambda_0 = \inf\{t : K_X + B + tA \text{ is nef}\}$. Suppose we can find a $(K_X + B)$-negative extremal ray R_0 which satisfies $(K_X + B + \lambda_0 A) \cdot R_0 = 0$. (In general, it is possible that there is no such extremal ray.) This is the first ray we contract in our MMP. If the contraction is a Mori fiber contraction, we stop. Otherwise, let X_1 be the result of the divisorial contraction or flip. Then $K_{X_1} + B_{X_1} + \lambda_0 A_{X_1}$ is also nef, where B_{X_1} and A_{X_1} denote the birational transforms on X_1 of B and A, respectively. We define $\lambda_1 = \inf\{t : K_{X_1} + B_{X_1} + tA_{X_1} \text{ is nef}\}$. The next step in our MMP is chosen to be a $(K_{X_1} + B_{X_1})$-negative extremal ray R_1 which is $(K_{X_1} + B_{X_1} + \lambda_1 A_{X_1})$-trivial. So long as we can always find the appropriate extremal rays, contractions, and flips, we can continue this process.

Proposition 2.21. *Let (X, B) be a $\mathbb{Q}$-factorial projective lc pair of dimension 3 over an algebraically closed field k of characteristic $p > 3$, and let W be an effective $\mathbb{Q}$-divisor such that $K_X + B + W$ is nef. Then either:*

(1) *there is a $(K_X + B)$-negative extremal ray which is $(K_X + B + W)$-trivial; or*
(2) *$K_X + B + (1 - \varepsilon)W$ is nef for any sufficiently small rational $\varepsilon > 0$.*

Proof. This is an adaptation of [KMM94, Lemma 5.1]. Note that the proof there only uses the fact that for any $(K_X + B)$-negative extremal ray R there is a rational curve C such that C generates R and $-(K_X + B) \cdot C \leq 6$, which holds in our setting by [HNT17, Theorem 1.3] and [HW22b] since k is an algebraically closed field of characteristic $p > 3$. □

Corollary 2.22. *Let (X, B) be a $\mathbb{Q}$-factorial projective lc pair of dimension 3 over an algebraically closed field k of characteristic $p > 3$ and A be an effective $\mathbb{Q}$-divisor such that $(X, B + A)$ is lc and $K_X + B + A$ is nef. If $K_X + B$ is not nef, then we can run a $(K_X + B)$-MMP with scaling of A.*

Proof. Let $\lambda := \inf\{t : K_X + B + tA \text{ is nef}\}$. It suffices to show that we can find a $(K_X + B)$-negative extremal ray R such that $(K_X + B + \lambda A) \cdot R = 0$. We apply Proposition 2.21 by letting $W := \lambda A$. □

In this paper, we use the following construction.

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DEFINITION 2.23. Let (X, B) be a $\mathbb{Q}$-factorial projective lc pair of dimension 3 over an algebraically closed field k of characteristic $p > 3$, and let A be an effective $\mathbb{Q}$-divisor such that $(X, B + A)$ is lc and $K_X + B + A$ is nef. We can run a partial $(K_X + B)$-MMP with scaling of A as follows.

Let $\lambda_0 = \inf\{t : K_X + B + tA \text{ is nef}\}$. If $\lambda_0 < 1$, then we stop. Otherwise, by Proposition 2.21 there exists a $(K_X + B)$-negative extremal ray R_0 which satisfies $(K_X + B + A) \cdot R_0 = 0$. We contract this extremal ray. If the contraction is a Mori fiber contraction, we stop. Otherwise, let $\mu_0 : X \dashrightarrow X_1$ be the divisorial contraction or flip. Repeat this process for $(X_1, \mu_{0*}B)$, $\mu_{0*}A$, and so on.

We call this construction a *$(K_X + B)$-MMP which is $(K_X + B + A)$-trivial*.

The following lemma tells us what the output of this construction is if it terminates.

LEMMA 2.24. *Let (X, B) be a $\mathbb{Q}$-factorial projective lc pair of dimension 3 over an algebraically closed field k of characteristic $p > 3$, and let A be an effective $\mathbb{Q}$-divisor such that $(X, B + A)$ is lc and $K_X + B + A$ is nef.*

If a $(K_X + B)$-MMP which is $(K_X + B + A)$-trivial terminates, then its output is a $\mathbb{Q}$-factorial projective lc pair $(X', B' + A')$, and either:

(1) *X' is a Mori fiber space over a variety Y, $K_{X'} + B' + A'$ is the pullback of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor from Y, and* Supp A' *dominates Y; or*

(2) *$K_{X'} + B' + (1 - \varepsilon)A'$ is nef for any sufficiently small rational $\varepsilon > 0$.*

Moreover, $K_{X'} + B' + A'$ is semi-ample if and only if $K_X + B + A$ is.

Proof. By Definition 2.23, if a $(K_X + B)$-MMP which is $(K_X + B + A)$-trivial does not terminate with a Mori fiber space, then we get a $\mathbb{Q}$-factorial projective lc pair $(X', B' + A')$ such that

$$\lambda := \inf\{t : K_{X'} + B' + tA' \text{ is nef}\} < 1.$$

That is to say $K_{X'} + B' + (1 - \varepsilon)A'$ is nef for any sufficiently small rational $\varepsilon > 0$. We are in case (2) above. If the $(K_X + B)$-MMP which is $(K_X + B + A)$-trivial terminates with a Mori fiber space $f : (X', B' + A') \to Y$, then, by [HNT17, Theorem 1.4] and [HW22b], $K_{X'} + B' + A'$ is the pullback of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor from Y since f contracts a $(K_{X'} + B' + A')$-trivial extremal ray. Moreover, Supp A' dominates Y since f only contracts curves which have positive intersections with A'. We are in case (1). The last assertion holds since $K_{X'} + B' + A'$ and $K_X + B + A$ coincide after being pulled back to a common resolution of X' and X. □

We use the following results on termination of flips.

THEOREM 2.25 ([Wal18, Theorem 1.6] and [HW22b]). *Let (X, B) be a projective lc pair of dimension 3 over a perfect field k of characteristic $p > 3$. If M is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, then any sequence of $(K_X + B)$-flips which are also M-flips terminates.*

LEMMA 2.26. *Let (X, B) be a $\mathbb{Q}$-factorial projective lc pair of dimension 3 over an algebraically closed field k of characteristic $p > 3$ such that $K_X + B$ is nef. If X is terminal, then any K_X-MMP which is $(K_X + B)$-trivial terminates.*

Proof of Lemma 2.26. Since every step of a K_X-MMP which is $(K_X + B)$-trivial is a step of a K_X-MMP, the assertion follows from [KM98, Theorem 6.17]. □

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2.4 Divisorial log terminal modifications

The following result helps us to reduce some problems for lc pairs to $\mathbb{Q}$-factorial divisorial log terminal (dlt) pairs.

Theorem 2.27. *Let (X, B) be an lc pair of dimension 3 over a perfect field k of characteristic $p > 3$. Then (X, B) has a crepant $\mathbb{Q}$-factorial dlt model. Moreover, we can modify X so that it is terminal.*

Proof. The first assertion is proven in the case when $p > 5$ in [Bir16, Theorem 1.6], and the proof there holds in the case when $p = 5$ by [HW22b]. Let us prove that we can make X terminal. We take a crepant $\mathbb{Q}$-factorial dlt model $g : (X', B') \to (X, B)$ by the first assertion. Hence, by replacing (X, B) by (X', B'), we may assume that (X, B) is $\mathbb{Q}$-factorial and dlt. Let $U \subseteq X$ be the largest open set such that $(U, B|_U)$ is a simple normal crossing (snc) pair. Then $\operatorname{codim}_X(X \setminus U) \geq 2$. Note that $(X, 0)$ is klt. By [Bir16, Theorem 1.7] we take a terminal model $f : (X', \Theta') \to (X, 0)$ of $(X, 0)$ such that $K_{X'} + \Theta' = f^*K_X$. Then f is an isomorphism over the smooth locus of X; in particular, f is an isomorphism over U. Let $Z = X \setminus U$. Define $B' := \Theta' + f^*B$ on X' so that

$$K_{X'} + B' = f^*(K_X + B)$$

and (X', B') is lc.

It remains to show that (X', B') is a dlt pair. Let $U' = f^{-1}(U)$ and $Z' = X' \setminus U'$. Then $(U', B'|_{U'})$ is a snc pair. If E is an exceptional divisor with center in Z', then its center in X is contained in Z. Hence $a(E; X', B') = a(E; X, B) > -1$. This completes the proof. □

2.5 Some known results on the abundance

The following theorem collects recent results toward the abundance conjecture in positive characteristics. For condition (1) see [Wal17, Theorem 1.3], [Zha19, Theorem 3.1], [DW19, Theorem A], and [Xu24, Theorem 5.1]. For (2) see [Wit21, Theorem 5] and [Xu24, Theorem 6.2]. For (3) see [Zha20, Theorem 1.1], [Wit21, Corollary 4.13], and [Xu24, Theorem 6.3].

Theorem 2.28. *Let (X, B) be a projective lc pair of dimension 3 over an algebraically closed field k of characteristic $p > 3$ such that $K_X + B$ is nef. Assume that one of the following conditions holds:*

(1) *$\kappa(X, K_X + B) \geq 1$;*
(2) *the nef dimension $n(X, K_X + B) \leq 2$; or*
(3) *the Albanese map $a_X : X \to \operatorname{Alb}(X)$ is non-trivial.*

Then $K_X + B$ is semi-ample.

Moreover, the non-vanishing theorem for threefolds in characteristic $p > 3$ has been proved. See [XZ19, Theorem 1.1], [Wit21, Theorem 3], and [Xu24, Theorem 4.4].

Theorem 2.29. *Let (X, B) be a projective lc pair of dimension 3 over a perfect field k of characteristic $p > 3$. If $K_X + B$ is pseudo-effective, then $\kappa(K_X + B) \geq 0$.*

2.6 Slope stability in positive characteristic

In this subsection, we recall some results on slope stability in positive characteristic. Let k be an algebraically closed field of characteristic $p > 0$.

Definition 2.30. Let X be a smooth projective variety of dimension $n \geq 2$ over k and let $D_1, \ldots, D_{n-1}$ be nef divisors on X such that the 1-cycle $D_1 \cdots D_{n-1}$ is not numerically trivial.

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Let $\mathcal{E}$ be a torsion-free coherent sheaf on X. The *slope* of $\mathcal{E}$ with respect to $(D_1, \dots, D_{n-1})$ is defined by

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot D_1 \cdots D_{n-1}}{\operatorname{rk}(\mathcal{E})}.$$

We say that $\mathcal{E}$ is $(D_1, \dots, D_{n-1})$-*semistable* (respectively $(D_1, \dots, D_{n-1})$-*stable*) if for every non-trivial subsheaf $\mathcal{E}' \subseteq \mathcal{E}$ we have

$$\mu(\mathcal{E}') \leq \mu(\mathcal{E}) \quad (\text{respectively} \mu(\mathcal{E}') < \mu(\mathcal{E})).$$

We say that $\mathcal{E}$ is *strongly* $(D_1, \dots, D_{n-1})$-*semistable* (respectively *strongly* $(D_1, \dots, D_{n-1})$-*stable*) if for every integer $e \geq 0$ the pullback $F^{e*}\mathcal{E}$ is $(D_1, \dots, D_{n-1})$-semistable (respectively $(D_1, \dots, D_{n-1})$-stable), where F is the Frobenius morphism.

The following theorem is an analogy of the Bogomolov inequality in positive characteristic.

THEOREM 2.31 [Lan04, Theorem 3.2]. *Let X be a smooth projective variety of dimension $n \geq 2$ over k and let $D_1, \dots, D_{n-1}$ be nef divisors on X such that the 1-cycle $D_1 \cdots D_{n-1}$ is not numerically trivial. Let $\mathcal{E}$ be a strongly $(D_1, \dots, D_{n-1})$-semistable torsion-free sheaf on X. Then*

$$\Delta(\mathcal{E}) \cdot D_2 \cdots D_{n-1} \geq 0,$$

where $\Delta(\mathcal{E}) := 2 \operatorname{rk}(\mathcal{E}) c_2(\mathcal{E}) - (\operatorname{rk}(\mathcal{E}) - 1) c_1(\mathcal{E})^2$.

We will use the following result to prove our main result (see Proposition 5.6).

THEOREM 2.32. *Let X be a smooth projective variety of dimension $n \geq 2$ over k and let $D_1, \dots, D_{n-1}$ be nef divisors on X such that the 1-cycle $D_1 \cdots D_{n-1}$ is not numerically trivial. If $c_1(T_X) \cdot D_1 \cdots D_{n-1} = 0$, then T_X is strongly $(D_1, \dots, D_{n-1})$-semistable if and only if it is $(D_1, \dots, D_{n-1})$-semistable. Moreover, if $p \geq (n-1)(n-2)$ and T_X is not strongly $(D_1, \dots, D_{n-1})$-semistable, then the maximal $(D_1, \dots, D_{n-1})$-destabilizing subsheaf of T_X is a 1-foliation.*

Proof. The former assertion follows directly from [MR83, Theorem 2.1], once we notice that $\mu_{\min}(T_X) = \mu(T_X) = 0$ when T_X is $(D_1, \dots, D_{n-1})$-semistable. The latter assertion follows from the former assertion and [Lan15, Theorem 3.2]. □

2.7 Properties of 1-foliations

We recall the notion of 1-foliations.

DEFINITION 2.33. Let X be a smooth variety over an algebraically closed field k of characteristic $p > 0$. A 1-*foliation* on X is a saturated subsheaf $\mathcal{F} \subseteq T_X$ which is involutive (i.e., $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$) and p-closed (i.e., $\xi^p \in \mathcal{F},\ \forall\, \xi \in \mathcal{F}$). We say that $\mathcal{F}$ is *smooth* if $\mathcal{F}$ and $T_X/\mathcal{F}$ are locally free.

The basic properties of 1-foliations are as follows.

PROPOSITION 2.34 [Eke87]. *Let X be a smooth variety over an algebraically closed field k of characteristic $p > 0$ and let $\mathcal{F}$ be a 1-foliation on X. Then the following assertions hold.*

(1) *The variety $Y := X/\mathcal{F} = \operatorname{Spec}_X \operatorname{Ann}(\mathcal{F}) := \{a \in \mathcal{O}_X \mid \xi(a) = 0,\ \forall\, \xi \in \mathcal{F}\}$ is a normal variety, and we have the commutative diagram*

$$\begin{array}{ccc} X & & \\ {\scriptstyle \pi}\downarrow & \searrow^{F} & \\ Y & \longrightarrow & X^{(1)} \end{array}$$

where F is the Frobenius morphism, and π is a morphism with $\deg \pi = p^{\operatorname{rk} \mathcal{F}}$.

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(2) *There is a one-to-one correspondence between* 1*-foliations and normal varieties between* X *and* $X^{(1)}$*. The correspondence is given by* $\mathcal{F} \mapsto X/\mathcal{F}$ *and the inverse correspondence by* $Y \mapsto \mathrm{Ann}(\mathcal{O}_Y) := \{\xi \in T_X \mid \xi(a) = 0,\ \forall\, a \in \mathcal{O}_Y\}$.

(3) *The variety* Y *is regular if and only if* $\mathcal{F}$ *is smooth.*

(4) *If* Y_0 *denotes the regular locus of* Y *and* $X_0 = \pi^{-1}(Y_0)$*, then*

$$K_{X_0} \sim \pi^* K_{Y_0} + (p-1)\det \mathcal{F}|_{X_0}.$$

The following criterion helps us to construct 1-foliations.

Lemma 2.35 [Eke87, Lemma 4.2]. *Let* X *be a smooth variety over an algebraically closed field* k *of characteristic* $p > 0$ *and* $\mathcal{F}$ *be a saturated* $\mathcal{O}_X$*-submodule of* T_X*. If*

$$\mathrm{Hom}(\bigwedge^2 \mathcal{F}, T_X/\mathcal{F}) = \mathrm{Hom}(F^*\mathcal{F}, T_X/\mathcal{F}) = 0,$$

then $\mathcal{F}$ *is a* 1*-foliation.*

The following theorem tells us how to do 'bend and break' by 1-foliations.

Theorem 2.36 [Lan15, Theorem 2.1]. *Let* X *be a smooth variety over an algebraically closed field* k *of characteristic* $p > 0$ *and* L *be a nef* $\mathbb{R}$*-divisor on* X*. Let* $f : C \to X$ *be a non-constant morphism from a smooth projective curve* C*. Assume that* $\mathcal{F} \subseteq T_X$ *is a* 1*-foliation and smooth along* $f(C)$*. If*

$$c_1(\mathcal{F}) \cdot C > \frac{K_X \cdot C}{p-1},$$

then for every $x \in f(C)$ *there is a rational curve* $B_x \subseteq X$ *passing through* x *such that*

$$L \cdot B_x \leq 2 \dim X \frac{pL \cdot C}{(p-1)c_1(\mathcal{F}) \cdot C - K_X \cdot C}.$$

3. Bertini-type theorems in positive characteristic

In this section, we give some Bertini-type results in positive characteristic. We use these results in § 5. It is well-known that the Bertini theorem fails for basepoint-free linear systems in positive characteristic. However, we have the following special Bertini-type result.

Proposition 3.1 [Lui98, Corollary 4.3 and Proposition 2.4 (1)]. *Let* $f : X \to \mathbb{P}^n$ *be a finite-type morphism from a smooth variety over any infinite field* k*. Assume that* f *induces separable residue field extensions, i.e., for any point* $x \in X$*, the residue field of* $\mathbb{P}^n$ *at* $y := f(x)$ *is a separable field extension of the residue field of* X *at* x*. Then a general section of* $|f^*\mathcal{O}_{\mathbb{P}^n}(1)|$ *is smooth.*

Remark 3.2. Note that we do not require f to be dominant, nor $x \in X$ to be a closed point. The assumption that f induces separable field extensions is a very strong condition. For instance, even the projection $g : \mathbb{A}^2_{u,v} \to \mathbb{A}^1_u$ does not induce separable residue field extensions: if we take x to be the generic point of the closed subvariety $\mathrm{Spec}(k[u,v]/(u - v^p))$, then the induced residue field extension $k(u) \to k(u^{1/p})$ is not separable. We apply this proposition in the case where f is a quasi-finite map (see Theorem 3.6 and Lemma 4.10).

Next, we follow the strategy in [Sat25] to prove some Bertini-type results in low dimensions.

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DEFINITION 3.3 [Sat25, Definition 4.1]. Let (X, x) be a normal surface singularity, and let $f : Y \to X$ be a proper birational morphism from an integral scheme Y with $\mathrm{Exc}(f) = \sum_{i=1}^{n} E_i$ such that $f(\mathrm{Exc}(f)) = x$. We say that f satisfies the $(*)$-*condition* if:

(1) E_i is a smooth curve over the residue field $k(x)$ for every i; and
(2) the scheme theoretic intersection $E_i \cap E_j$ is smooth over $k(x)$ for every $i \neq j$.

LEMMA 3.4. *Let (X, x) be a normal surface singularity, and let $f : Y \to X$ be the minimal resolution of (X, x) with $\mathrm{Exc}(f) = \sum_{i=1}^{n} E_i$ such that $f(\mathrm{Exc}(f)) = x$. Assume that the following conditions hold:*

(1) *the characteristic of the residue field $k(x)$ is greater than 3;*
(2) *(X, x) is a klt singularity; and*
(3) *E_i is geometrically irreducible over $k(x)$ for every i.*

Then f satisfies the $(*)$-condition.

Proof. Note that (X, x) is an lc and rational singularity (see [Sat25, Theorem A.3] for example). Then the assertion follows from the proof of [Sat25, Proposition 4.4]. □

LEMMA 3.5. *Let X be a normal klt surface over a field k of characteristic $p > 3$, and let x be a singular point of X. Let $f : Y \to X$ be the minimal resolution of X. We write E_i for the exceptional prime divisors of f lying over x. Assume that the following conditions hold:*

(1) *X is smooth over k outside the singular points;*
(2) *the residue field $k(x)$ is separable over k; and*
(3) *E_i is geometrically irreducible over $k(x)$ for every i.*

Then the surface $X_{\overline{k}} := X \times_k \overline{k}$ is klt at $\widetilde{x}$, where $\widetilde{x}$ is any point in the preimage of x along the base change. Moreover, the dual graph of $(X_{\overline{k}}, \widetilde{x})$ is the same as the dual graph of (X, x).

Proof. We may assume that x is the unique singular point of X since X is normal. By condition (3) and Lemma 3.4, we have that f satisfies the $(*)$-condition. It follows that E_i is smooth over $k(x)$ for every i. By (2), we know that the residue field of E_i at one of its closed points is separable over k for every i. Together with (1), this implies that Y is smooth over k by [Sta26, 00TV] since Y is regular. Hence we have a commutative diagram

$$\begin{array}{ccccc} Y_{\overline{k}} & \xrightarrow{f'} & X_{\overline{k}} & \longrightarrow & \mathrm{Spec}(\overline{k}) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} & & \downarrow \\ Y & \xrightarrow{f} & X & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

such that the following conditions hold:

(1) $Y_{\overline{k}} = Y \times_k \overline{k}$ and f' is the base change of f along φ;
(2) f' is a desingularization; and
(3) $K_{Y_{\overline{k}}/X_{\overline{k}}} \sim_{\mathbb{Q}} \psi^* K_{Y/X}$;

where (2) follows from the smoothness of Y, and (3) follows by [Sta26, Lemma 0AWD].

Since f is the minimal resolution of X, $K_{Y/X}$ is nef. Hence by (3), $K_{Y_{\overline{k}}/X_{\overline{k}}}$ is nef. Together with (2), this implies that f' is the minimal resolution of $X_{\overline{k}}$. We write $\widetilde{x}$ for a point in the preimage of x along φ and let $F_i \subseteq \psi^{-1}(E_i)$ be the exceptional prime divisors over $\widetilde{x}$. Then by

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Definition 2.7 the dual graph of $(X_{\overline{k}}, \widetilde{x})$ is determined by $\dim_{k(\widetilde{x})} H^0(F_i, \mathcal{O}_{F_i})$, $g(F_i)$, and $F_i \cdot F_j$. Since E_i are smooth over $k(x)$, the coefficients of $\psi^* E_i$ at F_i are 1. Since the base change

$$\mathrm{Spec}(k(\widetilde{x})) = \mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k(x))$$

is flat and

$$F_i = E_i \times_{k(x)} k(\widetilde{x}),$$

we have

$$\dim_{k(\widetilde{x})} H^0(F_i, \mathcal{O}_{F_i}) = \dim_{k(x)} H^0(E_i, \mathcal{O}_{E_i})$$

and, by Definition 2.3,

$$g(F_i) = \frac{\dim_{k(\widetilde{x})} H^1(F_i, \mathcal{O}_{F_i})}{\dim_{k(\widetilde{x})} H^0(F_i, \mathcal{O}_{F_i})} = \frac{\dim_{k(x)} H^1(E_i, \mathcal{O}_{E_i})}{\dim_{k(x)} H^0(E_i, \mathcal{O}_{E_i})} = g(E_i).$$

Moreover, by Definition 2.6,

$$F_i \cdot F_j = F_i \cdot \psi^* E_j = \deg_{F_i/k(\widetilde{x})}(\psi^* E_j) = \deg_{E_i/k(x)}(E_j) = E_i \cdot E_j.$$

Therefore, the assertion holds. □

The following theorem is a Bertini-type result which tells us the singularities that general hyperplane sections of klt quasi-projective threefolds have in characteristic $p > 3$. See also [Sat25, ST20].

Theorem 3.6. *Let X be a klt quasi-projective threefold over an algebraically closed field k of characteristic $p > 3$. Let C be a curve contained in the singular locus of X, and let H be a general hyperplane section of X. Then, for a closed point $x \in C \cap H$, we have that (H, x) is klt. Moreover, there exists a finite cover $g : V \to X$ such that g is étale over general points in C, and the dual graph of (H, x) is the same as the dual graph of (V_γ, γ), where γ is the generic point of a curve $\Gamma \subseteq V$ lying over C, and V_γ is the localization of V at γ.*

Proof. Note that, once we have a finite étale cover V_U of an open subset $U \subseteq X$ containing the generic point of C, we may take V to be the normalization of X in the functional field $k(V_U)$. Hence the problem is local, and we can always shrink X around the generic point of C if necessary. We assume that X is affine and the singular locus of X is C. Let δ be the generic point of C, and let $f : Y_\delta \to X_\delta$ be the minimal resolution of (X_δ, δ), where $\mathrm{Exc}(f) = \sum_i F_i$ and X_δ is the localization of X at δ. Let $\pi : X \to T$ be a projection induced by a general pencil of hyperplane sections, where T is an open set of $\mathbb{P}^1$. Then, C dominates T, and $\delta = \mathrm{Spec}(k(C))$ is mapped to $\eta := \mathrm{Spec}(k(T))$. Let $L_1/k(C)$ be a finite separable field extension such that every irreducible component of $F_i \times_{k(C)} L_1$ is geometrically irreducible over L_1 for every i. After shrinking T, the normalization $\varphi : T' \to T$ of T in L_1 is an étale map. Let

$$\pi' : V \to T'$$

be the base change of π along φ. Let $\Gamma \subseteq V$ be a curve lying over C, and let γ denote the generic point of Γ. Then we have that $\mathrm{Spec}(k(\Gamma))$ is a point in

$$\mathrm{Spec}(k(C) \otimes_{k(T)} L_1).$$

Since $k(C) \otimes_{k(T)} L_1$ is a vector space over L_1, it follows that

$$L_1 \subseteq k(\Gamma).$$

Note that V is finite over X. Hence up to replacing X by V, T by T', π by π', C by Γ, and δ by γ, we can assume that every exceptional prime divisor F_i of the minimal resolution f is

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geometrically irreducible over $k(C)$. Moreover, to prove the assertion, it suffices to show that the dual graph of (H, x) is the same as the dual graph of (X_δ, δ).

Let X_η be the generic fiber of π. By [Sat25, Theorem 5.2], X_η is geometrically klt over $k(T)$. Note that δ is a closed point in X_η. Moreover, the natural map

$$X_\delta \to X_\eta$$

induced by the universal property of X_η is isomorphic at δ. This implies that the dual graph of (X_δ, δ) is the same as the dual graph of (X_η, δ). Since $X \setminus C$ is smooth over k, by Proposition 3.1 general fibers of $\pi : X \setminus C \to T$ are smooth over k. By [Lui98, Proposition 2.4 (1)] it follows that $X_\eta \setminus \delta$ is smooth over $k(T)$. Since π is induced by a general pencil of hyperplane sections and an étale base change, the finite field extension $k(C)/k(T)$ is separable. Moreover, every exceptional prime divisor F_i of the minimal resolution f is geometrically irreducible over $k(C)$ by our assumption. Hence by Lemma 3.5 the dual graph of (X_η, δ) is the same as the dual graph of $(X_{\overline{\eta}}, \widetilde{\delta})$, where $X_{\overline{\eta}} := X_\eta \times_{k(T)} \overline{k(T)}$ and $\widetilde{\delta}$ is any point in the preimage of δ along the base change. This implies that the dual graph of (X_δ, δ) is the same as the dual graph of $(X_{\overline{\eta}}, \widetilde{\delta})$.

We prove that the dual graph of $(X_{\overline{\eta}}, \widetilde{\delta})$ is the same as the dual graph of (H, x). After shrinking $X_{\overline{\eta}}$, we can assume that $\widetilde{\delta}$ is the unique closed point in $X_{\overline{\eta}}$ lying over δ. Let $\rho_{\overline{\eta}} : Y_{\overline{\eta}} \to X_{\overline{\eta}}$ be the minimal resolution of $X_{\overline{\eta}}$. Then there exist a finite field extension L_2 of $k(T)$ and a Cartesian diagram

$$\begin{array}{ccc} Y_{\overline{\eta}} & \xrightarrow{\rho_{\overline{\eta}}} & X_{\overline{\eta}} \\ \downarrow & & \downarrow \\ Y_{L_2} & \xrightarrow{\rho_{L_2}} & X_{L_2} := X_\eta \times_{k(T)} L_2 \end{array}$$

where ρ_{L_2} is the minimal resolution of X_{L_2} and horizontal arrows are base changes of f along the base field extensions. Moreover, by Lemma 3.4 we can assume that every stratum of

$$\mathrm{Exc}(\rho_{L_2}) = \sum_{i=1}^{n} E_{i,L_2}$$

is smooth over L. Then by Definition 2.7 the dual graph of $(X_{\overline{\eta}}, \widetilde{\delta})$ is determined by $E_{i,L_2} \cdot E_{j,L_2}$. We replace $k(T)$ by L_2, T by its normalization in L_2, and X by its base change along the normalization. Since H is a fiber of π, (H, x) does not change under this replacement. Then by taking integral models there exists the commutative diagram

$$\begin{array}{ccc} Y_\eta & \xrightarrow{\rho_\eta} & X_\eta \\ \downarrow & & \downarrow \\ Y_{U_1} & \xrightarrow{\rho_{U_1}} & X_{U_1} \end{array}$$

where $U_1 \subseteq T$ is an open and dense subset, $X_{U_1} := \pi^{-1}(U_1)$, Y_{U_1} is a quasi-projective threefold such that ρ_{U_1} is a projective morphism, and vertical arrows are natural injections. After shrinking X and T, we can assume that $U_1 = T$ and $Y := Y_T$ is smooth over T, since Y is smooth over η. Moreover, by the following claim, we can assume that $\rho_t : Y_t \to X_t$ is a minimal resolution for every $t \in T$, where Y_t and X_t are fibers of Y and X over t.

CLAIM 3.7. *There exists an open and dense subset $U_2 \subseteq T$ such that $K_{(\pi \circ \rho_T)^{-1}(U_2)}$ is nef over $\pi^{-1}(U_2)$.*

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Proof of the claim. By Theorem 2.20, we can run a K_Y-MMP over X to get a relative minimal model Y'. Then $K_{Y'/X}$ is nef. Since Y' is terminal, it has isolated singularities. Hence after shrinking X and T, we can assume that Y' is smooth over T. Note that the natural map $Y' \to X$ gives the minimal resolution of X_η. Hence it is isomorphic to ρ_η over η since $\rho_\eta : Y_\eta \to X_\eta$ is the minimal resolution. This implies that at every step the MMP only contracts subvarieties which do not dominate T. Hence, the assertion holds. □

We write

$$\mathrm{Exc}(\rho_T) = \sum_{i=1}^{n} E_i$$

such that $E_i|_{Y_\eta} = E_{i,k(T)}$. After shrinking X and T, we can assume that E_i is smooth over T for every i since $E_{i,k(T)}$ is smooth over η. Then $E_{i,t}$ are all prime ρ_t-exceptional divisors for all $t \in T$, where $E_{i,t}$ are the restrictions of E_i on the fibers Y_t. This implies that

$$E_{i,t} \cdot E_{j,t} = E_i|_{Y_t} \cdot E_j|_{Y_t} = (E_i \cdot E_j)|_{Y_t} = (E_i \cdot E_j)|_{Y_\eta} = E_i|_{Y_\eta} \cdot E_j|_{Y_\eta} = E_{i,k(T)} \cdot E_{j,k(T)}$$

for all $t \in T$, where the third equality holds since both $(E_i \cdot E_j)|_{Y_t}$ and $(E_i \cdot E_j)|_{Y_\eta}$ are equal to the degree of $E_i \cdot E_j$ over T as a 1-cycle. Hence the dual graph of $(X_{\overline{\eta}}, \widetilde{\delta})$ is the same as the dual graph of (H, x). In conclusion, the assertion holds. □

Corollary 3.8. *Let (X, B) be an lc quasi-projective threefold over an algebraically closed field k of characteristic $p > 3$ where B is reduced. Let $C \subseteq B$ be a curve contained in the singular locus of X and let H be a general hyperplane section of X. Assume that (X, B) is plt at δ and (X_δ, δ) is a cyclic quotient singularity whose index is m, where δ is the generic point of C and X_δ is the localization of X at δ. Then for a closed point $x \in C \cap H$, we have (H, x) is a cyclic quotient singularity whose index is m.*

Proof. Since $\delta \in B \neq 0$, by Proposition 2.10, every exceptional prime divisor of the minimal resolution of (X_δ, δ) is isomorphic to $\mathbb{P}^1_{k(C)}$. Hence it is geometrically irreducible over $k(C)$. This implies that the dual graph of (X_δ, δ) does not change under any étale base change. Moreover, since X is klt along C, we can apply Theorem 3.6 to conclude that the dual graph of (X_δ, δ) is as same as the dual graph of (H, x). Hence, the assertion holds. □

Since we do not know whether the Bertini theorem holds for the pullback of a very ample linear system along a birational morphism, we prove the following special result. We use it in the proof of Proposition 5.3.

Proposition 3.9. *Let X be a klt quasi-projective threefold over an algebraically closed field k of characteristic $p > 3$. Then there exists a desingularization $\rho : V \to X$ such that, for a general hyperplane section H of X, we have that $S := \rho^{-1}(H)$ is smooth, and $\rho|_S : S \to H$ is the minimal resolution of H.*

Proof. Let $\rho_1 : V_1 \to X$ be a desingularization such that ρ_1 is an isomorphism outside the singular locus of X. By Theorem 2.20, we can run a K_{V_1}-MMP over X to get a relative minimal model V_2. Then V_2 is terminal and K_{V_2} is nef over X. There is a morphism $\rho_2 : V_2 \to X$. Let $\rho_3 : V \to V_2$ be a desingularization such that ρ_3 is an isomorphism outside the singular locus of V_2. We prove that

$$\rho : V \xrightarrow{\rho_3} V_2 \xrightarrow{\rho_2} X$$

satisfies the condition of the assertion. Let H be a general hyperplane section of X. Then by Theorem 3.6, H has klt singularities. Let C be a curve contained in the singular locus of X

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and $h \in H \cap C$ be a closed point. Then for a sufficiently small open neighborhood $h \in U \subseteq X$, we have that

$$\rho^{-1}(U) = \rho_2^{-1}(U)$$

is smooth since V_2 is terminal, H is general, and ρ_3 is an isomorphism outside the singular locus of V_2. Hence, it suffices to show that

$$S := \rho_2^{-1}(U \cap H)$$

is smooth. This problem is local on U. After shrinking U, by Theorem 3.6 we can replace U by its étale base change so that the dual graph of (H, h) is the same as the dual graph of (U_δ, δ), where δ is the generic point of C, and U_δ is the localization of U at δ. We replace X by U, H by $H \cap U$, and V_2 by $\rho_2^{-1}(U)$, and we define $\rho' := \rho_2|_S$.

Shrinking X if necessary, we assume $H \sim_{\mathbb{Q}} 0$. Thus $S = \rho_2^* H \sim_{\mathbb{Q}} 0$ on V_2. It follows that

$$K_{V_2}|_S \sim_{\mathbb{Q}} (K_{V_2} + S)|_S \sim_{\mathbb{Q}} K_S.$$

Hence K_S is nef over H since K_{V_2} is nef over X. We write

$$K_S \sim_{\mathbb{Q}} \rho'^* K_H + \sum_i a_i E_i.$$

Then by the negativity lemma [KM98, Lemma 3.39], we have

$$a_i \leq 0.$$

This says that ρ' does not contract any exceptional prime divisor E over H with the discrepancy $a(E; H) > 0$. To prove that S is smooth, i.e., terminal, it suffices to show that ρ' contracts all exceptional prime divisors E over H with the discrepancies $a(E; H) \leq 0$. Note that ρ_2 restricts to the minimal resolution

$$\rho_\delta : \rho_2^{-1}(X_\delta) \to X_\delta$$

of X_δ with the exceptional divisor $\mathrm{Exc}(\rho_\delta) = \sum_{j=1}^r F_{\delta,j}$. For every $F_{\delta,j}$, there exists an exceptional prime divisor F_j of ρ_2 which restricts to $F_{\delta,j}$. It follows that ρ_2 and hence ρ' contract at least r exceptional prime divisors. By our assumption, the dual graph of (H, h) is the same as the dual graph of (X_δ, δ). This implies that ρ' contracts all exceptional prime divisors E over H with the discrepancies $a(E; H) \leq 0$. Therefore, the assertion holds. □

4. Chern classes on normal varieties in positive characteristic

In this section, we recall Langer's definition of the Chern classes of reflexive sheaves on normal varieties in positive characteristic (see §§ 4.1 and 4.2). Moreover, using the Lefschetz–Riemann–Roch theorem, we calculate local contributions of some quotient singularities to the Riemann–Roch formula for normal surfaces. Then we give lower bounds of some relative second Chern classes (see § 4.3).

4.1 Local relative Chern classes

In this subsection, we recall the notion of local relative Chern classes for resolutions of normal surfaces. The main reference of this subsection is [Lan25, § 3]. Let k be an algebraically closed field and let A be an excellent normal two-dimensional henselian local k-algebra. Let $X = \mathrm{Spec}\, A$ and let $x \in X$ be the closed point of X. Let $f : \widetilde{X} \to X$ be a desingularization of X with $\mathrm{Exc}(f) = \sum_i E_i$.

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Definition 4.1. Let $\mathcal{F}$ be a vector bundle on $\widetilde{X}$. We define the *first relative Chern class* of $\mathcal{F}$ with respect to f to be the unique $\mathbb{Q}$-divisor $c_1(f,\mathcal{F})$ supported on E such that, for every irreducible component E_i of E, we have

$$c_1(f,\mathcal{F})\cdot E_i = \deg \mathcal{F}|_{E_i}.$$

Remark 4.2. The existence and uniqueness of $c_1(f,\mathcal{F})$ follow from the fact that the intersection matrix $[E_i\cdot E_j]$ of the exceptional divisors is negative definite (see [Lan25, 3.1]).

Let $\widetilde{\pi}:\widetilde{Y}\to\widetilde{X}$ be a generically finite proper morphism from a regular surface $\widetilde{Y}$. The Stein factorization of $f\circ\widetilde{\pi}$ gives a proper birational morphism $g:\widetilde{Y}\to Y$. Then we have the following commutative diagram.

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle \widetilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ \widetilde{X} & \xrightarrow{f} & X \end{array}$$

Fix a rank-r vector bundle $\mathcal{F}$ on $\widetilde{X}$. By [Kle69, Theorem 4.7 (ii)], possibly after further blowing up $\widetilde{Y}$, there exists a filtration

$$0=\mathcal{F}_0\subseteq\mathcal{F}_1\subseteq\cdots\subseteq\mathcal{F}_r=\widetilde{\pi}^*\mathcal{F}$$

such that all quotients $\mathcal{L}_i=\mathcal{F}_i/\mathcal{F}_{i-1}$ are line bundles on $\widetilde{Y}$. We define $L_i:=c_1(g,\mathcal{L}_i)$.

Definition 4.3. The *second relative Chern class* $c_2(f,\mathcal{F})$ of $\mathcal{F}$ with respect to f is defined as the real number

$$c_2(f,\mathcal{F}):=\inf\left(\frac{\sum_{i<j}L_iL_j}{\deg\widetilde{\pi}}\right),$$

where the infimum is taken over all $\widetilde{\pi}$ and filtrations as above.

Definition 4.4. We define the *relative Euler characteristic* $\chi(f,\mathcal{F})$ by

$$\chi(f,\mathcal{F}):=\dim H^0(X,f_{[*]}\mathcal{F}/f_*\mathcal{F})+\dim H^0(X,R^1f_*\mathcal{F})$$

and we set

$$a(f,\mathcal{F}):=\chi(f,\mathcal{F})-r\chi(f,\mathcal{O}_{\widetilde{X}})+\tfrac{1}{2}c_1(f,\mathcal{F})(c_1(f,\mathcal{F})-K_{\widetilde{X}})-c_2(f,\mathcal{F}).$$

Remark 4.5. The invariant $a(f,\mathcal{F})$ is important since it does not depend on the choice of desingularization f (see [Lan25, Proposition 3.7]). Hence, for a reflexive sheaf $\mathcal{E}$ on X, we can set

$$a(x,\mathcal{E}):=a(f',\mathcal{F}'),$$

where $f':X'\to X$ is any desingularization of X and $\mathcal{F}'$ is any vector bundle on X' such that $f'_{[*]}\mathcal{F}'\cong\mathcal{E}$.

We use the following result in the proof of Proposition 5.3.

Proposition 4.6. *Assume that the base field k has characteristic $p>0$, and that $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ are vector bundles on $\widetilde{X}$. If the exact sequence*

$$0\to\mathcal{F}_1\to\mathcal{F}_2\to\mathcal{F}_3\to 0$$

exists, then

$$\mathrm{ch}_2(f,\mathcal{F}_2)=\mathrm{ch}_2(f,\mathcal{F}_1)+\mathrm{ch}_2(f,\mathcal{F}_3),$$

where $\mathrm{ch}_2(f,\cdot):=\frac{1}{2}c_1(f,\cdot)^2-c_2(f,\cdot)$.

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Proof. By [Lan25, Theorem 3.15], for $i = 1, 2, 3$ and any positive integer m, we have

$$\mathrm{ch}_2(f, \mathcal{F}_i) = \frac{1}{2}c_1(f, \mathcal{F}_i)^2 - c_2(f, \mathcal{F}_i) = -\lim_{m\to\infty} \frac{\chi(f, (F^m)^*\mathcal{F}_i)}{p^{2m}}.$$

We consider the exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$. Since $\widetilde{X}$ is smooth, the maps F^m are flat. Hence we have the exact sequence

$$0 \to (F^m)^*\mathcal{F}_1 \to (F^m)^*\mathcal{F}_2 \to (F^m)^*\mathcal{F}_3 \to 0.$$

Thus the assertion follows from the additivity of the relative Euler characteristic. □

4.2 Chern classes of reflexive sheaves

In this subsection, we recall the definition of Chern classes of reflexive sheaves on normal varieties. The main reference of this subsection is [Lan25, §§ 4 and 5]. Let X be a proper normal surface defined over an algebraically closed field k, and let $\mathcal{E}$ be a reflexive sheaf on X. Let $f : \widetilde{X} \to X$ be a desingularization with $\mathrm{Exc}(f) = E$. For every $x \in f(E)$ we consider the map $\nu_x : \mathrm{Spec}\, \mathcal{O}^h_{X,x} \to X$ from the spectrum of the henselization of the local ring of X at x and the base change

$$f_x = \nu_x^* f : \widetilde{X}_x \to \mathrm{Spec}\, \mathcal{O}^h_{X,x} \tag{4.1}$$

of f via ν_x. Note that the group of divisors on $\widetilde{X}_x$ that are supported on the exceptional locus E_x of f_x embeds into $A_1(\widetilde{X})$, where $A_1(\widetilde{X})$ is the group of 1-cycles modulo rational equivalence on $\widetilde{X}$. For any vector bundle $\mathcal{F}$ on $\widetilde{X}$ we write $c_1(f_x, \mathcal{F}) \in A_1(\widetilde{X}) \otimes \mathbb{Q}$ for the image of $c_1(f_x, \mathcal{F}|_{\widetilde{X}_x})$. Then we have

$$c_1(\mathcal{F}) = f^*c_1(f_{[*]}\mathcal{F}) + \sum_{x\in f(E)} c_1(f_x, \mathcal{F}) \in A_1(\widetilde{X}) \otimes \mathbb{Q}.$$

We write $a(x, \mathcal{E})$ for $a(x, \nu_x^*\mathcal{E})$ (see Remark 4.5).

DEFINITION 4.7. Let $\mathcal{F}$ be a vector bundle on $\widetilde{X}$ such that $f_{[*]}\mathcal{F} \cong \mathcal{E}$. Using the homomorphism $f_* : A_0(\widetilde{X}) \to A_0(X)$, we define the *second Chern class* of $\mathcal{E}$ to be

$$c_2(\mathcal{E}) := f_*c_2(\mathcal{F}) - \sum_{x\in f(E)} c_2(f_x, \mathcal{F})[x] \in A_0(X) \otimes \mathbb{R},$$

where $A_0(\widetilde{X})$ and $A_0(X)$ denote the groups of 0-cycles modulo rational equivalence on $\widetilde{X}$ and X, respectively. Moreover, we define the *discriminant*

$$\Delta(\mathcal{E}) := 2r\, c_2(\mathcal{E}) - (r-1)c_1(\mathcal{E})^2 \in A_0(X) \otimes \mathbb{R}$$

where r is the rank of $\mathcal{E}$. For a more detailed explanation of these definitions, see [Lan25, 4.1].

In higher dimensions, we can define Chern classes by reducing to the case when X is a surface (see [Lan25, § 5]). We will use the following result in Proposition 5.6.

THEOREM 4.8 [Lan24, Theorem 3.4]. *Let X be a normal projective threefold over an algebraically closed field of characteristic $p > 0$. Let L_1 and L_2 be nef line bundles on X such that $L_1 \cdot L_2$ is numerically non-trivial, and let $\mathcal{E}$ be a strongly (L_1, L_2)-semistable reflexive sheaf on X. Then*

$$\Delta(\mathcal{E}) \cdot L_2 := 2\,\mathrm{rk}(\mathcal{E}) \int_X c_2(\mathcal{E}) \cdot L_2 - (\mathrm{rk}(\mathcal{E}) - 1) \int_X c_1(\mathcal{E})^2 \cdot L_2 \geq 0,$$

where $\int_X$ is the degree map of 0-cycles on X (see [Ful98, Definition 1.4]).

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4.3 Computations of some relative second Chern classes

In this subsection, we compute relative second Chern classes of some cyclic quotient singularities.

Setting 4.9. Let k be an algebraically closed field of characteristic $p>0$, and let $X=\widehat{\mathbb{A}}^2_k/\mathbb{Z}_m$ (see Remark 2.12), where m is a positive integer not divisible by p. Let $f:\widetilde{X}\to X$ be a desingularization of X with $\mathrm{Exc}(f)=\sum_i E_i$ such that $\mathrm{Exc}(f)$ is snc. We write $\Omega^{[1]}_X:=(\Omega_X)^{**}$. We will compute $a(0,\Omega^{[1]}_X)$ (see Remark 4.5), where 0 is the closed point of X, to give lower bounds of $c_2(f,\Omega_{\widetilde{X}})$.

Lemma 4.10. *There is a smooth projective variety T over k and a group monomorphism $\mathbb{Z}_m\hookrightarrow \mathrm{Aut}\,T$ such that the following assertions hold:*

(1) $|\mathrm{Fix}(\mathbb{Z}_m)|<+\infty$, *where* $\mathrm{Fix}(\mathbb{Z}_m)$ *is the set of fixed points for the action of* $\mathbb{Z}_m$; *and*
(2) *for any* $t\in\mathrm{Fix}(\mathbb{Z}_m)$ *we have* $\widehat{T}_t/\mathbb{Z}_m\cong X$, *where* $\widehat{T}_t$ *is the completion of* T *at* t.

Proof. The proof is similar to [Rei87, p. 405]. By the definition, $X=\widehat{\mathbb{A}}^2_k/\mathbb{Z}_m$ comes from a linear and diagonalized action of $\mathbb{Z}_m$ on $\mathbb{A}^2$. This action gives an action of $\mathbb{Z}_m$ on $\mathbb{A}^5$ by

$$g\in\mathbb{Z}_m:(x_1,x_2)\in\mathbb{A}^2\times\mathbb{A}^3\cong\mathbb{A}^5\mapsto(g(x_1),x_2).$$

It induces a natural action of $\mathbb{Z}_m$ on $\mathbb{P}^4$. Then it is clear that the fixed point set

$$\mathrm{Fix}(\mathbb{Z}_m)\subseteq(x_1=0)\cup(x_2=0)\subset\mathbb{P}^4.$$

Consider the quotient map $\pi:\mathbb{P}^4\to\mathbb{P}^4/\mathbb{Z}_m$. We use two general hyperplane sections to cut out a subvariety $Y\subseteq\mathbb{P}^4/\mathbb{Z}_m$. By Proposition 3.1, $T:=\pi^{-1}(Y)$ is smooth. Since $\dim(\mathrm{Fix}(\mathbb{Z}_m))=2$, we have that assertion (1) holds. Moreover, (2) holds by our construction. □

In order to compute $a(0,\Omega^{[1]}_X)$, we need to apply the Lefschetz fixed point formula in positive characteristic. Let $W(k)$ be the Witt ring of k. For $c\in k$, we set $w(c):=(c,0,0,\ldots)\in W(k)$. Let ψ be an endomorphism of a finite-rank k-vector space. Recall that the *Brauer trace* $\mathrm{tr}_{\mathrm{B}}(\psi)$ of ψ is defined to be $\sum_i w(c_i)$, where c_i are eigenvalues of ψ, with the appropriate multiplicities. We refer to [Ser77, 18.1] for more details.

Lemma 4.11. *Let T be a smooth projective surface over k with an action of $\mathbb{Z}_m$ on T. Assume that $|\mathrm{Fix}(\mathbb{Z}_m)|<+\infty$, where $\mathrm{Fix}(\mathbb{Z}_m)$ is the set of fixed points for the action of $\mathbb{Z}_m$. For any $\mathbb{Z}_m$-equivariant locally free sheaf $\mathcal{F}$ on T and $0\neq g\in\mathbb{Z}_m$, we have*

$$\sum_i(-1)^i\,\mathrm{tr}_{\mathrm{B}}(g^*|_{H^i(T,\mathcal{F})})=\sum_{t\in\mathrm{Fix}(\mathbb{Z}_m)}\frac{\mathrm{tr}_{\mathrm{B}}(g|_{\mathcal{F}_t})}{(1-w(c_{g,1}))(1-w(c_{g,2}))},$$

where $\mathcal{F}_t$ is the fiber of $\mathcal{F}$ at t and $c_{g,i}$ are the eigenvalues of g on $\mathbb{A}^2$.

Proof. The proof follows from the Lefschetz fixed point formula in positive characteristic (see [BFQ79, 0.6] for example). □

Lemma 4.12 [Ser77, 18.1]. *Let V be a vector space over k and let $\mathbb{Z}_m\to\mathrm{GL}(V)$ be a representation. Then we have*

$$\dim(V^{\mathbb{Z}_m})=\frac{1}{m}\sum_{g\in\mathbb{Z}_m}\mathrm{tr}_{\mathrm{B}}(g),$$

where $V^{\mathbb{Z}_m}$ is the subspace of V which is fixed by $\mathbb{Z}_m$.

The following result is an adaptation of [Lan00, Theorem 5.4].

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THEOREM 4.13. *Let X be as stated in Setting 4.9. Then we have*

$$a(0, \Omega_X^{[1]}) = \frac{1}{m} - 1.$$

Proof. Let T be a smooth projective surface as constructed in Lemma 4.10, and let $\pi : T \to Y := T/\mathbb{Z}_m$ be the quotient map. We set $N := |\mathrm{Fix}(\mathbb{Z}_m)|$ and $y := \pi(t)$, where $t \in \mathrm{Fix}(\mathbb{Z}_m)$ is a fixed point for the action of $\mathbb{Z}_m$. Note that by Lemma 4.10 (2), we have that $a(y, \Omega_Y^{[1]}) = a(0, \Omega_X^{[1]})$ is independent of the choice of t. It follows from [Lan25, Proposition 4.2 (4)] that $\int_Y c_2(\Omega_Y^{[1]}) = (1/m)\int_Y c_2(\Omega_T)$. Moreover, by [Lan25, Theorem 4.4], we have

$$\chi(Y, \Omega_Y^{[1]}) = -\int_Y c_2(\Omega_Y^{[1]}) + 2\chi(Y, \mathcal{O}_Y) + Na(y, \Omega_Y^{[1]})$$

and

$$\int_Y c_2(\Omega_T) = -\chi(T, \Omega_T) + 2\,\chi(T, \mathcal{O}_T).$$

This implies that

$$\begin{aligned}
a(y, \Omega_Y^{[1]}) &= \frac{1}{N}\left(\chi(Y, \Omega_Y^{[1]}) + \int_Y c_2(\Omega_Y^{[1]}) - 2\,\chi(Y, \mathcal{O}_Y)\right) \\
&= \frac{1}{N}\left(\sum_i (-1)^i \dim(H^i(Y, \Omega_Y^{[1]})) + \frac{1}{m}\int_Y c_2(\Omega_T) - 2\sum_i (-1)^i \dim(H^i(Y, \mathcal{O}_Y))\right) \\
&= \frac{1}{N}\left(\sum_i (-1)^i \dim(H^i(T, \Omega_T)^{\mathbb{Z}_m}) + \frac{1}{m}(-\chi(T, \Omega_T) + 2\,\chi(T, \mathcal{O}_T))\right. \\
&\quad \left. - 2\sum_i (-1)^i \dim(H^i(T, \mathcal{O}_T)^{\mathbb{Z}_m})\right),
\end{aligned}$$

where $H^i(T, \cdot)^{\mathbb{Z}_m}$ is the subspace of $H^i(T, \cdot)$ fixed by $\mathbb{Z}_m$. The third equality holds since $\pi_*(\Omega_T)^{\mathbb{Z}_m} = \Omega_Y^{[1]}$ and $\pi_*(\mathcal{O}_T)^{\mathbb{Z}_m} = \mathcal{O}_Y$, where $\pi_*(\Omega_T)^{\mathbb{Z}_m}$ and $\pi_*(\mathcal{O}_T)^{\mathbb{Z}_m}$ are the subsheaves of $\pi_*(\Omega_T)$ and $\pi_*(\mathcal{O}_T)$, respectively, fixed by $\mathbb{Z}_m$. Note that, by Lemma 4.12,

$$\dim(H^i(T, \Omega_T)^{\mathbb{Z}_m}) = \frac{1}{m}\sum_{g \in \mathbb{Z}_m} \mathrm{tr}_{\mathrm{B}}(g^*|_{H^i(\Omega_T)}), \quad \dim(H^i(T, \mathcal{O}_T)^{\mathbb{Z}_m}) = \frac{1}{m}\sum_{g \in \mathbb{Z}_m} \mathrm{tr}_{\mathrm{B}}(g^*|_{H^i(\mathcal{O}_T)}).$$

Then by Lemma 4.11, we have

$$\begin{aligned}
a(y, \Omega_Y^{[1]}) &= \frac{1}{mN}\left(\sum_i (-1)^i \sum_{g \in \mathbb{Z}_m} \mathrm{tr}_{\mathrm{B}}(g^*|_{H^i(\Omega_T)}) - \chi(T, \Omega_T) + 2\,\chi(T, \mathcal{O}_T)\right. \\
&\quad \left. - 2\sum_i (-1)^i \sum_{g \in \mathbb{Z}_m} \mathrm{tr}_{\mathrm{B}}(g^*|_{H^i(\mathcal{O}_T)})\right) \\
&= \frac{1}{mN}\left(\sum_i (-1)^i \sum_{g \in \mathbb{Z}_m} \mathrm{tr}_{\mathrm{B}}(g^*|_{H^i(\Omega_T)}) - \left(\sum_i (-1)^i \sum_{g = \mathrm{id}} \mathrm{tr}_{\mathrm{B}}(g^*|_{H^i(\Omega_T)})\right)\right. \\
&\quad \left. + 2\left(\sum_i (-1)^i \sum_{g = \mathrm{id}} \mathrm{tr}_{\mathrm{B}}(g^*|_{H^i(\mathcal{O}_T)})\right) - 2\sum_i (-1)^i \sum_{g \in \mathbb{Z}_m} \mathrm{tr}_{\mathrm{B}}(g^*|_{H^i(\mathcal{O}_T)})\right) \\
&= \frac{1}{mN}\left(\sum_i (-1)^i \sum_{\mathrm{id} \neq g \in \mathbb{Z}_m} \mathrm{tr}_{\mathrm{B}}(g^*|_{H^i(\Omega_T)}) - 2\sum_i (-1)^i \sum_{\mathrm{id} \neq g \in \mathbb{Z}_m} \mathrm{tr}_{\mathrm{B}}(g^*|_{H^i(\mathcal{O}_T)})\right)
\end{aligned}$$

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$$= \frac{1}{mN}\left(N \sum_{\mathrm{id}\neq g\in\mathbb{Z}_m} \frac{\mathrm{tr}_{\mathrm{B}}(g|_{\Omega_{T,t}})}{(1-w(c_{g,1}))(1-w(c_{g,2}))} - 2N \sum_{\mathrm{id}\neq g\in\mathbb{Z}_m} \frac{1}{(1-w(c_{g,1}))(1-w(c_{g,2}))}\right)$$
$$= \frac{1}{m} \sum_{\mathrm{id}\neq g\in\mathbb{Z}_m} \frac{\mathrm{tr}_{\mathrm{B}}(g|_{\Omega_{T,t}}) - 2}{(1-w(c_{g,1}))(1-w(c_{g,2}))},$$

where the fourth equality follows from Lemma 4.10 (2). Then by [Lan00, Example 5.6 and Corollary 5.9], we have $a(0,\Omega_X^{[1]}) = a(y,\Omega_Y^{[1]}) = 1/m - 1$, and hence the assertion holds. □

COROLLARY 4.14. *Let X and f be as stated in Setting 4.9. We have*

$$c_2(f,\Omega_{\widetilde{X}}) \geq 2 - \frac{1}{m}.$$

Proof. By Theorem 4.13, it suffices to show that $c_2(f,\Omega_{\widetilde{X}}) \geq 1 - a(0,\Omega_X^{[1]})$. Note that, by Definition 4.4,

$$c_2(f,\Omega_{\widetilde{X}}) = \chi(f,\Omega_{\widetilde{X}}) - 2\,\chi(f,\mathcal{O}_{\widetilde{X}}) + \tfrac{1}{2}c_1(f,\Omega_{\widetilde{X}})\cdot(c_1(f,\Omega_{\widetilde{X}}) - K_{\widetilde{X}}) - a(0,\Omega_X^{[1]}).$$

Since $c_1(f,\Omega_{\widetilde{X}})$ is supported on E and $c_1(f,\Omega_{\widetilde{X}}) - K_{\widetilde{X}}$ is f-trivial, we have

$$c_1(f,\Omega_{\widetilde{X}})\cdot(c_1(f,\Omega_{\widetilde{X}}) - K_{\widetilde{X}}) = 0.$$

Moreover, since X has rational singularities, we have

$$\chi(f,\mathcal{O}_{\widetilde{X}}) = h^0(f_{[*]}\mathcal{O}_{\widetilde{X}}/\mathcal{O}_X) + h^0(R^1f_*\mathcal{O}_{\widetilde{X}}) = 0.$$

Therefore,

$$c_2(f,\Omega_{\widetilde{X}}) = \chi(f,\Omega_{\widetilde{X}}) - a(0,\Omega_X^{[1]}).$$

Since

$$\chi(f,\Omega_{\widetilde{X}}) = h^0(f_{[*]}\Omega_{\widetilde{X}}/f_*\Omega_{\widetilde{X}}) + h^0(R^1f_*\Omega_{\widetilde{X}}),$$

we only need to show that $h^0(R^1f_*\Omega_{\widetilde{X}}) \geq 1$. By the exact sequence

$$0 \to \Omega_{\widetilde{X}}(-E) \to \Omega_{\widetilde{X}} \to \Omega_{\widetilde{X}}|_E \to 0,$$

and $R^2f_*\Omega_{\widetilde{X}}(-E) = 0$ since $\mathrm{Exc}(f)$ is one-dimensional, we know that the map

$$R^1f_*\Omega_{\widetilde{X}} \to R^1f_*(\Omega_{\widetilde{X}}|_E) \cong H^1(E,\Omega_{\widetilde{X}}|_E)$$

is surjective. Moreover, there exists the following exact sequence:

$$\mathcal{O}_E(-E) \to \Omega_{\widetilde{X}}|_E \to \Omega_E \to 0.$$

It follows that the map

$$H^1(E,\Omega_{\widetilde{X}}|_E) \to H^1(E,\Omega_E)$$

is surjective. By Serre duality, $h^1(E,\Omega_E) = h^0(E,\mathcal{O}_E) = 1$. Therefore $h^0(R^1f_*\Omega_{\widetilde{X}}) \geq 1$, and hence the assertion holds. □

5. Proof of abundance when $\nu = 2$

In this section, we prove the abundance conjecture in the case of $\nu(K_X + B) = 2$. We first prove a special case (see Proposition 5.6). Then we prove the general case by reducing it to the special case (see Theorem 5.8).

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5.1 A key inequality

SETTING 5.1. Let (X, B) be an lc projective pair of dimension 3 over an algebraically closed field k of characteristic $p > 3$. We assume that the following hold:

(1) X is $\mathbb{Q}$-factorial, B is reduced, and $X \setminus B$ is terminal;
(2) there exists a divisor $D \in |m(K_X + B)|$ such that $D_{\mathrm{red}} = B$, where m is a positive integer such that $L := m(K_X + B)$ is Cartier;
(3) $K_X + B$ is nef and $\nu(K_X + B) = 2$; and
(4) if C is a curve in X with $C \cdot (K_X + B) > 0$, then (X, B) is plt at the generic point of C.

LEMMA 5.2. *Let S be a component of B on which L is not numerically trivial, and let $\lambda : S^\lambda \to S$ be the normalization. Let Θ be the $\mathbb{Q}$-divisor defined by adjunction $(K_X + B)|_{S^\lambda} = K_{S^\lambda} + \Theta$. Then the following assertions hold.*

(1) *The Iitaka fibration f associated to $K_{S^\lambda} + \Theta$ is a morphism to a smooth curve.*
(2) *Let Θ_h be the horizontal part of Θ with respect to f. We write $\Theta_h = \sum c_q \Gamma_q + \sum d_l \Delta_l$, where the Γ_q map under λ to the singular locus of X and the Δ_l to the non-normal locus of B. Then we have that f is a generically smooth map and one of the following holds.*
   (i) *The generic fiber of λ is a smooth elliptic curve and $\Theta_h = 0$,*
   (ii) *The generic fiber of λ is $\mathbb{P}^1$, $d_l = 1$, and $c_q = 1 - (1/m_q)$, where m_q are the indices of cyclic quotient singularities of X at the generic point of Γ_q (see Theorem 2.11). Moreover, $(d_1, \ldots; m_1, \ldots)$ must be one of the following: $(1, 1;)$, $(1; 2)$, $(1; 2, 2)$, $(; 2)$, $(; 2, 2)$, $(; 2, 2, 2)$, $(; 2, 2, 2, 2)$, $(; 2, 3, 6)$, $(; 2, 4)$, $(; 2, 4, 4)$, $(; 3)$, $(; 3, 3)$, or $(; 3, 3, 3)$.*

Proof. For assertion (1), note that

$$0 \leq (K_{S^\lambda} + \Theta)^2 = S \cdot (K_X + B)^2 = 0,$$

where the last equality holds since, by (2) and (3) in Setting 5.1, $(K_X + B)^2$ is numerically trivial on B. Then (S^λ, Θ) is an slc pair with $\nu(K_{S^\lambda} + \Theta) = 1$ since L is not numerically trivial on S. It follows from Theorem 2.18 that $K_{S^\lambda} + \Theta$ is semi-ample, and hence the Iitaka fibration f associated to $K_{S^\lambda} + \Theta$ is a morphism to a smooth curve. For assertion (2), since $p > 3$, by [Wit21, Proposition 2.2], f is a generically smooth map. Let F be the generic fiber of f and consider the adjunction

$$(K_{S^\lambda} + \Theta)|_F \sim_{\mathbb{Q}} K_F + \Theta_h|_F.$$

If $\Theta_h|_F = 0$, then $\Theta_h = 0$. We get (i). Otherwise, we have $\deg K_F < 0$. Since F is smooth, $F = \mathbb{P}^1$. This implies that $\deg \Theta_h|_F = 2$. Now by (4) in Setting 5.1, Proposition 2.10, and Theorem 2.11, we have that (ii) holds. □

The following proposition is the key inequality for proving Proposition 5.6.

PROPOSITION 5.3. *Let $\rho : V \to X$ be a desingularization. Then*

$$\rho^* L \cdot (K_V^2 + c_2(V)) \geq L \cdot (K_X^2 + c_2(\Omega_X^{[1]})).$$

Moreover, if the equality holds, then we have $(K_X + B)|_{B'} \sim_{\mathbb{Q}} K_{B'} + \Delta_{B'}$ such that $(K_X + B)|_{B'}$ is numerically trivial on $\operatorname{Supp} \Delta_{B'}$ where B' is the S_2-fication of B.

Proof. By [Zha23, Corollary 2.2] we can choose a sufficiently ample divisor A such that $nL + A$ are very ample for all $n \gg 0$. We take a general section $H_n \in |nL + A|$. Note that $\rho^* L \cdot (K_V^2 + c_2(V))$ does not depend on the choice of ρ. Hence by Proposition 3.9, we can assume that $\rho : V \to X$ is a desingularization such that $\widetilde{H}_n := \rho^{-1}(H_n) = \rho^* H_n$ is smooth, and $\rho|_{\widetilde{H}_n} : \widetilde{H}_n \to H_n$

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is the minimal resolution of H_n. Then by [Lan25, Theorem 0.4] we have

$$\begin{aligned}\rho^*(nL+A)\cdot(K_V^2+c_2(V)) &= \widetilde{H}_n\cdot(K_V^2+c_2(V))\\ &= (K_V|_{\widetilde{H}_n})^2+c_2(\Omega_V|_{\widetilde{H}_n})\end{aligned}$$

and

$$\begin{aligned}(nL+A)\cdot(K_X^2+c_2(\Omega_X^{[1]})) &= H_n\cdot(K_X^2+c_2(\Omega_X^{[1]}))\\ &= (K_X|_{H_n})^2+c_2(\Omega_X^{[1]}|_{H_n}).\end{aligned}$$

We claim that

$$\rho_*(K_V|_{\widetilde{H}_n})^2+\rho_*c_2(\Omega_V|_{\widetilde{H}_n})-(K_X|_{H_n})^2-c_2(\Omega_X^{[1]}|_{H_n})=\rho_*K_{\widetilde{H}_n}^2+\rho_*c_2(\Omega_{\widetilde{H}_n})-K_{H_n}^2-c_2(\Omega_{H_n}^{[1]})$$

as 0-cycles. Consider the exact sequence

$$0\to\mathcal{O}_{\widetilde{H}_n}(-\widetilde{H}_n)\to\Omega_V|_{\widetilde{H}_n}\to\Omega_{\widetilde{H}_n}\to 0. \tag{5.1}$$

Let x be a point contained in the singular locus of H_n. Let ρ_x be the map over $\operatorname{Spec}\mathcal{O}_{H_n,x}^h$ defined by (4.1). By Proposition 4.6 and [Lan25, Proposition 3.2 (1)], we have

$$\begin{aligned}&\tfrac{1}{2}c_1(\rho_x,\Omega_V|_{\widetilde{H}_n})^2-c_2(\rho_x,\Omega_V|_{\widetilde{H}_n})\\ &=\tfrac{1}{2}c_1(\rho_x,\Omega_{\widetilde{H}_n})^2-c_2(\rho_x,\Omega_{\widetilde{H}_n})+\tfrac{1}{2}c_1(\rho_x,\mathcal{O}_{\widetilde{H}_n}(-\widetilde{H}_n))^2-c_2(\rho_x,\mathcal{O}_{\widetilde{H}_n}(-\widetilde{H}_n))\\ &=\tfrac{1}{2}c_1(\rho_x,\Omega_{\widetilde{H}_n})^2-c_2(\rho_x,\Omega_{\widetilde{H}_n})+\tfrac{1}{2}c_1(\rho_x,\mathcal{O}_{\widetilde{H}_n}(-\widetilde{H}_n))^2\\ &=\tfrac{1}{2}c_1(\rho_x,\Omega_{\widetilde{H}_n})^2-c_2(\rho_x,\Omega_{\widetilde{H}_n}),\end{aligned} \tag{5.2}$$

where the last equality holds since

$$\mathcal{O}_{\widetilde{H}_{n,x}}(\widetilde{H}_{n,x})\sim\rho_x^*\mathcal{O}_{H_n}(H_n)$$

and by Definition 4.1

$$c_1(\rho_x,\mathcal{O}_{\widetilde{H}_n}(\widetilde{H}_n))\sim_{\mathbb{Q}}\mathcal{O}_{\widetilde{H}_{n,x}}(\widetilde{H}_{n,x})-\rho_x^*\mathcal{O}_{H_n}(H_n)\sim_{\mathbb{Q}}0. \tag{5.3}$$

By Definitions 4.1 and 4.7, we have

$$\begin{aligned}\sum_{x\in H_n\text{ singular}}c_1(\rho_x,\Omega_V|_{\widetilde{H}_n}) &= K_V|_{\widetilde{H}_n}-\rho^*c_1(\Omega_X^{[1]}|_{H_n}),\\ \sum_{x\in H_n\text{ singular}}c_2(\rho_x,\Omega_V|_{\widetilde{H}_n})[x] &= \rho_*c_2(\Omega_V|_{\widetilde{H}_n})-c_2(\Omega_X^{[1]}|_{H_n}),\\ \sum_{x\in H_n\text{ singular}}c_1(\rho_x,\Omega_{\widetilde{H}_n}) &= K_{\widetilde{H}_n}-\rho^*c_1(\Omega_{H_n}^{[1]}),\end{aligned}$$

and

$$\sum_{x\in H_n\text{ singular}}c_2(\rho_x,\Omega_{\widetilde{H}_n})[x]=\rho_*c_2(\Omega_{\widetilde{H}_n})-c_2(\Omega_{H_n}^{[1]}).$$

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These imply that

$$\begin{aligned}
&\rho_*(K_V|_{\widetilde{H}_n})^2 + \rho_* c_2(\Omega_V|_{\widetilde{H}_n}) - (K_X|_{H_n})^2 - c_2(\Omega^{[1]}_X|_{H_n}) \\
&= \rho_*\left(\rho^*(K_X|_{H_n}) + \sum_{x\in H_n \text{ singular}} c_1(\rho_x, \Omega_V|_{\widetilde{H}_n})\right)^2 - (K_X|_{H_n})^2 + \sum_{x\in H_n \text{ singular}} c_2(\rho_x, \Omega_V|_{\widetilde{H}_n})[x] \\
&= \sum_{x\in H_n \text{ singular}} c_1(\rho_x, \Omega_V|_{\widetilde{H}_n})^2[x] + \sum_{x\in H_n \text{ singular}} c_2(\rho_x, \Omega_V|_{\widetilde{H}_n})[x] \\
&= \sum_{x\in H_n \text{ singular}} \frac{3}{2} c_1(\rho_x, \Omega_V|_{\widetilde{H}_n})^2[x] + \sum_{x\in H_n \text{ singular}} c_2(\rho_x, \Omega_V|_{\widetilde{H}_n})[x] \\
&\quad - \sum_{x\in H_n \text{ singular}} \frac{1}{2} c_1(\rho_x, \Omega_V|_{\widetilde{H}_n})^2[x].
\end{aligned}$$

Note that by (5.1), (5.3), and Definition 4.1, we have

$$c_1(\rho_x, \Omega_V|_{\widetilde{H}_n}) = c_1(\rho_x, \Omega_{\widetilde{H}_n}) + c_1(\rho_x, \mathcal{O}_{\widetilde{H}_n}(-\widetilde{H}_n)) = c_1(\rho_x, \Omega_{\widetilde{H}_n}).$$

Hence by (5.2) we have

$$\begin{aligned}
&\rho_*(K_V|_{\widetilde{H}_n})^2 + \rho_* c_2(\Omega_V|_{\widetilde{H}_n}) - (K_X|_{H_n})^2 - c_2(\Omega^{[1]}_X|_{H_n}) \\
&= \sum_{x\in H_n \text{ singular}} \frac{3}{2} c_1(\rho_x, \Omega_{\widetilde{H}_n})^2[x] + \sum_{x\in H_n \text{ singular}} c_2(\rho_x, \Omega_{\widetilde{H}_n})[x] - \sum_{x\in H_n \text{ singular}} \frac{1}{2} c_1(\rho_x, \Omega_{\widetilde{H}_n})^2[x] \\
&= \sum_{x\in H_n \text{ singular}} c_1(\rho_x, \Omega_{\widetilde{H}_n})^2[x] + \sum_{x\in H_n \text{ singular}} c_2(\rho_x, \Omega_{\widetilde{H}_n})[x] \\
&= \rho_*\left(\rho^* K_{H_n} + \sum_{x\in H_n \text{ singular}} c_1(\rho_x, \Omega_{\widetilde{H}_n})\right)^2 + \rho_* c_2(\Omega_{\widetilde{H}_n}) - K^2_{H_n} - c_2(\Omega^{[1]}_{H_n}) \\
&= \rho_* K^2_{\widetilde{H}_n} + \rho_* c_2(\Omega_{\widetilde{H}_n}) - K^2_{H_n} - c_2(\Omega^{[1]}_{H_n}).
\end{aligned}$$

Hence, the claim holds. Note that

$$\begin{aligned}
a_n &:= \rho_* K^2_{\widetilde{H}_n} + \rho_* c_2(\Omega_{\widetilde{H}_n}) - K^2_{H_n} - c_2(\Omega^{[1]}_{H_n}) \\
&= \sum_{x\in H_n \text{ singular}} c_1(\rho_x, \Omega_{\widetilde{H}_n})^2[x] + \sum_{x\in H_n \text{ singular}} c_2(\rho_x, \Omega_{\widetilde{H}_n})[x]
\end{aligned}$$

is a 0-cycle supported on the singular locus of H_n. Since $X \setminus B$ is terminal by (1) in Setting 5.1, a_n is supported on $H_n \cap C$, where $C := C_1 + C_2$, $C_1 \subseteq B$ is the union of curves contained in the singular locus of X on which L are numerically trivial, and $C_2 \subseteq B$ is the union of curves contained in the singular locus of X on which L are positive.

We claim that the degrees of a_n on C_1 are bounded from below. Note that, for any irreducible component Γ of C_1,

$$\Gamma \cdot H_n = \Gamma \cdot (nL + A) = \Gamma \cdot A$$

are independent of n. Let $z_{1,n}$ be a point in $H_n \cap \Gamma$. It suffices to show that the degrees of a_n at $z_{1,n}$ are bounded from below. By Theorem 3.6, the dual graphs of $(H_n, z_{1,n})$ are independent of n. Hence, $c_1(\rho_{z_{1,n}}, \Omega_{\widetilde{H}_n})^2$ are bounded from below since $\rho|_{\widetilde{H}_n}$ are minimal resolutions and $c_1(\rho_{z_{1,n}}, \Omega_{\widetilde{H}_n})^2$ are determined by the dual graphs of $(H_n, z_{1,n})$. By [Lan25, Proposition 3.2],

$$c_1(\rho_{z_{1,n}}, \Omega_{\widetilde{H}_n})^2 + c_2(\rho_{z_{1,n}}, \Omega_{\widetilde{H}_n}) \geq \tfrac{5}{4} c_1(\rho_{z_{1,n}}, \Omega_{\widetilde{H}_n})^2.$$

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Hence the claim follows. Therefore, to prove that $\rho^*L\cdot(K_V^2+c_2(V))-L\cdot(K_X^2+c_2(\Omega_X^{[1]}))\geq 0$, it suffices to show that $(nL+A)\cdot(\rho_*(K_V^2+c_2(V))-(K_X^2+c_2(\Omega_X^{[1]})))=a_n$ are of non-negative degrees on C_2 for all $n\gg 0$. Fix an $n\gg 0$. Let $z_2\in C_2$ be a singular point of H_n. Note that z_2 is in an irreducible component S of B, on which L is not numerically trivial. Then we can apply Lemma 5.2 and Corollary 3.8 to conclude that (H_n,z_2) is a cyclic quotient singularity whose index m is one of 2, 3, 4, or 6. Then it follows from Corollary 4.14 that, if z_2 is a Du Val singularity, the degree of a_n at z_2, which is equal to $c_2(\rho_{z_2},\Omega_{\widetilde{H}_n})$, is at least $\frac{3}{2}$, $\frac{5}{3}$, $\frac{7}{4}$, or $\frac{11}{6}$ when m is 2, 3, 4, or 6, respectively. Otherwise, by [Kol13, Definition 3.33], m is one of 3, 4, or 6, and $c_1(\rho_{z_2},\Omega_{\widetilde{H}_n})^2$ is $-\frac{1}{3}$, -1, or $-\frac{8}{3}$ when m is 3, 4, or 6, respectively. Hence, by Corollary 4.14, the degree of a_n at z_2 is at least $\frac{4}{3}$, $\frac{3}{4}$, or $-\frac{5}{6}$ when m is 3, 4, or 6, respectively. By Lemma 5.2 and Corollary 3.8, an index-6 point is always accompanied by an index-2 point and an index-3 point. Moreover, different index-6 points give different index-2 points and index-3 points. Hence we always have that a_n is of positive degree on C_2. Therefore, the inequality holds. Moreover, if the equality holds, then there is no such singular point z_2. It is to see that (ii) holds in (2) of Lemma 5.2. Hence, the second assertion holds by Lemma 2.19. □

5.2 A special case

In this subsection, we prove a special case of the abundance conjecture in the case of $\nu(K_X+B)=2$ (see Proposition 5.6). Before proving Proposition 5.6, we give some results which help us to calculate some cohomology.

Lemma 5.4. *Let B be a reduced equidimensional quasi-projective scheme of dimension 2 over a field and $g:B'\to B$ be its S_2-fication. Assume that L is a Cartier divisor on B. Then*

$$h^i(B',\mathcal{O}_{B'}(ng^*L))-h^i(B,\mathcal{O}_B(nL))$$

are bounded for all i and $n\geq 0$.

Proof. Since g is a finite morphism by Proposition 2.14, we have

$$h^i(B',\mathcal{O}_{B'}(ng^*L))=h^i(B,g_*\mathcal{O}_{B'}\otimes\mathcal{O}_B(nL)).$$

Note that there exists an exact sequence

$$0\to\mathcal{O}_B\to g_*\mathcal{O}_{B'}\to\mathcal{Q}\to 0$$

such that $\mathcal{Q}$ is a coherent sheaf supported on some points by Proposition 2.14. Tensoring it with $\mathcal{O}_B(nL)$, we get

$$0\to\mathcal{O}_B(nL)\to g_*\mathcal{O}_{B'}\otimes\mathcal{O}_B(nL)\to\mathcal{Q}\otimes\mathcal{O}_B(nL)\to 0.$$

They give the exact sequence

$$H^{i-1}(B,\mathcal{Q}\otimes\mathcal{O}_B(nL))\to H^i(B,\mathcal{O}_B(nL))\to H^i(B,g_*\mathcal{O}_{B'}\otimes\mathcal{O}_B(nL))\to H^i(B,\mathcal{Q}\otimes\mathcal{O}_B(nL)).$$

Since both $h^{i-1}(B,\mathcal{Q}\otimes\mathcal{O}_B(nL))$ and $h^i(B,\mathcal{Q}\otimes\mathcal{O}_B(nL))$ are constants for $n\geq 0$, the assertion holds. □

Lemma 5.5. *Let X be a klt projective threefold over an algebraically closed field of characteristic $p>3$. Assume that M is a nef $\mathbb{Q}$-Cartier Weil divisor on X such that $\nu(M)\geq 2$. Then we have $h^2(X,\mathcal{O}_X(K_X+nM))$ is bounded for $n\geq 0$.*

Proof. We take a general hyperplane section A on X and consider the exact sequence

$$0\to\mathcal{O}_X(K_X+nM+lA)\to\mathcal{O}_X(K_X+nM+(l+1)A)\to\mathcal{Q}_{n,l+1}\to 0,$$

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where $l \geq 0$ is an integer. Since both $\mathcal{O}_X(K_X + nM + lA)$ and $\mathcal{O}_X(K_X + nM + (l+1)A)$ are S_2, by [Kol13, Lemma 2.60] $\mathcal{Q}_{n,l+1}$ are S_1 and supported on A. Hence they are torsion-free. Note that the sheaves $\mathcal{O}_A((K_X + nM + (l+1)A)|_A)$ are reflexive. They give the exact sequence

$$0 \to \mathcal{Q}_{n,l+1} \to \mathcal{O}_A((K_X + nM + (l+1)A)|_A) \to \mathcal{G}_{n,l+1} \to 0,$$

where $\mathcal{G}_{n,l+1}$ are supported on some points. Moreover, since M is $\mathbb{Q}$-Cartier, $\mathcal{G}_{n,l+1}$ are of bounded rank at every point. Hence there is the exact sequence

$$H^0(A, \mathcal{G}_{n,l+1}) \to H^1(A, \mathcal{Q}_{n,l+1}) \to H^1(A, \mathcal{O}_A((K_X + nM + (l+1)A)|_A)).$$

Since A is a klt surface by Theorem 3.6, for $n \gg 0$ and any fixed l, we have

$$H^1(A, \mathcal{O}_A((K_X + nM + (l+1)A)|_A)) = 0$$

by [Tan15, Theorem 2.11]. Thus, both $h^0(A, \mathcal{G}_{n,l+1})$ and $h^1(A, \mathcal{O}_A((K_X + nM + (l+1)A)|_A))$ are bounded for $n \geq 0$ and any fixed l. This implies that there exists a constant C_{l+1} such that

$$h^1(X, \mathcal{Q}_{n,l+1}) \leq C_{l+1}$$

for $n \geq 0$ and any l. By [Kee03, Theorem 1.5], there exists an integer $l_0 > 0$ such that, for $l \geq l_0$ and all $n \geq 0$, we have $h^2(X, \mathcal{O}_X(K_X + nM + (l+1)A)) = 0$. It follows that

$$\begin{aligned} h^2(X, \mathcal{O}_X(K_X + nM)) &\leq h^1(X, \mathcal{Q}_{n,1}) + h^2(X, \mathcal{O}_X(K_X + nM + A)) \\ &\leq \cdots \leq \sum_{i=1}^{l_0+1} h^1(X, \mathcal{Q}_{n,i}) \leq \sum_{i=1}^{l_0+1} C_i. \end{aligned}$$

□

PROPOSITION 5.6. *Let (X, B) be an lc projective pair of dimension 3 over an algebraically closed field k of characteristic $p > 3$. We assume that the following hold:*

(1) *X is $\mathbb{Q}$-factorial, B is reduced, and $X \setminus B$ is terminal;*
(2) *there exists a divisor $D \in |m(K_X + B)|$ such that $D_{\mathrm{red}} = B$, where m is a positive integer such that $L := m(K_X + B)$ is Cartier;*
(3) *$K_X + (1-\varepsilon)B$ is nef for any sufficiently small $\varepsilon \geq 0$;*
(4) *$\nu(K_X + B) = 2$; and*
(5) *if C is a curve in X with $C \cdot (K_X + B) > 0$, then (X, B) is plt at the generic point of C.*

Then $K_X + B$ is semi-ample.

Proof. By Theorem 2.28 it suffices to show that $\kappa(K_X + B) > 0$. Since $K_X + (1-\varepsilon)B$ is nef for any sufficiently small $\varepsilon \geq 0$, by Theorem 2.29 there exists an effective $\mathbb{Q}$-divisor $\Delta_\varepsilon \sim_{\mathbb{Q}} K_X + (1-\varepsilon)B$ for any sufficiently small rational $\varepsilon \geq 0$. Moreover, by assumption (2) we can assume that $B \subseteq \operatorname{Supp} \Delta_\varepsilon$ and Δ_ε have the same support for all sufficiently small rational $\varepsilon \geq 0$. Therefore by [Kol92, Lemma 11.3.3] we have

$$\nu(K_X + (1-\varepsilon)B) = \nu(K_X + B) = 2$$

for any sufficiently small rational $\varepsilon > 0$. This implies that

$$(K_X + (1-\varepsilon)B)^3 = (K_X + B)^3 - 3\varepsilon(K_X + B)^2 \cdot B + 3\varepsilon^2(K_X + B) \cdot B^2 - \varepsilon^3 B^3 = 0.$$

Hence, we have

$$(K_X + B)^3 = K_X \cdot (K_X + B)^2 = K_X^2 \cdot (K_X + B) = B^3 = 0. \tag{5.4}$$

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Now let $\rho : V \to X$ be a desingularization of X. By the Riemann–Roch theorem, we have

$$\chi(V, \mathcal{O}_V(n\rho^*L)) = \frac{n^3}{6}(\rho^*L)^3 - \frac{n^2}{4}K_V \cdot (\rho^*L)^2 + \frac{n}{12}(c_2(V) + K_V^2) \cdot \rho^*L + \chi(\mathcal{O}_V).$$

Note that by the projection formula and (5.4),

$$(\rho^*L)^3 = K_V \cdot (\rho^*L)^2 = 0.$$

Hence we have

$$\chi(V, \mathcal{O}_V(n\rho^*L)) = \frac{n}{12}(K_V^2 + c_2(V)) \cdot \rho^*L + \chi(\mathcal{O}_V).$$

If T_V is not strongly (ρ^*L, ρ^*L)-semistable, then by Theorem 2.32 there is a 1-foliation $\mathcal{F} \subseteq T_V$ such that $\mu(\mathcal{F}) > 0 = \mu(T_V)$ with respect to (ρ^*L, ρ^*L). By [Zha23, Corollary 2.2] we can take a sufficiently ample line bundle H on V such that $a\rho^*L + H$ are very ample for all integers $a \geq 0$. Then we can use two general divisors in $|a\rho^*L + H|$ to cut out a smooth curve C_a such that $\mathcal{F}$ is smooth along C_a. Note that

$$\begin{aligned} c_1(\mathcal{F}) \cdot C_a &= c_1(\mathcal{F}) \cdot (a\rho^*L + H)^2 \\ &= a^2 c_1(\mathcal{F}) \cdot (\rho^*L)^2 + 2ac_1(\mathcal{F}) \cdot \rho^*L \cdot H + c_1(\mathcal{F}) \cdot H^2, \\ K_V \cdot C_a &= K_V \cdot (a\rho^*L + H)^2 \\ &= 2aK_V \cdot \rho^*L \cdot H + K_V \cdot H^2, \end{aligned}$$

and

$$\begin{aligned} \rho^*L \cdot C_a &= \rho^*L \cdot (a\rho^*L + H)^2 \\ &= 2a(\rho^*L)^2 \cdot H + \rho^*L \cdot H^2. \end{aligned}$$

Since $\mu(\mathcal{F}) = (c_1(\mathcal{F}) \cdot (\rho^*L)^2)/\operatorname{rk}\mathcal{F} > 0$, $c_1(\mathcal{F}) \cdot C_a$ grows with a^2, while $K_V \cdot C_a$ and $\rho^*L \cdot C_a$ grow at most linearly in a. In particular, for $a \gg 0$,

$$c_1(\mathcal{F}) \cdot C_a > \frac{K_V \cdot C_a}{p-1}.$$

By Theorem 2.36, for general $y \in C_a$, we know that there exists a rational curve B_y such that B_y passes through y and

$$\rho^*L \cdot B_y \leq \frac{6p\rho^*L \cdot C_a}{(p-1)c_1(\mathcal{F}) \cdot C_a - K_V \cdot C_a}.$$

Note that $\rho^*L \cdot B_y$ is a non-negative integer, while the denominator of the right-hand side above grows with a^2, and the numerator grows at most linearly in a. Hence $\rho^*L \cdot B_y = 0$ for $a \gg 0$. Since C_a and y are general, we obtain a family of curves B_y covering V such that $\rho^*L \cdot B_y = 0$. It follows that

$$n(K_X + B) = n(L) = n(\rho^*L) \leq 2.$$

By Theorem 2.28, we have that $K_X + B$ is semi-ample.

From now on, we can assume that T_V is strongly (ρ^*L, ρ^*L)-semistable. Then we have that $\Omega_X^{[1]}$ is strongly (L, L)-semistable since ρ is birational. Hence by Theorem 4.8,

$$\Delta(\Omega_X^{[1]}) \cdot L = (6c_2(\Omega_X^{[1]}) - 2c_1(\Omega_X^{[1]})^2) \cdot L \geq 0.$$

Since $K_X^2 \cdot L = K_X^2 \cdot (K_X + B) = 0$ by (5.4), we have

$$c_2(\Omega_X^{[1]}) \cdot L \geq 0. \tag{5.5}$$

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Note that by assumptions (1)–(5) of the proposition, the assumptions in Setting 5.1 are satisfied. Hence we can apply Proposition 5.3 and (5.4) to conclude that

$$\begin{aligned}\chi(V, \mathcal{O}_V(n\rho^*L)) &= \frac{n}{12}(K_V^2 + c_2(V)) \cdot \rho^*L + \chi(\mathcal{O}_V) \\ &\geq \frac{n}{12}(K_X^2 + c_2(\Omega_X^{[1]})) \cdot L + \chi(\mathcal{O}_V) \\ &= \frac{n}{12}c_2(\Omega_X^{[1]}) \cdot L + \chi(\mathcal{O}_V). \end{aligned} \tag{5.6}$$

We claim that there exists a constant t_1 such that

$$\chi(X, \mathcal{O}_X(nL)) \geq \chi(V, \mathcal{O}_V(n\rho^*L)) + t_1. \tag{5.7}$$

Note that $R^i\rho_*\mathcal{O}_V$ are supported on the closed subsets of X which are of dimension at most $2 - i$ for $i = 1$ or 2. By the Leray spectral sequence, we have

$$\begin{aligned}&\chi(V, \mathcal{O}_V(n\rho^*L)) \\ &\quad = h^0(V, \mathcal{O}_V(n\rho^*L)) - h^1(V, \mathcal{O}_V(n\rho^*L)) + h^2(V, \mathcal{O}_V(n\rho^*L)) - h^3(V, \mathcal{O}_V(n\rho^*L)) \\ &\quad = h^0(X, \mathcal{O}_X(nL)) - (h^1(X, \mathcal{O}_X(nL)) + h^0(X, R^1\rho_*\mathcal{O}_V \otimes \mathcal{O}_X(nL))) \\ &\qquad + (h^2(X, \mathcal{O}_X(nL)) + h^1(X, R^1\rho_*\mathcal{O}_V \otimes \mathcal{O}_X(nL)) + h^0(X, R^2\rho_*\mathcal{O}_V \otimes \mathcal{O}_X(nL))) \\ &\qquad - h^3(X, \mathcal{O}_X(nL)) \\ &\quad = \chi(X, \mathcal{O}_X(nL)) - h^0(X, R^1\rho_*\mathcal{O}_V \otimes \mathcal{O}_X(nL)) + h^1(X, R^1\rho_*\mathcal{O}_V \otimes \mathcal{O}_X(nL)) \\ &\qquad + h^0(X, R^2\rho_*\mathcal{O}_V \otimes \mathcal{O}_X(nL)).\end{aligned}$$

It follows that

$$\begin{aligned}&\chi(X, \mathcal{O}_X(nL)) + h^1(X, R^1\rho_*\mathcal{O}_V \otimes \mathcal{O}_X(nL)) + h^0(X, R^2\rho_*\mathcal{O}_V \otimes \mathcal{O}_X(nL)) \\ &\quad \geq \chi(V, n\rho^*L) + h^0(X, R^1\rho_*\mathcal{O}_V \otimes \mathcal{O}_X(nL)).\end{aligned}$$

Note that $h^0(X, R^2\rho_*\mathcal{O}_V \otimes \mathcal{O}_X(nL))$ is bounded for $n \geq 0$ since $R^2\rho_*\mathcal{O}_V$ are supported on some points. Since $R^1\rho_*\mathcal{O}_V$ is a coherent sheaf supported on a closed subset Z_1 of X which is of dimension at most 1, $h^1(X, R^1\rho_*\mathcal{O}_V \otimes \mathcal{O}_X(nL))$ is bounded for $n \geq 0$ by [Laz04, Theorem 1.4.40]. Hence (5.7) holds.

Now let us consider the exact sequence

$$0 \to \mathcal{O}_X(nL - B) \to \mathcal{O}_X(nL) \to \mathcal{O}_B(nL|_B) \to 0. \tag{5.8}$$

We claim that $h^2(X, nL)$ is bounded for $n \geq 0$. We write $g : B' \to B$ for the S_2-fication of B. By Lemma 2.19, we have $(K_X + B)|_{B'} \sim_{\mathbb{Q}} K_{B'} + \Delta_{B'}$ and $(B', \Delta_{B'})$ is an slc pair, where $|_{B'}$ denotes the pullback to B'. Note that by (5.4) we have

$$(K_{B'} + \Delta_{B'})^2 = B \cdot (K_X + B)^2 = 0,$$

and by assumptions (2) and (4) of the proposition we have

$$(K_{B'} + \Delta_{B'}) \cdot A|_{B'} = B \cdot (K_X + B) \cdot A > 0,$$

where A is an ample divisor on X. Hence $\nu(K_{B'} + \Delta_{B'}) = 1$. By Theorem 2.18,

$$\kappa(L|_{B'}) = \kappa(K_{B'} + \Delta_{B'}) = \nu(K_{B'} + \Delta_{B'}) = 1.$$

Hence

$$h^0(B', \mathcal{O}_{B'}(nm(K_{B'} + \Delta_{B'}))) = h^0(B', \mathcal{O}_{B'}(nL|_{B'}))$$

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grows with n. By Serre duality,

$$\begin{aligned} h^2(B', \mathcal{O}_{B'}(nL|_{B'})) &= h^0(B', \mathcal{O}_{B'}(K_{B'} - nL|_{B'})) \\ &= h^0(B', \mathcal{O}_{B'}(K_{B'} - nm(K_X + B)|_{B'})) \\ &= h^0(B', \mathcal{O}_{B'}(K_{B'} - nm(K_{B'} + \Delta_{B'}))) \\ &= 0 \end{aligned}$$

for $n \gg 0$. It follows that

$$h^2(B, \mathcal{O}_B(nL|_B))$$

is bounded for $n \geq 0$ by Lemma 5.4. Moreover, since

$$nL - B \sim_{\mathbb{Q}} K_X + (nm-1)(K_X + B)$$

and $K_X + B$ is a nef Weil divisor such that $\nu(K_X + B) = 2$, by Lemma 5.5 we have that $h^2(X, \mathcal{O}_X(nL - B))$ is bounded for $n \geq 0$. Therefore, by (5.8), $h^2(X, \mathcal{O}_X(nL))$ is bounded for $n \geq 0$.

If the inequality in (5.6) is strict, then by (5.5)

$$(K_V^2 + c_2(V)) \cdot \rho^* L > c_2(\Omega_X^{[1]}) \cdot L \geq 0.$$

Hence $\chi(V, \mathcal{O}_V(n\rho^* L))$ grows with n. By (5.7), $\chi(X, \mathcal{O}_X(nL))$, and hence $h^0(X, \mathcal{O}_X(nL))$, grow with n. This implies that $\kappa(K_X + B) > 0$. The assertion holds. Otherwise, we have $(K_X + B)|_{B'}$ is numerically trivial on $\operatorname{Supp} \Delta_{B'}$ by Proposition 5.3. Since $h^2(X, \mathcal{O}_X(nL))$ is bounded for $n \geq 0$, by (5.5), (5.6), and (5.7) there exists a constant t_2 such that

$$h^0(X, \mathcal{O}_X(nL)) \geq h^1(X, \mathcal{O}_X(nL)) + t_2. \tag{5.9}$$

Note that, by Theorem 2.16 and (5.4),

$$\begin{aligned} \chi(B', \mathcal{O}_{B'}(nL|_{B'})) &= \tfrac{1}{2}(nL|_{B'}) \cdot (nL|_{B'} - K_{B'}) + \chi(B', \mathcal{O}_{B'}) \\ &= \tfrac{1}{2}(nL|_{B'}) \cdot (-K_{B'} - \Delta_{B'} + \Delta_{B'}) + \chi(B', \mathcal{O}_{B'}) \\ &= \tfrac{1}{2}(nL|_{B'}) \cdot \Delta_{B'} + \chi(B', \mathcal{O}_{B'}) \\ &= \chi(B', \mathcal{O}_{B'}). \end{aligned}$$

Since $h^0(B', \mathcal{O}_{B'}(nL|_{B'}))$ grows with n and $h^2(B', \mathcal{O}_{B'}(nL|_{B'}))$ is bounded for $n \geq 0$, we have that $h^1(B', \mathcal{O}_{B'}(nL|_{B'}))$ grows with n. By Lemma 5.4, $h^1(B, \mathcal{O}_B(nL|_B))$ grows with n. Since $h^2(X, \mathcal{O}_X(nL - B))$ is bounded for $n \geq 0$, by (5.8), $h^1(X, \mathcal{O}_X(nL))$ grows with n. Hence, by (5.9), $h^0(X, \mathcal{O}_X(nL))$ grows with n. In conclusion, we have

$$\kappa(K_X + B) = \kappa(X, L) > 0,$$

and hence $K_X + B$ is semi-ample. This completes the proof of Proposition 5.6. □

5.3 The general case

We recall a standard lemma from [Kol92, Lemma 14.2].

Lemma 5.7. *Let (X, B) be a $\mathbb{Q}$-factorial dlt projective pair of dimension 3 over an algebraically closed field k of characteristic $p > 3$. Assume that X is terminal, $K_X + B$ is nef, and there exists an effective $\mathbb{Q}$-divisor D such that $D \sim_{\mathbb{Q}} K_X + B$ and $\operatorname{Supp} B \subseteq \operatorname{Supp} D$. Then there exists a $\mathbb{Q}$-factorial lc pair (Y, B_Y) such that the following hold:*

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(1) $(Y \setminus \operatorname{Supp} B_Y) \cong (X \setminus \operatorname{Supp} D)$ *is terminal, and* B_Y *is reduced;*

(2) $K_Y + B_Y \sim_{\mathbb{Q}} D_Y$ *for an effective* $\mathbb{Q}$*-divisor* D_Y *with* $\operatorname{Supp} D_Y = B_Y$*;*

(3) $K_Y + (1-\varepsilon)B_Y$ *is nef for any sufficiently small* $\varepsilon > 0$*;*

(4) $\kappa(K_X + B) = \kappa(K_Y + B_Y)$ *and* $\nu(K_X + B) = \nu(K_Y + B_Y)$*; and*

(5) *if* C *is a curve in* X *with* $C \cdot (K_X + B) > 0$*, then* (X, B) *is plt at the generic point of* C*.*

Proof. Let $g : W \to X$ be a log smooth resolution of $(X, \operatorname{Supp} D)$ with the reduced exceptional divisor E. Set $B_W := \operatorname{Supp}(g_*^{-1}D) + E$. Since (X, B) is dlt, we can write $K_W + B_W \sim_{\mathbb{Q}} D_W$, where $D_W = g^*D + (K_W + B_W - g^*(K_X + B))$ is an effective $\mathbb{Q}$-divisor with

$$\operatorname{Supp} D_W = \operatorname{Supp}(g_*^{-1}D) + E = B_W.$$

By replacing W by the output of a $(K_W + B_W)$-MMP over X by Theorem 2.20 and using again that (X, B) is dlt, we can assume that $W \setminus \operatorname{Supp} D_W \cong X \setminus \operatorname{Supp} D$ and $\operatorname{Supp} D_W = \operatorname{Supp} g^*D$.

Now, since $K_W + B_W$ is pseudo-effective, by Theorem 2.20 we can run a $(K_W + B_W)$-MMP which terminates with a minimal model $h : W \dashrightarrow Y$. In particular $K_Y + B_Y$ is nef for $B_Y := h_*B_W$. Furthermore, $K_Y + B_Y \sim_{\mathbb{Q}} D_Y$ and $\operatorname{Supp} D_Y = B_Y$, where $D_Y := h_*D_W$. Hence, (2) holds. Since h and g are isomorphisms outside of D_W, we get (1). Note that both $K_X + B$ and $K_Y + B_Y$ are effective and nef $\mathbb{Q}$-divisors. Applying [Kol92, Lemma 11.3.3] to pullbacks of $K_X + B$ and $K_Y + B_Y$ on any common resolution of X and Y, we get

$$\kappa(K_X + B) = \kappa(K_Y + B_Y), \quad \nu(K_X + B) = \nu(K_Y + B_Y).$$

Hence (4) holds. To get (3), note that, since $K_Y + B_Y \sim_{\mathbb{Q}} D_Y$ and $\operatorname{Supp} D_Y = B_Y$, we have that $K_Y + (1-\varepsilon)B_Y$ is effective for any sufficiently small $\varepsilon \geq 0$. Hence by Theorem 2.25, we can run a K_Y-MMP which is $(K_Y + B_Y)$-trivial and it terminates. Replacing (Y, B_Y) by the output, we get (3). Finally, since cyclic quotient surface singularities are klt (see [Sat25, Theorem A.3] for example), (5) follows from Lemma 5.2 and the proof of [Kol92, Lemma 14.2]. □

We can now prove our main result.

THEOREM 5.8. *Let* (X, B) *be an lc projective pair of dimension* 3 *over a perfect field* k *of characteristic* $p > 3$*. Assume that* $K_X + B$ *is nef and* $\nu(K_X + B) = 2$*. Then* $K_X + B$ *is semi-ample.*

Proof. By [GNT19, Remark 2.7], we may assume that k is algebraically closed. By Theorem 2.27, replacing (X, B) by a dlt model, we can assume that X is terminal, and (X, B) is $\mathbb{Q}$-factorial and dlt. By Definition 2.23 we run a K_X-MMP which is $(K_X + B)$-trivial. Since X is terminal, it terminates by Lemma 2.26. If it terminates with a Mori fiber space, then we have $n(K_X + B) \leq 2$. By Theorem 2.28, $K_X + B$ is semi-ample. Hence by Lemma 2.24 we can assume that the K_X-MMP which is $(K_X + B)$-trivial terminates with an lc pair (X', B') such that $K_{X'} + (1-\varepsilon)B'$ is nef for any sufficiently small $\varepsilon > 0$. By Theorem 2.29, we have

$$\kappa(K_X + (1-\varepsilon)B) = \kappa(K_{X'} + (1-\varepsilon)B') \geq 0$$

for any sufficiently small rational $\varepsilon > 0$. Hence there exists an effective $\mathbb{Q}$-divisor $\Delta \sim_{\mathbb{Q}} K_X + B$ such that $\operatorname{Supp} B \subseteq \operatorname{Supp} \Delta$. By Lemma 5.7, we can reduce the assertion to the case when (X, B) is an lc projective pair of dimension 3 and the following assertions hold:

(1) X is $\mathbb{Q}$-factorial, B is reduced, and $X \setminus B$ is terminal;

(2) $K_X + B \sim_{\mathbb{Q}} D$ for an effective $\mathbb{Q}$-divisor D with $\operatorname{Supp} D = B$;

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(3) $K_X + (1-\varepsilon)B$ is nef for any sufficiently small $\varepsilon > 0$;
(4) $\nu(K_X + B) = 2$; and
(5) if C is a curve in X with $C \cdot (K_X + B) > 0$, then (X, B) is plt at the generic point of C.

Then by Proposition 5.6, we have that $K_X + B$ is semi-ample. □

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