Ergod. Th. & Dynam. Sys. (1997), **17**, 1257–1266
Printed in the United Kingdom © *1997 Cambridge University Press*

The dynamics of contractions

A. F. BEARDON

Department of Pure Mathematics & Mathematical Statistics, 16 Mill Lane, University of Cambridge, Cambridge CB2 1SB, UK
(e-mail: A.F.Beardon@dpmms.cam.ac.uk)

(*Received* 17 *November* 1995 *and accepted in revised form* 20 *March* 1996)

Abstract. We show that, providing a metric space X has a boundary that is in some sense similar to the boundary of hyperbolic space, the iterates of a contraction $f : X \to X$ converge locally uniformly to a point in, or on the boundary of, X. This generalises the Denjoy–Wolff theorem for analytic self-maps of the unit disc in the complex plane, and also shows that if D is a bounded strictly convex subdomain of $\mathbb{R}^n$, then any contraction of D with respect to the Hilbert metric of D converges to a point in the closure of D.

1. *Introduction*

A *contraction* of a metric space (X, d) is a map $f : X \to X$ that satisfies

$$d(fx, fy) < d(x, y). \tag{1.1}$$

The contraction mapping theorem is that if f maps a complete metric space (X, d) into itself and satisfies the stronger condition

$$d(fx, fy) \leq kd(x, y), \tag{1.2}$$

where $0 \leq k < 1$, then the iterates f^n of f converge locally uniformly on X to the unique fixed point ζ of f in X. The Denjoy–Wolff theorem describes the dynamics of a contraction which satisfies (1.1) but not (1.2), and our aim in this paper is to generalise this so as to describe the dynamics of a contraction of a metric space (X, d) which, in a certain sense, resembles hyperbolic space. We shall embed X in a larger space $\overline{X}$ and be content with the conclusion that the iterates f^n of f converge to some point ζ in $\overline{X}$.

We begin by describing the Denjoy–Wolff theorem. The hyperbolic plane is the open unit disc Δ in the complex plane endowed with the Poincaré (or hyperbolic) metric ρ derived from the line element

$$ds = \frac{2|dz|}{1 - |z|^2}.$$

The Schwarz–Pick lemma is that if $f : \Delta \to \Delta$ is analytic, then either f is a ρ-isometry or a ρ-contraction in the sense of (1.1). Even though the contraction mapping theorem is not applicable, it is easy to see that if a ρ-contraction f has a fixed point ζ in Δ,

https://doi.org/10.1017/S0143385797086434 Published online by Cambridge University Press

then $f^n \to \zeta$ locally uniformly on Δ. Now a contraction $f : \Delta \to \Delta$ need not have a fixed point in Δ and when it does not, one can still conclude that there is a point ζ in the closed unit disc such that $f^n \to \zeta$ locally uniformly on Δ: this is the Denjoy–Wolff theorem.

In [**2**] the author showed that the Denjoy–Wolff theorem is a purely geometric result, and that analyticity is irrelevant, by showing that, for a large class of negatively curved Riemannian spaces (of any dimension), the iterates of contractions satisfying (1.1) do converge to a point in the closure of the manifold. In this paper, we give a generalisation in a different direction and establish a general theorem, the key ingredients of which are, roughly speaking:

(1) X is embedded in a reasonably good topological space $\overline{X}$;
(2) the boundary of X in $\overline{X}$ is metrically similar to the boundary of hyperbolic space;
(3) f is the pointwise limit of contractions f_j, each of which has a fixed point in X. It would be desirable to avoid (3) but this seems difficult.

We now state our result and, after the proof, we give some applications. In particular, we show that our result includes the classical Denjoy–Wolff theorem as a very simple special case. We shall also see that the result applies to a class of metric spaces introduced by Hilbert. In the Klein model of the hyperbolic plane, the hyperbolic metric is given by a cross-ratio and the geodesics are chords of Δ. Hilbert noted that this cross-ratio provides a metric for any bounded convex domain D of $\mathbb{R}^k$ and our result yields a Denjoy–Wolff theorem for these spaces.

The general nature of the assumptions made in (1) are not surprising; the innovation lies in the recognition that condition (2) can be given in a very general form *which avoids any smoothness assumptions on the space or the metric*. The precise content of (1) and (2) on the metric space (X, d), and the larger topological space $\overline{X}$ are as follows.

Assumption 1. We require:

(1a) every closed bounded subset of (X, d) is compact;
(1b) $\overline{X}$ is a compact Hausdorff topological space; and
(1c) X is an open dense subset of $\overline{X}$ whose relative topology coincides with the metric topology.

With these assumptions, the boundary ∂X of X is $\overline{X} - X$.

Assumption 2. Given that Assumption 1 holds, we require:

(2a) if $w \in X$, and if x_n in X converges to x in ∂X, then $d(x_n, w) \to +\infty$;
(2b) if x_n and y_n in X converge to distinct points x and y, respectively, in ∂X then, for every w in X,

$$d(x_n, y_n) - \max\{d(x_n, w), d(y_n, w)\} \to +\infty.$$

Several remarks may be helpful. First, (1b), (1c) and (2a) imply (1a). Next, a simple application of the triangle inequality shows that neither (2a) nor (2b) depend on w in the sense that they hold either for all w in X, or for no w in X. Finally, providing that there are at least two points in ∂X, (2b) implies (2a) because (from the triangle inequality)

$$d(x_n, y_n) \leq \max\{d(x_n, w), d(y_n, w)\} + \min\{d(x_n, w), d(y_n, w)\}.$$

https://doi.org/10.1017/S0143385797086434 Published online by Cambridge University Press

Thus, in effect, it suffices to assume that (1b), (1c) and (2b) hold. Our theorem on the dynamics of a contraction now follows.

THEOREM 1. *Suppose that the metric space (X, d) and the topological space $\overline{X}$ satisfy Assumptions 1 and 2, and that the contraction $f : X \to X$ satisfies condition (3). Then the iterates f^n of f converge locally uniformly on X to some point ζ in $\overline{X}$.*

§§2 and 3 contain two simple special cases of Theorem 1, and §4 contains its proof. §5 is concerned with the more substantial special case of Hilbert's metric, and for the proof of this we require an interesting variant of the classical intersecting chord theorem of Euclidean geometry; this is given in §6.

2. *Euclidean contractions of the real line*

Let $X = \mathbb{R}$ with the Euclidean metric d, and let $\overline{X}$ be $[-\infty, +\infty]$ with the usual topology. Assumption 1 holds as does Assumption 2 because if x_n and y_n converge to distinct boundary points x and y, then x and y can only be $+\infty$ and $-\infty$, so that for sufficiently large n,

$$d(x_n, y_n) - \max\{d(x_n, w), d(y_n, w)\} = \min\{d(x_n, w), d(y_n, w)\} \to +\infty.$$

Given a contraction $f : \mathbb{R} \to \mathbb{R}$ with respect to the Euclidean metric, we put $f_n = (1 - n^{-1})f$. Then $f_n \to f$ pointwise on X and, by the contraction mapping theorem, f_n has a fixed point in $\mathbb{R}$. We deduce that the iterates of a Euclidean contraction $f : \mathbb{R} \to \mathbb{R}$ converge locally uniformly on $\mathbb{R}$ to a point in $[-\infty, +\infty]$.

3. *The classical Denjoy–Wolff theorem*

Let X be the unit disc Δ in the complex plane with hyperbolic metric d, and let $\overline{X}$ be the Euclidean closure of X with the Euclidean metric; these satisfy Assumption 1. Next, the hyperbolic metric is known to satisfy (2b); indeed, if x_n and y_n are sequences in X converging to distinct points x and y in ∂X, then, using the cosine formula in hyperbolic geometry, we obtain

$$\cosh d(x_n, y_n) = \cosh d(x_n, w) \cosh d(y_n, w) - \cos\theta_n \sinh d(x_n, w) \sinh d(y_n, w),$$

where θ_n (the angle at w in the triangle with vertices w, x_n and y_n) converges to some angle θ with $\cos\theta \neq 1$. This shows that

$$d(x_n, y_n) = d(x_n, w) + d(y_n, w) + O(1), \quad d(x_n, w) \to +\infty, \quad d(y_n, w) \to +\infty \tag{3.1}$$

so that Assumption 2 also holds.

Now consider an analytic map $f : X \to X$ and exclude the case when f is a (Möbius) isometry of (X, d); then f is a d-contraction. We take $f_j = (1 - j^{-1})f$ and as this is an analytic map of X into (but not onto) itself, this is also a d-contraction. Further, as f_j maps X $(= \Delta)$ into a compact subset of itself, we see that $|z| > |f_j(z)|$ on the unit circle; thus, by Rouché's theorem, f_j has a fixed point in X and (3) holds. Theorem 1 is now applicable and this gives the classical Denjoy–Wolff theorem.

https://doi.org/10.1017/S0143385797086434 Published online by Cambridge University Press

A *horoball based at* ζ is a Euclidean disc in Δ internally tangent to the unit circle at ζ; these play an important role in all proofs of the Denjoy–Wolff theorem. Indeed, if an analytic contraction f does not have a fixed point in Δ, then there is a unique point ζ in the closed unit disc such that, for each horoball H based at ζ, $f(H) \subset H$ (Julia's lemma) and it is this that guarantees that if an orbit $f(z), f^2(z), \ldots$ accumulates only on the boundary of Δ, then it must converge to ζ. The same idea will appear in our proof of Theorem 1.

4. *The proof of Theorem 1*

By assumption, each f_j has a (necessarily unique) fixed point a_j in X, and the proof divides into two cases according to whether the sequence a_j has (i) a limit point in X, or (ii) a limit point in ∂X. As $\overline{X}$ is compact one of these cases must arise, and we shall see that the two cases correspond to the cases $\zeta \in X$ and $\zeta \in \partial X$, respectively. We begin the proof of Theorem 1 with several propositions, the first of which shows that ∂X reflects the nature of the boundary of hyperbolic space. *In each of these propositions we assume, without explicit mention, that the hypotheses in Theorem 1 hold.*

PROPOSITION 4.1.

(i) *If x_n and y_n are sequences in X converging to distinct points x and y, respectively, in ∂X then, $d(x_n, y_n) \to +\infty$.*

(ii) *If x_n and y_n are sequences in X, if x_n converges to a point x in ∂X, and if for all n, $d(x_n, y_n) \leq M$, then $y_n \to x$.*

Proof. Of course, (i) is an immediate consequence of Assumption 2 and it is given here to compare it with (ii). If (ii) fails, then there is some subsequence of y_n which converges to some point v in $\overline{X}$, where $v \neq x$. If $v \in X$, this contradicts (2a); if $v \in \partial X$, this contradicts (i). □

PROPOSITION 4.2. *Suppose that $x_j \to x$, where x_j and x are in X. Then $f_j(x_j) \to f(x)$.*

Proof. This follows directly from the assumption that $f_j \to f$ and the inequalities

$$d(f_j(x_j), f(x)) \leq d(f_j(x_j), f_j(x)) + d(f_j(x), f(x)) \leq d(x_j, x) + d(f_j(x), f(x)). \quad \square$$

PROPOSITION 4.3. *If $x \in X$ and some subsequence of $f^n(x)$ converges to ζ in X, then $f(\zeta) = \zeta$.*

Proof. Because f is a contraction, the sequence $d(f^n(x), f^{n+1}(x))$ is decreasing with n and so converges to some limit δ. By letting n tend to ∞ through the subsequence $\{n_1, n_2, \ldots\}$ on which $f^n(x) \to \zeta$, we see that $\delta = d(\zeta, f(\zeta))$. By letting n tend to ∞ through the subsequence $\{n_1 + 1, n_2 + 1, \ldots\}$ we see that $\delta = d(f(\zeta), f^2(\zeta))$, so that

$$d(\zeta f(\zeta)) = d(f(\zeta), f^2(\zeta)).$$

As f is a contraction, this implies that $f(\zeta) = \zeta$. □

PROPOSITION 4.4. *Suppose that some subsequence of (a_j) converges to ζ in X. Then $f(\zeta) = \zeta$ and $f^n \to \zeta$ locally uniformly on X.*

https://doi.org/10.1017/S0143385797086434 Published online by Cambridge University Press

Proof. We may assume that $a_j \to \zeta$ as $j \to \infty$ through some subsequence $J = \{j_1, j_2, \ldots\}$ of the positive integers. Taking the limit as $j \to \infty$ through J we have, from Proposition 4.2,

$$f(\zeta) = \lim f_j(a_j) = \lim a_j = \zeta.$$

Next, we show that $f^n \to \zeta$ pointwise on X. Consider any x in X and let $\overline{B}$ be the closed ball with centre ζ and radius $d(x, \zeta)$. As f is a contraction that fixes ζ, f maps $\overline{B}$ into itself. Further, $x \in \overline{B}$ so that the orbit $\mathcal{O}(x)$ of x (that is the sequence $x, f(x), f^2(x), \ldots$) lies in $\overline{B}$. Now $\overline{B}$ is compact, so $\mathcal{O}(x)$ certainly has limit points (in $\overline{B}$) and from Proposition 4.3, any such limit point is a fixed point of f. As f is a contraction, it has at most one fixed point and so we conclude that the sequence $\mathcal{O}(x)$ converges to ζ. This shows that $f^n \to \zeta$ pointwise on X.

To prove that the convergence is locally uniform on X, take any positive ε, and any compact subset K of X and cover K with a finite number of open balls $B(w_j, \varepsilon/2)$, $j = 1, \ldots, r$, of radius $\varepsilon/2$. If n is sufficiently large then, for each j, $f^n(w_j)$ lies within a distance $\varepsilon/2$ of ζ, and hence $f(B(w_j, \varepsilon/2)) \subset B(\zeta, \varepsilon)$. We deduce that, for these n, $f(K) \subset B(\zeta, \varepsilon)$ and this completes the proof. □

Proposition 4.4 show that if the sequence a_j has a limit point ζ in X, then ζ is a fixed point of f and $f^n \to \zeta$ locally uniformly on X. Proposition 4.3 shows that if, for some x in X, the sequence $f^n(x)$ has a limit point ζ in X, then again $f(\zeta) = \zeta$, and (as in the proof of Proposition 4.4), $f^n \to \zeta$ locally uniformly on X. *It follows that for the remainder of the proof we may assume*:

(a) that the sequence a_j accumulates only on ∂X; and

(b) for all x in X the sequence $f^n(x)$ accumulates only on ∂X.

We come now to the definition of horoballs in this abstract situation. From (a), we may suppose that there is some subsequence a_j of a_n, where $j \in J$ and $J = \{j_1, j_2, \ldots\}$, that converges to the point ζ in ∂X. We also select a point y in X and, for j in J, we let B_j be the closed ball in X with centre a_j and radius $d(a_j, y)$. Given this initial data (namely a_j converging to ζ, and the point y) we define the horoball $\mathcal{H}$ by saying that $x \in \mathcal{H}$ if and only if x (in X) is the limit of some sequence x_k, where $x_k \in B_k$, and $k \in J$. For clarity, we suppress the notational dependence of $\mathcal{H}$ on a_j, ζ and y.

PROPOSITION 4.5. *With the notation above, $f(\mathcal{H}) \subset \mathcal{H}$.*

Proof. Take x in $\mathcal{H}$ with $x = \lim x_k$ and $x_k \in B_k$. As f_j is a contraction that fixes a_j, and as x_j lies in the ball B_j with centre a_j, we see that $f_j(x_j) \in B_j$. By Proposition 4.2, $f_j(x_j) \to f(x)$, so that $f(x) \in \mathcal{H}$ as required. □

PROPOSITION 4.6. *If a sequence v_n in $\mathcal{H}$ converges to α on ∂X, then $\alpha = \zeta$.*

Proof. As $v_1 \in \mathcal{H}$, it is the limit of some sequence u_m, where $u_m \in B_m$ and $m \in J$, so that we can pick some u_m satisfying $d(u_m, v_1) < 1$. Put $m = k(1)$ and relabel u_m as v_1'; thus

$$d(v_1', v_1) < 1, \quad v_1' \in B_{k(1)}.$$

https://doi.org/10.1017/S0143385797086434 Published online by Cambridge University Press

As $v_2 \in \mathcal{H}$, it is the limit of some sequence t_m, where $t_m \in B_m$, so that we can pick some t_m satisfying $d(t_m, v_1) < 1$, where $m > k(1)$. Put $m = k(2)$ and relabel t_m as v_2'; then

$$d(v_2', v_2) < 1, \quad v_2' \in B_{k(2)}, \quad k(2) > k(1).$$

Continuing in this way, we construct a strictly increasing sequence $k(1), k(2), \ldots$ of positive integers, and a sequence $v_1', v_2', \ldots$ such that for all n,

$$d(v_n', v_n) < 1, \quad v_n' \in B_{k(n)}.$$

As $v_n \to \alpha$, Proposition 4.1(ii) implies that $v_n' \to \alpha$. As $v_n' \in B_{k(n)}$, we see that

$$d(v_n', a_{k(n)}) < d(y, a_{k(n)}) \leq \max\{d(y, a_{k(n)}), d(y, v_n')\},$$

so that

$$d(v_n', a_{k(n)}) - \max\{d(y, a_{k(n)}), d(y, v_n')\} \leq 0.$$

This contradicts Assumption 2(b) unless $\alpha = \zeta$ $(= \lim a_j)$; thus, as asserted, $\alpha = \zeta$. □

We can now complete the proof of Theorem 1. Take any y in X and construct the horoball $\mathcal{H}$ from the initial data of a_j, where $j \in J$, ζ and the point y. As $y \in B_n$ for every n, we see that $y \in \mathcal{H}$, and by Proposition 4.5, the entire sequence $f^n(y)$, $n = 1, 2, \ldots$ lies in $\mathcal{H}$. From (b) this sequence has all its limit points on ∂X and, from Proposition 4.6, such a limit point can only be ζ. As $\overline{X}$ is compact, it follows that $\lim_n f^n(y) = \zeta$. It is now clear that $f^n \to \zeta$ pointwise on X since, given any v in X, we have $d(f^n(y), f^n(v)) < d(y, v)$, and Proposition 4.1(ii) now shows that $f^n(v) \to \zeta$.

It remains to show that this convergence is locally uniform on X and it suffices to show that if $\mathcal{N}$ is any $\overline{X}$-open set containing ζ, and if B is a closed ball in X with centre y, then $f^n(B) \subset \mathcal{N}$ for all sufficiently large n. However, if this fails, then (on a subsequence) there are points x_n in B for which $f^n(x_n)$ lies in the compact set $\overline{X} - \mathcal{N}$ and so converges to some v in ∂X, where $v \neq \zeta$. As this violates Proposition 4.1(ii), the proof of Theorem 1 is complete. □

5. *Contractions in Hilbert's geometry*

From now on we shall be working in Euclidean space $\mathbb{R}^k$. The Euclidean distance between x and y will be denoted by $|x - y|$, and the cross-ratio $[x_1, x_2, x_3, x_4]$ of the four distinct points a, x, y, b in $\mathbb{R}^k$ is given by

$$[a, x, y, b] = \frac{|a - y| \cdot |x - b|}{|a - x| \cdot |y - b|}. \tag{5.1}$$

For distinct points x and y in $\mathbb{R}^k$, L_{xy} will denote the Euclidean line through x and y, and (x, y) and $[x, y]$ will denote the open and closed segments, respectively, with endpoints x and y. If x and y lie in a convex domain D in $\mathbb{R}^k$, and $x \neq y$, the chord of D determined by x and y is the intersection $L_{xy} \cap \overline{D}$. We shall adopt the convention that when we say that $\langle a, x, y, b\rangle$ is a chord of D, we mean that x and y are in D, a and b are in ∂D, and that a, x, y, b occur *in this order* along L_{xy}; note that, in this case, $|a - y| > |a - x|$ and $|x - b| > |y - b|$ so that $\log[a, x, y, b] > 0$.

https://doi.org/10.1017/S0143385797086434 Published online by Cambridge University Press

We can now define Hilbert's metric d for a bounded convex domain D in $\mathbb{R}^k$. Distinct points x and y in D determine a chord $\langle a, x, y, b\rangle$ of D and, following Klein's construction of a model of hyperbolic geometry, Hilbert noticed that the function d defined by

$$d(x, y) = \begin{cases} \log[a, x, y, b] & \text{if } x \neq y \\ 0 & \text{if } x = y, \end{cases}$$

is a metric on D; this is the Hilbert metric of D, and more details can be found in [**1–6**].

Next, a subdomain D of $\mathbb{R}^k$ is *strictly convex* if, for all a and b on the boundary ∂D of D, the open segment (a, b) lies in D. With this, we can state the particular form that Theorem 1 takes when applied to Hilbert's metric on a strictly convex subdomain.

THEOREM 1a. *Let D be a bounded, strictly convex, subdomain of $\mathbb{R}^k$, and suppose that $f: D \to D$ is a contraction with respect to the Hilbert metric d of D. Then the iterates f^n of f converge locally uniformly on D to some point ζ in the Euclidean closure of D.*

In our application of Theorem 1 to Hilbert's geometry, we take X to be bounded, strictly convex, domain D in $\mathbb{R}^k$ equipped with its Hilbert metric d, and $\overline{X}$ to be the Euclidean closure of X equipped with the Euclidean topology. Clearly, Assumption 1 is satisfied provided that the d-topology on D agrees with the Euclidean topology there, and this is easily verified. Indeed, using the (trivial) relation

$$|a - y| \cdot |b - x| = |b - y| \cdot |a - x| + |x - y| \cdot |a - b|,$$

we find that if $\langle a, x, y, b\rangle$ is a chord of D, then

$$\exp d(x, y) = 1 + \frac{|x - y| \cdot |a - b|}{|b - y| \cdot |a - x|}, \tag{5.2}$$

and this shows that the two topologies agree on D. We conclude that *Assumption 1 is satisfied*.

Next, we turn our attention to condition (3). Any simply connected proper subdomain D of the complex plane inherits a hyperbolic metric ρ_D from Δ, and this metric satisfies a monotonicity principle: if $D_1 \subset D_2$ then $\rho_2 \leq \rho_1$, where ρ_j is the hyperbolic metric on D_j. Motivated by this and the connection that Hilbert's metric has with hyperbolic geometry, we observe that Hilbert's metric also satisfies this monotonicity property.

LEMMA 5.1. *Let D_1 and D_2 be bounded convex domains in $\mathbb{R}^k$ with Hilbert metrics d_1 and d_2, respectively, and suppose that $D_1 \subset D_2$. Then $d_2 \leq d_1$. Further, for distinct points x and y in D_1, $d_1(x, y) = d_2(x, y)$ if and only if $L_{xy} \cap D_1 = L_{xy} \cap D_2$.*

Proof. Take x and y in D_1, and let these determine the chords $\langle a_j, x, y, b_j\rangle$ of D_j, $j = 1, 2$. Then a_2, a_1, x, y, b_1, b_2 occur in this order along the line L_{xy}, and

$$[a, x, y, b] \leq [a', x, y, b'],$$

with equality if and only if $[a, b] = [a', b']$. □

Lemma 5.1 has a useful corollary.

LEMMA 5.2. *Suppose that $x_0 \in D$ and $0 \leq l < 1$. Then $g : x \mapsto x_0 + l(x - x_0)$ is a d-contraction of D.*

https://doi.org/10.1017/S0143385797086434 Published online by Cambridge University Press

Proof. As g preserves cross-ratios and straight lines, it is an isometry of D onto $g(D)$; let these domains have Hilbert metrics d and d', respectively. As $g(D)$ lies in a compact subset of D, Lemma 5.1 implies that $d < d'$ so that $d(gx, gy) < d'(gx, gy) = d(x, y)$. □

Lemma 5.2 enables us to show that condition (3) is satisfied. Given a d-contraction f, we select a point x_0 in D and, for $n = 1, 2, \ldots$, define

$$f_n(x) = g_n(f(x)), \quad g_n(x) = x_0 + (1 - n^{-1})(x - x_0).$$

As the composition of two contractions is again a contraction, f_n is a d-contraction of D. Further, as $f_n(D)$ lies in a compact subset of D, the argument used to prove Proposition 4.4 shows that f_n has a fixed point in D; thus *condition (3) is satisfied.*

We now consider Assumption 2. We know that (2a) will follow from (2b) but it seems that a significant effort is required to establish (2b) and for this we need the following result.

LEMMA 5.3. *Suppose that D is strictly convex. Then the end-points $a(x, y)$ and $b(x, y)$ of the chord $\langle a, x, y, b\rangle$ are continuous functions of (x, y) on $\overline{D} \times \overline{D} - \{(x, y) : x = y\}$. In particular, if $x_n \to \bar{x}$, $y_n \to \bar{y}$, $\bar{x} \neq \bar{y}$ and $\bar{x} \in \partial D$, then $a(x_n, y_n) \to \bar{x}$.*

Remark. The assumption of strict convexity is necessary. For example, if $D = (-2, 2) \times (0, 1)$, $x_n = n^{-1} + in^{-2}$ and $y_n = 1 + in^{-1}$, then $x_n \to 0$, $y_n \to 1$ and $a(x_n, y_n) = 0$ whereas $a(0, 1) = -2$.

Proof of Lemma 5.3 We consider two sequences (x_n) and (y_n) in $\overline{D}$ which converge to distinct points $\bar{x}$ and $\bar{y}$, respectively, and let $\langle a_n, x_n, y_n, b_n\rangle$ and $\langle a, \bar{x}, \bar{y}, b\rangle$ be chords of D. We have to show that $a_n \to a$ and $b_n \to b$, and it suffices to show that $a_n \to a$. As $x_n \to \bar{x}$, $y_n \to \bar{y}$ and $\bar{x} \neq \bar{y}$, it is evident that any accumulation point α of the sequence a_n lies on that half-line L^+ of $L_{\bar{x},\bar{y}}$ which has end-point $\bar{x}$ and which does not contain $\bar{y}$. As a_n lies in the closed set ∂D, so does α; thus $\alpha \in \partial D \cap L^+$. Now, by the strict convexity of D, $[a, b] \cap \partial D = \{a, b\}$ and $\partial D \cap L^+ = \{a\}$; thus $\alpha = a$ and $a_n \to a$ as required. The last assertion in the lemma follows because $a(x_n, y_n) \to a(\bar{x}, \bar{y})$ and, when $\bar{x} \in \partial D$, we have $a(\bar{x}, \bar{y}) = \bar{x}$. The proof of Lemma 5.3 is complete. □

We return now to the problem of verifying Assumption 2 in the situation described in Theorem 1a, and consider points w, x_n and y_n in D with x_n and y_n converging to $\bar{x}$ and $\bar{y}$, respectively, on ∂D. Without loss of generality, we may assume that

$$d(x_n, w) \leq d(y_n, w) = \max\{d(x_n, w), d(y_n, w)\}. \tag{5.3}$$

Next, we denote the chords of D through w and y_n, and through x_n and y_n by

$$\langle u_n, w, y_n, v_n\rangle, \quad \langle a_n, x_n, y_n, b_n\rangle,$$

respectively (the reader is urged to draw a diagram). From the definition of d, we see that

$$\exp[d(x_n, y_n) - d(w, y_n)] = \frac{|a_n - y_n| \cdot |x_n - b_n| \cdot |u_n - w| \cdot |y_n - v_n|}{|a_n - x_n| \cdot |y_n - b_n| \cdot |u_n - y_n| \cdot |w - v_n|}. \tag{5.4}$$

https://doi.org/10.1017/S0143385797086434 Published online by Cambridge University Press

Recalling that $x_n \to \bar{x}$ and $y_n \to \bar{y}$, where $\bar{x}$ and $\bar{y}$ are *distinct* points of ∂D, and (from Lemma 5.3) that $a_n \to \bar{x}$ and $b_n \to \bar{y}$, we see that there is some number M (depending only on w and D) such that

$$\exp[d(x_n, y_n) - d(w, y_n)] \geq \frac{M}{|a_n - x_n|}\left(\frac{|y_n - v_n|}{|y_n - b_n|}\right). \tag{5.5}$$

Because of (5.3), (2b) is equivalent to

$$d(x_n, y_n) - d(w, y_n) \to +\infty,$$

so we seek a lower bound of the right-hand side of (5.5) that tends to $+\infty$ as $n \to \infty$. As

$$|a_n - x_n| \leq |a_n - \bar{x}| + |\bar{x} - x_n| \to 0,$$

it suffices to show that for some positive number m, depending only on w and D,

$$\frac{|y_n - v_n|}{|y_n - b_n|} \geq m. \tag{5.6}$$

If D were a Euclidean ball, the intersecting chord theorem would imply that

$$\frac{|y_n - v_n|}{|y_n - b_n|} = \frac{|y_n - a_n|}{|y_n - u_n|},$$

and as a_n is approximately $\bar{x}$, and y_n is approximately $\bar{y}$, the ratio $|y_n - a_n|/|y_n - u_n|$ would be bounded below by a positive quantity and (5.6) would follow (this shows that a Denjoy–Wolff theorem holds for the Klein model of hyperbolic space). Of course, the intersecting chord theorem fails for many strictly convex domains, but we can and shall establish a version of it that does imply the validity of (5.6). The next section contains our version of the intersecting chord theorem, stated and proved as an independent result, and with this the proof of Theorem 1a will be complete.

6. *The intersecting chord theorem*

The intersecting chord theorem states that if two intersecting chords of a circle are divided into segments of lengths d_1 and d_2, and lengths ℓ_1 and ℓ_2, respectively, then $d_1 d_2 = \ell_1 \ell_2$. In this section we are concerned with the extent to which an analogous result is true, or false, when the circle is replaced by a curve C bounding a bounded convex plane domain D, where, of course, we only ask that, for some positive number M,

$$M^{-1} \leq \frac{d_1 d_2}{\ell_1 \ell_2} \leq M. \tag{6.1}$$

It is easy to see that we must restrict our attention to strictly convex domains, and even then it is clear that no such result holds without further restrictions.

We shall now show that if D is a bounded, strictly convex domain then (6.1) holds for all pairs of chords whose lengths are bounded below by some fixed positive number. While this may not at first seem surprising, we emphasise that it is true without being given any algebraic information (for example, the equation of ∂D), and without any analytic information (for example, the smoothness, or curvature, of ∂D).

https://doi.org/10.1017/S0143385797086434 Published online by Cambridge University Press

THE INTERSECTING CHORD THEOREM. *Let D be a bounded strictly convex plane domain, and let α and β be intersecting chords of D with d_1, d_2, ℓ_1 and ℓ_2 as above. For each positive δ, there is a positive number M such that, if α and β each have length at least δ, then (6.1) holds.*

Proof. Our proof uses the Hilbert metric for D. Take two chords $[a, b]$ and $[u, v]$ of D that meet at a point x in D. We may assume that these are labelled so that

$$|a - x| \geq |x - b|, \quad |u - x| \geq |x - v|,$$

and we let c and w be the mid-points of $[a, b]$ and $[u, v]$, respectively; the reader should draw a diagram, noting that $c \in [a, x]$ and $w \in [u, x]$.

The triangle inequality for Hilbert's metric yields

$$-d(c, w) \leq d(c, x) - d(x, w) \leq d(c, w),$$

and hence

$$\exp(-d(c, w)) \leq \frac{[a, c, x, b]}{[u, w, x, v]} \leq \exp(d(c, w)).$$

By the symmetry of the situation, it suffices to establish the upper bound only. Expressing these cross-ratios as ratios of distances, and using $|a-c| = |b-c|$ and $|u-w| = |w-v|$, we easily see that

$$\frac{|x - v|}{|x - b|} \leq \left(\frac{|x - u|}{|x - a|}\right) \exp d(c, w),$$

and hence that

$$\frac{|x - v| \cdot |x - u|}{|x - b| \cdot |x - a|} \leq \left(\frac{|x - u|}{|x - a|}\right)^2 \exp d(c, w) \leq \left(\frac{|D|}{\delta}\right)^2 \exp d(c, w), \tag{6.2}$$

where $|D|$ is the Euclidean diameter of D.

Now each chord is parametrised by its endpoints. The space of chords can therefore be identified with the compact space $\partial D \times \partial D$, and the space $\mathcal{C}$ of pairs of chords, each of length at least δ, is also compact. The strict convexity of D ensures that c and w (the mid-points of the chords) are in D, and as the function $d(c, w)$ is defined (that is, is finite) and continuous on $\mathcal{C}$, it is bounded above there; thus by (6.2), the proof of the theorem is complete. □

REFERENCES

[1] H. S. Bear. Part metric and hyperbolic metric. *Amer. Math. Monthly* **98** (1991), 109–123.

[2] A. F. Beardon. Iteration of contractions and analytic maps. *J. London Math. Soc.* **41** (1990), 141–150.

[3] H. Busemann. *The Geometry of Geodesics.* Academic Press, 1955.

[4] H. Busemann and P. J. Kelly. *Projective Geometry and Projective Metrics*. Academic Press, 1953.

[5] P. de la Harpe. Hilbert's metric for simplices. *Geometric Group Theory (London Math. Soc. Lecture Notes 182)*. Eds G. A. Niblo and M. A. Roller. Cambridge University Press, 1993, pp. 97–119.

[6] D. Hilbert. Ueber die gerade Linie als kürzeste Verbindung zweier Punkte. *Math. Ann.* **46** (1895), 91–96.

https://doi.org/10.1017/S0143385797086434 Published online by Cambridge University Press