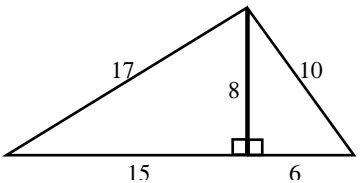
103.29 Heronian triangles and squares in arithmetic progression

Introduction

There are many known results concerning Heronian triangles i.e. triangles with integer sides and area. An introduction to these ideas can be found in [1].

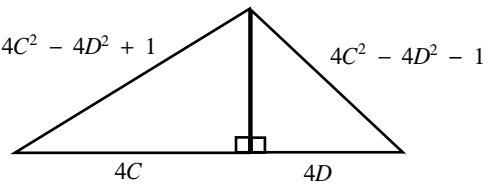
An aspect of Heronian triangles which seems to have received little attention is the ratio of the lengths into which a side of the triangle is divided by an altitude.



For example, the ratio 5 : 2 is easily obtained as shown. However. is it possible to construct a triangle to give the ratio 2 : 1?

It should be noted that for triangles with integer sides (i.e. if we drop the requirement for the area to be an integer) all integer ratios are possible.

The diagram shows how to obtain the ratio $C : D$ for any two positive integers C and D, with $C > D$.



However, for Heronian triangles we shall see that finding possible ratios is of a different level of complexity.

The ratio 2 : 1

After 1 : 1, arguably the simplest ratio is 2 : 1. This ratio also has added interest because of a connection with a classical problem of number theory. First we shall prove that 2 : 1 is not a possible ratio for a Heronian triangle. We shall use Fermat's method of infinite descent, written out as a proof by minimal counterexample.

Theorem 1: An altitude of a Heronian triangle cannot divide the base in the ratio of 2 : 1.

Proof: The altitudes of Heronian triangles are rational so the existence of a Heronian triangle with a ratio of 2 : 1 implies that there is a triangle as shown in Figure 1 with H, L, U and V positive integers.

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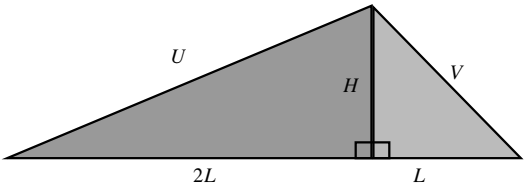


FIGURE 1

Any common factor of H and L is a factor of all the side-lengths and can be removed.

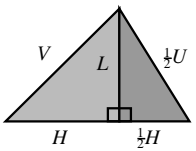


FIGURE 2

If H is even then so is U and we can then consider the triangle in Figure 2.

We can therefore suppose that H and L are coprime and that H is odd. For triangles as shown in Figure 1, with H, L, U and V positive integers, H and L coprime and H odd, let Figure 1 represent the triangle with the smallest possible base.

By Euclid's formula for Pythagorean triples, there are positive integers a, b, c and d, with a and b of opposite parity, such that

$$H = a^2 - b^2 = c^2 - d^2,\ 2L = 2ab,\ L = 2cd$$
$$\Rightarrow a^2 + d^2 = b^2 + c^2,\ ab = 2cd.$$

Assume that a is odd. (If not then interchange $a \leftrightarrow b$ and $c \leftrightarrow d$ in the following argument.) Then

$$a^2\left(a^2 + d^2\right) = a^2\left(b^2 + c^2\right) = c^2\left(a^2 + 4d^2\right)$$
$$\Rightarrow a^2 + d^2 = Tx^2,\ a^2 + 4d^2 = Ty^2$$

where x^2 and y^2 are integer squares and T is square-free.

If $T = 1$ then we would have a new Heronian triangle with odd height a and base $3d < 6cd = 3L$. This contradicts the minimality of the triangle in Figure 1.

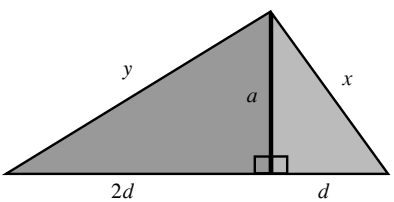


FIGURE 3

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So suppose $T \neq 1$. Any common prime divisor of a and d would also be a divisor of both x and y and can be cancelled. We now have a and d coprime and

$$3a^2 = T\left(4x^2 - y^2\right),\ 3d^2 = T\left(y^2 - x^2\right).$$

Any prime divisor of T other than 3 would be a common factor of a and d and so $T = 3$. Then, $a^2 + d^2 \equiv 0 \,(\mathrm{mod}\, 3)$. However, all squares are 0 or 1 modulo 3 and so 3 is a factor of a and d, a contradiction which completes the proof of Theorem 1.

It is perhaps worth noting that Theorem 1 can be restated in purely algebraic terms: there are no non-zero integers a and d such that both $a^2 + d^2$ and $a^2 + 4d^2$ are squares.

Squares in arithmetic progression

Number theoretical problems can be divided into those amenable to elementary methods, for example factorisation and modular arithmetic, and those of greater 'depth'. Many deeper results require some form of infinite descent. In modern proofs this descent may take the form of a proof by minimal counterexample or an analysis involving elliptic curves whereas Fermat, on the rare occasions when he gave specific details of his proofs, used arguments similar to those used above in Theorem 1, [2].

One of the major results of Leonardo Pisano (Fibonacci) concerned a general rule for finding three squares in arithmetic progression, such as 1, 25, 49 and 49, 169, 289. This is Proposition 14 of *Liber quadratorum*, [3].

A particularly nice rule was referred to by Diophantus when solving Problem 19 of Book 3 and Problem 7 of Book 5 of the *Arithmetica*, [4]. For a given Pythagorean triangle, Diophantus gave the square on the hypotenuse as the middle number of the progression and the common difference as twice the product of the other two sides. The two progressions given above are formed by this method from the (3, 4, 5) and (5, 12, 13) triangles, respectively.

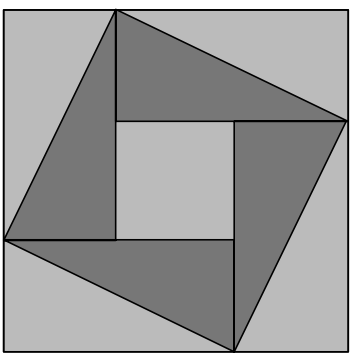


FIGURE 4: A 'proof without words' for the method used by Diophantus

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The result of Diophantus is an elementary one based upon the algebra of Pythagoras' Theorem. However, Fermat claimed the much deeper result that it is not possible to have four integer squares in arithmetic progression. This result is of the non-elementary type and is one where the primes 2 and 3 have to be dealt with carefully during the descent and where some equations will lead to an infinite loop (corresponding to arithmetic progressions of common difference 0).

Nevertheless, a 'strong' version of Fermat's result is an almost immediate corollary of Theorem 1.

Theorem 2: Suppose four positive integers form a strictly increasing arithmetic progression. Then the product of the integers cannot be a square. Hence four integer squares cannot be in arithmetic progression.

Proof: Suppose such a progression exists. We can scale the four terms so that they are all even integers. They can then be expressed as $r - 3s$, $r - s$, $r + s$ and $r + 3s$ for two positive integers r and s. Then for some positive integer x the product of these terms is

$$r^4 - 10r^2s^2 + 9s^4 = x^2.$$

Theorem 1 would then be contradicted by this Heronian triangle:

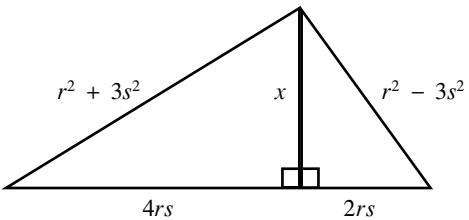


FIGURE 5

Concluding remarks

We have seen how the impossibility of constructing one particular ratio in a Heronian triangle solves a non-elementary problem in number theory. The general problem of determining which ratios can and cannot be constructed should be a rich area for further investigation.

References

1. https://en.wikipedia.org/wiki/Heronian_triangle
2. Sir T. L. Heath, *Diophantus of Alexandria*, Dover (1964) pp. 293-297.
3. L. E. Sigler, *The book of squares*, Academic Press (1987) pp. 53-74.
4. Sir T. L. Heath, *Diophantus of Alexandria*, Dover (1964) pp. 166,205.

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STAN DOLAN
126A Harpenden Road, St Albans AL3 6BZ
e-mail: *stan@standolan.co.uk*

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