Combinatorics, Probability and Computing (2012) 21, 113–127. © Cambridge University Press 2012
doi:10.1017/S0963548311000575

On Even-Degree Subgraphs of Linear Hypergraphs

D. DELLAMONICA Jr.[1†], P. HAXELL[2†], T. ŁUCZAK[1,3†],
D. MUBAYI[4†], B. NAGLE[5†], Y. PERSON[6†],
V. RÖDL[1†], M. SCHACHT[7†] and J. VERSTRAËTE[8†]

[1]Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322, USA
(e-mail: ddellam@mathcs.emory.edu, rodl@mathcs.emory.edu)
[2]Department of Combinatorics and Optimization, University of Waterloo, Waterloo, ON N2L 3G1, Canada
(e-mail: pehaxell@math.uwaterloo.ca)
[3]Department of Discrete Mathematics, Faculty of Mathematics and Computer Science,
Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland
(e-mail: tomasz@amu.edu.pl)
[4]Department of Mathematics, Statistics, and Computer Science, University of Illinois, Chicago, IL 60607, USA
(e-mail: mubayi@math.uic.edu)
[5]Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA
(e-mail: bnagle@math.usf.edu)
[6]Institute of Mathematics, Freie Universität Berlin, 14195 Berlin, Germany
(e-mail: person@math.fu-berlin.de)
[7]Fachbereich Mathematik, Universität Hamburg, Bundesstraße 55, D-20146 Hamburg, Germany
(e-mail: schacht@math.uni-hamburg.de)
[8]Department of Mathematics, University of California, San Diego (UCSD), La Jolla, CA 92093-0112, USA
(e-mail: jacques@ucsd.edu)

Received 13 April 2011; revised 12 October 2011; first published online 2 February 2012

Dedicated to the memory of Professor Richard H. Schelp

A subgraph of a hypergraph H is *even* if all its degrees are positive even integers, and *b-bounded* if it has maximum degree at most b. Let $f_b(n)$ denote the maximum number of edges in a *linear* n-vertex 3-uniform hypergraph which does not contain a b-bounded even

† D.D. was supported by a CAPES/Fulbright scholarship. P.H. was partially supported by NSERC. T.Ł. was partially supported by the Foundation for Polish Science. D.M. was supported in part by NSF grants DMS 0653946 and DMS 0969092. B.N. was supported by NSF grant DMS 1001781. Y.P. was supported by GIF grant I-889-182.6/2005. V.R. was supported by NSF grant DMS 0800070. M.S. was supported by the Heisenberg-Programme of the Deutsche Forschungsgemeinschaft (DFG Grant SCHA 1263/4-1). J.V. was supported by an Alfred P. Sloan Research Fellowship.
This research was done when the first eight authors participated in the BIRS Focused Research Group *Sparse Pseudorandom Objects* in May 2010 (Banff, Canada).

https://doi.org/10.1017/S0963548311000575 Published online by Cambridge University Press

subgraph. In this paper, we show that if $b \geqslant 12$, then

$$\frac{n \log n}{3b \log \log n} \leqslant f_b(n) \leqslant Bn(\log n)^2$$

for some absolute constant B, thus establishing $f_b(n)$ up to polylogarithmic factors. This leaves open the interesting case $b = 2$, which is the case of 2-regular subgraphs. We are able to show for some constants $c, C > 0$ that

$$cn \log n \leqslant f_2(n) \leqslant Cn^{3/2}(\log n)^5.$$

We conjecture that $f_2(n) = n^{1+o(1)}$ as $n \to \infty$.

1. Introduction

A *k-uniform hypergraph* or simply *k-graph* is a pair (V, E) where V is a set of vertices and E is a set of k-subsets of V (the edges of the hypergraph). We identify a hypergraph H with its edge set and denote by $|H|$ its number of edges. The degree $\deg_H(v)$ of a vertex v in a hypergraph is the number of edges of the hypergraph containing v. A hypergraph is *even* if all of its vertices have positive even degree. A hypergraph is *b-bounded* if it has maximum degree at most b and *r-regular* if all of its vertices have degree r. A hypergraph is *linear* if every pair of its edges meet in at most one vertex. In this paper, we are interested in the following extremal question: determine the maximum number of edges $f_b(n)$ in a linear n-vertex 3-uniform hypergraph that does not contain a b-bounded even subgraph. Note that $f_b(n) \leqslant f_{b-1}(n)$ for all b.

1.1. Bounded degree even subgraphs

An elementary result in graph theory states that the extremal graphs with no even subgraphs are trees. Given a hypergraph with more edges than vertices, the characteristic vectors of the edges form a linear dependency over $\mathbb{F}_2$, which implies that the edges corresponding to those characteristic vectors form an even subgraph. The extremal problem for b-bounded subgraphs can therefore also be viewed as an extremal problem involving linear dependencies. We obtain bounds on $f_b(n)$ which are tight up to polylogarithmic factors provided $b \geqslant 12$.

Theorem 1.1. *Let $b \geqslant 12$. Then there exists an absolute constant B such that*

$$\frac{n \log n}{3b \log \log n} \leqslant f_b(n) \leqslant Bn(\log n)^2.$$

We give the proof of Theorem 1.1 in Section 5. The problem of determining $f_b(n)$ can be viewed as an extremal problem for a 'sparse linear dependency'. This problem is motivated by the work of Feige [2] on certain randomized algorithms for the SAT refutation problem, in which one of the key ingredients is determining the extremal function in hypergraphs for an even subgraph with few edges.

1.2. Small even subgraphs

Feige [2] conjectured that for some $c > 0$, any 3-uniform hypergraph on n vertices with more than $(\log n)^c m^{-1/2} n^{3/2}$ edges has an even subgraph of size at most m. In the language

https://doi.org/10.1017/S0963548311000575 Published online by Cambridge University Press

of linear dependencies, we are asking for the maximum size of an *m-wise independent* set of vectors – no set of at most m of the vectors is linearly dependent – of Hamming weight three in an n-dimensional vector space over $\mathbb{F}_2$. This question comes up naturally in coding theory in the context of parity check matrices and the minimum distance of a code in $\mathbb{F}_2^n$. In [7], it was shown that the largest size of an m-wise independent set of vectors in a vector space of dimension n over a finite field is $n^{3/2+\Theta(1/m)}$ as $m \to \infty$ by seeking a certain type of even subgraph with at most m edges which produces field-independent linear dependencies. One may ask for an analogue of Theorem 1.1 for small b-bounded even subgraphs under the additional condition of linearity. Let $f_b(n;m)$ denote the maximum number of edges in a linear 3-uniform hypergraph not containing a b-bounded even subgraph with at most m edges. In Section 4 we prove the following.

Theorem 1.2. *For any $b \geqslant 4$,*

$$f_b(n;m) = n^{3/2+\Theta(1/m)} \quad \text{as } m \to \infty.$$

The lower bound in this theorem is the standard probabilistic argument, given in [7, Theorem 1.2], whereas the upper bound is a counting argument. This theorem would also be implied by the truth of the following conjecture for 2-regular subgraphs.

Conjecture 1.3. *For any $m \in \mathbb{N}$, there is a constant $c > 0$ such that $f_2(n;m) = O(n^{3/2+c/m})$.*

This conjecture is tight by the same probabilistic construction which gives the lower bound in Theorem 1.2. We turn next to the case of estimating $f_2(n)$.

1.3. 2-regular subgraphs

The case of 2-regular subgraphs (namely the case $b = 2$ in the last section) appears to be substantially more difficult. We are able to prove the following theorem regarding $f_2(n)$ in linear 3-uniform hypergraphs.

Theorem 1.4. *There exist constants $c, C > 0$ such that*

$$cn \log n \leqslant f_2(n) \leqslant Cn^{3/2}(\log n)^5.$$

We prove Theorem 1.4 in Section 3, using a 'regularization lemma' for hypergraphs. We remark that if we relax the condition of linearity, then it was shown in [6] that any n-vertex 3-uniform hypergraph with no 2-regular subgraphs has at most $\binom{n-1}{2} + O(n)$ edges as $n \to \infty$, and if k is even, it was shown that if n is large enough, then any k-uniform n-vertex hypergraph without 2-regular subgraphs has at most $\binom{n-1}{k-1}$ edges, with equality only for the hypergraph consisting of all edges containing a vertex. Despite the large gap between the upper and lower bounds for $f_2(n)$ in Theorem 1.4, we make the following conjecture, which is supported by Theorem 1.1.

Conjecture 1.5. $f_2(n) = n^{1+o(1)}$ *as $n \to \infty$.*

https://doi.org/10.1017/S0963548311000575 Published online by Cambridge University Press

1.4. Organization

We begin with the proof of Theorem 1.4 in Section 3. Thereafter, we prove Theorem 1.2 in Section 4, and finally we prove Theorem 1.1 in Section 5. We end with some concluding remarks on a few related results for the extremal problem of subgraphs in which all degrees are small multiples of a prime p.

1.5. Notation

We use standard graph theory notation. In particular, for a graph $G = (V, E)$ we denote by $\delta(G)$ the minimum degree of G and by $\Delta(G)$ the maximum degree of G.

Throughout this paper, a hypergraph refers to a linear 3-uniform hypergraph, unless otherwise specified. If H is a hypergraph, then $V(H)$ denotes its vertex set. We write $\deg_H(x)$ for the degree of x in H, which is the number of edges that contain x. The minimum degree of H, denoted by $\delta(H)$, is a minimum taken over all $\deg_H(x)$ with $x \in V(H)$. A hypergraph H is 3-*partite* if we may write $V(H) = X \dot\cup Y \dot\cup Z$ and all edges of H are of the form $\{x, y, z\}$ with $x \in X, y \in Y$ and $z \in Z$. We refer to X, Y, and Z as the *parts* of H. We denote by $H[X, Y, Z]$ a 3-partite hypergraph H with parts X, Y, and Z. It will be convenient to identify (hyper)graphs with their edge sets, *i.e.*, $|H|$ stands for the number of edges in the hypergraph H.

2. A regularization lemma

A 3-partite hypergraph $G[X, Y, Z]$ is defined to be t-*balanced* if, for $W \in \{X, Y, Z\}$,

$$\max_{w \in W} \deg_G(w) \leqslant t \cdot \frac{|G|}{|W|}.$$

The following lemma will be used to prove the upper bound in Theorem 1.4.

Lemma 2.1. *Let $H = H[X, Y, Z]$ be a (not necessarily linear) 3-partite hypergraph of maximum degree $\Delta \geqslant 2$, and let $t = \lceil \log_2 \Delta \rceil$. Then H has a $2t^2$-balanced subgraph with at least $|H|/t^3$ edges.*

Proof. We may assume H has no isolated vertices. For sets $A \subseteq X, B \subseteq Y, C \subseteq Z$, let H_{ABC} denote the subgraph induced by $A \cup B \cup C$. By averaging, for some $a \in [t]$, the set

$$A = \{x \in X \ : \ 2^{a-1} \leqslant \deg_H(x) < 2^a\}$$

has the property that $|H_{AYZ}| \geqslant |H|/t$. We repeat the same procedure for Y and H_{AYZ}. For some $b \in [t]$, the set

$$B = \{y \in Y \ : \ 2^{b-1} \leqslant \deg_{H_{AYZ}}(y) < 2^b\}$$

has the property that $|H_{ABZ}| \geqslant |H_{AYZ}|/t \geqslant |H|/t^2$. For some $c \in [t]$, the set

$$C = \{z \in Z \ : \ 2^{c-1} \leqslant \deg_{H_{ABZ}}(z) < 2^c\}$$

has the property that $|H_{ABC}| \geqslant |H_{ABZ}|/t \geqslant |H_{AYZ}|/t^2 \geqslant |H|/t^3$. We prove that $G = H_{ABC}$ is $2t^2$-balanced. By definition, $|G| \geqslant 2^{c-1}|C|$, $|G| \geqslant 2^{b-1}|B|/t$, and $|G| \geqslant 2^{a-1}|A|/t^2$. Since

https://doi.org/10.1017/S0963548311000575 Published online by Cambridge University Press

the maximum degrees of vertices in A, B, C are at most $2^a, 2^b$ and 2^c in G, we have, for $W \in \{A, B, C\}$,

$$\max_{w \in W} \deg_G(w) \leqslant 2t^2 \cdot \frac{|G|}{|W|}.$$

Therefore G is $2t^2$-balanced. Since $|G| \geqslant |H|/t^3$, this completes the proof. □

3. Proof of Theorem 1.4

For the upper bound in Theorem 1.4, we use a key observation of Lovász [5] that the symmetric difference of two matchings in a hypergraph with the same vertex set gives a 2-regular subgraph, together with Lemma 2.1 from the last section.

3.1. Proof of $f_2(n) \leqslant Cn^{3/2}(\log n)^5$

Let H be a linear hypergraph on n vertices containing no 2-regular subgraphs. We shall show $|H| < 150n^{3/2}\lceil \log_2 n \rceil^5$. It is well known that H contains a 3-partite subgraph F with at least $\frac{2}{9}|H|$ edges – for instance, the expected number of edges in a random 3-partition is $\frac{2}{9}|H|$. Suppose F has maximum degree Δ and let $t = \lceil \log_2 \Delta \rceil$. By Lemma 2.1, F has a $2t^2$-balanced subgraph G and

$$|G| \geqslant \frac{|F|}{t^3} \geqslant \frac{2|H|}{9t^3}. \tag{3.1}$$

Let X, Y, and Z be the parts of G. Set

$$n' = \min\{|X|, |Y|, |Z|\} \quad \text{and} \quad m = \frac{n'}{12t^2}. \tag{3.2}$$

For future reference, let us note that since G is linear, $\Delta \leqslant (n-1)/2$, and hence

$$n' \geqslant \frac{|G|}{\Delta} > \frac{2}{n}|G| \overset{(3.1)}{\geqslant} \frac{4|H|}{9nt^3}. \tag{3.3}$$

An *m-matching* in G is a set of m pairwise vertex-disjoint edges of G.

Claim 3.1. *Let $\mathcal{M}$ denote the set of m-matchings in G. Then*

$$|\mathcal{M}| \leqslant \binom{n'}{m}\binom{n}{m}^2. \tag{3.4}$$

This claim is proved as follows. Suppose that $|\mathcal{M}|$ is larger than the bound in the claim. Every m-matching of G intersects each part X, Y and Z in precisely m elements. Hence the number of sets supporting some m-matching in G is at most $\binom{n'}{m}\binom{n}{m}^2$. Thus, if the inequality (3.4) does not hold then there exist m-matchings $M_1 \neq M_2 \in \mathcal{M}$ for which $V(M_1) = V(M_2)$. Consider the symmetric difference $M = M_1 \triangle M_2$ (of edges), which is non-empty as $M_1 \neq M_2$. Since every $v \in V(M)$ is contained in either 0 or 2 edges of M, the hypergraph M is 2-regular. This contradicts that H has no such subgraph, and proves the claim.

https://doi.org/10.1017/S0963548311000575 Published online by Cambridge University Press

It remains to find a lower bound for $|\mathcal{M}|$. The following greedy procedure renders an m-matching in G: pick an arbitrary $e_0 \in G$, and after choosing $e_0, e_1, \ldots, e_j \in G$, pick $e_{j+1} \in G$ which is disjoint from each e_i, $0 \leqslant i \leqslant j$. Let Δ_X, Δ_Y and Δ_Z be the maximum degrees in X, Y, and Z. We claim that, provided $j < m$, there are at least $|G|/2$ choices for e_{j+1}. Indeed, since G is $2t^2$-balanced, the number of edges intersecting $\bigcup_{1 \leqslant i \leqslant j} e_i$ is at most

$$j \cdot \big(\Delta_X + \Delta_Y + \Delta_Z\big) < m \cdot 2t^2 \left(\frac{|G|}{|X|} + \frac{|G|}{|Y|} + \frac{|G|}{|Z|} \right) \leqslant m \cdot 2t^2 \cdot \frac{3|G|}{n'} \overset{(3.2)}{\leqslant} \frac{|G|}{2}.$$

Consequently, using (3.1),

$$|\mathcal{M}| \geqslant \frac{1}{m!} \left(\frac{|G|}{2} \right)^m \geqslant \frac{1}{m!} \left(\frac{|H|}{9t^3} \right)^m. \tag{3.5}$$

By (3.5) and (3.4),

$$|H| \leqslant 9t^3 n' \left(\frac{en}{m} \right)^2. \tag{3.6}$$

By the definition of m, and (3.3), we have

$$\begin{aligned} n' \left(\frac{en}{m} \right)^2 &= n' \left(\frac{12t^2 en}{n'} \right)^2 = \frac{(12t^2 ne)^2}{n'} \\ &\leqslant \frac{(12t^2 ne)^2}{4|H|/9nt^3} = \frac{(18e)^2 n^3 t^7}{|H|}. \end{aligned} \tag{3.7}$$

It follows together with (3.6) that $|H| \leqslant (54e) n^{3/2} t^5 < 150 n^{3/2} \lceil \log_2 n \rceil^5$, as required.

3.2. Proof of $f_2(n) \geqslant c \log n$

We give a recursive construction of linear 3-partite 3-uniform hypergraphs $H_i = (V_i, E_i)$, where $|V_i| = n_i$ and $|E_i| = m_i$, with vertex partition $V_i = A_i \dot\cup B_i \dot\cup C_i$, $i \geqslant 0$, satisfying

(i) $|E_i| = \Omega\big(n_i \log n_i\big)$, and
(ii) H_i contains no 2-regular subgraph.

First, let H_0 consist of three vertices and one edge. For $i \geqslant 1$, construct H_i from H_{i-1} as follows. Let H'_{i-1} be a (vertex-disjoint) copy of H_{i-1} with 3-partition $V'_{i-1} = A'_{i-1} \dot\cup B'_{i-1} \dot\cup C'_{i-1}$. The 3-uniform hypergraph H_i will contain $H_{i-1} \cup H'_{i-1}$, together with the following additional edges. Fix $Z_{i-1} \in \{A_{i-1}, B_{i-1}, C_{i-1}\}$ achieving $|Z_{i-1}| \geqslant n_{i-1}/3$, and let Z'_{i-1} denote its copy. Add a new vertex x_i and all triples of the form $\{x_i, z, z'\}$, where $z' \in Z'_{i-1}$ is the copy of $z \in Z_{i-1}$.

Clearly, H_i is linear. Observe that it is also 3-partite. Indeed, if (without loss of generality) we assume $Z_{i-1} = A_{i-1}$, then a 3-partition of H_i is given by $A_i = A_{i-1} \cup C'_{i-1}$, $B_i = B_{i-1} \cup B'_{i-1} \cup \{x_i\}$, and $C_i = A'_{i-1} \cup C_{i-1}$. Moreover, it is easy to see that H_i satisfies property (i). Indeed, the construction of H_i implies the following recursive formulas for $i \geqslant 1$:

$$n_i = 2n_{i-1} + 1 \quad \text{and} \quad m_i \geqslant 2m_{i-1} + \tfrac{n_{i-1}}{3}.$$

https://doi.org/10.1017/S0963548311000575 Published online by Cambridge University Press

A simple induction gives $n_i = 2^{i+2} - 1$, and similarly $m_i \geqslant (i+1)2^{i-1}$, since

$$m_i \geqslant 2m_{i-1} + \tfrac{1}{3}n_{i-1} \geqslant 2(i2^{i-2}) + \tfrac{1}{3}(2^{i+1} - 1) \geqslant i2^{i-1} + 2^{i-1}.$$

It follows as required that $m_i = \Omega\big(n_i \log n_i\big)$.

Now we need to verify property (ii). We proceed by induction and show, in fact, the following stronger statement for every $i \geqslant 0$:

($\mathcal{S}_i$) Every non-empty subgraph $G \subset H_i$ with maximum degree $\Delta(G) \leqslant 2$ is either a single edge, or contains at least four vertices of degree one.

Clearly, 3.2 holds for $i = 0$, so let $i \geqslant 1$. Let G be a non-empty subgraph of H_i with $\Delta(G) \leqslant 2$, and for sake of the argument, assume that G is not just a single edge.

Let $G_1 \subseteq H_{i-1}$ and $G_2 \subseteq H'_{i-1}$ denote the (possibly empty) induced subgraphs of G contained in H_{i-1} and H'_{i-1}, respectively. Let $\ell(G)$ denote the number of vertices of degree one in G, and let $\ell_r = \ell(G_r)$, $r = 1, 2$, denote the number of vertices of degree one in G_r. The statement 3.2 follows from a simple case analysis according to $\deg_G(x_i)$ of x_i in G.

Case 1: $\deg_G(x_i) = 0$. At least one of $G_1, G_2 \neq \emptyset$. If, without loss of generality, $G_2 = \emptyset$, then $\ell(G) = \ell_1 \geqslant 4$ (since $|G_1| = |G| > 1$). Otherwise, $\ell(G) = \ell_1 + \ell_2 \geqslant 6$.

Case 2: $\deg_G(x_i) = 1$. At least one of $G_1, G_2 \neq \emptyset$. If, without loss of generality, $G_2 = \emptyset$, then $\ell(G) \geqslant (\ell_1 - 1) + 2 \geqslant 4$ (the edge of G incident to x_i has two vertices of degree 1; its third vertex may be counted by $\ell_1 = \ell(G_1)$). Otherwise, $\ell(G) \geqslant (\ell_1 - 1) + (\ell_2 - 1) + 1 \geqslant 5$.

Case 3: $\deg_G(x_i) = 2$. We show that in this case G has at least two vertices of degree one in each of H_{i-1} and H'_{i-1}. Indeed, let f_1, f_2 be the two edges of G containing x_i. Note that by linearity $f_1 \cap f_2 = \{x_i\}$. If, say, $G_1 = \emptyset$, then the two ends of f_1 and f_2 in H_{i-1} are the two vertices of G of degree one. If $G_1 = e$, then e has only one vertex in the set of the tripartition of H_{i-1} which is intersected by f_1 and f_2. Consequently, in H_{i-1} there are $|e \setminus (f_1 \cup f_2)| \geqslant 2$ vertices of G of degree one. Finally, if $|G_1| \geqslant 2$, then H_{i-1} contains at least $\ell_1 - 2 \geqslant 2$ vertices of G of degree one.

This concludes the proof of the induction step and, therefore, (ii) and Theorem 1.4 follow. □

4. Small even subgraphs

In this section, we prove Theorem 1.2. Recall that $f_b(n;m)$ is the largest number of edges in a linear hypergraph on n vertices containing no b-bounded even subgraphs with at most m edges. The lower bound in Theorem 1.2 is proved by taking a random hypergraph on $[n]$ whose edges are chosen from all 3-element sets in $[n]$ independently and with probability $n^{-3/2+c/m}$ for an appropriate constant $c > 0$. The details are given in [7]. We turn now to the proof that $f_b(n;m) = n^{(3/2)+O(1/m)}$ for $b \geqslant 4$. It is enough to prove this for $b = 4$. We begin with a sketch of the proof. For $\varepsilon > 0$, let H be a linear 3-graph with $n \geqslant n_0$ vertices and at least $n^{(3/2)+\varepsilon}$ edges. Showing that H contains a (small) even 4-bounded subgraph will depend on two observations. The first is that $|H| \geqslant n^{(3/2)+\varepsilon}$ will imply that H contains 'many' *cherries*, *i.e.*, pairs of edges meeting in a single vertex, or

https://doi.org/10.1017/S0963548311000575 Published online by Cambridge University Press

equivalently, a subgraph consisting of one 'degree 2' vertex and four 'degree 1' vertices. More strongly, H will contain many 'short' (of length less than $1/\varepsilon$) paths of cherries, where two adjoined cherries on such a path connect along two 'degree 1' vertices. The second observation is that there will be so many of these paths that there must be a pair of distinct paths sharing identical ends. The symmetric difference (of edges) of these two paths will result in a 4-bounded even subgraph of H. We now give the precise details.

For a given $\varepsilon > 0$, define

$$m = \lceil 4/\varepsilon \rceil \quad \text{and} \quad n_0 = \lceil (5/\varepsilon)^{1/(2\varepsilon)} \rceil. \tag{4.1}$$

Let H be a linear 3-uniform hypergraph with $n \geqslant n_0$ vertices and at least $n^{(3/2)+\varepsilon}$ edges, where we set V to be the vertex set of H. Regarding the first observation in the sketch, the linearity of H implies it has precisely $\sum_{v\in V} \binom{\deg_H(v)}{2}$ many cherries, which equals

$$\frac{1}{2}\left(\sum_{v\in V} \deg_H^2(v) - \sum_{v\in V} \deg_H(v)\right) = \frac{1}{2}\left(\sum_{v\in V} \deg_H^2(v) - 3|H|\right).$$

Using the Cauchy–Schwarz inequality, the number of cherries of H is at least

$$\frac{1}{2}\left(\frac{1}{n}\left(\sum_{v\in V} \deg_H(v)\right)^2 - 3|H|\right) = \frac{1}{2}\left(\frac{9|H|^2}{n} - 3|H|\right) > 4n^{2+2\varepsilon},$$

where the last inequality follows from the hypothesis that $|H| \geqslant n^{(3/2)+\varepsilon}$.

We now prepare for the second observation from the sketch (which corresponds to Claim 4.1 below). Define the following auxiliary graph G to have vertex set $V(G) = \{uv \in V \times V \,:\, u \neq v\}$, consisting of all ordered pairs of distinct vertices of H, and edge set

$$E(G) = \{\{uv, xy\} \,:\, \exists z \in V \text{ such that } \{u, z, y\} \neq \{v, z, x\} \in H\}.$$

Note that each edge in G corresponds to a unique cherry in H (since vertices of G are ordered pairs and H is linear). In other words, there is an injective map from the set of cherries of H to the edge set of G. Consequently, G contains at least $4n^{2+2\varepsilon}$ edges (on $n^2 - n$ vertices). Now, delete vertices from G that have degree less than $3n^{2\varepsilon}$ to form a subgraph G' with $\delta(G') \geqslant 3n^{2\varepsilon}$ and $|E(G')| \geqslant n^{2+2\varepsilon}$. As in the sketch, we consider a (hyper)path (of cherries) in H: suppose $u_1v_1, u_2v_2, \ldots, u_kv_k$ is the vertex sequence of a (graph) path in G', where $z_1, z_2, \ldots, z_{k-1}$ satisfy that z_i is the intersection point of the cherry corresponding to the edge $\{u_iv_i, u_{i+1}v_{i+1}\}$ of G'. We say such a path is *faithful* (in G') if

$$|\{u_1, v_1, z_1, \ldots, u_{k-1}, v_{k-1}, z_{k-1}, u_k, v_k\}| = 3k - 1.$$

In other words, all these vertices are distinct (see Figure 1).

Claim 4.1. *For every $uv \in V(G')$, there exists $wz \in V(G')$, $\{u, v\} \cap \{w, z\} = \emptyset$, and faithful paths Q_1, Q_2, $Q_1 \neq Q_2$, of length $< 1/\varepsilon$, connecting uv to wz.*

Before we verify Claim 4.1, we use it to finish the proof of Theorem 1.2.

Fix an arbitrary $uv \in V(G')$, and let wz, Q_1, Q_2 be given by Claim 4.1. Let $\mathcal{P}_1, \mathcal{P}_2 \subset H$ be the subgraphs of H corresponding to Q_1, Q_2, respectively, *i.e.*, $\mathcal{P}_i$ is the union of the cherries corresponding to the edges of Q_i, $i = 1, 2$. Note that every vertex of $\mathcal{P}_i$ has

https://doi.org/10.1017/S0963548311000575 Published online by Cambridge University Press

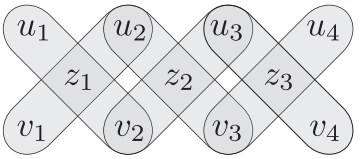


Figure 1. A faithful path $(u_1v_1, u_2v_2, u_3v_3, u_4v_4)$ of length 3 in the auxiliary graph G' corresponds to the above subgraph of H.

degree 2, except for u, v, z, w, which have degree 1. Then $\mathcal{C} = \mathcal{P}_1 \triangle \mathcal{P}_2 \neq \emptyset$ is a 4-bounded even hypergraph on at most $4/\varepsilon \leqslant m$ edges, and so deleting the isolated vertices from $\mathcal{C}$ renders the subgraph of H promised by Theorem 1.2.

Proof of Claim 4.1. Fix $uv \in V(G')$, and let $S(uv, k)$ be the set of vertices in $V(G')$ that are reachable in G' by a faithful path of length exactly k (where $S(uv, 0) = \{uv\}$). Note that if a path is faithful, then every subpath is also faithful. In particular, if $wz \in S(uv, k)$, $k \geqslant 1$, then there exists $xy \in S(uv, k-1)$ such that a faithful path from uv to xy can be extended to a faithful path from uv to wz by adding the edge $\{xy, wz\} \in E(G')$. Conversely, fix $xy \in S(uv, k-1)$ and fix a faithful path from uv to xy. We assert that all but $9(k-1)$ many $wz \in N_{G'}(xy)$ satisfy that the fixed path from uv to xy can be extended to a faithful path from uv to wz by adding the edge $\{xy, wz\} \in E(G')$.

Indeed, let $uv = u_1v_1, u_2v_2, \ldots, u_kv_k = xy$ be the vertices of a faithful path from uv to xy of length $k-1$ in G'. For each $i = 1, \ldots, k-1$, let z_i be the intersection vertex of the cherry corresponding to the edge $\{u_iv_i, u_{i+1}v_{i+1}\}$ of G', and set $B = \{u_1, v_1, z_1, \ldots, u_{k-1}, v_{k-1}, z_{k-1}\}$. Note that any $wz \in N_{G'}(xy)$ belongs to $S(uv, k)$ if $\{w, z\} \cap B = \emptyset$ and if the intersection point z' of the cherry corresponding to $\{xy, wz\}$ satisfies $z' \notin B$. Our assertion is that at most $3|B| = 9(k-1)$ vertices $wz \in N_{G'}(xy)$ violate this condition. Indeed, at most $|B|$ many $wz \in N_{G'}(xy)$ will not belong to $S(uv, k)$ because their intersection point z' belongs to B since, by the linearity of H, xy together with z' uniquely determine $wz \in N_{G'}(xy)$. On the other hand, if $w \in B$, then z' is determined ($\{y, z', w\} \in H$) and z is determined ($\{x, z', z\} \in H$). A similar conclusion holds in the case $z \in B$. Consequently, at most $3|B|$ vertices $wz \in N_{G'}(xy)$ will not belong to $S(uv, k)$, as asserted.

To conclude the proof of Claim 4.1, consider the directed bipartite graph D_k with vertex bipartition $S(uv, k-1) \cup S(uv, k)$, and arcs (xy, wz), $xy \in S(uv, k-1)$, $wz \in S(uv, k)$, whenever there is a faithful path of length k from uv to wz which uses the edge $\{xy, wz\} \in G'$. From the assertion above (and the minimum degree of G'), $|E(D_k)| \geqslant |S(uv, k-1)|(3n^{2\varepsilon} - 9k)$. Consequently, a simple induction yields that for every $k \geqslant 1$, either

(i) $\exists j \leqslant k$ and $wz \in S(uv, j)$ with in-degree $\text{i.d.}_{D_j}(wz) \geqslant 2$, or
(ii) $|S(uv, k)| \geqslant (3n^{2\varepsilon} - (9/\varepsilon))^k \geqslant n^{2k\varepsilon}$.

Case (i) would yield the conclusion of the claim, and case (ii) is impossible when $k \geqslant 1/\varepsilon$, since $|S(uv, k)| \leqslant |V(G)| < n^2$. □

https://doi.org/10.1017/S0963548311000575 Published online by Cambridge University Press

5. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1, starting with 3-graphs establishing the lower bound of Theorem 1.1.

5.1. Proof of $f_b(n) \geqslant n \log n/(3b \log\log n)$

To construct the hypergraphs establishing the lower bound, we will use an explicit family of graphs constructed by Lazebnik and Ustimenko [4]. For every prime power q and $k \geqslant 3$, [4] provides a q-regular bipartite graph $G_{q,k}$ on $2q^k$ vertices with girth $g \geqslant k+5$. Let X and Y be the classes of $G_{q,k}$. Since $G_{q,k}$ is q-regular and bipartite, it is possible to decompose its edge set into q disjoint perfect matchings $G_{q,k} = M_1 \cup \cdots \cup M_q$.

Let q be a (large) prime power and $k = bq - 1$. Set $n = 2q^k + q$ and consider a 3-partite 3-graph H_n with classes X, Y, and $Z = [q]$ constructed as follows. For each $e = \{u, v\} \in M_j$, $j \in Z$, let $e \cup \{j\} = \{u, v, j\} \in H_n$. Notice that H_n is linear since the matchings M_j are disjoint. Suppose that H_n contains a b-bounded even subgraph F with vertex set $X' \cup Y' \cup Z'$. Notice that

$$|F| \leqslant b|Z'| \leqslant bq. \tag{5.1}$$

Let $F' \subset G_{q,k}$ be the shadow of F, that is, $F' = \{e \setminus Z' \ : \ e \in F\}$. Because $\delta(F') \geqslant \delta(F) \geqslant 2$, the graph F' contains a cycle of length at most $|F'| = |F|$. The girth of $G_{q,k}$ then implies that $|F| \geqslant k+5$. By our choice of k, this is a contradiction with (5.1) and hence H_n does not contain an even b-bounded subgraph. In fact, it does not contain a subgraph with all degrees in $[2, b]$.

Notice that $|H_n| = q^{k+1} = q(n-q)/2 > qn/3$. Moreover, $q > \frac{\log n}{b \log\log n}$, since otherwise $n < q^{k+1} < (\log n)^{bq} = n$. Therefore H_n establishes the lower bound. □

5.2. Proof of $f_b(n) \leqslant Bn(\log n)^2$

In the rest of this section, for convenience we use log to denote $\log_2$. For a sufficiently large integer n_0, let H be a linear hypergraph on $n \geqslant n_0$ vertices with at least $1000n(\log n)^2$ edges. We show that H contains a 12-bounded even subgraph, and begin by introducing some notation. Let V be the vertex set of H. Set

$$\psi_n(x) = \frac{\log x}{\log n}.$$

For a set $S \subseteq V$, define

$$\partial_H(S) = \{e \in H \ : \ e \cap S \neq \emptyset\}.$$

Let $I = I(H) \subset [n]$ be the set of all numbers $s > 1$ such that there exists a set $S \subseteq V$ with $|S| = s$ satisfying

$$|\partial_H(S)| \geqslant \psi_n(s) \cdot |H|.$$

Clearly, $I \neq \emptyset$ since $n \in I$. Denote by r the smallest element from I. Let $R \subseteq V$ correspond to r, that is, $|R| = r$ and $|\partial_H(R)| \geqslant \psi_n(r) \cdot |H|$. It is not difficult to see that $r \geqslant 2000 \log n$.

https://doi.org/10.1017/S0963548311000575 Published online by Cambridge University Press

Indeed, since H is linear, its maximum degree is at most $(n-1)/2$, and so

$$|\partial_H(R)| \leqslant \sum_{v \in R} \deg_H(v) \leqslant \frac{rn}{2}.$$

On the other hand, by definition, $|\partial_H(R)| \geqslant |H| \log r / \log n$, and so

$$\frac{r}{\log r} \geqslant \frac{2|H|}{n \log n} \geqslant 2000 \log n.$$

Now, let $G_0 = (V_0, E_0)$ be a graph with $V_0 \subseteq V$ obtained from the edges in $\partial_H(R)$ by removing from each hyperedge f an arbitrary vertex contained in $f \cap R$. The edges of G_0 are then naturally R-coloured by the mapping $\chi : E_0 \to R$, where $\{u, v, \chi(uv)\} \in \partial_H(R)$ for all $uv \in E_0$. Since H is linear, $|E_0| = |\partial_H(R)|$ and χ is a proper edge-colouring of G_0. The proof of Theorem 1.1 will rest on the upcoming Claim 5.1, for which we need the following definition.

Definition. We say a subgraph $F \subset G_0$ is 2-*nice*, if F is 4-bounded and even, and if no colour of $\chi : E_0 \to R$ appears on more than two edges of F.

We may now state Claim 5.1.

Claim 5.1. *For some $\ell \geqslant r/15$, the graph G_0 contains at least $(|H|/n)^{\ell/6}$ 2-nice subgraphs on ℓ edges.*

Our proof of Claim 5.1 is unfortunately quite technical, so we postpone it for a minute in favour of concluding the proof of Theorem 1.1.

Indeed, Claim 5.1 ensures there are at least $(|H|/n)^{\ell/6}$ 2-nice subgraphs $F \subset G_0$ of size $\ell \geqslant r/15$. For each such F, let χ_F denote the multi-set of colours on the edges of F, where we recall that a colour may appear at most twice in χ_F. The number of multi-sets from R of size ℓ, where each element has multiplicity at most 2, is at most $\binom{2r}{\ell}$. Since

$$\binom{2r}{\ell} \leqslant \left(\frac{2er}{\ell}\right)^{\ell} \leqslant (30e)^{\ell} \ll \left(\frac{|H|}{n}\right)^{\ell/6},$$

there exist 2-nice subgraphs $F' \neq F'' \subset G_0$ with $\chi_{F'} = \chi_{F''}$. Consider $F^* = F' \triangle F'' \neq \emptyset$. Since F', F'' are 4-bounded and even, F^* is 8-bounded and even, and the colours on the edges of F^* appear either 2 or 4 times. Hence, the corresponding 3-uniform hypergraph $H^* = \{\{u, v, \chi(uv)\} \in H \ : \ uv \in E(F^*)\}$ is 12-bounded and even. Indeed, a vertex $v \in V(H^*) \setminus R$ has $\deg_{H^*}(v) = \deg_{F^*}(v)$, while a vertex $v \in V(H^*) \cap R$ has $\deg_{H^*}(v) = \deg_{F^*}(v) + |\{e \in H \ : \ \chi(e) = v\}| \in \{0, 2, 4, 6, 8, 10, 12\}$. Removing the isolated vertices from H^* gives the 12-bounded even subgraph of H promised by Theorem 1.1. □

5.3. Proof of Claim 5.1

Our proof of Claim 5.1 consists largely of iteratively applying the following further claim.

https://doi.org/10.1017/S0963548311000575 Published online by Cambridge University Press

Claim 5.2. *Suppose $G \subset G_0$ and $z \in \mathbb{N}$ satisfy $\delta(G) \geqslant z \geqslant 1000 \log n$. Then for some $\ell \in \left[\frac{z}{25}, \frac{z}{25} + 2\log n + 2\right]$, the graph G contains at least $z^{0.9\ell}$ 2-nice connected subgraphs $F \subseteq G$ with $|E(F)| = \ell$.*

We prove Claim 5.2 after we use it to complete the proof of Claim 5.1. We will also need the following fact (which will allow us to apply Claim 5.2 in the context of proving Claim 5.1).

Fact 5.3. *Suppose $S \subset V_0$ and $X \subset R$ satisfy $s' = |S \cup X| < r$. Then the graph $G_1 = (V_1, E_1)$ with $V_1 = V_0 \setminus (S \cup X)$ and $E_1 = \{e \in E_0 \,:\, e \subseteq V_1 \text{ and } \chi(e) \notin X\}$ contains at least $\psi_n(r/s')|H|$ edges.*

Proof of Fact 5.3. Every $e \in E_0 \setminus E_1$ satisfies $f = e \cup \{\chi(e)\} \in H$ and $f \cap (S \cup X) \neq \emptyset$, which means $f \in \partial_H(S \cup X)$. Since $|S \cup X| < r$, the minimality assumption on R yields $|\partial_H(S \cup X)| < \psi_n(s')|H|$. In particular, $|E_0 \setminus E_1| \leqslant |\partial_H(S \cup X)| < \psi_n(s')|H|$. By the choice of R and by the definition of G_0, $|E_0| = |\partial_H(R)| \geqslant \psi_n(r)|H|$. Thus, $|E_1| \geqslant \big(\psi_n(r) - \psi_n(s')\big)|H| = \psi_n(r/s')|H|$. □

Now, to prove Claim 5.1, set

$$z = \frac{|H|}{n \log n} \geqslant 1000 \log n \quad \text{and} \quad t = \left\lceil \frac{2r}{z} \right\rceil. \tag{5.2}$$

We assert that, by repeated applications of Claim 5.2, we can obtain at least $z^{zt/30}$ sequences of vertex-disjoint connected 2-nice subgraphs $F_1, \ldots, F_t$, satisfying $|E(F_1)|, \ldots, |E(F_t)| \in [\frac{z}{25}, \frac{z}{25} + 2\log n + 2]$ and $\chi(E(F_i)) \cap \chi(E(F_j)) = \emptyset$ for all $1 \leqslant i < j \leqslant t$. Indeed, since every graph contains a subgraph whose minimum degree is at least half of the average, we start with a subgraph $G^0_* \subset G_0$ with

$$\delta(G^0_*) \geqslant \frac{|E_0|}{|V_0|} \geqslant \frac{\partial_H(R)}{n} \geqslant \frac{|H| \log r}{n \log n} \geqslant \frac{|H|}{n \log n} = z \geqslant 1000 \log n,$$

and apply Claim 5.2, which yields a 2-nice connected graph F_1. Suppose now that F_i has been obtained for every $i < j$, with $j \geqslant 2$. Let $S^j = \bigcup_{i<j} V(F_i)$ and $X^j = \bigcup_{i<j} \chi(E(F_i))$. Define $G^j = (V^j, E^j)$ with $V^j = V_0 \setminus (S^j \cup X^j)$ and $E^j = \{e \in E_0 \,:\, e \subseteq V^j \text{ and } \chi(e) \notin X^j\}$. Since, for all $1 \leqslant i < j \leqslant t$,

$$|V(F_i)|, |\chi(E(F_i))| \leqslant |E(F_i)| \leqslant \frac{z}{25} + 2\log n + 2 < \frac{z}{20},$$

it follows by our choice of t (see (5.2)) that

$$|S^j|, |X^j| \leqslant (j-1)\frac{z}{20} \leqslant (t-1)\frac{z}{20} < \frac{r}{10}.$$

From Fact 5.3, we conclude that for every j,

$$|E^j| \geqslant \frac{\log(r/|S^j \cup X^j|)}{\log n}|H| \geqslant \frac{\log 5}{\log n}|H| \geqslant \frac{2|H|}{\log n} = 2zn,$$

and therefore there exists a subgraph $G^j_* \subset G^j$ with $\delta(G^j_*) \geqslant z \geqslant 1000 \log n$. We apply Claim 5.2 to G^j_* to obtain at least $z^{0.9\ell_j}$ graphs $F_j \subset G^j_* \subset G^j$, for some $\ell_j \in \left[\frac{z}{25}, \frac{z}{25} +\right.$

https://doi.org/10.1017/S0963548311000575 Published online by Cambridge University Press

$2\log n + 2\big]$. In particular, we always obtain at least $z^{z/30}$ possible graphs F_j, and it follows from the construction that all those F_j are vertex-disjoint from (the earlier fixed) $F_1, \dots, F_{j-1}$, and also that $\chi(E(F_i)) \cap \chi(E(F_j)) = \emptyset$ for all $i < j$. Thus, the number of distinct (ordered) sequences $F_1, \dots, F_t$ obtained by this process is at least $z^{zt/30}$.

To complete the proof of Claim 5.1, consider the set of all unions $F = F_1 \cup F_2 \cup \cdots \cup F_t$ obtained from the sequences above. Note that any such union $\bigcup_{j=1}^{t} F_j$ is a 2-nice but disconnected graph. We now estimate the number of unions F. Since each F_j is connected and vertex-disjoint from the other F_i, a graph F may be represented by at most $t!$ such sequences. Thus, the number of such F is at least

$$z^{zt/30}/t! \geqslant (z^{z/30}/t)^t > (z^{z/30}/n)^t > z^{zt/50}.$$

Every graph F obtained satisfies

$$\frac{r}{15} \overset{(5.2)}{\leqslant} \frac{tz}{25} \leqslant |E(F)| \leqslant t\left(\frac{z}{25} + 2\log n + 2\right) < t\frac{z}{20}.$$

Therefore, there exists some ℓ with $r/15 \leqslant \ell \leqslant tz/20$ such that there are at least

$$\frac{z^{zt/50}}{tz/20} \geqslant z^{zt/60} \geqslant z^{\ell/3} \overset{(5.2)}{=} \left(\frac{|H|}{n\log n}\right)^{\ell/3} \geqslant \left(\frac{|H|}{n}\right)^{\ell/6}$$

2-nice graphs of size ℓ in G_0. All that remains is to prove Claim 5.2.

Proof of Claim 5.2. Let $v \in V(G)$ be arbitrary. Our first goal is to inductively construct a tree T rooted in v with the property that every vertex of the tree is connected to the root by a rainbow path (that is, by a path whose edges are coloured with distinct colours). To that end, set $T_0 = (\{v\}, \emptyset)$. For $i \geqslant 0$, and from an inductively constructed T_i, we construct T_{i+1} as follows. Let L_i denote the set of leaves of T_i at depth i, where we set $L_0 = \{v\}$. For every $u \in L_i$, let N_u denote the set of $w \in V(G) \setminus V(T_i)$ for which the (rainbow) path in T_i connecting v to u can be extended to a rainbow path connecting v to w by adding the edge $uw \in E(G)$. We define T_{i+1} by adding, for each $w \in \bigcup_{u\in L_i} N_u$, some edge uw, where $w \in N_u$ and $u \in L_i$. (If there is more than one u for which $w \in N_u$, choose arbitrarily.) Note that T_{i+1} satisfies that $L_{i+1} = \bigcup_{u\in L_i} N_u$. To define the promised tree T, it remains to define the last iteration i we perform in the process above.

Observe that every $u \in L_i$ has the property that all but (at most) i of its neighbours in G are contained in $V(T_{i+1}) = V(T_i) \cup L_{i+1}$. Indeed, a neighbour w of $u \in L_i$ which fails to be in $V(T_{i+1})$ must be such that the edge uw has the same colour as an edge of the rainbow path (of length i) connecting v to u in T_i. Since χ is a proper edge-colouring, at most i such edges are incident to u. (Note that all the other neighbours are either already in the tree T_i or will be included in the tree T_{i+1}.) Let k be the smallest index for which $|L_{k+1}| < 2|L_k|$. Finally, set $T = T_{k+1}$.

The discussion above implies that the number M of edges of $G[V(T)]$ incident to L_k is

$$\begin{aligned} M &= e_G(L_k, V(T_k) \setminus L_k) + e_G(L_k) + e_G(L_k, L_{k+1}) \\ &\geqslant \frac{\delta(G)}{2}|L_k| - k|L_k| \geqslant \left[\frac{z}{2} - k\right]|L_k|. \end{aligned} \tag{5.3}$$

https://doi.org/10.1017/S0963548311000575 Published online by Cambridge University Press

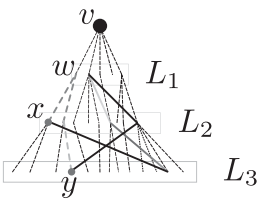


Figure 2. The dashed path corresponds to $P_2 \cup P_3 \subset E(T)$ and the continuous path corresponds to $P_1 \subset E\big(G[V(T)]\big) \setminus E(T)$.

Observe that $|E(G[V(T)])| \geqslant M$, and that $|V(T)|$ equals

$$|L_0| + \cdots + |L_k| + |L_{k+1}| < (2^{-k} + 2^{-k+1} + \cdots + 1)|L_k| + 2|L_k| < 4|L_k|.$$

Hence, the average degree in $G[V(T)]$ is at least $2M/(4|L_k|) \geqslant z/4 - k/2$, and the average degree of $G[V(T)] \setminus E(T)$ is therefore at least $z/4 - k/2 - 2$. Since $|L_i| \geqslant 2^i$ for every $i = 0, 1, \ldots, k$, we have $k \leqslant \log n$, and so $z \geqslant 1000 \log n \geqslant 1000k$. Consequently, $z/4 - k/2 - 2 \geqslant z/5$ (with n sufficiently large). Thus, there exists a subgraph $G' = (V', E')$ of $G[V(T)] \setminus E(T)$ with minimum degree $\delta(G') \geqslant z/10$.

We need the following consideration to conclude the proof of Claim 5.2. Let $x \neq v \in V(G')$ be fixed, and let P_0 denote the rainbow T-path connecting v to x. Let P_1 be a G'-path (importantly, not T-path) from x to some vertex y such that $P_0 \cup P_1$ is a rainbow path from v to y. (Below, we estimate how many such paths P_1 will exist.) Let w be the first common ancestor of both x and y in T. Let $P_2 \subset P_0$ be the T-path from w to x, and let P_3 be the T-path from w to y (see Figure 2). By construction, both paths $P_2 \cup P_1$ and P_3 are edge-disjoint rainbow paths with the same end vertices w and y. Hence, the union $F = P_1 \cup P_2 \cup P_3$ is a connected 2-nice graph. We bound the number of graphs F which can be thus created.

The number N of rainbow paths extending P_0 from x by a path $P_1 \subset G'$ of length $z/25$ is at least

$$\prod_{j=1}^{z/25} \big(\delta(G') - 2(k+j)\big) \geqslant \big(z/10 - 2(k + z/25)\big)^{z/25} \geqslant (z/50)^{z/25}.$$

Every such P_1 yields a distinct graph $F = P_1 \cup P_2 \cup P_3$ with

$$z/25 + 1 \leqslant |E(F)| \leqslant z/25 + 2(k+1)$$

(these graphs F are distinct since $P_1 \subset G' \subset G \setminus E(T)$ and $P_2 \cup P_3 \subset E(T)$). By averaging, there exists some ℓ, $z/25 + 1 \leqslant \ell \leqslant z/25 + 2(k+1)$, such that the number of graphs F of size ℓ is at least

$$\frac{N}{2(k+1)} \geqslant \frac{(z/50)^{z/25}}{2\log n + 2} \geqslant z^{0.9\ell}$$

for n sufficiently large. This concludes the proof of Claim 5.2. □

https://doi.org/10.1017/S0963548311000575 Published online by Cambridge University Press

6. Concluding remarks

6.1. Even subgraphs of r-graphs

We have mainly studied extremal problems for subgraphs with small even degrees in linear 3-uniform hypergraphs. It is possible to ask similar questions for r-uniform hypergraphs with $r > 3$. Let $f_b^r(n)$ denote the maximum number of edges in a linear n-vertex r-graph with no b-bounded even subgraphs. Theorem 1.4 can be extended to r-graphs by repeating the matching counting proof given here. One can show that $f_2^r(n) = O(n^{2-1/(r-1)}(\log n)^{O(r)})$ using that proof. It is likely to be very difficult to determine the correct order of magnitude of $f_2^r(n)$ for any $r > 2$, and in particular we conjectured $f_2^3(n) = f_2(n) = n^{1+o(1)}$. The problem of finding small even subgraphs of r-graphs was studied in [7].

6.2. Degrees in residue classes

More generally, one can consider subgraphs in which the degrees are multiples of an integer p. If p is prime, then Alon, Friedland and Kalai [1] showed that any graph of average degree more than $2p-2$ contains a non-empty subgraph in which the degrees are zero modulo p. Using this result, Pyber, Rödl and Szemerédi [8] showed that the maximum number of edges in an n-vertex graph with no p-regular subgraph is $O(n \log n)$. The proof of the result of Alon, Friedland and Kalai uses the Chevalley–Warning theorem, and extends to r-graphs easily: in an r-graph of average degree more than $r(p-1)$, there is a non-empty subgraph in which all the degrees are zero modulo p. The question of determining $f_p(n)$, the maximum number of edges in a linear n-vertex 3-graph with no p-regular subgraph, appears to be very difficult. In fact, it appears difficult to show that every sufficiently large Steiner triple system contains a 3-regular subgraph, so we leave it as an open problem to show $f_3(n) = o(n^2)$.

References

[1] Alon, N., Friedland, S. and Kalai, G. (1984) Regular subgraphs of almost regular graphs. *J. Combin. Theory Ser. B* **37** 79–91.

[2] Feige, U. (2008) Small linear dependencies for binary vectors of low weight. In *Building Bridges: Between Mathematics and Computer Science*, Vol. 19 of *Bolyai Society Mathematical Studies*, Springer, pp. 283–307.

[3] Frankl, P. and Rödl, V. (1985) Near perfect coverings in graphs and hypergraphs. *European J. Combin.* **6** 317–326.

[4] Lazebnik, F. and Ustimenko, V. A. (1995) Explicit construction of graphs with an arbitrary large girth and of large size. *Discrete Appl. Math.* **60** 275–284.

[5] Lovász, L. Personal communication.

[6] Mubayi, D. and Verstraëte, J. (2009) Two-regular subgraphs of hypergraphs. *J. Combin. Theory Ser. B* **99** 643–655.

[7] Naor, A. and Verstraëte, J. (2008) Parity check matrices and product representations of squares. *Combinatorica* **28** 163–185.

[8] Pyber, L., Rödl, V. and Szemerédi, E. (1995) Dense graphs without 3-regular subgraphs. *J. Combin. Theory Ser. B* **63** 41–54.

https://doi.org/10.1017/S0963548311000575 Published online by Cambridge University Press