*Math. Proc. Camb. Phil. Soc.* (1992), **112**, 589
*Printed in Great Britain*



# Spectral inequalities for compact integral operators on Banach function spaces

BY ROMAN DRNOVŠEK

*Institute of Mathematics, Physics and Mechanics, Jadranska* 19, 61111 *Ljubljana, Slovenia*

(*Received* 27 *September* 1991; *revised* 31 *January* 1992)

*Abstract*

This article generalizes some spectral inequalities for non-negative matrices (see [2] and [3]) to compact integral operators with non-negative kernels defined on Banach function spaces. The spectral radius of a sum of such operators is estimated under certain conditions and a generalization of this inequality is given. An inequality for the spectral radius of a compact integral operator with the kernel equal to a weighted geometric mean of non-negative kernels is also proved.

## 1. *Introduction*

In this section we will first describe Banach function spaces, and then recall some theorems for compact integral operators with non-negative kernels defined on such function spaces.

*Banach function spaces.* Let $\mu$ be a positive $\sigma$-finite measure on a $\sigma$-algebra $\mathscr{M}$ on a set $X$. Suppose that the $\sigma$-algebra $\mathscr{M}$ contains at least one measurable set $A$ such that $\mu(A) > 0$ and $\mu(A^c) > 0$. Let $\mathscr{P}$ be the collection of all $\mu$-measurable non-negative functions $f(x)$ defined on $X$, and let $d(f)$ denote a metric function (with $0 \leqslant d(f) \leqslant \infty$) on $\mathscr{P}$ with the following properties:

1. $d(f) = 0$ if and only if $f(x) = 0$ a.e. on $X$,
   $d(f_1+f_2) \leqslant d(f_1)+d(f_2)$, and
   $d(af) = ad(f)$ for any non-negative number $a$.
2. If $f_n(x) \in \mathscr{P}$ $(n = 1, 2, \ldots)$, and $f_n(x) \uparrow f(x)$ a.e. on $X$, then $d(f_n) \uparrow d(f)$.
3. If $E$ is a measurable set with $\mu(E) < \infty$ and $\chi_E(x)$ is its characteristic function, then $d(\chi_E) < \infty$.

Let $L$ denote the space of all (equivalence classes of) measurable real functions $f(x)$ on $X$ for which $\|f\| = d(|f|)$ is finite. It turns out (see [5]) that the function space $L$ is a Banach space, and indeed it is easy to verify that it is a Banach lattice. Let $M$ be the space of all measurable functions $g$ such that

$$\phi(f) = \int_X f(x)\, g(x)\, d\mu(x)$$

defines a bounded linear functional on $L$. The corresponding space of all such functionals $L'$ is a closed subspace of the dual Banach lattice $L^*$. It turns out easily

https://doi.org/10.1017/S0305004100071279 Published online by Cambridge University Press

that $b: M \to L'$, defined by $b(g) = \phi$, provides an isometric lattice isomorphism between two Banach lattices. Here, the vector lattice $M$ is equipped with the natural partial ordering and the norm defined by $\|g\| = \|b(g)\|$. Thus the associated space $L'$ may be considered as a function space as well as a Banach lattice (see [5]).

Let $S$ and $S'$ denote the cones of non-negative functions of $L$ and $L'$, respectively. For typographical convenience only, we shall write

$$(f, g) = \int_X f(x)\, g(x)\, d\mu(x)$$

for all functions $f \in L$ and $g \in L'$. In the special case when $L = L^2(X, \mu)$ this reduces to the usual scalar product. For brevity, integration over the whole set $X$ will be denoted by $\int d\mu(x)$ or even $\int dx$.

*Compact integral operators with non-negative kernels.* Let $\mathscr{K}(L)$ be the space of all compact integral operators on $L$, and $\mathscr{K}^+(L)$ the cone of such operators with non-negative kernels. The spectral radius of an operator $K \in \mathscr{K}(L)$ is denoted by $\rho(K)$.

It is not difficult to show that the adjoint operator $K^*$ of an operator $K \in \mathscr{K}^+(L)$ with the kernel $k$ can be restricted to the function space $L'$: the result is a compact integral operator on $L'$ with kernel $k^*(x, y) = k(y, x)$ for $x, y \in X$.

In general, the following inequality holds

$$\rho(K^*|_{L'}) \leqslant \rho(K). \tag{1}$$

For this reason, let $\mathscr{K}^+_=(L)$ denote the set of all operators from $\mathscr{K}^+(L)$ for which the equality holds in (1). The notation $K^*$ will be used throughout instead of $K^*|_{L'}$.

Since any operator $K \in \mathscr{K}^+(L)$ leaves the cone $S$ invariant, the Krein–Rutman theorem may be used (see [4, 6]).

THEOREM 1·1. *If an operator $K \in \mathscr{K}^+(L)$ is not quasi-nilpotent, then the spectral radius $\rho(K)$ is an eigenvalue with at least one non-negative eigenfunction. If, in addition, $K \in \mathscr{K}^+_=(L)$, then the operator $K^*$ has at least one eigenfunction of $S'$ belonging to the eigenvalue $\rho(K)$.*

Finally, we recall Jentzsch's theorem for irreducible kernel operators (see [6]).

*Definition* 1·2. An operator $K \in \mathscr{K}^+(L)$ with a kernel $k$ is *reducible*, if there exists a set $A \in \mathscr{M}$ such that $\mu(A) > 0$, $\mu(A^c) > 0$ and $k = 0$ a.e. on $A \times A^c$. Otherwise, if there is no such set, the operator $K$ is *irreducible*.

THEOREM 1·3. *The spectral radius of any irreducible operator $K \in \mathscr{K}^+(L)$ is a positive eigenvalue. There exists a corresponding eigenfunction $f$ which is positive almost everywhere on $X$, and any other eigenfunction belonging to the eigenvalue $\rho(K)$ is a constant multiple of $f$.*

We conclude this section by stating the following useful proposition.

PROPOSITION 1·4. (*a*) *If $|f| \leqslant |g|$, then $\|f\| \leqslant \|g\|$, for all $f, g \in L$.*
(*b*) $\|K\| = \sup\{\|Kf\|, f \in S, \|f\| \leqslant 1\}$, *for all $K \in \mathscr{K}^+(L)$.*
(*c*) *If $K_1 \leqslant K_2$, then $\|K_1\| \leqslant \|K_2\|$ and $\rho(K_1) \leqslant \rho(K_2)$, for all $K_1, K_2 \in \mathscr{K}^+(L)$.*

*Proof.* Only the proof of (*c*) is not obvious. Since $K_1 f \leqslant K_2 f$ for all $f \in S$, we have $\|K_1\| \leqslant \|K_2\|$ by (*a*) and (*b*). This also gives $\|K_1^n\| \leqslant \|K_2^n\|$ for all $n \in \mathbb{N}$, and hence $\rho(K_1) \leqslant \rho(K_2)$. ∎

https://doi.org/10.1017/S0305004100071279 Published online by Cambridge University Press

## 2. *The spectral radius of a sum of operators*

In this section, sums of compact integral operators with non-negative kernels will be studied. In particular, we will be able to give an estimate of the spectral radius of this sum in terms of spectral radii of the summands. Let us start with the following definition.

*Definition* 2·1. The *Hadamard product* $h = u \circ v$ of two non-negative functions $u \in S$ and $v \in S'$ is the non-negative function $h(x) = u(x)\, v(x)$.

From the fact that $\int h\, d\mu = \int uv\, d\mu = (b(v))(u) < \infty$ it follows that $h \in L^1(X, \mathscr{M}, \mu)$. In the special case when $L = \mathbb{R}^n$, the Hadamard product of vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ is the vector $z = x \circ y = (x_1 y_1, \ldots, x_n y_n)$. By the Krein–Rutman theorem, for any operator $K \in \mathscr{K}^+_=(L)$ with positive spectral radius $\rho(K)$ there exist non-negative eigenfunctions $f \in S$ and $g \in S'$ such that $Kf = \rho(K) f$ and $K^* g = \rho(K) g$. If $h \neq 0$ on a set of positive measure $\mu$, we may assume that

$$\int_X h(x)\, d\mu(x) = 1 \tag{2}$$

by multiplying either $f$ or $g$ by a certain constant if necessary; here we have used the fact that $h \in L^1(X, \mathscr{M}, \mu)$.

LEMMA 2·2. *Let $K \in \mathscr{K}^+_=(L)$ be an irreducible operator with non-negative kernel $k$ and with spectral radius equal to $\rho(K) = \rho$. Let $f \in S$ and $g \in S'$ be positive eigenfunctions satisfying $Kf = \rho f$, $K^* g = \rho g$ and be normalized so that for the Hadamard product $h = f \circ g$ the equation (2) holds. Then*

$$\rho = (Kf, g) \leqslant \exp\left( \int_X h(x) \log\left( \frac{(Ku)(x)}{u(x)} \right) d\mu(x) \right) \leqslant (Ku, v)$$

*for all positive functions $u \in S$ and $v \in S'$ satisfying the condition $u \circ v = h$.*

*Proof.* For almost every $x \in X$ let us define a positive measure on $\mathscr{M}$

$$\nu_x(A) = \frac{1}{\rho f(x)} \int_A k(x, y) f(y)\, dy.$$

Clearly $\nu_x(X) = (Kf)(x)/(\rho f(x)) = 1$ for almost all $x \in X$. Using next the obvious estimates $\log^+(t) < t$ and $\log^-(t) < 1/t$ for $t > 0$, one can show that

$$h \log\left( \frac{u}{f} \right) \in L^1(X, \mu). \tag{3}$$

An application of Fubini's theorem gives

$$\begin{aligned}
\int h(y) \log\left( \frac{u(y)}{f(y)} \right) dy &= \int f(y) \log\left( \frac{u(y)}{f(y)} \right) \frac{1}{\rho} (K^* g)(y)\, dy \\
&= \int f(y) \log\left( \frac{u(y)}{f(y)} \right) \frac{1}{\rho} \left( \int k(x, y)\, g(x)\, dx \right) dy \\
&= \int f(x)\, g(x) \left( \int \frac{k(x, y) f(y)}{\rho f(x)} \log\left( \frac{u(y)}{f(y)} \right) dy \right) dx \\
&= \int h(x) \left( \int \log\left( \frac{u(y)}{f(y)} \right) d\nu_x(y) \right) dx.
\end{aligned}$$

https://doi.org/10.1017/S0305004100071279 Published online by Cambridge University Press

Now, Jensen's inequality shows that

$$\int h(y)\log\left(\frac{u(y)}{f(y)}\right)dy \leqslant \int h(x)\log\left(\int\frac{u(y)}{f(y)}d\nu_x(y)\right)dx$$

$$= \int h(x)\log\left(\frac{1}{\rho f(x)}\int k(x,y)f(y)\frac{u(y)}{f(y)}dy\right)dx = \int h(x)\log\left(\frac{(Ku)(x)}{\rho f(x)}\right)dx$$

$$= \int h(x)\log\left(\frac{(Ku)(x)}{f(x)}\right)dx - \int h(x)\log(\rho)\,dx.$$

Since $|\int h\log(u/f)\,d\mu| < \infty$ by (3), we have

$$\int h(x)\log\left(\frac{(Ku)(x)}{u(x)}\right)dx = \int h(x)\log\left(\frac{(Ku)(x)}{f(x)}\right)dx - \int h(x)\log\left(\frac{u(x)}{f(x)}\right)dx$$

$$\geqslant \log(\rho)\int h(x)\,dx = \log(\rho).$$

This completes the proof of the left inequality.

To prove the right inequality, let $\lambda$ be the positive measure on $\mathscr{M}$ defined by

$$\lambda(A) = \int_A h(x)\,d\mu(x).$$

Obviously, $\lambda(X) = \int_X h(x)\,d\mu(x) = 1$ by (2). Applying Jensen's inequality again, the desired inequality is quickly obtained:

$$\log((Ku,v)) = \log\left(\int_X\frac{(Ku)(x)}{u(x)}d\lambda(x)\right)$$

$$\geqslant \int_X\log\left(\frac{(Ku)(x)}{u(x)}\right)d\lambda(x) = \int_X h(x)\log\left(\frac{(Ku)(x)}{u(x)}\right)d\mu(x). \quad \blacksquare$$

LEMMA 2·3. *(a) Let $K\in\mathscr{K}_{-}^{+}(L)$ be an irreducible operator with a kernel of the form*

$$k(x,y) = a_1(x)\,b_1(y) + a_2(x)\,b_2(y) + \ldots + a_n(x)\,b_n(y), \tag{4}$$

*where the functions $a_i\in S$, $b_i\in S'$ $(i = 1,2,\ldots,n)$ are not almost everywhere equal to zero. Then a positive eigenfunction $f$ which corresponds to the spectral radius $\rho(K) > 0$ is of the form*

$$f(x) = c_1\,a_1(x) + \ldots + c_n\,a_n(x),$$

*where $c_i > 0$ $(i = 1,2,\ldots,n)$.*

*(b) Let the operators $K_1\in\mathscr{K}_{-}^{+}(L)$ and $K_2\in\mathscr{K}^{+}(L)$ have kernels $0\leqslant k_1\leqslant k_2$. Suppose that the operator $K_1$ is irreducible with kernel of the form* (4). *(Thus $K_2$ is also irreducible.) Let $f_i(i = 1,2)$ denote positive eigenfunctions of operators $K_i$ belonging to the spectral radii $\rho(K_i)$. Then there exists a positive real number $c$ so that almost everywhere the inequality $c.f_1\leqslant f_2$ holds.*

*Proof.* (a) The case $n = 1$ is obvious. Therefore we suppose that $n\geqslant 2$. Obviously, any eigenfunction of the operator $K$ is of the form $f(x) = c_1\,a_1(x) + \ldots + c_n\,a_n(x)$, where $c_1,\ldots,c_n$ are real numbers. The eigenfunction equation $Kf = \rho(K)f$ now becomes

$$\sum_{j=1}^{n}\sum_{i=1}^{n}(a_i,b_j)\,c_i\,a_j(x) = \rho(K).\sum_{j=1}^{n}c_j\,a_j(x).$$

https://doi.org/10.1017/S0305004100071279 Published online by Cambridge University Press

Define a matrix $A$ and a vector $c$ by

$$A = \begin{pmatrix} (a_1, b_1) & \dots & (a_n, b_1) \\ \dots & \dots & \dots \\ (a_1, b_n) & \dots & (a_n, b_n) \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

The lemma follows provided that there is a positive vector $c$ satisfying $Ac = \rho(K)c$. Let us prove first that the matrix $A$ is irreducible. If it is not, then we may assume that there is a number $k \in \{1, 2, \dots, n-1\}$ such that $(a_i, b_j) = 0$ for all $i = 1, 2, \dots, k$ and $j = k+1, \dots, n$. Define for all $i, j = 1, 2, \dots, n$

$$A_i = \{x \in X, a_i(x) = 0\}, \quad B_j = \{x \in X, b_i(x) = 0\}$$

and

$$A = \bigcap_{i=1}^{k} A_i, \quad B = \bigcap_{j=k+1}^{n} B_j.$$

Since the functions $a_i$ and $b_j$ are not almost everywhere equal to zero, $\mu(A^c) > 0$ and $\mu(B^c) > 0$. Since $(a_i, b_j) = 0$ we have

$$\mu((A_i \cup B_j)^c) = 0, \quad (i = 1, 2, \dots, k, \quad j = k+1, \dots, n).$$

Since

$$(A \cup B)^c = \bigcup_{i=1}^{k} \bigcup_{j=k+1}^{n} (A_i \cup B_j)^c,$$

we also have $\mu((A \cup B)^c) = 0$. Thus $\mu(A \cup B) = \mu(X) > 0$. Therefore, either $\mu(A) > 0$ or $\mu(B) > 0$. Suppose that $\mu(A) > 0$ (the case $\mu(B) > 0$ is similar). Since

$$\mu(A^c \backslash B) = \mu((A \cup B)^c) = 0$$

and

$$k(x, y) = \sum_{i=1}^{k} a_i(x)\, b_i(y) + \sum_{j=k+1}^{n} a_j(x)\, b_j(y) = 0$$

for all $(x, y) \in A \times B$, it follows that $k = 0$ a.e. on $A \times A^c$; hence $K$ is reducible. This contradiction shows that the matrix $A$ is irreducible. Thus by the Perron–Frobenius theorem there exists a positive eigenvector $c$, i.e. $c_i > 0$ $(i = 1, \dots, n)$, which corresponds to the spectral radius $\rho(A)$. Since $K$ is irreducible, the function defined by $f(x) = c_1 a_1(x) + \dots + c_n a_n(x)$ is almost everywhere positive. It is also the eigenfunction of the operator $K$ corresponding to the eigenvalue $\rho(A)$. Let $g \in S'$ be a positive eigenfunction of $K^*$ belonging to the spectral radius $\rho(K)$. Hence

$$\rho(K) \cdot (f, g) = (f, K^* g) = (Kf, g) = \rho(A) \cdot (f, g).$$

Thus $\rho(K) = \rho(A)$, since $f$ and $g$ are positive functions.

(*b*) By (*a*) the operator $K_1$ has an eigenfunction corresponding to $\rho(K_1) > 0$

$$f_1(x) = c_1 a_1(x) + \dots + c_n a_n(x), \quad (c_i > 0, i = 1, 2, \dots, n).$$

Defining a positive number $d = \min\{(b_i, f_2)/c_i, i = 1, 2, \dots, n\}$, we have

$$\begin{aligned} \rho(K_2) f_2(x) &= \int_X k_2(x, y) f_2(y)\, dy \\ &\geqslant \int_X k_1(x, y) f_2(y)\, dy = \sum_{i=1}^{n} (b_i, f_2)\, a_i(x) \\ &\geqslant d \sum_{i=1}^{n} c_i a_i(x) = d f_1(x). \end{aligned}$$

https://doi.org/10.1017/S0305004100071279 Published online by Cambridge University Press

This gives the desired conclusion, with $c = d/\rho(K_2)$. ∎

The main result of this section follows.

THEOREM 2·4. *Let $K_i \in \mathscr{K}_+^+(L)$ $(i = 1, 2, \ldots, n)$ be irreducible operators with spectral radii $\rho(K_i)$. Let $f_i \in S$ and $g_i \in S'$ be corresponding positive eigenfunctions satisfying*

$$K_i f_i = \rho(K_i) f_i, \quad K_i^* g_i = \rho(K_i) g_i$$

*and be normalized so that*

$$f_i \circ g_i = h \ (i = 1, 2, \ldots, n) \quad \text{and} \quad \int_X h(x)\, d\mu(x) = 1.$$

*Suppose also that at least one operator has either*

1. *kernel of the form* (4), *or*

2. *kernel bounded in the sense that there exist positive numbers $c$ and $C$ such that $c \leqslant k(x, y) \leqslant C$ for almost all $(x, y) \in X \times X$.*

*Then for all non-negative numbers $t_i$ $(i = 1, 2, \ldots, n)$ the following inequality holds:*

$$\rho(t_1 K_1 + \ldots + t_n K_n) \geqslant t_1 \rho(K_1) + \ldots + t_n \rho(K_n).$$

*Proof.* Clearly we may assume that all $t_i$ are positive. Let $u$ denote the positive eigenfunction of the operator $K = t_1 K_1 + \ldots + t_n K_n$ belonging to the spectral radius. Define a function $v > 0$ such that $u \circ v = h$. It will be shown that $v \in S'$.

*Case* 1. If $K_i$ has the kernel of the form (4), then the operator $t_i K_i$ also has such a kernel. Since $t_i K_i \leqslant K$, then by Lemma 2·3(*b*) there exists number $c > 0$ such that $c f_i \leqslant u$. Then

$$v(x) = h(x)/u(x) \leqslant h(x)/(c f_i(x)) = g_i(x)/c \in S',$$

which implies $v \in S'$.

*Case* 2. Let $K_i$ be an operator with bounded kernel. From the equation

$$\int_X k_i(x, y) f_i(y)\, d\mu(y) = \rho(K_i) f_i(x)$$

it follows first that

$$c \int_X f_i(y)\, d\mu(y) \leqslant \rho(K_i) f_i(x) \quad \text{or} \quad \int_X f_i(y)\, d\mu(y) < \infty,$$

and then that

$$f_i(x) \leqslant (C/\rho(K_i)) \cdot \int_X f_i(y)\, d\mu(y) = M$$

for almost every $x \in X$. Similarly, the fact that $\int_X k(x, y)\, u(y)\, d\mu(y) = \rho(K)\, u(x)$ implies that

$$u(x) \geqslant t_i c/\rho(K) \cdot \int_X u(y)\, d\mu(y) = m > 0.$$

Finally, from $\quad v(x) = h(x)/u(x) = f_i(x)\, g_i(x)/u(x) \leqslant (M/m)\, g_i(x) \in S'$

it follows that $v \in S'$.

https://doi.org/10.1017/S0305004100071279 Published online by Cambridge University Press

By Lemma 2·2 we have $\rho(K_i) = (K_i f_i, g_i) \leqslant (K_i u, v)$ for $i = 1, 2, \ldots, n$. Multiplying these inequalities by $t_i \geqslant 0$ respectively for $i = 1, \ldots, n$, and summing gives the desired inequality

$$\sum_{i=1}^{n} t_i \rho(K_i) \leqslant \left(\left(\sum_{i=1}^{n} t_i K_i\right) u, v\right) = \rho\left(\sum_{i=1}^{n} t_i K_i\right). \quad \blacksquare$$

COROLLARY 2·5. *Let $K \in K^+(L^2(X, \mu))$ be an irreducible operator with a kernel either of the form* (4) *or with kernel bounded (in the sense of Theorem* 2·4*). Then*

$$\rho(tK + (1-t)K^*) \geqslant \rho(K)$$

*for any number $0 \leqslant t \leqslant 1$.*

This Corollary, which follows immediately from Theorem 2·4, is obviously a generalization of Levinger's inequality for non-negative matrices.

We shall now give a generalization of the previous theorem. First, let $\mathscr{D}$ be the set of all positive measurable functions $d$ for which there exist positive numbers $c(d)$ and $C(d)$ satisfying $c(d) \leqslant d(x) \leqslant C(d)$ for almost every $x \in X$. If $d \in \mathscr{D}$, then the operator $D$ defined by $(Df)(x) = d(x) f(x)$ is obviously a linear bounded operator on $L$.

THEOREM 2·6. *Under the assumptions of Theorem* 2·4 *let $d_i \in \mathscr{D}$, and let $D_i$ be the corresponding operators $(i = 1, 2, \ldots, n)$. Then*

$$\rho\left(\sum_{i=1}^{n} D_i K_i\right) \geqslant \sum_{i=1}^{n} \rho(K_i) \cdot \exp\left(\int_X h(x) \log (d_i(x))\, d\mu(x)\right).$$

*Proof.* Let $u$ be a positive eigenfunction of the operator $K = \sum_{i=1}^{n} D_i K_i$ corresponding to the spectral radius. Define a positive function $v$ by $u \circ v = h$. A verification similar to those above shows that $v \in S'$. Using Jensen's inequality for the measure $\lambda$, defined in the proof of Lemma 2·2, we obtain

$$\begin{aligned}((D_i K_i) u, v) &= \int_X (K_i u)(x)\, d_i(x)\, v(x)\, d\mu(x) = \int_X \frac{(K_i u)(x)}{u(x)} d_i(x)\, d\lambda(x) \\ &\geqslant \exp\left(\int_X \log\left(\frac{(K_i u)(x)}{u(x)} d_i(x)\right) d\lambda(x)\right) \\ &= \exp\left(\int_X h(x) \log\left(\frac{(K_i u)(x)}{u(x)}\right) d\mu(x) + \int_X h(x) \log (d_i(x))\, d\mu(x)\right).\end{aligned}$$

By Lemma 2·2 it follows that

$$((D_i K_i) u, v) \geqslant \rho(K_i) \cdot \exp\left(\int_X h(x) \log (d_i(x))\, d\mu(x)\right).$$

Finally, we have

$$\begin{aligned}\rho\left(\sum_{i=1}^{n} D_i K_i\right) = \left(\rho\left(\sum_{i=1}^{n} D_i K_i\right) u, v\right) &= \sum_{i=1}^{n} ((D_i K_i) u, v) \\ &\geqslant \sum_{i=1}^{n} \rho(K_i) \cdot \exp\left(\int h(x) \log (d_i(x))\, dx\right). \quad \blacksquare\end{aligned}$$

https://doi.org/10.1017/S0305004100071279 Published online by Cambridge University Press

### 3. *The spectral radius of a weighted geometric mean of operators*

In the previous section the spectral radius of a sum of operators is estimated; we added up kernels of operators of $\mathscr{K}^+(L)$. We shall now multiply kernels of such operators. However it will be assumed that the space $L$ has the following property:

if an operator $K_1 \in \mathscr{K}^+(L)$ has a kernel $k_1$, and $K_2$ is bounded integral operator on $L$ with non-negative kernel $k_2$ such that $0 \leqslant k_2 \leqslant k_1$, then the operator $K_2$ is also compact.

By the theorem of Dodds and Fremlin[1] the space $L$ has this property if the Banach lattices $L$ and $L^*$ both have order continuous norms.

*Definition* 3·1. The *Hadamard (or Schur) product* of operators $K_1$ and $K_2$ of $\mathscr{K}^+(L)$ with kernels $k_1$ and $k_2$ respectively is the integral operator $K = K_1 \circ K_2$ with kernel $k(x,y) = k_1(x,y)\, k_2(x,y)$ for $x, y \in X$. The condition $K \in \mathscr{K}^+(L)$ must be satisfied.

In the special case of non-negative matrices, the Hadamard product of two matrices $A = (a_{ij})_{i,j=1}^n$ and $B = (b_{ij})_{i,j=1}^n$ is the matrix $A \circ B = (a_{ij} b_{ij})_{i,j=1}^n$.

*Definition* 3·2. Let $K \in \mathscr{K}^+(L)$ be an operator with kernel $k$ and let $\alpha$ be a non-negative real number. If the operator with kernel equal to $k^\alpha$ is a compact integral operator on $L$, then it is denoted by $K^{(\alpha)}$. (Here we adopt the convention that $0^0 = 1$.)

In the proof of the following spectral inequality the following lemma is needed.

LEMMA 3·3. (*a*) *Suppose that the space $\mathscr{K}^+(L)$ contains an irreducible operator, and let $K \in \mathscr{K}^+(L)$. Then there exists a sequence of irreducible operators $K_n \in \mathscr{K}^+(L)$ so that*

$$K_1 \geqslant K_2 \geqslant \ldots \geqslant K_n \geqslant \ldots \geqslant K \quad \text{and} \quad \rho(K) = \lim_{n\to\infty} \rho(K_n).$$

(*b*) *The space $\mathscr{K}^+(L)$ contains an irreducible operator if and only if the spaces $L$ and $L'$ both contain positive functions.*

*Proof.* (*a*) Let $A \in \mathscr{K}^+(L)$ be irreducible. The operators $K_n = K + (1/n) A$ are obviously irreducible operators of $\mathscr{K}^+(L)$. Clearly $\|K_n - K\| \to 0$ as $n \to \infty$ and $K_1 \geqslant K_2 \geqslant \ldots \geqslant K_n \geqslant \ldots \geqslant K$. Thus

$$\rho(K_1) \geqslant \rho(K_2) \geqslant \ldots \geqslant \rho(K_n) \geqslant \ldots \geqslant \rho(K)$$

and hence $\rho(K) \leqslant \lim_{n\to\infty} \rho(K_n)$. Since the spectral radius is upper semicontinuous, we have finally

$$\rho(K) = \lim_{n\to\infty} \rho(K_n).$$

(*b*) If $f \in S$ and $g \in S'$ are positive functions, then the operator with kernel equal to $k(x,y) = f(x)\, g(y)$ is an irreducible operator of $\mathscr{K}^+(L)$.

On the other hand, if $K \in \mathscr{K}^+(L)$ is an irreducible operator, then there exists a positive eigenfunction $f \in S$ belonging to the spectral radius. Clearly $K^*$ is also an irreducible operator. Therefore there exists a positive eigenfunction $g \in S'$ corresponding to the spectral radius. ∎

Now we state the main result of this section. However in accordance with Lemma 3·3 we assume that the spaces $L$ and $L'$ both contain positive functions.

https://doi.org/10.1017/S0305004100071279 Published online by Cambridge University Press

THEOREM 3·4. *Let $K_1$ and $K_2$ be operators of $\mathscr{K}^+(L)$ with kernels $k_1$ and $k_2$ respectively, and $0 \leqslant \alpha \leqslant 1$. Then*

$$K_1^{(\alpha)} \circ K_2^{(1-\alpha)} \in \mathscr{K}^+(L) \quad \text{and} \quad \rho(K_1^{(\alpha)} \circ K_2^{(1-\alpha)}) \leqslant \rho(K_1)^{\alpha} . \rho(K_2)^{(1-\alpha)}.$$

*Proof.* The well-known inequality between the arithmetic and geometric means implies

$$0 \leqslant K = K_1^{(\alpha)} \circ K_2^{(1-\alpha)} \leqslant \alpha K_1 + (1-\alpha) K_2,$$

and hence $K$ is a bounded integral operator on $L$ with non-negative kernel. Using also our assumption on the space $L$, it follows that $K$ is compact.

To prove the inequality, we assume first that $K_1$ and $K_2$ are irreducible operators. Obviously, since they are not quasi-nilpotent, it is enough to prove the inequality only in the case when $\rho(K_1) = \rho(K_2) = 1$. Recall that the operators $K_1^*$ and $K_2^*$ are just the restrictions of corresponding adjoint operators. Thus $0 < \rho(K_1^*) \leqslant 1$ and $0 < \rho(K_2^*) \leqslant 1$. Let $g_1$ and $g_2$ denote positive eigenfunctions of $K_1^*$ and $K_2^*$, respectively, corresponding to the spectral radii. If $\rho(K) = 0$ there is nothing to prove. Otherwise, let $f$ be a non-negative eigenfunction of the operator $K$ belonging, as usual, to the spectral radius. Then we have $K_1^* g_1 = \rho(K_1^*) g_1$, $K_2^* g_2 = \rho(K_2^*) g_2$, $Kf = \rho(K) f$, $g_1, g_2 \in S'$, $f \in S$. Applying the inequality between the arithmetic and geometric means again, we conclude that $g_1^{\alpha} g_2^{1-\alpha} \in S'$, and therefore that $h = g_1^{\alpha} g_2^{1-\alpha} f \in L^1(X, \mu)$. Fubini's theorem for non-negative functions implies

$$\rho(K) . \int h(x)\, dx = \int g_1^{\alpha}(x)\, g_2^{1-\alpha}(x)\, \rho(K) f(x)\, dx = \int g_1^{\alpha}(x)\, g_2^{1-\alpha}(x) \left( \int k(x, y) f(y)\, dy \right) dx$$

$$= \int f(y) \left( \int k(x, y)\, g_1^{\alpha}(x)\, g_2^{1-\alpha}(x)\, dx \right) dy.$$

An application of Hőlder's inequality gives

$$\rho(K) . \int h(x)\, dx = \int f(y) \left( \int [k_1(x, y)\, g_1(x)]^{\alpha} [k_2(x, y)\, g_2(x)]^{1-\alpha} dx \right) dy$$

$$\leqslant \int f(y) \left( \int k_1^*(y, x)\, g_1(x)\, dx \right)^{\alpha} \left( \int k_2^*(y, x)\, g_2(x)\, dx \right)^{1-\alpha} dy$$

$$= \rho(K_1^*)^{\alpha} \rho(K_2^*)^{1-\alpha} . \int f(y)\, g_1^{\alpha}(y)\, g_2^{1-\alpha}(y)\, dy \leqslant \int h(x)\, dx.$$

Since $0 < \int h(x)\, dx < \infty$, the desired conclusion that $\rho(K) \leqslant 1$ follows.

In the general case, by the previous lemma there exist non-increasing sequences of irreducible operators $A_n, B_n \in \mathscr{K}^+(L)$ such that

$$\rho(K_1) = \lim_{n \to \infty} \rho(A_n) \quad \text{and} \quad \rho(K_2) = \lim_{n \to \infty} \rho(B_n).$$

Since the inequality holds for the operators $A_n$ and $B_n$, we have

$$\rho(K) \leqslant \lim_{n \to \infty} \rho(A_n^{(\alpha)} \circ B_n^{(1-\alpha)}) \leqslant \lim_{n \to \infty} (\rho(A_n)^{\alpha} . \rho(B_n)^{(1-\alpha)}) = \rho(K_1)^{\alpha} . \rho(K_2)^{(1-\alpha)}.$$

This completes the proof of the theorem. ∎

By a simple induction we can generalize the previous result.

https://doi.org/10.1017/S0305004100071279 Published online by Cambridge University Press

THEOREM 3·5. *Let $K_1, K_2, \dots, K_n \in \mathscr{K}^+(L)$ be operators with kernels $k_1, k_2, \dots, k_n$, and let $\alpha_1, \alpha_2, \dots, \alpha_n$ be non-negative numbers such that*

$$\sum_{k=1}^{n} \alpha_k = 1. \tag{5}$$

*Then $K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \dots \circ K_n^{(\alpha_n)} \in \mathscr{K}^+(L)$ and*

$$\rho(K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \dots \circ K_n^{(\alpha_n)}) \leqslant \rho(K_1)^{\alpha_1} \rho(K_2)^{\alpha_2} \dots \rho(K_n)^{\alpha_n}.$$

*Proof.* If $n = 1$, there is nothing to prove; if $n = 2$, the inequality follows immediately from Theorem 3·4. We may suppose that $\alpha_n < 1$. Assume now that the theorem holds with $n$ replaced by $(n-1)$ and define $\beta_k = \alpha_k/(1-\alpha_n)$ for $k = 1, \dots, n-1$. Since $\sum_{k=1}^{n-1} \beta_k = 1$, the operator $K \in \mathscr{K}^+(L)$ defined by $K = K_1^{(\beta_1)} \circ K_2^{(\beta_2)} \circ \dots \circ K_{n-1}^{(\beta_{n-1})}$ satisfies the inequality

$$\rho(K) \leqslant \rho(K_1)^{\beta_1} \dots \rho(K_{n-1})^{\beta_{n-1}}.$$

Then Theorem 3·4 implies

$$\begin{aligned} \rho(K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \dots \circ K_n^{(\alpha_n)}) &= \rho(K^{(1-\alpha_n)} \circ K_n^{(\alpha_n)}) \leqslant \rho(K)^{1-\alpha_n} \rho(K_n)^{\alpha_n} \\ &\leqslant (\rho(K_1)^{\beta_1} \dots \rho(K_{n-1})^{\beta_{n-1}})^{1-\alpha_n} \rho(K_n)^{\alpha_n} \\ &= \rho(K_1)^{\alpha_1} \rho(K_2)^{\alpha_2} \dots \rho(K_n)^{\alpha_n} \end{aligned}$$

and the proof is completed. ∎

The corresponding inequality for non-negative matrices [3] holds under a more general hypothesis: condition (5) may be replaced by

$$\sum_{k=1}^{n} \alpha_k \geqslant 1.$$

Let us show that this condition does not suffice for operators on $L^2[0, 1]$, even if we assume in addition that $K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \dots \circ K_n^{(\alpha_n)} \in \mathscr{K}^+(L)$ which is not implied by this condition. For example, let us define a kernel $k$ of an operator $K$ by

$$k(x, y) = e^x, \quad \text{for} \quad (x, y) \in [0, 1] \times [0, 1].$$

A simple computation gives

$$\rho(K) = e - 1 \quad \text{and} \quad \rho(K^{(2)}) = \tfrac{1}{2}(e^2 - 1),$$

and convinces us that

$$\rho(K^{(2)}) > \rho(K)^2.$$

The author would like to express his gratitude to Professor Matjaž Omladič for all his helpful advice.

## REFERENCES

[1] P. G. DODDS and D. H. FREMLIN. Compact operators in Banach lattices. *Israel J. Math.* **34** (1979), 287–320.

[2] L. ELSNER and C. R. JOHNSON. Nonnegative matrices, zero patterns, and spectral inequalities. *Linear Algebra Appl.* **120** (1989), 225–236.

[3] L. ELSNER, C. R. JOHNSON and J. A. DIAS DA SILVA. The Perron root of a weighted geometric mean of nonnegative matrices. *Linear and Multilinear Algebra* **24** (1988), 1–13.

[4] M. G. KREIN and M. A. RUTMAN. Linear operators leaving invariant a cone in a Banach space. In *Functional Analysis and Measure Theory* (American Mathematical Society, 1962), pp. 199–325.

[5] W. A. LUXEMBURG. *Banach Function Spaces* (Van Gorcum and Company, 1955).

[6] A. C. ZAANEN. *Riesz Spaces*, vol. 2 (North Holland, 1983).

https://doi.org/10.1017/S0305004100071279 Published online by Cambridge University Press