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Entropy of AT(n) systems

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Abstract. In this paper we show that any ergodic measure preserving transformation of a standard probability space which is AT(n) for some positive integer n has zero entropy. We show that for every positive integer n any Bernoulli shift is not AT(n). We also give an example of a transformation which has zero entropy but does not have property AT(n) for any integer $n \geq 1$.

1. *Introduction*

In order to answer some questions from ergodic theory closely related to the theory of von Neumann algebras, Giordano and Handelman [**8**] reformulated matrix-valued random walks and their associated group actions in terms of dimension spaces. Their approach leads to a notion of rank called AT(n), for integers $n \geq 1$. This new concept generalizes approximate transitivity (AT), a property of ergodic actions introduced by Connes and Woods [**1**] in the theory of von Neumann algebras, which occurs for $n = 1$.

Throughout this paper $(X, \mathfrak{B}, \mu, T)$ denotes a dynamical system, where T is a measure preserving automorphism of a standard probability space $(X, \mathfrak{B}, \mu)$. For an integer $n \geq 1$, we say that the dynamical system $(X, \mathfrak{B}, \mu, T)$ (or simply T) is AT(n) if for any $\varepsilon > 0$, for any finite set of functions $\{f_i\}_{i=1}^{k}$ from $L^1_+(X, \mu)$ there exist n functions $\{g_m\}_{m=1,\ldots,n} \in L^1_+(X, \mu)$, a positive integer N, non-negative reals $\{\alpha_{i,j}^{(m)}\}_{i=1,2,\ldots,k,\, j=1,2,\ldots,N}^{m=1,2,\ldots,n}$ and integers $\{t_j^{(m)}\}_{j=1,\ldots,N}^{m=1,\ldots,n}$, such that

$$\left\| f_i - \sum_{m=1}^{n} \sum_{j=1}^{N} \alpha_{i,j}^{(m)} g_m \circ T^{t_j^{(m)}} \right\| < \varepsilon, \tag{1}$$

for $i = 1, 2, \ldots, k$, where $\| \cdot \|$ denotes the L^1 norm.

Note that it is sufficient to require equation (1) to hold for $k = n + 1$. Also, one can demand that $\|g_m\| = 1$ for all m and that $\sum_{m=1}^{n} \sum_{j=1}^{N} \alpha_{i,j}^{(m)} = \|f_i\|$, for all i.

Remark that a rank-n transformation (see [**5**]) is AT(n) and every AT(n) system enjoys the AT($n + 1$) property. The techniques developed by Giordano and Handelman in [**8**]

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allowed the authors to construct examples of AT(2) transformations which are not AT. In [**4**], it was proved that the measure preserving automorphism corresponding to the Rudin–Shapiro substitution, which has rank 4 (and therefore is AT(4)), is not AT.

Dynamical entropy is an invariant of measure-theoretic dynamical systems introduced by Kolmogorov [**9**] and put in its contemporary form by Sinai [**11**]. In 1970, Ornstein [**10**] showed that Kolmogorov–Sinai entropy completely classifies Bernoulli shifts, a basic problem which could not be solved for many decades. It was proved by Connes and Woods [**1**] that any dynamical system which is AT has zero entropy. Different proofs of this result can be found in [**2–4**]. It is natural to ask whether AT(n) dynamical systems have also zero entropy.

In this paper we give a necessary condition for shift maps to be AT(n) (Theorem 2.1). For such transformations, this is a generalization of the necessary condition for an action to be AT from [**3**].

We use this condition to prove that, for any positive integer n, Bernoulli shifts are not AT(n), for $n \geq 1$. An important consequence of this result is Corollary 3.1, which shows that any finite measure preserving transformation which is AT(n), for some positive integer n, has zero entropy. In [**3**] Dooley and Quas proved that zero entropy is not sufficient for AT; they gave an example of a zero-entropy transformation which is not approximately transitive. In this paper we prove that the zero entropy and not AT transformation from [**3**] is not AT(n), for any $n \geq 1$.

2. *A necessary condition for shift maps to be* AT(n)

Let $\Sigma_k = \{1, 2, \ldots, k\}^{\mathbb{Z}}$ be the shift space over the alphabet $\{1, 2, \ldots, k\}$. A cylinder set in Σ_k is a set of the form $[y_0, y_1, \ldots, y_m]_n = \{x \in \Sigma_k; x_n = y_0, x_{n+1} = y_1, \ldots, x_{n+m} = y_m\}$, where $m, n \in \mathbb{Z}$, $m \geq 0$ and $y_i \in \{1, 2, \ldots, k\}$. The cylinders of the form $[y_0, y_1, \ldots, y_{2n}]_{-n}$ with $n \geq 0$ are called centered cylinders. We denote by $\mathfrak{B}_k$ the σ-algebra generated by the cylinder sets of the shift space Σ_k. The map $S_k : \Sigma_k \to \Sigma_k$ defined by

$$(S_k x)_n = x_{n+1} \quad \text{for } x = (x_n)_{n\in\mathbb{Z}}$$

is called the k-shift map.

Let $p = (p(1), \ldots, p(k))$ be a probability vector with non-zero entries, i.e. $p(i) > 0$ for $i = 1, 2, \ldots, k$ and $\sum_{i=1}^{k} p(i) = 1$. Let μ_p be the unique probability measure on $(\Sigma_k, \mathfrak{B}_k)$, which on cylinder sets is given by

$$\mu_p([y_0, y_1, \ldots, y_m]_n) = p(y_0)p(y_1)\cdots p(y_m).$$

This measure is called the Bernoulli measure determined by p. The dynamical system $(\Sigma_k, \mathfrak{B}_k, \mu_p, S_k)$ is called the Bernoulli shift associated to the probability vector p.

Let Λ be a finite subset of the integers. A funny word on the alphabet $\{1, 2, \ldots, k\}$ based on Λ is a finite sequence $W = (W_n)_{n\in\Lambda}$ with $W_n \in \{1, 2, \ldots, k\}$. The Hamming distance between two funny words W, W' based on the same set Λ is given by

$$d_\Lambda(W, W') = \frac{1}{|\Lambda|}|\{n \in \Lambda : W_n \neq W'_n\}|.$$

If $x \in \Sigma_k$ and Λ is a finite set in $\mathbb{Z}$ we denote by $x|_\Lambda$ the funny word $(x_n)_{n\in\Lambda}$.

The following theorem provides a necessary condition for shift maps to be AT(n).

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THEOREM 2.1. *Let S_k be the shift map on the space Σ_k and ν be a non-atomic shift invariant probability measure on Σ_k. Assume that S_k is* AT(n) *but not* AT($n-1$)*, for some $n \geq 2$. Then for every $\varepsilon > 0$ and every $\delta > 0$ there exist finite sets $\Lambda^1, \Lambda^2, \ldots, \Lambda^n$, with $\min\{|\Lambda^i|; i = 1, 2, \ldots, n\}$ arbitrarily large, and funny words W^i based on Λ^i for $i = 1, 2, \ldots, n$ such that*

$$\sum_{i=1}^{n} |\Lambda^i| \nu(\{x \in \Sigma_k : d_{\Lambda^i}(x|_{\Lambda^i}, W^i) < \varepsilon\}) > 1 - \delta.$$

Proof. Let $(\delta_m)_{m\geq 1}$ be a sequence decreasing to zero. The theorem will result if we prove that for every $\varepsilon > 0$ there exist finite sets $\Lambda_m^i \subset \mathbb{Z}$ and funny words W_m^i based on Λ_m^i for $i = 1, 2, \ldots, n$ such that $\sup_{m\geq 1} \min\{|\Lambda_m^i|, i = 1, 2, \ldots, n\} = \infty$ and

$$\sum_{i=1}^{n} |\Lambda_m^i| \nu(\{x \in \Sigma_k : d_{\Lambda^i}(x|_{\Lambda}, W_m^i) < \varepsilon\}) > 1 - \delta_m.$$

Let $\varepsilon > 0$. For $m \geq 0$, denote by $\mathcal{C}_m$ the set of all centered cylinders of the form $[y_0, y_1, \ldots, y_{2m}]_{-m}$ which have positive measure. Since ν is non-atomic, for each $C \in \mathcal{C}_m$, one can find a measurable partition $\mathcal{P}_C$ of C such that $\nu(A) < \delta_m$, for every $A \in \mathcal{P}_C$. Let

$$\mathcal{J}_m = \{A \in \mathcal{P}_C : C \in \mathcal{C}_m\}.$$

Notice that $\{A : \exists m \geq 1 \text{ such that } A \in \mathcal{J}_m\}$ generates (up to null sets) the sigma algebra $\mathfrak{B}_k$.

For $A \in \mathcal{J}_m$ let $g_A = (1/\nu(A))\mathbf{1}_A$ be the normalized indicator function corresponding to A. Let $m \geq 1$ and $A \in \mathcal{J}_m$. Since, by assumption, S_k is AT(n), there exist $f_{1,m}, f_{2,m}, \ldots, f_{n,m} \in L^1_+(\Sigma_k, \nu)$ of norm 1, sequences of non-negative numbers $\{a^1_{A,j}\}_{j\in\mathbb{Z}}, \{a^2_{A,j}\}_{j\in\mathbb{Z}}, \ldots, \{a^n_{A,j}\}_{j\in\mathbb{Z}}$ with finitely many non-zero elements such that $\sum_j a^1_{A,j} + \sum_j a^2_{A,j} + \cdots + \sum_j a^n_{A,j} = 1$ and

$$\left\| g_A - \sum_{i=1}^{n} \sum_j a^i_{A,j} f_{i,m} \circ S_k^{-j} \right\| < \frac{\varepsilon \delta_m^2}{36}.$$

It then follows that for any $A \in \mathcal{J}_m$ we have

$$\int_{\Sigma_k \setminus A} \sum_{i=1}^{n} \sum_j a^i_{A,j} f_{i,m} \circ S_k^{-j} \, d\nu < \frac{\varepsilon \delta_m^2}{36}.$$

Let $A \in \mathcal{J}_m$. For $i = 1, 2, \ldots, n$, define

$$P_A^i = \left\{ j : \int_{\Sigma_k \setminus A} f_{i,m} \circ S_k^{-j} \, d\nu \geq \varepsilon \delta_m / 6 \right\}.$$

It can easily be seen that $\sum_{j\in P_A^1} a^1_{A,j} + \sum_{j\in P_A^2} a^2_{A,j} + \cdots + \sum_{j\in P_A^n} a^n_{A,j} < \delta_m/6$. By setting the $a^i_{A,j}$ to be 0 for $j \in P_A^i$, and rescaling the remaining $a^i_{A,j}$, we obtain coefficients $b^1_{A,j}, b^2_{A,j}, \ldots, b^n_{A,j}$ with $\sum_j b^1_{A,j} + \sum_g b^2_{A,j} + \sum_j b^n_{A,j} = 1$, satisfying

$$\left\| g_A - \sum_{i=1}^{n} \sum_j b^i_{A,j} f_{i,m} \circ S_k^{-j} \right\| < \frac{\delta_m}{2},$$

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and such that

$$\int_{\Sigma_k \setminus A} f_{i,m} \circ S_k^{-j} \, d\nu < \frac{\varepsilon \delta_m}{6}$$

if $b^i_{A,j} > 0$. For $i = 1, 2, \ldots, n$, let

$$\Lambda^i_m = \{j \in \mathbb{Z} : \exists A \in \mathcal{J}_m, b^i_{A,j} > 0\}.$$

We claim that

$$\sup_{m \geq 1} \min\{|\Lambda^i_m|, i = 1, 2, \ldots, n\} = \infty. \tag{2}$$

We will prove the claim by contradiction. Let us suppose that

$$\min\{|\Lambda^i_m|, 1 \leq i \leq n\} \leq M < \infty \quad \text{for all } m \geq 1.$$

Let $i(m)$ be such that $|\Lambda^{i(m)}_m| = \min\{|\Lambda^i_m|, 1 \leq i \leq n\}$; then $|\Lambda^{i(m)}_m| \leq M$, for all $m \geq 1$. Let

$$\mathcal{L}_m = \{A \in \mathcal{J}_m : \exists j \in \mathbb{Z} \text{ such that } b^{i(m)}_{A,j} > 0\}.$$

Remark that $|\mathcal{L}_m| \leq |\Lambda^{i(m)}_m|$. Since $\nu(A) < \delta_m$ for every $A \in \mathcal{J}_m$, it follows that $\lim_{m\to\infty} \nu(\bigcup_{A\in\mathcal{L}_m} A) = 0$. Hence, any $f \in L^1_+(\Sigma_k, \nu)$ can be approximated arbitrarily closely in L^1-norm by step functions of the form $\sum_{A\in\mathcal{J}_m\setminus\mathcal{L}_m} \alpha_A g_A$ with $\alpha_A \geq 0$ and $\sum_{A\in\mathcal{J}_m\setminus\mathcal{L}_m} \alpha_A = \|f\|$, by choosing m sufficiently large.

Let $f_l \in L^1_+(\Sigma_k, \nu)$, $l = 1, 2, \ldots, n$, be functions of norm 1 and $\eta > 0$. Choose $m \geq 1$ sufficiently large such that $\delta_m < \eta$ and such that there exist non-negative numbers α^l_A with $\sum_{A\in\mathcal{J}_m\setminus\mathcal{L}_m} \alpha^l_A = 1$ satisfying

$$\left\| f_l - \sum_{A\in\mathcal{J}_m\setminus\mathcal{L}_m} \alpha^l_A g_A \right\|_1 < \frac{\eta}{2},$$

for $l = 1, 2, \ldots, n$. Note that if $A \in \mathcal{J}_m \setminus \mathcal{L}_m$ then

$$\left\| g_A - \sum_{i=1, i\neq i(m)}^{n} \sum_j b^i_{A,j} f_{i,m} \circ S_k^{-j} \right\| < \frac{\delta_m}{2}.$$

We then obtain non-negative coefficients $c^{i,l}_j$, $i, l \in \{1, 2, \ldots n\}$, $i \neq i(m)$, with finitely many of them different from zero such that

$$\sum_{i=1, i\neq i(m)}^{n} \sum_j c^{i,l}_j = 1$$

and

$$\left\| f_l - \sum_{i=1, i\neq i(m)}^{n} \sum_j c^{i,l}_j f_{i,m} \circ S_k^{-j} \right\| < \eta.$$

Since such an approximation can be done for any $\eta > 0$, it follows that S_k is AT($n-1$). But this contradicts our hypothesis, and therefore, the claim (2) holds.

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Notice that, for $i = 1, 2, \ldots, n$, we have

$$\sum_{A\in\mathcal{J}_m}\sum_{j:b^i_{A,j}>0}\int_{\Sigma_k\setminus A} f_{i,m}\circ S_k^{-j}\,d\nu < \frac{\varepsilon\delta_m|\Lambda^i_m|}{6},$$

and then

$$\begin{aligned}\frac{\varepsilon\delta_m|\Lambda^i_m|}{6} &> \sum_{A\in\mathcal{J}_m}\sum_{j:b^i_{A,j}>0}\int f_{i,m}\circ S_k^{-j}\cdot 1_{\Sigma_k\setminus A}\,d\nu\\ &= \int f_{i,m}\sum_{A\in\mathcal{J}_m}\sum_{g:b^i_{A,j}>0} 1_{\Sigma_k\setminus A}\circ S_k^{j}\,d\nu.\end{aligned}$$

Let

$$H^i_m = \frac{1}{|\Lambda^i_m|}\sum_{A\in\mathcal{J}_m}\sum_{j:b^i_{A,j}>0} 1_{\Sigma_k\setminus A}\circ S_k^{j}.$$

For $j\in\Lambda^i_m$, there exists a unique set $A\in\mathcal{J}_m$ such that $\int_A f_{i,m}\circ S_k^{-j}\,d\nu > 1-\varepsilon\delta_m/6$. Let $[z]_0 = \{x\in\Sigma_k : x_0 = z\}$ be the unique cylinder set from $\mathcal{C}_0$ containing A and define $W^i_{m,j}$ to be this z. Denote by W^i_m the funny word $(W^i_{m,j})_{j\in\Lambda^i_m}$ based on Λ^i_m. Since the above inequality demonstrates that $\int H^i_m f_{i,m}\,d\nu < \varepsilon\delta_m/6$, it follows that $\int_{\{z:H^i_m(z)>\varepsilon\}} f_{i,m}(x)\,d\nu < \delta_m/6$. Let $\widetilde{f}_{i,m}$ be the function defined by

$$\widetilde{f}_{i,m}(x) = \begin{cases} 0 & \text{if } H^i_m(x) > \varepsilon,\\ f(x)\Big/\displaystyle\int_{\{z:H^i_m(z)<\varepsilon\}} f_{i,m}\,d\nu & \text{otherwise.}\end{cases}$$

Clearly,

$$H^i_m(x) \ge \frac{1}{|\Lambda^i_m|}|\{j\in\Lambda^i_m : W^k_{m,j}\neq x_j\}|.$$

Therefore, the support of $\widetilde{f}_{i,m}$ is contained in $\{x\in\Sigma_k : d(x|_{\Lambda^i_m}, W^i_m) < \varepsilon\}$. Since $\|\widetilde{f}_{i,m} - f_{i,m}\| < \delta_m/2$ for $i = 1, 2, \ldots, n$, for $A\in\mathcal{J}_m$ we have that

$$\left\| g_A - \sum_{i=1}^n\sum_j b^i_{A,j}\widetilde{f}_{i,m}\circ S_k^{-j}\right\| < \delta_m.$$

Hence, summing over $A\in\mathcal{J}_m$, we get

$$\left\| 1 - \sum_{i=1}^n\sum_{A\in\mathcal{J}_m}\sum_j \nu(A)b^i_{A,j}\widetilde{f}_{i,m}\circ S_k^{-j}\right\| < \delta_m.$$

Since the support of each $\widetilde{f}_{i,m}$ is contained in $\{x\in\Sigma_k : d_{\Lambda^i_m}(x|_{\Lambda^i_m}, W^i_m) < \varepsilon\}$, it follows that the function

$$\sum_{i=1}^n\sum_{A\in\mathcal{J}_m}\sum_j \nu(A)b^i_{A,j}\widetilde{f}_{i,m}\circ S_k^{-j}$$

is supported on a set of measure at most

$$\sum_{i=1}^n |\Lambda^i_m|\{x\in\Sigma_k : d_{\Lambda^i_m}(x|_{\Lambda^i_m}, W^i_m) < \varepsilon\}.$$

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Therefore

$$\sum_{i=1}^{n} |\Lambda_m^i| \nu(\{x \in \Sigma_k : d_{\Lambda_m^i}(x|_\Lambda, W_m^i) < \varepsilon\}) > 1 - \delta_m$$

and the proof of the theorem is complete. □

3. AT(n) *systems have zero entropy*

In this section we show that for any positive integer n, Bernoulli shifts are not AT(n). We also show that AT(n) systems have zero entropy.

By taking the expected values of L^1-functions with respect to a factor σ-algebra in the definition of AT(n), we obtain the following lemma.

LEMMA 3.1. *A factor of an* AT(n) *system is* AT(n).

PROPOSITION 3.1. *Let* $(\Sigma_k, \mathfrak{B}_k, \mu_p, S_k)$ *be the Bernoulli shift associated to the probability vector* $p = (p(1), \ldots, p(k))$. *Then, for any* $n \geq 1$, *the shift map* S_k *is not* AT(n).

Proof. We prove the proposition by induction on n. It is well known that S_k has positive entropy and therefore is not AT(1). Let us assume now that for some $n \geq 2$, S_k is not AT($n-1$). Let $r = \max\{p(i) : i = 1, 2, \ldots, k\}$. If W is a funny word based on Λ then

$$\mu_p(\{x \in \Sigma_k : d_\Lambda(x|_W, W)\}) \leq \binom{m}{[m\varepsilon]} r^{m-[m\varepsilon]},$$

where $|\Lambda| = m$.

Notice that if ε is sufficiently small then

$$\frac{r(1-\varepsilon)^\varepsilon}{(1-\varepsilon)\varepsilon^\varepsilon r^\varepsilon} < 1.$$

For m sufficiently large, we have

$$\begin{aligned}\binom{m}{[m\varepsilon]} r^{m-[m\varepsilon]} &< \frac{2(m/e)^m\sqrt{2\pi m}}{(m\varepsilon/e)^{m\varepsilon}\sqrt{2\pi m\varepsilon}((m(1-\varepsilon))/e)^{m(1-\varepsilon)}\sqrt{2\pi m(1-\varepsilon)}} r^{m-[m\varepsilon]} \\ &< \frac{1}{\sqrt{2\pi m\varepsilon(1-\varepsilon)}}\left(\frac{r(1-\varepsilon)^\varepsilon}{(1-\varepsilon)\varepsilon^\varepsilon r^\varepsilon}\right)^m < \frac{1-\varepsilon}{n \cdot m}.\end{aligned}$$

It then follows that if $|\Lambda|$ sufficiently large, $\mu_p(\{x \in \Sigma_k : d_\Lambda(x|_W, W) < \varepsilon\}) < (1-\varepsilon)/n \cdot |\Lambda|$. Then Theorem 2.1 implies that S_k is not AT(n). □

In view of Proposition 3.1, Lemma 3.1 and Sinai's theorem, we obtain the following corollary.

COROLLARY 3.1. *Let* n *be a positive integer and let* T *be an* AT(n) *ergodic measure preserving transformation of a standard probability space* $(X, \mathfrak{B}, \mu)$. *Then* T *has zero entropy.*

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In the last part of this section we will show that there exists a zero-entropy dynamical system which is not AT(n) for any $n \geq 1$. Let α be an irrational number and let T be the transformation of the 2-torus $\mathbb{T}^2$ defined by

$$T(s, t) = (s + \alpha, 2s + t + \alpha) \pmod 1.$$

This transformation, studied by Furstenberg [**6**, **7**], is measure preserving, uniquely ergodic and has zero entropy. It was proved by Dooley and Quas in [**3**] that T is not approximately transitive.

PROPOSITION 3.2. *The zero-entropy transformation T defined above does not have property* AT(n) *for any positive integer n.*

Proof. In order to prove the proposition it is enough to show that T is not AT($k-1$) for any integer $k \geq 3$. Fix $k \geq 3$ and let $\mathcal{P}$ be the partition of $\mathbb{T}^2$ consisting of the sets $A_i = \mathbb{T} \times [(i-1)/k, i/k)$, for $i = 1, 2, \ldots, k$. Let $\pi : \mathbb{T}^2 \to \Sigma_k$ be the natural map from $\mathbb{T}^2$ onto the shift space Σ_k, defined by $\pi(z) = (x_m)_{m \in \mathbb{Z}}$, where $x_m = i$ if $T^m(z) \in A_i$.

If μ is the Haar measure of $\mathbb{T}^2$, denote by ν the measure $\pi^{-1} \circ \mu$ induced by μ on Σ_k. We show first that, for all $l \neq m$,

$$\nu(\{x \in \Sigma_k : x_l = i, x_m = j\}) = \frac{1}{k^2}. \tag{3}$$

Since ν is shift invariant it is enough to show that for every $m \in \mathbb{Z}$, $\nu(\{x \in \Sigma_k : x_0 = i, x_m = j\}) = 1/k^2$. This is the Haar measure of the set of points (s, t) such that $t \in [(i-1)/k, i/k)$ and $\pi_2(T^n(s, t)) \in [(j-1)/k, j/k)$ (here $\pi_2(s, t) = t$ for $(s, t) \in \mathbb{T}^2$). In other words, this is the set of all (s, t) such that $t \in [(i-1)/k, i/k)$ and $t + m^2\alpha + 2ms \in [(j-1)/k, j/k)$, where the addition is modulo 1. The measure of this set of points is indeed $1/k^2$, since for fixed t, the transformation of the 1-torus $\mathbb{T}$ defined by $s \mapsto t + m^2\alpha + 2ms \pmod 1$ preserves the Lebesgue measure.

Let Λ be an arbitrary subset of integers of cardinality $|\Lambda|$ and let $W = (W_j)_{j \in \Lambda}$ be a funny word based on Λ. For $j \in \Lambda$, define $\Lambda_j : \Sigma_k \to \mathbb{C}$ by

$$\Lambda_j(x) = \begin{cases} 1 & \text{if } x_j = W_j, \\ \varepsilon & \text{if } x_j = W_j + 1, \\ \ldots & \\ \varepsilon^{k-1} & \text{if } x_j = W_j + k - 1, \end{cases}$$

where $\varepsilon = e^{2\pi i/k}$ and the equalities are modulo k. Let $S(x) = \sum_{j \in \Lambda} \Lambda_j(x)$. Since the sum of the kth roots of unity is zero and the measure of any cylinder of the form $[z]_m$ is $1/k$, we have $\mathbb{E}(\Lambda_j) = 0$, and then, using (3), it follows that $\mathbb{E}(|S|^2) = |\Lambda|$. By Markov inequality we obtain

$$\begin{aligned} \nu\left(|S| > \frac{2k-1}{2k} \cdot |\Lambda|\right) &= \nu\left(|S|^2 > \frac{(2k-1)^2}{4k^2} \cdot |\Lambda|^2\right) \\ &\leq \frac{\mathbb{E}(|S|^2)}{|\Lambda|^2} \cdot \frac{4k^2}{(2k-1)^2} = \frac{4k^2}{(2k-1)^2} \cdot \frac{1}{|\Lambda|}. \end{aligned}$$

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For $x \in \Sigma_k$, define

$$a_i(x) = |\{j \in \Lambda : x_j = W_j + i - 1\}| \quad \text{for } i = 1, 2, \ldots, k.$$

Then

$$S = a_1 + \varepsilon a_2 + \varepsilon^2 a_3 + \cdots + \varepsilon^{k-1} a_k.$$

Now consider the sets

$$B_{1/4k,W} = \left\{ x \in \Sigma_k : d_\Lambda(x|_\Lambda, W) < \frac{1}{4k} \right\},$$
$$C = \left\{ x \in \Sigma_k : |S(x)| > \frac{2k-1}{2k} \cdot |\Lambda| \right\},$$

and, for $i = 1, 2, \ldots, k$,

$$C_i = \{x \in C : a_i(x) > a_j(x) \ \forall j \neq i\}.$$

Notice that $B_{1/4k,W} \subset C_1$. Indeed, if $x \in B_{1/4k,W}$, then $a_1(x) > (4k-1)/4k \cdot |\Lambda|$, and consequently,

$$|S(x)| \geq |a_1(x)| - |\varepsilon a_2(x) + \cdots + \varepsilon^k a_{k+1}(x)| > \left(\frac{4k-1}{4k} - \frac{1}{4k} \right) \cdot |\Lambda| = \frac{2k-1}{2k} \cdot |\Lambda|.$$

Denote by R be the transformation of $\mathbb{T}^2$ defined by $R(s, t) = (s, t + 1/k) \pmod 1$, which clearly commutes with T. It is easy to see that $a_i(\pi(s, t)) = a_{i+1}(\pi(R(s, t)))$ for all $(s, t) \in \mathbb{T}^2$ and $i \in \{1, 2, \ldots, k-1\}$, and then $R(\pi^{-1}(C_i)) = \pi^{-1}(C_{i+1})$ for every $i \in \{1, 2, \ldots, k-1\}$. Since R is a measure preserving transformation, it follows that $\nu(C_i) = \mu(\pi^{-1}(C_i)) = \mu(R(\pi^{-1}(C_i))) = \mu(\pi^{-1}(C_{i+1})) = \nu(C_{i+1})$ for every $i \in \{1, 2, \ldots, k-1\}$, and then $\nu(C_1) \leq 1/k \cdot \nu(C)$. Hence,

$$|\Lambda| \cdot \nu(B_{1/4k,W}) \leq |\Lambda| \cdot \nu(C_1) \leq \frac{4k}{4k^2 - 4k + 1}. \tag{4}$$

This clearly implies that

$$|\Lambda| \cdot \nu(B_{1/4k,W}) \leq 1 - \frac{1}{4k^2 - 4k + 1},$$

and then, applying [**3**, Theorem 2.1], it follows that the system $(\Sigma_k, \mathfrak{B}_k, \nu, S_k)$ is not AT(1).

Now let $m \in \{2, 3, \ldots, k-1\}$. It is straightforward to see that from (4) we have

$$|\Lambda^1| \cdot \nu(B_{1/4k,W^1}) + |\Lambda^2| \cdot \nu(B_{1/4k,W^2}) + \cdots + |\Lambda^m| \cdot \nu(B_{1/4k,W^m}) \leq 1 - \frac{1}{4k^2 - 4k + 1}$$

whenever $\Lambda^1, \Lambda^2, \ldots, \Lambda^m$ are arbitrary finite subsets of $\mathbb{Z}$ and W_i are funny words based on Λ^i for $i = 1, 2, \ldots, m$. Then, if the dynamical system $(\Sigma_k, \mathfrak{B}_k, \nu, S_k)$ is not AT($m-1$), Theorem 2.1 implies that the system is not AT(m). By applying this argument successively for m between 2 and $k-1$, we obtain that the system is not AT($k-1$). Now using Proposition 3.1, we conclude that T is not AT($k-1$). □

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