Math. Model. Nat. Phenom.
Vol. 8, No. 1, 2013, pp. 106–121
DOI: 10.1051/mmnp/20138107

Identifiability for Linearized Sine-Gordon Equation

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Abstract. The paper presents theoretical and numerical results on the identifiability, i.e. the unique identification for the one-dimensional sine-Gordon equation. The identifiability for nonlinear sine-Gordon equation remains an open question. In this paper we establish the identifiability for a linearized sine-Gordon problem. Our method consists of a careful analysis of the Laplace and Fourier transforms of the observation of the system, conducted at a single point. Numerical results based on the best fit to data method confirm that the identification is unique for a wide choice of initial approximations for the sought test parameters. Numerical results compare the identification for the nonlinear and the linearized problems.

Keywords and phrases: Identification, identifiability, sine-Gordon equation

Mathematics Subject Classification: 35R30, 93B30

1. Introduction

Let $q = (\alpha, \beta, \gamma) \in P \subset \mathbb{R}^3$ be a triple of parameters. The corresponding damped one-dimensional sine-Gordon problem is

$$\begin{array}{ll} y_{tt} + 2\alpha y_t - \beta y_{xx} + \gamma \sin y = f(t,x), & (t,x) \in Q = (0,T) \times (0,\pi), \\ y(t,0) = y(t,\pi) = 0, & t \in (0,T), \\ y(0,x) = g(x),\ y_t(0,x) = h(x), & x \in (0,\pi). \end{array} \tag{1.1}$$

Equation (1.1) describes the dynamics of a Josephson junction driven by a current source taking into account the damping effect [1]. This model has attracted an additional interest since it is known to exhibit chaotic behavior for certain values of the governing parameters α, β and γ. For details we refer to [10] and [19].

We study the case of $f(t,x) \equiv 0$ in Q. Then the corresponding linearized sine-Gordon problem is

$$\begin{array}{ll} y_{tt} + 2\alpha y_t - \beta y_{xx} + \gamma y = 0, & (t,x) \in Q = (0,T) \times (0,\pi), \\ y(t,0) = y(t,\pi) = 0, & t \in (0,T), \\ y(0,x) = g(x),\ y_t(0,x) = h(x), & x \in (0,\pi). \end{array} \tag{1.2}$$

The linearized equation is important in physics and electronics. In fact, equation (1.2) is a linear Klein-Gordon equation describing the effect of the shed dispersive radiation on the evolving soliton [17]. In

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https://doi.org/10.1051/mmnp/20138107 Published online by Cambridge University Press

signal analysis of transmission and propagation of electrical signals equation (1.2) is referred to as a telegraph equation with constant coefficients, [12]. Necessary conditions for the coefficients α and γ that imply periodic or bounded solutions of (1.1)-(1.2) are discussed in [14].

The parameter identification problem for (1.1) was discussed in [7]. The authors have established the existence of an optimal parameter triple $q^* = (\alpha^*, \beta^*, \gamma^*) \in P$ satisfying

$$J(q^*) = \inf_P J(q) = \inf_P \|y(q) - z_d\|^2_{L^2(Q)}, \tag{1.3}$$

where $y(q)$ is the solution of (1.1), $z_d \in L^2(Q)$ is the targeted behavior, and a closed and convex set $P \subset \mathbb{R}^3$ is an admissible set. Furthermore, it was shown in [7] that such an optimal set of parameters q^* can be characterized by a bang-bang type of optimality condition. In [6] this problem was studied from a computational perspective.

In this paper our goal is to present theoretical and numerical results on the **identifiability**, i.e. the unique identification for the one-dimensional linearized sine-Gordon problem. In general, there are many optimal parameters q^* in the admissible set P. The identifiability problem is to determine under what conditions on the admissible set P and the observations of the system (1.2) one can conclude that there is a unique optimal parameter set q^*. For various identifiability results see [4, 5, 8, 13, 15] and [9]. The identifiability for the nonlinear sine-Gordon problem (1.1) is still an open question.

Our results show that under some mild conditions on the initial data in (1.2) the parameters α, β, γ are identifiable from an observation of the solution $y(t, x; q), t \geq 0$ at a single point $p \in (0, \pi)$. The problem setup is given in section 2. Our main identifiability result Theorem 4.3 is presented in section 4. Computational algorithms are discussed in section 5. In section 6 we give numerical examples of the identifiability for (1.1) and (1.2) using a one-point observation data.

2. Problem setup

Let Hilbert space $H = L_2(0, \pi)$ have the norm $|\cdot|$ and the inner product $(\cdot, \cdot)$. Let $V = H_0^1(0, \pi)$ have the norm $\|\cdot\|$ and the inner product $((\cdot, \cdot))$. Denote by V' the dual space of V. We consider the standard triple $V \subset H \subset V'$, where the inner product $(\cdot, \cdot)$ extends to the duality mapping between V and V'.

The parameters α, β, γ are required to be in the admissible set

$$P = \{(\alpha, \beta, \gamma) \ : \ \alpha \geq 0, \ \beta \geq \beta_{min} > 0, \ \gamma \geq 0\}. \tag{2.1}$$

Within this framework problem (1.2) is restated as an abstract differential equation

$$y'' + 2\alpha y' + Ay = 0, \quad y(0) = g \in V, \quad y'(0) = h \in H, \tag{2.2}$$

where the operator A is defined by the relation $(Ay, v) = \beta(y_x, v_x) + \gamma(y, v)$ for $v \in V$. According to [3], [11], problem (2.2) has a unique weak solution $y \in C([0, T]; V)$.

We need a more detailed information about the solution $y(t, x)$. First, we introduce some notation.

Definition 2.1. Let $q = (\alpha, \beta, \gamma) \in P$. Define the following subsets of $\mathbb{N} = \{1, 2, \ldots\}$

(i)
$$\Pi_- = \{k \in \mathbb{N} \ : \ k^2\beta + \gamma - \alpha^2 < 0\},$$

(ii)
$$\Pi_0 = \{k \in \mathbb{N} \ : \ k^2\beta + \gamma - \alpha^2 = 0\},$$

(iii)
$$\Pi_+ = \{k \in \mathbb{N} \ : \ k^2\beta + \gamma - \alpha^2 > 0\}.$$

Note that Π_- is at most finite, Π_0 contains at most one element, and Π_+ always contains infinitely many elements. Depending on the triple $(\alpha, \beta, \gamma) \in P$ it may happen that Π_- and Π_0 are empty.

https://doi.org/10.1051/mmnp/20138107 Published online by Cambridge University Press

Theorem 2.2. *Let $D = [0,\infty) \times [0,\pi]$, and $w_k(x) = \sqrt{\frac{2}{\pi}} \sin kx$, $k = 1, 2, \ldots$ Then*

(i) The weak solution $y(t,x)$ of (2.2) on D is given by

$$y(t,x) = e^{-\alpha t} \sum_{k \in \Pi_-} (A_k e^{\alpha_k t} + B_k e^{-\alpha_k t}) w_k(x) + e^{-\alpha t}(A_m + tB_m) w_m(x) + e^{-\alpha t} \sum_{k \in \Pi_+} (A_k \cos \alpha_k t + B_k \sin \alpha_k t) w_k(x), \quad (2.3)$$

where α_k, A_k, B_k are given as follows:

If $k \in \Pi_-$, then $\alpha_k = \sqrt{\alpha^2 - \beta k^2 - \gamma}$,

$$A_k = \frac{1}{2}\left(1 + \frac{\alpha}{\alpha_k}\right)(g, w_k) + \frac{1}{2\alpha_k}(h, w_k),$$

$$B_k = \frac{1}{2}\left(1 - \frac{\alpha}{\alpha_k}\right)(g, w_k) - \frac{1}{2\alpha_k}(h, w_k).$$

If $\Pi_0 = \{m\}$, $m \in \mathbb{N}$, then $A_m = (g, w_m)$, and $B_m = \alpha A_m + (h, w_m)$.

If $k \in \Pi_+$, then $\alpha_k = \sqrt{\beta k^2 + \gamma - \alpha^2}$,

$$A_k = (g, w_k), \quad B_k = \frac{1}{\alpha_k}[\alpha A_k + (h, w_k)].$$

(ii) The series

$$v(t,x) = \sum_{k \in \Pi_+} (A_k \cos \alpha_k t + B_k \sin \alpha_k t) w_k(x) \quad (2.4)$$

converges uniformly on D. Function $v(t,x)$ is a continuous and bounded function on D.

(iii) The series in (2.3) converges uniformly on D. The solution $y(t,x)$ is a continuous and bounded function on D.

Proof. Since functions $\{w_k\}_{k=1}^{\infty}$ form an orthonormal basis in H, we can define Galerkin approximations $y^N(t,x)$ to the solution $y(t,x)$ of (2.2) to be functions of the form

$$y^N(t,x) = \sum_{k=1}^{N} b_k(t) w_k(x) \quad (2.5)$$

satisfying conditions

$$\left(\frac{d^2}{dt^2} y^N + 2\alpha \frac{d}{dt} y^N + A y^N, w_k\right) = 0,$$

$$(y^N(0), w_k) = (g, w_k), \quad \left(\frac{d}{dt} y^N(0), w_k\right) = (h, w_k)$$

for $k = 1, 2, ..., N$. Therefore

$$b_k'' + 2\alpha b_k' + \beta k^2 b_k + \gamma b_k = 0, \quad (2.6)$$

and we conclude that $y^N(t,x)$ is given by the partial sum of the series (2.3) with $k \le N$. Furthermore, the coefficients A_k and B_k, $k = 1, 2..., N$ are defined as in (i) of the Theorem.

Multiply both sides in (2.6) by b_k'. Then this equation can be rewritten as

$$\frac{1}{2}\frac{d}{dt}\left[|b_k'|^2 + (\beta k^2 + \gamma)|b_k|^2\right] + 2\alpha |b_k'|^2 = 0. \quad (2.7)$$

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Therefore

$$|b_k'|^2 + (\beta k^2 + \gamma)|b_k|^2 + 4\alpha \int_0^t |b_k'(s)|^2 ds = |(h, w_k)|^2 + (\beta k^2 + \gamma)|(g, w_k)|^2. \tag{2.8}$$

Since

$$\|y^N\|^2 = \left| \sum_{k=1}^N b_k(t) k \sqrt{\frac{2}{\pi}} \cos(kx) \right|_H^2 = \sum_{k=1}^N |b_k|^2 k^2, \tag{2.9}$$

and

$$\|g\|^2 = \sum_{k=1}^\infty k^2 |(g, w_k)|^2, \tag{2.10}$$

equation (2.8) implies

$$\left| \frac{dy^N}{dt} \right|^2 + \beta \|y^N\|^2 + \gamma |y^N|^2 + 4\alpha \int_0^t \left| \frac{dy^N(s)}{ds} \right|^2 ds \le |h|^2 + \beta \|g\|^2 + \gamma |g|^2. \tag{2.11}$$

Now the convergence of the Galerkin approximations y^N to the solution y defined in (2.3) can be established using energy estimate (2.11) as in [11] or [19]. Passing to the limit in (2.11), and accounting for (2.8) gives

$$\left| \frac{dy}{dt} \right|^2 + \beta \|y\|^2 + \gamma |y|^2 + 4\alpha \int_0^t \left| \frac{dy(s)}{ds} \right|^2 ds = |h|^2 + \beta \|g\|^2 + \gamma |g|^2. \tag{2.12}$$

Equation (2.12) implies the uniqueness of the solution y.

To prove assertion (ii) of the Theorem we need a more detailed information about the coefficients $A_k, B_k,\ k \in \Pi_+$.

Since $A_k = (g, w_k)$ for $k \in \Pi_+$ we have

$$\sum_{k \in \Pi_+} |A_k|^2 \le |g|^2, \quad \text{and} \quad \sum_{k \in \Pi_+} k^2 |A_k|^2 \le \|g\|^2. \tag{2.13}$$

The coefficients B_k are defined by $B_k = \frac{1}{\alpha_k}[\alpha A_k + (h, w_k)],\ k \in \Pi_+$ Therefore, according to (2.4)

$$\begin{aligned} |v(t,x)| \le \sqrt{\frac{2}{\pi}} \sum_{k \in \Pi_+} (|A_k| + |B_k|) &= \sqrt{\frac{2}{\pi}} \sum_{k \in \Pi_+} \left(k|A_k| \frac{1}{k} + \alpha_k |B_k| \frac{1}{\alpha_k} \right) \\ &\le \frac{1}{\sqrt{2\pi}} \sum_{k \in \Pi_+} \left(k^2 |A_k|^2 + \frac{1}{k^2} \right) + \frac{1}{\sqrt{2\pi}} \sum_{k \in \Pi_+} \left(\alpha_k^2 |B_k|^2 + \frac{1}{\alpha_k^2} \right). \end{aligned} \tag{2.14}$$

The first series in (2.14) is convergent because of (2.13). For the second series we have

$$\sum_{k \in \Pi_+} \alpha_k^2 |B_k|^2 = \sum_{k \in \Pi_+} [\alpha A_k + (h, w_k)]^2 \le 2\alpha^2 \sum_{k \in \Pi_+} |A_k|^2 + 2 \sum_{k \in \Pi_+} |(h, w_k)|^2 \le 2\alpha^2 |g|^2 + 2|h|^2, \tag{2.15}$$

and

$$\sum_{k \in \Pi_+} \frac{1}{\alpha_k^2} = \sum_{k \in \Pi_+} \frac{1}{\beta k^2 + \gamma - \alpha^2} < \infty. \tag{2.16}$$

Estimates (2.14)-(2.16) show that the series defining $v(t,x)$ in (2.4) converges uniformly on D according to Weierstrass M-test. Thus $v(t,x)$ is a bounded continuous function on D as claimed in the Theorem.

To establish (iii) of the Theorem we argue as follows. If $\alpha = 0$, then $\Pi_- = \Pi_0 = \emptyset$. So $y(t,x) = v(t,x)$ in this case. If $\alpha > 0$, let $N = \max\{k \in \Pi_- \cup \Pi_0\}$. Then, by (2.3), the approximation $y^N(t, \cdot) : [0, \infty) \to V$ is a continuous and bounded function on $[0, \infty)$. The Π_+ part of the solution is $e^{-\alpha t} v(t,x)$, so the result follows from part (ii) of the Theorem. □

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3. Laplace and Fourier transforms of the observation $z_p(t)$

Let $p \in (0,\pi)$. Then the observation of the solution $y(t,x),\ t \geq 0$ of the system (1.2) at the point p is given by

$$z_p(t;q) = y(t,p\,;q) = e^{-\alpha t}\sum_{k\in\Pi_-}(A_k e^{\alpha_k t} + B_k e^{-\alpha_k t})w_k(p) + e^{-\alpha t}(A_m + tB_m)w_m(p) + e^{-\alpha t}\sum_{k\in\Pi_+}(A_k\cos\alpha_k t + B_k\sin\alpha_k t)w_k(p), \quad (3.1)$$

where the coefficients A_k and B_k are defined in (i) of Theorem 2.2. By Theorem 2.2(iii) the observation $z_p(t;q)$ is a bounded continuous function for $t \geq 0$.

The main tools in our arguments are the Laplace and the Fourier transforms of the observation $z_p(t) = z_p(t;q)$. While these transforms are related through a complex argument, it is more convenient for us to separate them.

The Laplace transform $\mathcal{L}f$ of a piecewise continuous function $f(t)$ satisfying $|f(t)| \leq Me^{at},\ t \geq 0$ for some $M, a \in \mathbb{R}$ is defined for any real $s > a$ by

$$\mathcal{L}f(s) = \int_0^\infty e^{-st}f(t)\,dt.$$

The Fourier transform $\mathcal{F}f$ of a function $f \in L^1(\mathbb{R})$ is defined by

$$\mathcal{F}f(w) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(t)e^{-iwt}dt, \quad w \in \mathbb{R}.$$

It can be extended to tempered distributions $T \in \mathcal{S}'(\mathbb{R})$ as described in [18], for which we refer for relevant definitions and results. In particular, we need the Fourier transform of the Heaviside function $H(t)$ defined by $H(t) = 1$ for $t \geq 0$, and $H(t) = 0$ for $t < 0$. The Sokhatsky–Weierstrass theorem states that

$$\mathcal{F}\{H(t)\}(w) = \frac{1}{\sqrt{2\pi i}}pv\left(\frac{1}{w}\right) + \sqrt{\frac{\pi}{2}}\delta, \quad (3.2)$$

where the distribution for $1/w$ is defined by the principal value integral

$$\left\langle pv\left(\frac{1}{w}\right), \phi(w)\right\rangle = \lim_{\epsilon\to 0+}\int_{|w|>\epsilon}\frac{\phi(w)}{w}dw, \quad \phi \in \mathcal{S}(\mathbb{R}),$$

see [18], section 3.4.

Lemma 3.1. *Let $p \in (0,\pi)$, $v(t,x)$ be defined by (2.4), and $z_p(t;q) = y(t,p;q)$ be defined in (3.1). Then*

(i) The Laplace transform of v is given by

$$V(s) = \mathcal{L}\{v(t,p)\}(s) = \sum_{k\in\Pi_+}\left(\frac{A_k s}{s^2+\alpha_k^2} + \frac{B_k}{s^2+\alpha_k^2}\right)w_k(p). \quad (3.3)$$

Function $V(s)$ is continuous and bounded for $s > 0$.

(ii) The Laplace transform of z_p is given by

$$Z(s) = \mathcal{L}\{z_p(t;q)\}(s) = \sum_{k\in\Pi_-}\left(\frac{A_k}{s+\alpha-\alpha_k} + \frac{B_k}{s+\alpha+\alpha_k}\right)w_k(p) + \left(\frac{A_m}{s+\alpha} + \frac{B_m}{(s+\alpha)^2}\right)w_m(p) + V(s+\alpha), \quad (3.4)$$

where $m \in \Pi_0$. If $\Pi_- \neq \emptyset$, then function $Z(s)$ is defined for $s > -\alpha+\alpha_1$. Otherwise $Z(s)$ is defined for $s > -\alpha$.

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(iii) The Fourier transform of v is given by

$$\mathcal{F}v(w) = -\frac{1}{2\sqrt{2\pi}} \sum_{k\in\Pi_+}^{\infty} \Big\{ A_k i \Big[pv\left(\frac{1}{w-\alpha_k}\right) + pv\left(\frac{1}{w+\alpha_k}\right) + \sqrt{\frac{\pi}{2}}\delta(w-\alpha_k) + \sqrt{\frac{\pi}{2}}\delta(w+\alpha_k)\Big] + B_k \Big[pv\left(\frac{1}{w-\alpha_k}\right) - pv\left(\frac{1}{w+\alpha_k}\right) + \sqrt{\frac{\pi}{2}}\delta(w-\alpha_k) - \sqrt{\frac{\pi}{2}}\delta(w+\alpha_k)\Big] w_k(p) \Big\}, \quad (3.5)$$

where the principal value integrals are taken around the singularities.

(iv) For $\eta > 0$ define

$$u_\eta(t) = \begin{cases} e^{-\eta t} v(t,p), & t \geq 0, \\ 0, & t < 0. \end{cases} \quad (3.6)$$

Then

$$\mathcal{F}u_\eta(w) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\Pi_+}^{\infty} \frac{A_k(\eta + iw) + B_k\alpha_k}{\alpha_k^2 + (\eta+iw)^2} w_k(p). \quad (3.7)$$

Function $\mathcal{F}u_\eta(w)$ is bounded for $w \in \mathbb{R}$.

(v) We have

$$\lim_{\eta\to 0+} \mathcal{F}u_\eta = \mathcal{F}v \quad (3.8)$$

as distributions.

Proof. The partial sums for $V(s)$ are obtained by taking the Laplace transform of the partial sums of $v(t,p)$ in (2.4). By Theorem 2.2 the series in (2.4) converges uniformly for $t \geq 0$, therefore expression (3.3) follows.

Let $\xi > 0$. The maximum of the function $s \to s/(s^2+\xi^2)$ on $s \in [0,\infty)$ is attained at $s = \xi$, and the maximum value is $1/2\xi$. Using estimates (2.13)-(2.16) we get

$$|V(s)| \leq \sum_{k\in\Pi_+} \left(\frac{|A_k|s}{s^2+\alpha_k^2} + \frac{|B_k|}{s^2+\alpha_k^2} \right) |w_k(p)| \leq \sqrt{\frac{2}{\pi}} \sum_{k\in\Pi_+} |A_k| \frac{1}{2\alpha_k} + \sqrt{\frac{2}{\pi}} \sum_{k\in\Pi_+} |B_k| \frac{1}{\alpha_k^2} < \infty. \quad (3.9)$$

Weierstrass M-test implies that $V(s)$ is bounded and continuous on $(0,\infty)$. Expression (3.4) follows from formula (3.1) for $z_p(t;q)$, and (3.3).

By a direct computation

$$\mathcal{F}\{e^{-\eta t}H(t)\}(w) = \frac{1}{\sqrt{2\pi}} \frac{1}{\eta + iw}. \quad (3.10)$$

Using $\mathcal{F}\{e^{iat}f(t)\} = \mathcal{F}f(w-a)$,

$$\mathcal{F}\{e^{-\eta t}H(t)\cos\alpha_k t\}(w) = \frac{1}{\sqrt{2\pi}} \frac{1}{2} \left(\frac{1}{\eta + i(w-\alpha_k)} + \frac{1}{\eta + i(w+\alpha_k)} \right), \quad (3.11)$$

and

$$\mathcal{F}\{e^{-\eta t}H(t)\sin\alpha_k t\}(w) = \frac{1}{\sqrt{2\pi}} \frac{1}{2i} \left(\frac{1}{\eta + i(w-\alpha_k)} - \frac{1}{\eta + i(w+\alpha_k)} \right). \quad (3.12)$$

By Theorem 2.2 the series defining $v(t,x)$ in (2.4) converges uniformly on D. Therefore the series

$$u_\eta(t) = \sum_{k\in\Pi_+}^{\infty} (A_k e^{-\eta t}H(t)\cos\alpha_k t + B_k e^{-\eta t}H(t)\sin\alpha_k t) \quad (3.13)$$

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defining $u_\eta(t)$ for $t \in \mathbb{R}$ converges uniformly on $\mathbb{R}$. Since the Fourier transform is a continuous operator in the space of tempered distributions $\mathcal{S}'(\mathbb{R})$, we can apply it to this series termwise, and obtain (3.7).

By Theorem 2.2 function $v(t,p)$ is bounded for $0 \le t < \infty$. Therefore $u_\eta \in L^1(\mathbb{R})$. This implies that its Fourier transform $\mathcal{F}u_\eta(w)$ is a continuous and bounded function for $w \in \mathbb{R}$.

Since $\lim_{\eta\to 0+} u_\eta(t) = v(t,p)$ uniformly on bounded intervals in $\mathbb{R}$, and the Fourier transform is continuous in $\mathcal{S}'(\mathbb{R})$, we get (3.8). Accordingly, representation (3.5) can be obtained by passing to the limit in (3.7) as $\eta \to 0+$. However, this limit cannot be obtained by simply letting $\eta = 0$ in (3.7). Indeed, letting $\eta = 0$ in (3.10) gets an incorrect formula $\mathcal{F}\{H(t)\}(w) = 1/(\sqrt{2\pi}iw)$, because the function $w \to 1/w$ is not locally integrable in $\mathbb{R}$. The correct result for $\mathcal{F}\{H(t)\}(w)$ known as the Sokhatsky–Weierstrass theorem is stated in (3.2).

Therefore

$$\mathcal{F}\{H(t)\cos\alpha_k t\}(w) = \frac{1}{\sqrt{2\pi}}\frac{1}{2i}\Big[pv\left(\frac{1}{w-\alpha_k}\right) + pv\left(\frac{1}{w+\alpha_k}\right) + \sqrt{\frac{\pi}{2}}\delta(w-\alpha_k) + \sqrt{\frac{\pi}{2}}\delta(w+\alpha_k)\Big], \quad (3.14)$$

$$\mathcal{F}\{H(t)\sin\alpha_k t\}(w) = -\frac{1}{\sqrt{2\pi}}\frac{1}{2}\Big[pv\left(\frac{1}{w-\alpha_k}\right) - pv\left(\frac{1}{w+\alpha_k}\right) + \sqrt{\frac{\pi}{2}}\delta(w-\alpha_k) - \sqrt{\frac{\pi}{2}}\delta(w+\alpha_k)\Big], \quad (3.15)$$

where the principal value integrals are taken around the singularities. Expression (3.5) follows. □

Definition 3.2. Function $f : \mathbb{R} \to \mathbb{R}$ is called *almost-periodic*, if for any $\epsilon > 0$ there exists $T = T(\epsilon) > 0$ such that any interval $(t, t+T)$ contains at least one number τ such that $|f(y+\tau) - f(y)| < \epsilon$ for any $y \in \mathbb{R}$.

The main result in the theory of almost-periodic functions is that such functions are exactly the uniform limits of trigonometric polynomials $\sum_{k=1}^m c_k e^{i\alpha_k x}$, for complex c_k and real α_k, see [2, 20].

Let $v(t,x)$ be defined by (2.4). Then we have

Lemma 3.3. *Extend function $v(t,p)$ to any $t \in \mathbb{R}$ by letting $t \in \mathbb{R}$ in (2.4). Then $v(t,p)$ is an almost-periodic function on $\mathbb{R}$.*

Proof. By Theorem 2.2(ii) function $v(t,x)$ is a uniform limit of trigonometric polynomials. □

Lemma 3.4. *Suppose that $\mu > 0$, and $v(t,p) \not\equiv 0$ for $t \ge 0$. Then function $e^{\mu t}v(t,p)$ is unbounded on $[0,\infty)$.*

Proof. Extend $v(t,p)$ to all $t \in \mathbb{R}$ by formula (2.4). Since $v(t,p)$ is continuous and assumed to be nonzero, there exists $y > 0$ such that $v(y,p) \ne 0$. Let $\epsilon = |v(y,p)|/2$. By Lemma 3.3 function $v(t,p)$ is almost periodic on $\mathbb{R}$. Let $T = T(\epsilon) > 0$ be the corresponding number T from the definition 3.2. Then there exists a sequence of numbers $\tau_n \in \mathbb{R}$, $n \in \mathbb{N}$ such that $\tau_n \in ((n-1)T, nT)$, and $|v(y+\tau_n,p) - v(y,p)| < \epsilon$. We conclude that $\tau_n \to \infty$ as $n \to \infty$, and $|v(y+\tau_n,p)| \ge \epsilon/2 > 0$. Therefore $e^{\mu(y+\tau_n)}|v(y+\tau_n,p)| \to \infty$, as $n \to \infty$, which means that function $e^{\mu t}v(t,p)$ is unbounded on $[0,\infty)$. □

4. Identifiability for linearized sine-Gordon problem

The parameter identifiability for the system (1.2) in P means that the assumption $q_1, q_2 \in P$ and $z_p(t;q_1) = z_p(t;q_2)$ for all $t \ge 0$ implies $q_1 = q_2$. We establish it by showing that the observation $z_p(t;q)$, $t \ge 0$ uniquely defines its generating triple of the parameters $q = (\alpha, \beta, \gamma) \in P$.

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In some cases, such as trivial initial data $g = h = 0$, the observation $z_p(t) \equiv 0$ for any $q \in P$, and there is no hope to recover q uniquely. Thus, we have to make some assumptions on the initial data. We choose either

$$p \neq \frac{\pi}{2}, \quad (g, w_1) > 0, \quad (h, w_1) > 0, \quad (g, w_2) \neq 0, \quad (h, w_2) \neq 0, \tag{4.1}$$

or

$$p \neq \frac{\pi}{2}, \quad (g, w_1) \neq 0, \quad (g, w_2) \neq 0, \quad h = 0. \tag{4.2}$$

We will always assume (without loss of generality) that $(g, w_1) > 0$. The assumption $p \neq \pi/2$, is used to assure that $w_2(p) \neq 0$. Since $w_1(x) > 0$ on $(0, \pi)$, conditions $(g, w_1) > 0$, $(h, w_1) > 0$ can be achieved by taking $g(x) > 0$ and $h(x) > 0$ for $x \in (0, \pi)$. Conditions $(g, w_2) \neq 0$ and $(h, w_2) \neq 0$ are satisfied if functions g and h are not symmetric with respect to $x = \pi/2$.

Recall that given $q \in P$, the corresponding subsets Π_-, Π_0, and Π_+ were defined in Definition 2.1.

Lemma 4.1. *Suppose that conditions* (4.1) *or* (4.2) *are satisfied. Let* $q \in P$, $z_p(t; q) = y(t, p; q)$ *for* $t \geq 0$, *and* $Z(s) = \mathcal{L}\{z_p(t; q)\}(s)$.

(i) If $\Pi_- \neq \emptyset$, *then* $1 \in \Pi_-$,

$$\alpha > 0, \quad \alpha_1 = \sqrt{\alpha^2 - \beta - \gamma}, \quad 0 < \alpha_1 < \alpha, \quad A_1 > 0, \quad B_1 < 0.$$

The Laplace transform $Z(s)$ *is unbounded on interval* $(-\alpha + \alpha_1, \infty)$.

(ii) If $\Pi_- = \emptyset$, *but* $\Pi_0 \neq \emptyset$, *then* $\Pi_0 = \{1\}$, $\Pi_+ = \{2, 3, 4, \ldots\}$,

$$\alpha > 0, \quad A_1 > 0, \quad B_1 > 0,$$

$$z_p(t; q) = e^{-\alpha t}(A_1 + tB_1)w_1(p) + e^{-\alpha t}v(t, p).$$

The Laplace transform $Z(s)$ *is unbounded on interval* $(-\alpha, \infty)$.

(iii) If $\Pi_- = \Pi_0 = \emptyset$, *then* $\Pi_+ = \{1, 2, 3, \ldots\}$,

$$\alpha \geq 0, \quad \alpha_1 = \sqrt{\beta + \gamma - \alpha^2} > 0, \quad A_1 > 0, \quad B_1 \geq 0.$$

$$z_p(t; q) = e^{-\alpha t}v(t, p).$$

The Laplace transform $Z(s)$ *is bounded on interval* $(-\alpha, \infty)$.

Proof. Since $\beta > 0$ and $\gamma \geq 0$, sets Π_- and Π_0 can be nonempty only if $\alpha > 0$.

(i) If $\Pi_- \neq \emptyset$, then $\beta + \gamma - \alpha^2 < 0$. That is, $1 \in \Pi_-$. By Theorem 2.2(i), $\alpha_1 = \sqrt{\alpha^2 - \beta - \gamma}$. Thus $0 < \alpha_1 < \alpha$. By assumption $(g, w_1) > 0$, and $(h, w_1) \geq 0$. Therefore

$$A_1 = \frac{1}{2}\left(1 + \frac{\alpha}{\alpha_1}\right)(g, w_1) + \frac{1}{2\alpha_1}(h, w_1) > 0,$$

and

$$B_1 = \frac{1}{2}\left(1 - \frac{\alpha}{\alpha_1}\right)(g, w_1) - \frac{1}{2\alpha_1}(h, w_1) < 0.$$

By Lemma 3.1(ii), the Laplace transform $Z(s)$ is defined on $(-\alpha + \alpha_1, \infty)$. Its expression is given by (3.4). Lemma 3.1(i) shows that $V(s + \alpha)$ is bounded for $s \in (-\alpha + \alpha_1, \infty)$. All the other terms of $Z(s)$ are also bounded on $(-\alpha + \alpha_1, \infty)$, except for the very first term $A_1/(s + \alpha - \alpha_1)$. Since $A_1 \neq 0$, this term is unbounded on $(-\alpha + \alpha_1, \infty)$.

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(ii) Suppose that $\Pi_- = \emptyset$, but $\Pi_0 \neq \emptyset$. Then $\Pi_0 = \{1\}$. Therefore $\Pi_+ = \{2, 3, 4, \ldots\}$. Since $\Pi_0 \neq 0$, we must have $\alpha > 0$. From Theorem 2.2(i) we get the sought expression for $z_p(t; q)$, and

$$A_1 = (g, w_1) > 0, \quad B_1 = \alpha A_1 + (h, w_1) > 0.$$

The Laplace transform $Z(s)$, given in Lemma 3.1(ii), takes the form

$$Z(s) = \left(\frac{A_1}{s+\alpha} + \frac{B_1}{(s+\alpha)^2} \right) w_1(p) + V(s+\alpha). \tag{4.3}$$

It is defined for $s > -\alpha$. By Lemma 3.1(i), function $V(s+\alpha)$ is bounded on $(-\alpha, \infty)$. The remaining term is unbounded on $(-\alpha, \infty)$, since $A_1, B_1 > 0$.

(iii) If $\Pi_- = \Pi_0 = \emptyset$, then $k \in \Pi_+$ for $k = 1, 2, 3, \ldots$ In particular, $\beta + \gamma - \alpha^2 > 0$. By Theorem 2.2(i), $\alpha_1 = \sqrt{\beta + \gamma - \alpha^2} > 0$, and

$$A_1 = (g, w_1) > 0, \quad B_1 = \frac{1}{\alpha_1}[\alpha A_1 + (h, w_1)] \geq 0.$$

The Laplace transform $Z(s)$ takes the form $Z(s) = V(s+\alpha)$. It is bounded on $(-\alpha, \infty)$ by Lemma 3.1(i).

□

Lemma 4.2. *Suppose that conditions* (4.1) *or* (4.2) *are satisfied. Given* $z_p(t) = y(t, p; q) \not\equiv 0$, *there exists* $\xi \leq 0$ *such that* $\|e^{-\eta t} z_p(t)\|_\infty < \infty$ *for* $\eta > \xi$, *and* $\|e^{-\eta t} z_p(t)\|_\infty = \infty$ *for* $\eta < \xi$. *If* $\Pi_- \neq \emptyset$, *then* $\xi = \alpha_1 - \alpha$. *If* $\Pi_- = \emptyset$, *then* $\xi = -\alpha$.

Proof. By Theorem 2.2(iii), $y(t, x; q)$ is a bounded and continuous function on $D = [0, \infty) \times [0, \pi]$. Therefore $\xi \leq 0$. Also, by Theorem 2.2(ii), $\|v(t, p)\|_\infty < \infty$.

(i) If $\Pi_- \neq \emptyset$, then, by Lemma 4.1(i), $\alpha > 0$, $0 < \alpha_1 < \alpha$, and $A_1 > 0$. Therefore, using expression (3.1) for the observation $z_p(t)$, we can write

$$e^{-\eta t} z_p(t) = e^{(-\eta - \alpha + \alpha_1)t}(A_1 w_1(p) + o(t)),$$

where $o(t) \to 0$, as $t \to \infty$, with $A_1 \neq 0$ and $w_1(p) \neq 0$.
If $\eta \geq \alpha_1 - \alpha$, then $-\eta - \alpha + \alpha_1 \leq 0$, and $\|e^{-\eta t} z_p(t)\|_\infty < \infty$. If $\eta < \alpha_1 - \alpha$, then $-\eta - \alpha + \alpha_1 > 0$, and we have $\|e^{-\eta t} z_p(t)\|_\infty = \infty$. Thus we conclude that in this case $\xi = \alpha_1 - \alpha$.

(ii) Suppose that $\Pi_- = \emptyset$, but $\Pi_0 \neq \emptyset$. By Lemma 4.1(ii), $\Pi_0 = \{1\}$, $B_1 > 0$, and $\alpha > 0$. Then

$$e^{-\eta t} z_p(t) = t e^{(-\eta - \alpha)t}(B_1 w_1(p) + o(t)),$$

where $o(t) \to 0$, as $t \to \infty$, with $B_1 \neq 0$ and $w_1(p) \neq 0$.
If $\eta > -\alpha$, then $-\eta - \alpha < 0$, and $\|e^{-\eta t} z_p(t)\|_\infty < \infty$. If $\eta < -\alpha$, then $\|e^{-\eta t} z_p(t)\|_\infty = \infty$. Thus, in this case $\xi = -\alpha$.

(iii) Suppose that $\Pi_- = \Pi_0 = \emptyset$. By Lemma 4.1(iii),

$$e^{-\eta t} z_p(t) = e^{(-\eta - \alpha)t} v(t, p).$$

Since $z_p(t) \not\equiv 0$, we have $v(t, p) \not\equiv 0$. If $\eta > -\alpha$, then $\|e^{\eta t} z_p(t)\|_\infty < \infty$, since $v(t, p)$ is bounded on $[0, \infty)$. If $\eta < -\alpha$, then $\|e^{\eta t} z_p(t)\|_\infty = \infty$ by Lemma 3.4. Thus, in this case $\xi = -\alpha$.

□

Our main result is the next Theorem.

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Theorem 4.3. *Suppose that conditions* (4.1) *or* (4.2) *are satisfied. Given an observation* $z_p(t;q) = y(t,p\,;q) \not\equiv 0$ *for* $t \geq 0$*, the parameters* $q = (\alpha, \beta, \gamma)$ *of the system* (1.2) *can be uniquely identified.*

Proof. Let

$$\xi = \inf\{\eta \in \mathbb{R} \,:\, \|e^{-\eta t} z_p(t)\|_\infty < \infty\}. \tag{4.4}$$

By Lemma 4.2 such ξ is finite, and $\|e^{-\eta t} z_p(t)\|_\infty < \infty$ for any $\eta > \xi$. This implies that the Laplace transform $Z(s) = \mathcal{L}\{z_p(t;q)\}(s)$ exists for any $s > \xi$. Let

$$b = \lim_{s \to \xi +} Z(s)(s-\xi)^2. \tag{4.5}$$

Let us examine the boundedness of $Z(s)$ on (ξ, ∞), and whether $b = 0$, depending on the structure of the sets $\mathit{\Pi}_-$, $\mathit{\Pi}_0$, and $\mathit{\Pi}_+$.

If $\mathit{\Pi}_- = \mathit{\Pi}_0 = \emptyset$, then $\xi = -\alpha$ by Lemma 4.2. By Lemma 4.1(iii), the Laplace transform $Z(s)$ is bounded on (ξ, ∞).

If $\mathit{\Pi}_- = \emptyset$, but $\mathit{\Pi}_0 \neq \emptyset$, then $\xi = -\alpha$ by Lemma 4.2. By Lemma 4.1(ii), the Laplace transform $Z(s)$ is unbounded on (ξ, ∞), and $B_1 > 0$. The expression for $Z(s)$ in this case is given by (4.3). Recall that function $V(s+\alpha)$ is bounded on $(-\alpha, \infty)$ by Lemma 3.1(i). Therefore

$$b = \lim_{s \to -\alpha +} (s+\alpha)^2 \left[\left(\frac{A_1}{s+\alpha} + \frac{B_1}{(s+\alpha)^2} \right) w_1(p) + V(s+\alpha) \right] = B_1 w_1(p) > 0. \tag{4.6}$$

If $\mathit{\Pi}_- \neq \emptyset$, then $\xi = \alpha_1 - \alpha$ by Lemma 4.2. By Lemma 4.1(i), the Laplace transform $Z(s)$ is unbounded on (ξ, ∞), and $A_1 > 0$. The expression for $Z(s)$ is given by (3.4). Therefore

$$b = \lim_{s \to \alpha_1 - \alpha +} (s+\alpha-\alpha_1)^2 \Bigg[\sum_{k \in \mathit{\Pi}_-} \left(\frac{A_k}{s+\alpha-\alpha_k} + \frac{B_k}{s+\alpha+\alpha_k} \right) w_k(p) + \left(\frac{A_m}{s+\alpha} + \frac{B_m}{(s+\alpha)^2} \right) w_m(p) + V(s+\alpha) \Bigg] = 0. \tag{4.7}$$

Thus, the boundedness of $Z(s)$ on (ξ, ∞), and the value of b give us the following three distinct possibilities.

(i) $Z(s)$ is bounded on (ξ, ∞). This can happen only if $\mathit{\Pi}_- = \mathit{\Pi}_0 = \emptyset$. Therefore, in this case $\alpha = -\xi$. By Lemma 4.1(iii), $A_1 > 0$, and $z_p(t) = e^{-\alpha t} v(t,p)$. To determine the other two parameters β, and γ, consider the Fourier transform of $v(t,p) = e^{\alpha t} z_p(t)$. It is given by formula (3.5).
We have $A_1 > 0$, and $A_2 = (g, w_2) \neq 0$ by the assumption. Therefore the first two positive singularities of the distribution $\mathcal{F}v$ are at $w = \alpha_1 = \sqrt{\beta + \gamma - \alpha^2}$ and at $w = \alpha_2 = \sqrt{4\beta + \gamma - \alpha^2}$. Thus the unique identification for the parameters is given by

$$\alpha = -\xi, \quad \beta = \frac{1}{3}(\alpha_2^2 - \alpha_1^2), \quad \gamma = \alpha_1^2 - \beta + \alpha^2.$$

The conclusion is that if $Z(s)$ is bounded on (ξ, ∞), then the parameter $q \in P$ in $z_p(t) = y(t,p\,;q)$ is identifiable.

(ii) $Z(s)$ is unbounded on (ξ, ∞), and $b \neq 0$. As shown above, this can happen only if $\mathit{\Pi}_- = \emptyset$, but $\mathit{\Pi}_0 \neq \emptyset$. By Lemma 4.2, $\xi = -\alpha$. By Lemma 4.1(ii), $\mathit{\Pi}_0 = \{1\}$, $A_1 > 0$, $B_1 > 0$. Therefore $\beta + \gamma = \alpha^2$. Since $\alpha = -\xi$ is already found, it gives the first equation for the sum $\beta + \gamma$.
By (4.6), $b = B_1 w_1(p)$. Then the expression for the Laplace transform (4.3) can be used to find

$$a = \lim_{s \to -\alpha +} (s+\alpha) \left(Z(s) - \frac{b}{(s+\alpha)^2} \right) = A_1 w_1(p) \neq 0.$$

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Therefore, by Lemma 4.1(ii)

$$v(t,p) = e^{\alpha t} z_p(t) - (a + bt).$$

The first positive singularity of the Fourier transform $\mathcal{F}v$ occurs at $\alpha_2 = \sqrt{4\beta + \gamma - \alpha^2}$. Thus $4\beta + \gamma = \alpha_2^2$. Together with the previously found equation $\beta + \gamma = \alpha^2$ this gives the unique identification

$$\alpha = -\xi, \quad \beta = \frac{1}{3}(\alpha_2^2 - \alpha^2), \quad \gamma = \alpha^2 - \beta.$$

(iii) The last case to consider is when $Z(s)$ is unbounded on (ξ, ∞), and b, defined in (4.5), satisfies $b = 0$. This implies $\Pi_- \neq \emptyset$. By Lemma 4.1(i), $1 \in \Pi_-$, $0 < \alpha_1 < \alpha$, $A_1 > 0$, $B_1 < 0$. By Lemma 4.2, $\xi = \alpha_1 - \alpha$. Now we can find α, and α_1 as follows.
Define

$$a_1 = \lim_{s \to (\alpha_1 - \alpha)+} Z(s)(s - \alpha_1 + \alpha). \tag{4.8}$$

From (3.4) we conclude that $a_1 = A_1 w_1(p)$. Since $w_1(p) > 0$, and a_1 is found, it gives us the value of A_1. On the other hand, by Theorem 2.2

$$A_1 = \frac{1}{2}\left(1 + \frac{\alpha}{\alpha_1}\right)(g, w_1) + \frac{1}{2\alpha_1}(h, w_1).$$

Since $(g, w_1) > 0, (h, w_1) \geq 0$, and $0 < \alpha_1 < \alpha$, we conclude that $A_1 > (g, w_1)$. Substitute $\alpha = \alpha_1 - \xi$ into the equation for A_1, and find

$$\alpha_1 = \frac{1}{2}\frac{(h, w_1) - \xi(g, w_1)}{A_1 - (g, w_1)}.$$

Thus α_1 and $\alpha = \alpha_1 - \xi$ are found.
Now we eliminate the leading exponent from the expression (3.1) for $z_p(t)$. For $t \geq 0$ let

$$r(t) = e^{(\alpha - \alpha_1)t} z_p(t) - a_1.$$

By (3.1)

$$r(t) = B_1 e^{-2\alpha_1 t} w_1(p) + e^{-\alpha_1 t} \sum_{k \in \Pi_- \setminus \{1\}} (A_k e^{\alpha_k t} + B_k e^{-\alpha_k t}) w_k(p) + e^{-\alpha_1 t}(A_m + tB_m) w_m(p) + e^{-\alpha_1 t} v(t,p). \tag{4.9}$$

The analysis of $r(t)$ is done in the same manner as the one for $Z(s)$ considered above. Specifically, first we prove the existence of $\xi_1 \in \mathbb{R}$ such that $\|e^{-\eta t} r(t)\|_\infty < \infty$ for any $\eta > \xi_1$, and $\|e^{-\eta t} r(t)\|_\infty = \infty$ for any $\eta < \xi_1$. Then the Laplace transform $R(s)$ of $r(t)$ is defined for $s > \xi_1$. Our goal is to determine whether $2 \in \Pi_-$, $2 \in \Pi_0$, or $2 \in \Pi_+$, depending on the behavior of $R(s)$.
If $2 \in \Pi_+$, then $\Pi_0 = \emptyset$, and

$$r(t) = B_1 e^{-2\alpha_1 t} w_1(p) + e^{-\alpha_1 t} v(t,p). \tag{4.10}$$

Therefore $\xi_1 = -\alpha_1$, and $R(s)$ is bounded on (ξ_1, ∞).
If $2 \in \Pi_0$, then

$$r(t) = B_1 e^{-2\alpha_1 t} w_1(p) + e^{-\alpha_1 t}(A_2 + tB_2) w_2(p) + e^{-\alpha_1 t} v(t,p).$$

By Theorem 2.2, $A_2 = (g, w_2)$. By the assumptions (4.1) or (4.2), $A_2 \neq 0$. Therefore, irrespective of B_2, the term $e^{-\alpha_1 t}(A_2 + tB_2) w_2(p)$ is nontrivial. Thus we have $\xi_1 = -\alpha_1$, and function $R(s)$ is unbounded on (ξ_1, ∞).

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If $2 \in \Pi_-$, then $\xi_1 = -\alpha_1 + \alpha_2$, and $R(s)$ is unbounded on (ξ_1, ∞). Note that $0 < \alpha_2 < \alpha_1$, since $\alpha_k = \sqrt{\alpha^2 - \beta k^2 - \gamma}$ for $k \in \Pi_-$. In this case $\xi_1 > -\alpha_1$.
This investigation leads to the following three sub-cases of (iii).
(iii-1). $R(s)$ is bounded on (ξ_1, ∞). Then the implication is that $\Pi_- = \{1\}$, $\Pi_0 = \emptyset$, $2 \in \Pi_+$, and $\xi_1 = -\alpha_1$. Equation (4.10) gives $v(t,p) = e^{\alpha_1 t} r(t) - B_1 e^{-\alpha_1 t} w_1(p)$. The first positive singularity of the Fourier transform of $v(t,p)$ is at $\alpha_2 = \sqrt{4\beta + \gamma - \alpha^2}$. At this point we know $\alpha_1 - \alpha = \xi$, and $\alpha_1 = -\xi_1$. We have $\alpha_1^2 = \alpha^2 - \beta - \gamma$, and $\alpha_2^2 = 4\beta + \gamma - \alpha^2$. Therefore

$$\alpha_1 = -\xi_1, \quad \alpha = -\xi_1 - \xi, \quad \beta = \frac{1}{3}(\alpha_1^2 + \alpha_2^2), \quad \gamma = \alpha^2 - \alpha_1^2 - \beta.$$

(iii-2). $R(s)$ is unbounded on (ξ_1, ∞), and $\xi_1 = -\alpha_1$. Then $\Pi_- = \{1\}$, and $\Pi_0 = \{2\}$. Thus $4\beta + \gamma - \alpha^2 = 0$. In this case we have $\alpha_1 - \alpha = \xi$, $\alpha_1 = -\xi_1$, $\alpha^2 - \beta - \gamma = \alpha_1^2$, and $4\beta + \gamma - \alpha^2 = 0$. Thus the parameters are uniquely identified by

$$\alpha_1 = -\xi_1, \quad \alpha = -\xi_1 - \xi, \quad \beta = \frac{1}{3}\alpha_1^2, \quad \gamma = \alpha^2 - \alpha_1^2 - \beta.$$

(iii-3). $R(s)$ is unbounded on (ξ_1, ∞), and $\xi_1 \neq -\alpha_1$. Then $1 \in \Pi_-$, and $2 \in \Pi_-$. Parameters α and α_1 have already been determined above. In this case $\alpha_2 = \xi_1 + \alpha_1$. Since $\alpha_1^2 = \alpha^2 - \beta - \gamma$, and $\alpha_2^2 = \alpha^2 - 4\beta - \gamma$, this system can be solved to obtain

$$\beta = \frac{1}{3}(\alpha_1^2 - \alpha_2^2), \quad \gamma = \alpha^2 - \alpha_1^2 - \beta.$$

Thus the parameters (α, β, γ) are uniquely identifiable from $z_p(t)$ in all cases.

□

5. Computational algorithms

A numerical implementation of the theoretical method described in Theorem 4.3 does not produce satisfactory identification results. Therefore, we choose to approach the parameter identification as the best fit to data problem

$$J(q^*) = \inf_P J(q) = \inf_P \|y(q) - z_d\|^2_{L^2(Q)} \tag{5.1}$$

described in the Introduction.

Direct minimization (DM)

Let $N > 0$. Given $q = (\alpha, \beta, \gamma) \in P$ define the objective function

$$J_N(q) = \|z_d(t) - y_N(t, p\,; q)\|^2_{L^2(0,T)}. \tag{5.2}$$

For the linearized sine-Gordon problem (1.2) the approximate solution $y_N(t, p\,; q)$ is computed as in Theorem 2.2. For the nonlinear sine-Gordon problem (1.1) the approximate solution $y_N(t, p\,; q)$ is computed numerically, see [6].

The identifiability analysis of the previous sections did not depend on the boundedness of the admissible set P. However, for a minimization problem it is natural to assume that the admissible set P is bounded. To avoid an introduction of an additional notation we just redefine P as a bounded box in $\mathbb{R}^3$

$$P = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 \ : \ 0 \le \alpha \le \alpha_{max},\ \beta_{min} \le \beta \le \beta_{max},\ 0 \le \gamma \le \gamma_{max}\}. \tag{5.3}$$

The sought minimizer q^* is found from

$$q_N^* = (\alpha_N^*, \beta_N^*, \gamma_N^*) = argmin\ J_N(\alpha, \beta, \gamma), \quad (\alpha, \beta, \gamma) \in P. \tag{5.4}$$

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The continuity of the solution map $y_N : P \to L^2(0,T;H)$ for the sine-Gordon problem is established in [7]. The continuity proof remains valid for the linearized sine-Gordon problem. Therefore the minimization problem (5.4) has at least one solution. We can show that if q^* is a limit point of the sequence $q_N^*,\ N = 1, 2, \ldots$ then $J(q^*) = \min_{q\in P} J(q)$, [6].

The nonlinear minimization is carried out by a modification of Powell's minimization method, [16]. The advantage of the Powell's method is that it does not require the knowledge of the derivatives (gradients) to minimize functions of several variables, but it still achieves a quadratic convergence near the points of minima.

The method iteratively minimizes a function of several variables. Here $q = (\alpha, \beta, \gamma)$, so the minimization is carried out in a three-dimensional space. Steps 1-7 describe one iteration of the method.

Powell's minimization method

1. Initialize the set of directions $u_i \in \mathbb{R}^3$ to the standard basis vectors e_i in $\mathbb{R}^3$

$$u_i = \mathrm{e}_i\,, \quad i = 1, 2, 3.$$

2. Save your starting position as $q_0 \in \mathbb{R}^3$.
3. For $i = 1, 2, 3$ move from q_0 along the direction u_i and find the point of minimum p_i.
4. Re-index the directions u_i, so that (for the new indices) $J_N(p_1) \le J_N(p_2) \le J_N(p_3) \le J_N(q_0)$.
5. Move from q_0 along the new direction u_1 and find the point of minimum r_1. Move from r_1 along the direction u_2 and find the point of minimum r_2. Move from r_2 along the direction u_3 and find the point of minimum r_3.
6. Set $v = r_3 - q_0$.
7. Move from q_0 along the direction v and find the minimum. Call it q_0. It replaces q_0 from step 2.
8. Repeat the above steps until a stopping criterion is satisfied.

The minimization is further restricted to the admissible set $P \subset \mathbb{R}^3$, satisfying (5.3).

6. Numerical results

Let $T > 0$. Choose $M \in \mathbb{N}$. Define the time grid $t_j = jM/T,\ j = 0, 1, 2, ..., M$. The admissible set of parameters $q = (\alpha, \beta, \gamma)$ is taken to be

$$P = [0, 1] \times [0.2, 1] \times [0, 1].$$

We consider three numerical examples for the identifiability and the identification problems. Example 6.1 is for a linearized sine-Gordon equation. Two other examples are for a non-linear sine-Gordon equation.

The values $T = 8,\ M = 128,\ N = 20$, and the test parameters

$$q_{test} = (0.512, 0.934, 0.230) \in P$$

are common in all the examples. The observation points are defined by $p_j = j\pi/16 : j = 1, 2, \cdots, 15$. Given $p \in (0, \pi)$ the simulated data $z_d(t) = y_{40}(t, p\,; q_{test})$ is generated by solving the appropriate linearized, or non-linearized sine-Gordon problem.

In every example the identification was obtained by the Direct minimization utilizing Powell's minimization method as described in section 5. The initial approximation $q_0 \in P$ was chosen to be

$$q_0 = (0.5000, 0.6000, 0.5000).$$

Example 6.1. The governing system is the linearized sine-Gordon problem (2.2) with $g(x) = x^2(\pi - x)$ and $h(x) = x(\pi - x)$.

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Table 1 shows the identified parameters q_N^* and the corresponding values of the objective function J_N using different observations $z_{p_j}(t)$ with various observation points p_j. The table shows that almost precise identification was achieved by utilizing 12 iterations of the Powell's minimization method. The run time for the code execution was less than 1 second for each identification.

TABLE 1. Identification results for linearized sine-Gordon equation

p	Iterations	q_N^*	$J_N(q_N^*)$
p_1	12	$(0.5120, 0.9340, 0.2300)$	0.000021882
p_3	12	$(0.5120, 0.9340, 0.2300)$	0.000016025
p_5	12	$(0.5120, 0.9340, 0.2300)$	0.000011622
p_7	12	$(0.5120, 0.9340, 0.2300)$	0.000008510
p_9	12	$(0.5120, 0.9340, 0.2300)$	0.000007003
p_{10}	12	$(0.5120, 0.9340, 0.2300)$	0.000010157
p_{12}	12	$(0.5120, 0.9339, 0.2301)$	0.000009177
p_{14}	12	$(0.5120, 0.9339, 0.2301)$	0.000007986

Example 6.2. Homogeneous damped sine-Gordon system

$$\begin{aligned} &y_{tt} + 2\alpha y_t - \beta y_{xx} + \gamma \sin y = 0, \quad (t,x) \in Q = (0,8) \times (0,\pi), \\ &y(t,0) = y(t,\pi) = 0, \quad t \in (0,8), \\ &y(0,x) = x^2(\pi - x),\ y_t(0,x) = x(\pi - x), \quad x \in (0,\pi). \end{aligned} \tag{6.1}$$

This type of problems was considered by us in [6]. The test parameters q_{test} and the corresponding target behavior $z_d(t)$ are the same as in example 6.1. The nonlinear minimization is carried out by the Powell's minimization method. The stopping criterion for this iterative process was chosen to be

$$\frac{|J_N(q_m) - J_N(q_{m-1})|}{|J_N(q_0)|} < 10^{-6}. \tag{6.2}$$

Let $q_N^* = q_m$ be the set of parameters attained when the Powell's minimization method was terminated according to the stopping criterion (6.2). The identification results are shown in Table 2. It shows that the identification is successful for observations taken at various observation points p_j. The execution time for the sine-Gordon problem is about 22 seconds.

TABLE 2. Identification results for damped sine-Gordon equation (6.1)

p	Iterations	q_N^*	$J_N(q_N^*)$
p_1	6	$(0.51200, 0.93340, 0.23026)$	0.000043
p_3	10	$(0.51201, 0.93402, 0.23108)$	0.000033
p_5	4	$(0.51200, 0.93400, 0.22998)$	0.000000
p_7	5	$(0.51200, 0.93400, 0.23081)$	0.000000
p_9	7	$(0.51204, 0.93400, 0.23078)$	0.000000
p_{10}	7	$(0.51202, 0.93400, 0.23042)$	0.000000
p_{12}	6	$(0.51199, 0.93400, 0.23063)$	0.000000
p_{14}	6	$(0.51199, 0.93400, 0.23074)$	0.000000

Example 6.3. Non-homogeneous sine-Gordon equation (6.1) with a forcing term $f = 1$. Other assumptions are the same as in Example 6.2.

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This example is similar to examples considered in [6]. The identification results are shown in Table 3. It shows that the identification is successful for such more general equations. The execution time was about 22 seconds. A discussion of the performance of the method applied to noisy data can be found in [6].

TABLE 3. Identification results for damped sine-Gordon equation (6.1) with a forcing term

p	Iterations	q_N^*	$J_N(q_N^*)$
p_1	8	$(0.51200, 0.93399, 0.23017)$	0.000035
p_3	6	$(0.51208, 0.93403, 0.23182)$	0.000117
p_5	5	$(0.51200, 0.93400, 0.23010)$	0.000000
p_7	5	$(0.51198, 0.93400, 0.23054)$	0.000000
p_9	4	$(0.51200, 0.93400, 0.22992)$	0.000000
p_{10}	4	$(0.51200, 0.93400, 0.23001)$	0.000000
p_{12}	4	$(0.51201, 0.93400, 0.23005)$	0.000000
p_{14}	4	$(0.51199, 0.93390, 0.23018)$	0.000000

7. Conclusions

The identifiability problem for nonlinear damped sine-Gordon equation is an open problem. In this paper we have established the identifiability for linearized damped sine-Gordon problem. The results are also applicable to linearized Klein-Gordon equation since it has the same form.

Our method consists of a careful analysis of the Laplace and Fourier transforms of the observation $z_p(t),\ t > 0$ of the system, conducted at a single point $p \in (0, \pi)$. Clearly, the results are also valid on any other interval $I \subset \mathbb{R}$. In this work the analysis is carried out for equations with constant coefficients, zero boundary conditions, and with no forcing term. The question whether the system parameters can be identified from the observation $z_p(t)$, when the observation interval for t is bounded, remains open.

Numerical results based on the best fit to data method confirm that the identification is unique for a wide choice of initial approximation q_0 for the sought parameters q_{test}.

We have also compared the obtained numerical results with experiments conducted in [6], where the identification was carried out for nonlinear sine-Gordon system. The numerical results show that the identification for such system is qualitatively the same as for the linearized one. That is, the numerical computation identifies the unique set of parameters q^* which is practically identical to the test set q_{test}. Therefore, from the numerical evidence we can conclude that the nonlinear sine-Gordon problem may be identifiable under the same conditions as for the corresponding linearized problem.

Acknowledgements. The authors would like to thank the reviewer for valuable suggestions.

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