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# Complexity of injective piecewise contracting interval maps

E. CATSIGERAS†, P. GUIRAUD‡ and A. MEYRONEINC§

† *Instituto de Matemática y Estadística Rafael Laguardia, Universidad de la República, Montevideo, Uruguay*
*(e-mail: eleonora@fing.edu.uy)*
‡ *Centro de Investigación y Modelamiento de Fenómenos Aleatorios Valparaíso, Facultad de Ingeniería, Universidad de Valparaíso, Valparaíso, Chile*
*(e-mail: pierre.guiraud@uv.cl)*
§ *Departamento de Matemáticas, Instituto Venezolano de Investigaciones Científicas, Apartado 20632, Caracas 1020A, Venezuela*
*(e-mail: ameyrone@ivic.gob.ve)*

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*Abstract*. We study the complexity of the itineraries of injective piecewise contracting maps on the interval. We prove that for any such map the complexity function of any itinerary is eventually affine. We also prove that the growth rate of the complexity is bounded from above by the number, $N-1$, of discontinuities of the map. To show that this bound is optimal, we construct piecewise affine contracting maps whose itineraries all have the complexity $(N-1)n+1$. In these examples, the asymptotic dynamics take place in a minimal Cantor set containing all the discontinuities.

## 1. *Introduction*

Let $(X, d)$ be a compact metric space and let $\{X_i\}_{i=1}^N$ be a finite collection of $N \geqslant 2$ non-empty pairwise disjoint open subsets such that $X = \bigcup_{i=1}^N \overline{X_i}$. Let $f : X \to X$ and assume that $f$ is discontinuous on the set $\Delta := \{x \in \overline{X_i} \cap \overline{X_j},\ i \neq j \in \{1, \ldots, N\}\}$. If there exists a constant $\lambda \in (0, 1)$ such that, for any $i \in \{1, \ldots, N\}$, the map $f$ satisfies

$$d(f(x),\ f(y)) \leqslant \lambda d(x,\ y) \quad \text{for all } x,\ y \in X_i, \tag{1}$$

then $f$ is called a *piecewise contracting map* and each element of the collection $\{X_i\}_{i=1}^N$ is called a *contraction piece*.

In [**5**], we explored the diversity of asymptotic dynamics of these systems, and we proved that rich dynamics can appear if the attractor contains discontinuity points. In particular, we exhibited three-dimensional examples with exponential complexity and

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positive topological entropy. On the other hand, if the attractor does not contain discontinuity points, then its dynamics are simple, being composed of a just finite number of periodic orbits.

In the present paper, we remain interested in the diversity of the dynamics, but we restrict the study to a class of one-dimensional piecewise contracting maps. Our objective is to determine the range of all the possible complexity functions in that class. In particular, we are interested in the relationship between certain features of the discontinuity points and the complexity of the dynamics.

If a piecewise contracting map $f$ is defined on a compact interval and each contraction piece is an open interval, we say that $f$ is a *piecewise contracting interval map*. For these systems, it has been shown that, generically, the asymptotic dynamics are periodic, first for injective maps [**2**, **12**, **13**] and later for more general one-dimensional maps [**14**]. In this paper, we are instead interested in the non-periodic asymptotic dynamics. In dimension one, there are few known examples of piecewise contracting maps with non-periodic attractors [**4**, **7**, **9**, **16**], and none of them have orbits that accumulate on more than one discontinuity point. Therefore, little is known about the maximum complexity of the dynamics when the interval map has an arbitrary (finite) number of discontinuity points.

By complexity of a map, we refer to the complexity function of the itineraries of its orbits. To define the itineraries of a piecewise contracting map $f$, consider the set $\widetilde{X}$ of those points of $X$ whose orbit never intersects $\Delta$, that is,

$$\widetilde{X} := \bigcap_{n=0}^{+\infty} f^{-n}(X \backslash \Delta), \tag{2}$$

and assume that $\widetilde{X}$ is non-empty. We say that the sequence $\theta = \{\theta_t\}_{t\in\mathbb{N}} \in \{1, 2, \ldots, N\}^{\mathbb{N}}$ is the *itinerary* of $x \in \widetilde{X}$ if, for every $t \in \mathbb{N}$ and $i \in \{1, \ldots, N\}$, we have $\theta_t = i$ if and only if $f^t(x) \in X_i$. The *complexity function* of a sequence $\theta$ is the function defined for every $n \geqslant 1$ by

$$p(\theta, n) := \#L_n(\theta) \quad \text{where } L_n(\theta) := \{\theta_t \ldots \theta_{t+n-1},\, t \geqslant 0\} \text{ for all } n \geqslant 1,$$

that is, $p(\theta, n)$ gives the number of different words of length $n$ contained in $\theta$. Therefore, the complexity function of a sequence is a non-decreasing function of $n$. Also, if there exists $n_0 \geqslant 1$ such that $p(\theta, n_0 + 1) = p(\theta, n_0)$, then $p(\theta, n) = p(\theta, n_0)$ for all $n \geqslant n_0$. This implies that a symbolic sequence is eventually periodic if and only if its complexity function is eventually constant.

In this paper, we consider piecewise contracting maps satisfying a 'separation property'. To define this property, first note that, from inequality (1), for any $i \in \{1, \ldots, N\}$, the restriction $f|_{X_i}$ of $f$ to the piece $X_i$ admits a continuous extension $f_i : \overline{X_i} \to X$ which also satisfies inequality (1) on $\overline{X_i}$.

*Definition 1.1.* (Separation) We say that a piecewise contracting map $f$ satisfies the *separation property* if, for every $i \in \{1, \ldots, N\}$, the continuous extension $f_i : \overline{X}_i \to X$ is injective and $f_i(\overline{X}_i) \cap f_j(\overline{X}_j) = \emptyset$ for any $j \in \{1, \ldots, N\}$ such that $j \neq i$.

A map $f$ which satisfies the separation property is obviously injective on $X\backslash\Delta$, but not necessarily on the whole set $X$. A map $f$ which is injective on $X$ does not

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satisfy the separation property if and only if there are $i$ and $j$ in $\{1, \dots, N\}$ such that $\lim_{x\to y} f|_{X_i}(x) = \lim_{x\to z} f|_{X_j}(x)$ for some $y \in \Delta \cap \overline{X}_i$ and $z \in \Delta \cap \overline{X}_j$, with $y \neq z$ if $i = j$. It follows that not every injective map satisfies the separation property. Nevertheless, in dimension one, every injective map whose discontinuities are all of the first kind satisfies the separation property.

Our main result is the following Theorem 1.2. We will later complement its statement with the additional results given by Theorems 2.10 and 3.1 about the relationship between the complexity function and the dynamical asymptotic behaviour of the orbits near the discontinuity points.

THEOREM 1.2.

(1) *Let $\theta$ be the itinerary of an orbit of a piecewise contracting interval map which has $N$ contraction pieces and satisfies the separation property. Then there exist $m_0 \geqslant 1$, $\alpha \in \{0, 1, \dots, N-1\}$ and $\beta \in \{1, \dots, 1 + m_0(N - 1 - \alpha)\}$ such that the complexity function of $\theta$ satisfies*

$$p(\theta, n) = \alpha n + \beta \quad \text{for all } n \geqslant m_0. \tag{3}$$

(2) *For every $N \geqslant 2$, there exists a piecewise affine contracting interval map $f$ which has $N$ contraction pieces and satisfies the separation property, such that*

$$p(\theta, n) = (N-1)n + 1 \quad \text{for all } n \geqslant 1, \tag{4}$$

*for every itinerary $\theta$ of $f$.*

As mentioned above, generically, in the space of piecewise contracting interval maps, all the orbits are attracted by periodic orbits. Therefore, generically, the itinerary of any orbit is eventually periodic and has an eventually constant complexity function. In other words, $\alpha = 0$ in equality (3). Nevertheless, non-periodic attractors do appear when some orbits accumulate on discontinuity points [**5**]. Already in dimension two there are known examples of injective piecewise contracting maps that have itineraries with affine [**7**], polynomial [**11**] or exponential [**10**] complexity. However, for two- (and higher) dimensional piecewise contracting maps, there are no known conditions that guarantee a given type of complexity to the itineraries, other than eventually constant. Theorem 1.2 proves that in dimension one, in contrast to higher dimensions, the complexity of any non-periodic itinerary is eventually affine.

To prove Theorem 1.2, we will deduce equality (3) from precise results stated in Lemma 2.9 and Theorem 2.10, which relate the complexity of an itinerary with the recurrence properties of the corresponding orbit arbitrarily near the discontinuity points. In fact, the value of $\alpha$ is equal to the number of discontinuity points on which the orbit accumulates from both sides, and therefore is bounded above by the number of discontinuities contained in the attractor (which is at most $N - 1$). On the other hand, unless $\alpha = N - 1$, the constant $\beta$ depends on the transient behaviour of the dynamics and can be arbitrarily large; see Remark 2.11.

Part (1) of Theorem 1.2 states that the complexity of an itinerary is at most equal to $(N - 1)n + 1$. For $N = 2$, there are known examples of piecewise contracting maps whose itineraries have such Sturmian complexity [**3**, **4**, **6**, **7**, **9**, **16**]. In those examples,

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the attractor is a Cantor set supporting minimal dynamics. Part (2) of Theorem 1.2 states that, for any value of $N \geqslant 2$, there also exists a piecewise contracting map with 'full' complexity. Thereby, it establishes the optimality of the upper bound $N-1$ for the growth rate $\alpha$ of the complexity function for any number $N \geqslant 2$ of contracting pieces.

To prove the existence of those maps with full complexity, by induction on $N$, we construct, for any $N \geqslant 3$, a piecewise contracting map whose complexity function satisfies equality (4), using as a base case a known example with $N = 2$ and a Sturmian complexity. As a consequence, we will prove with Theorem 3.1 that the attractor of each of these maps inherits the Cantor structure and minimality of the attractor of the base case.

The proof of part (2) of Theorem 1.2 provides, for each $N \geqslant 2$, a map whose itineraries' complexity has the maximal growth rate $(N-1)$. But using Theorem 1.2, we can also construct examples with any given growth rate, as stated by the following Corollary 1.3.

COROLLARY 1.3. *For every $N \geqslant 2$ and $\alpha \in \{1, \ldots, N-1\}$, there exists a piecewise affine contracting interval map $f$ which has $N$ contraction pieces and satisfies the separation property, such that, for every itinerary $\theta$ of $f$, there exist $m_0 \geqslant 1$ and $\beta \in \{1, \ldots, 1 + m_0(N-1-\alpha)\}$ such that*

$$p(\theta, n) = \alpha n + \beta \quad \text{for all } n \geqslant m_0. \tag{5}$$

*Proof.* See §4. □

The paper is organized as follows. In §2, we give some preliminary results on itineraries and we prove part (1) of Theorem 1.2. In §3, we prove part (2) of Theorem 1.2. The proof of Corollary 1.3 is given in §4.

## 2. *Complexity of the itineraries*

### 2.1. *Preliminary general results on itineraries.*

In this subsection, we give some preliminary results that are not specific to piecewise contracting interval maps. In fact, here $X$ is not necessarily an interval and $f : X \to X$ may not satisfy inequality (1), provided it admits continuous extensions on each continuity piece $X_i$.

*Definition 2.1.* ((Atoms)) For every $i \in \{1, \ldots, N\}$, let $F_i : \mathcal{P}(X) \to \mathcal{P}(X)$ be defined by $F_i(A) = \overline{f(A \cap X_i)}$ for all $A \in \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the power set of $X$. Let $n \geqslant 1$ and $(i_1, \ldots, i_n) \in \{1, \ldots, N\}^n$. We say that $A_{i_1,\ldots i_n} := F_{i_n} \circ F_{i_{n-1}} \circ \cdots \circ F_{i_1}(X)$ is an *atom of generation n* if it is non-empty. We denote by $\mathcal{A}_n$ the set of all the atoms of generation $n$.

*Remark 2.2.* In the subsequent work, we will often use the following basic properties of the atoms. By construction, $A_{i_1 i_2 \ldots i_n} \subset A_{i_2 \ldots i_n} \subset \ldots \subset A_{i_n}$, and if $f$ is piecewise contracting, then $\max_{A \in \mathcal{A}_{n+1}} \operatorname{diam}(A) \leqslant \lambda \max_{A \in \mathcal{A}_n} \operatorname{diam}(A)$ for all $n \geqslant 1$, where $\operatorname{diam}(A)$ is the diameter of $A$.

As shown by the following Lemma 2.3, the separation property implies that the atoms of a same generation are pairwise disjoint.

LEMMA 2.3. *Suppose that $f$ satisfies the separation property. For every $n \geqslant 1$:*

(1) *if $A, B \in \mathcal{A}_n$ are such that $A \cap B \neq \emptyset$, then $A = B$; and*

(2) *if $A_{i_1 \ldots i_n}, A_{j_1 \ldots j_n} \in \mathcal{A}_n$ and $A_{i_1 \ldots i_n} = A_{j_1 \ldots j_n}$, then $(i_1, \ldots, i_n) = (j_1, \ldots, j_n)$.*

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*Proof.* It is easy to show that (1) is true for $n = 1$. Suppose now that it is true for some $n \geqslant 1$. Let $A,\ B \in \mathcal{A}_{n+1}$. Then there exist $C$ and $D \in \mathcal{A}_n$ and $i,\ j \in \{1, \dots N\}$ such that $A = f_i(\overline{C \cap X_i})$ and $B = f_j(\overline{D \cap X_j})$. Suppose that $A \cap B \neq \emptyset$. Since $f_i(\overline{X}_i) \cap f_j(\overline{X}_j) = \emptyset$ for $i \neq j$, it follows that $i = j$. Now, since $f_i$ is injective, if $A \cap B \neq \emptyset$, we have $(\overline{C \cap X_i}) \cap (\overline{D \cap X_i}) \neq \emptyset$, and $\overline{C} \cap \overline{D} \neq \emptyset$. Since $\overline{C} \cap \overline{D} = C \cap D$, using the induction hypothesis we deduce that $C = D$ and it follows that $A = B$.

By the separation property, (2) is true for $n = 1$. Suppose it is true for some $n \geqslant 1$ and suppose that $A_{i_1 \dots i_{n+1}} = A_{j_1 \dots j_{n+1}}$. Then $f_{i_{n+1}}(\overline{A_{i_1 \dots i_n} \cap X_{i_{n+1}}}) = f_{j_{n+1}}(\overline{A_{j_1 \dots j_n} \cap X_{j_{n+1}}})$ and $j_{n+1} = i_{n+1}$, since $f_{i_{n+1}}(\overline{X}_{i_{n+1}}) \cap f_{j_{n+1}}(\overline{X}_{j_{n+1}}) \neq \emptyset$ implies that $j_{n+1} = i_{n+1}$. On the other hand, if $f_{i_{n+1}}(\overline{A_{i_1 \dots i_n} \cap X_{i_{n+1}}}) = f_{i_{n+1}}(\overline{A_{j_1 \dots j_n} \cap X_{i_{n+1}}})$ then, by injectiveness, $A_{i_1 \dots i_n} \cap A_{j_1 \dots j_n} \neq \emptyset$, which implies by (1) that $A_{i_1 \dots i_n} = A_{j_1 \dots j_n}$. Using the induction hypothesis, it follows that $i_k = j_k$ for all $k \leqslant n$. □

The following Lemmas, 2.4 and 2.5, give the relationship between the itinerary of a point of $\widetilde{X}$ and the atoms visited by the orbit of that point.

LEMMA 2.4. *Let $x \in \widetilde{X}$ and let $\theta \in \{1, \dots, N\}^{\mathbb{N}}$ be its itinerary. Then $f^{t+n}(x) \in A_{\theta_t \theta_{t+1} \dots \theta_{t+n-1}}$ for every $t \geqslant 0$ and $n \geqslant 1$.*

*Proof.* Let $t = 0$. By the definitions of atom and itinerary, we have that $f(x) \in f(X_{\theta_0}) \subset A_{\theta_0}$ since $x \in X_{\theta_0}$. Assume that $f^n(x) \in A_{\theta_0 \theta_1 \dots \theta_{n-1}}$ for some $n \geqslant 1$. Then $f^{n+1}(x) = f(f^n(x)) \in f(A_{\theta_0 \theta_1 \dots \theta_{n-1}} \cap X_{\theta_n}) \subset A_{\theta_0 \theta_1 \dots \theta_n}$. Now suppose that $t \neq 0$, let $y = f^t(x)$ and let $\omega$ be the itinerary of $y$. Then $f^{t+n}(x) = f^n(y) \in A_{\omega_0 \dots \omega_{n-1}} = A_{\theta_t \theta_{t+1} \dots \theta_{t+n-1}}$. □

LEMMA 2.5. *Suppose that $f$ satisfies the separation property. Let $x \in \widetilde{X}$, $t \geqslant 0$, $n \geqslant 1$ and let $\theta$ be the itinerary of $x$. If $f^{t+n}(x) \in A_{i_1 i_2 \dots i_n}$, then $\theta_t \theta_{t+1} \dots \theta_{t+n-1} = i_1 i_2 \dots i_n$.*

*Proof.* Suppose that $t = 0$. By Lemma 2.4, we have $f^n(x) \in A_{\theta_0 \theta_1 \dots \theta_{n-1}}$, and therefore $A_{i_1 \dots i_n} \cap A_{\theta_0 \dots \theta_{n-1}} \neq \emptyset$. By Lemma 2.3, we have $A_{i_1 \dots i_n} = A_{\theta_0 \dots \theta_{n-1}}$ and $\theta_0 \dots \theta_{n-1} = i_1 \dots i_n$. Now suppose that $t \neq 0$, let $y = f^t(x)$ and let $\omega$ be the itinerary of $y$. Then $f^{t+n}(x) = f^n(y) \in A_{i_1 \dots i_n}$, which implies that $\omega_0 \dots \omega_{n-1} = i_1 \dots i_n$, that is, $\theta_t \dots \theta_{t+n-1} = i_1 \dots i_n$. □

COROLLARY 2.6. *Suppose that $f$ satisfies the separation property. Let $x \in \widetilde{X}$, $t \geqslant 0$, $n \geqslant 1$, let $\theta$ be the itinerary of $x$ and let $(i_1, i_2, \dots, i_n) \in \{1 \dots N\}^n$. Then $\theta_t \theta_{t+1} \dots \theta_{t+n-1} = i_1 i_2 \dots i_n$ if and only if $f^{t+n}(x) \in A_{i_1 \dots i_n}$.*

*Proof.* This follows directly from Lemmas 2.4 and 2.5. □

Let $x \in \widetilde{X}$, let $I := \{1, \dots, N\}$ and let $\theta \in I^{\mathbb{N}}$ be the itinerary of $x$. Now, for any $n \geqslant 1$ and $k \in I$, consider the set

$$L_n^k(\theta) := \{i_1 \dots i_n \in L_n(\theta) : \#\{j \in I : \exists t \geqslant 0 \text{ such that } f^{t+n}(x) \in A_{i_1 \dots i_n} \cap X_j\} = k\}.$$

A word of length $n$ of $\theta$ belongs to $L_n^k(\theta)$ if it is the label of an atom that intersects at least $k$ continuity pieces and if the orbit of $f^n(x)$ visits exactly $k$ of these intersections. The following Lemma 2.7 puts in relation the growth of the complexity function of an itinerary $\theta$ and the cardinality of the sets $L_n^k(\theta)$.

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LEMMA 2.7. *Let $x \in \widetilde{X}$ and let $\theta \in I^{\mathbb{N}}$ be the itinerary of $x$. Then*

$$p(\theta, n+1) \leqslant p(\theta, n) + \sum_{k=2}^{N} (k-1) \# L_n^k(\theta) \quad \text{for all } n \geqslant 1. \tag{6}$$

*If, moreover, $f$ satisfies the separation property, then (6) is an equality.*

*Proof.* Let $n \geqslant 1$, and observe that $L_n(\theta) = \bigcup_{k=1}^{N} L_n^k(\theta)$. First, we show the inclusion

$$L_{n+1}(\theta) \subset \bigcup_{k=1}^{N} B_n^k(\theta), \tag{7}$$

where

$$B_n^k(\theta) := \bigcup_{i_1 \dots i_n \in L_n^k(\theta)} \{i_1 \dots i_n i_{n+1} \in I^{n+1} : \exists t \geqslant 0 : f^{t+n}(x) \in A_{i_1 \dots i_n} \cap X_{i_{n+1}}\}.$$

Let $i_1 \dots i_{n+1} \in L_{n+1}(\theta)$. Then there exists $t \geqslant 0$ such that $i_1 \dots i_{n+1} = \theta_t \dots \theta_{t+n}$, which implies that $i_1 \dots i_n \in L_n^k(\theta)$ for some $k \in I$ (since $i_1 \dots i_n \in L_n(\theta)$ and $f^{t+n}(x) \in X_{i_{n+1}}$). On the other hand, by Lemma 2.4, we have $f^{t+n}(x) \in A_{i_1 \dots i_n}$. It follows that $i_1 \dots i_{n+1} \in B_n^k(\theta)$, and thus (7) is true.

If we suppose, moreover, that $f$ satisfies the separation property, we can deduce that (7) is an equality. Indeed, if $i_1 \dots i_{n+1} \in \bigcup_{k=1}^{N} B_n^k(\theta)$, then there exist $k \in I$ and $t \geqslant 0$ such that $i_1 \dots i_n \in L_n^k(\theta)$ and $f^{t+n}(x) \in A_{i_1 \dots i_n} \cap X_{i_{n+1}}$. The latter implies that $\theta_{t+n} = i_{n+1}$, and $i_1 \dots i_n = \theta_t \dots \theta_{t+n-1}$ by Lemma 2.5. It follows that $i_1 \dots i_{n+1} \in L_{n+1}(\theta)$.

To finish the proof, observe that, for any $k \in I$, the set $B_n^k(\theta)$ is defined by the union of disjoint sets that satisfy

$$\#\{i_1 \dots i_n i_{n+1} \in I^{n+1} : \exists t \geqslant 0 : f^{t+n}(x) \in A_{i_1 \dots i_n} \cap X_{i_{n+1}}\} = k \quad \text{for all } i_1 \dots i_n \in L_n^k(\theta),$$

by definition of $L_n^k(\theta)$. So we have $\# B_n^k(\theta) = k \# L_n^k(\theta)$. Moreover, since $L_n^k(\theta) \cap L_n^{k'}(\theta) = \emptyset$ if $k \neq k'$, on the one hand, $B_n^k(\theta) \cap B_n^{k'}(\theta) = \emptyset$ if $k \neq k'$, and on the other hand, $p(\theta, n) = \sum_{k=1}^{N} \# L_n^k(\theta)$. We deduce that

$$\begin{aligned} \# \bigcup_{k=1}^{N} B_n^k(\theta) &= \sum_{k=1}^{N} k \, \# L_n^k(\theta) = \sum_{k=1}^{N} \# L_n^k(\theta) + \sum_{k=1}^{N} (k-1) \, \# L_n^k(\theta) \\ &= p(\theta, n) + \sum_{k=2}^{N} (k-1) \, \# L_n^k(\theta). \end{aligned}$$

Now, from (7), we conclude that (6) is true, and it is an equality if $f$ satisfies the separation property. □

2.2. *Discontinuities of piecewise contracting interval maps and complexity.* From now on, we assume that the phase space $X$ of $f$ is a compact interval of $\mathbb{R}$ and that the contraction pieces are open intervals in $X$. This implies, in particular, that the atoms are closed intervals. Also, since the number of pieces is finite, the map $f$ has a finite number of discontinuities. We label the $N$ contraction pieces, $\{X_i\}_{i=1}^{N}$ of $f$, in such a way that $X_1 < X_2 < \cdots < X_N$.

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*Definition 2.8.* Let $x \in X$ and for any $n \geqslant 1$ denote $\mathcal{A}_n(x) := \{A \in \mathcal{A}_n : \exists t \in \mathbb{N} : f^{t+n}(x) \in A\}$. Let $c \in \Delta$ and $i \in \{1, \ldots, N-1\}$ be such that $c = \overline{X}_i \cap \overline{X}_{i+1}$. Let $n \geqslant 1$. We say that $c$ is *n-left-right visited* (in short *nlr-visited*) by $\{f^k(x)\}_{k\in\mathbb{N}}$ if there exists $A \in \mathcal{A}_n(x)$ such that $c \in A$ and

$$\{t \in \mathbb{N} : f^{t+n}(x) \in A \cap X_i\} \neq \emptyset \quad \text{and} \quad \{t \in \mathbb{N} : f^{t+n}(x) \in A \cap X_{i+1}\} \neq \emptyset.$$

We denote by $\Delta_{lr}^n(x)$ the set of the discontinuities that are $n$lr-visited. We say that $c$ is *left-right recurrently visited* (in short *lr-recurrently visited*) by $\{f^k(x)\}_{k\in\mathbb{N}}$ if $c \in \Delta_{lr}^n(x)$ for all $n \geqslant 1$. We denote by $\Delta_{lr}(x)$ the set of the discontinuities that are lr-recurrently visited.

Note that, for any $x \in X$ and $n \geqslant 1$, we have $\Delta_{lr}(x) \subset \Delta_{lr}^{n+1}(x) \subset \Delta_{lr}^n(x) \subset \Delta$, because any atom of generation $n+1$ is contained in an atom of generation $n$. Also, if $c \in \Delta_{lr}(x)$, then $c$ is an accumulation point (by the left and by the right) of the orbit of $x$, since the diameter of the atoms of a piecewise contracting map goes to zero as their generation goes to infinity (see Remark 2.2).

LEMMA 2.9. *Let $f$ be a piecewise contracting interval map satisfying the separation property. Let $x \in \widetilde{X}$ and let $\theta$ be its itinerary. Then*

$$\#\Delta_{lr}^n(x) \leqslant p(\theta, n+1) - p(\theta, n) \leqslant \#\Delta \quad \textit{for all } n \geqslant 1. \tag{8}$$

*Moreover, if $n_0 \geqslant 1$ is the smallest integer such that $\#A \cap \Delta \leqslant 1$ for any $A \in \mathcal{A}_n(x)$ with $n \geqslant n_0$, then*

$$p(\theta, n+1) = p(\theta, n) + \#\Delta_{lr}^n(x) \quad \textit{for all } n \geqslant n_0. \tag{9}$$

Note that $n_0$ exists and is bounded above by the smallest $n \geqslant 1$ such that

$$\max_{A\in\mathcal{A}_n} \operatorname{diam}(A) < \min_{i\in\{1,\ldots,N\}} \operatorname{diam}(X_i),$$

which, in turn, can be bounded above by a function of $\lambda$ and the diameters of the continuity pieces (see Remark 2.2).

*Proof.* Let $n \geqslant 1$. Suppose that $c \in \Delta_{lr}^n(x)$. Then there exists $A \in \mathcal{A}_n(x)$ such that $c \in A$. Moreover, there exist $i \in \{1, \ldots, N\}$ and $t \in \mathbb{N}$ such that $f^{t+n}(x) \in A \cap X_i$. Therefore, according to Lemma 2.4, we have $A \cap A_{\theta_t \ldots \theta_{t+n-1}} \neq \emptyset$, and after Lemma 2.3 we have that $A = A_{\theta_t \ldots \theta_{t+n-1}}$. As there exists also $t' \in \mathbb{N}$ such that $f^{t'+n}(x) \in A \cap X_{i+1}$, we have that $\theta_t \ldots \theta_{t+n-1} \in L_n^k(\theta)$ for some $k \geqslant 2$. We deduce that

$$\Delta_{lr}^n(x) \subset \bigcup_{k=2}^{N} \bigcup_{i_1 \ldots i_n \in L_n^k(\theta)} (A_{i_1 \ldots i_n} \cap \Delta_{lr}^n(x))$$

and it follows that

$$\#\Delta_{lr}^n(x) \leqslant \sum_{k=2}^{N} \sum_{i_1 \ldots i_n \in L_n^k(\theta)} \#(A_{i_1 \ldots i_n} \cap \Delta_{lr}^n(x)).$$

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Now, if $A \in \mathcal{A}_n(x)$ and $\#A \cap \Delta_{lr}^n(x) = q$, then $A$ intersects at least $q+1$ continuity pieces that are visited by the orbit of $f^n(x)$. It follows that, for any $k \geqslant 2$ and $i_1 \dots i_n \in L_n^k(\theta)$, we have that $\#(A_{i_1\dots i_n} \cap \Delta_{lr}^n(x)) \leqslant k-1$. We deduce that

$$\#\Delta_{lr}^n(x) \leqslant \sum_{k=2}^{N}(k-1)\#L_n^k(\theta) \quad \text{for all } n \geqslant 1. \tag{10}$$

Now, let $n \geqslant 1$ and $k \geqslant 2$. If $i_1 \dots i_n \in L_n^k(\theta)$, then $A_{i_1\dots i_n} \neq \emptyset$ and $A_{i_1\dots i_n}$ intersects at least $k$ continuity pieces. As $A_{i_1\dots i_n}$ is a closed interval and the continuity pieces are open intervals, it follows that $A_{i_1\dots i_n}$ contains at least $k-1$ elements of $\Delta$. Now, according to Lemma 2.3, if $i_1 \dots i_n$ and $i_1' \dots i_n'$ are two different words of $L_n(\theta)$, then $A_{i_1\dots i_n} \cap A_{i_1'\dots i_n'} = \emptyset$. It follows that

$$\#\Delta \geqslant \sum_{k=2}^{N}(k-1)\#L_n^k(\theta) \quad \text{for all } n \geqslant 1. \tag{11}$$

Then inequalities (8) follow from (10), (11) and Lemma 2.7.

Let $n \geqslant n_0$. Then, for any $i_1 \dots i_n \in L_n(\theta)$, the atom $A_{i_1\dots i_n}$ intersects at most two continuity pieces, and therefore $L_n^k(\theta) = \emptyset$ for all $k \geqslant 3$. Moreover, for any $i_1 \dots i_n \in L_n^2(\theta)$ the discontinuity contained in $A_{i_1\dots i_n}$ belongs to $\Delta_{lr}^n(x)$. We deduce that

$$\#\Delta_{lr}^n(x) \geqslant \#L_2^k(\theta) = \sum_{k=2}^{N}(k-1)\#L_n^k(\theta) \quad \text{for all } n \geqslant n_0, \tag{12}$$

which, together with (10) and Lemma 2.7, implies (9). □

THEOREM 2.10. *Let $f$ be a piecewise contracting map satisfying the separation property. Let $x \in \widetilde{X}$ and let $\theta$ be its itinerary. Then there exists $m_0 \geqslant 1$ such that*

$$p(\theta, n) = n\#\Delta_{lr}(x) + \beta(x) \quad \textit{for all } n \geqslant m_0, \tag{13}$$

*with*

$$p(\theta, 1) - \#\Delta_{lr}(x) \leqslant \beta(x) \leqslant p(\theta, 1) - \#\Delta + m_0(\#\Delta - \#\Delta_{lr}(x)). \tag{14}$$

*Proof.* For any $c \in \Delta$, either $c \in \Delta_{lr}(x)$ or there exists $\nu(c) := \min\{n \geqslant 1 : c \notin \Delta_{lr}^n(x)\}$. As $\Delta_{lr}^{n+1}(x) \subset \Delta_{lr}^n(x)$ for all $n \geqslant 1$, it follows that, for any $c \in \Delta \setminus \Delta_{lr}(x)$, we have that $c \notin \Delta_{lr}^n(x)$ for all $n \geqslant \nu(c)$. Therefore, if $n_1 = \max\{\nu(c), c \in \Delta \setminus \Delta_{lr}(x)\}$ if $\Delta \neq \Delta_{lr}(x)$, and $n_1 = 1$ otherwise. Then $\Delta_{lr}^n(x) = \Delta_{lr}(x)$ for all $n \geqslant n_1$.

Let $m_0 := \max\{n_0, n_1\}$. Then we can write (9) as

$$p(\theta, n+1) - p(\theta, n) = \#\Delta_{lr}(x) \quad \text{for all } n \geqslant m_0,$$

which implies that $p(\theta, n) = p(\theta, m_0) + (n - m_0)\#\Delta_{lr}(x)$ for all $n \geqslant m_0$. It follows that (13) is true with

$$\beta(x) = p(\theta, m_0) - m_0\#\Delta_{lr}(x). \tag{15}$$

Recalling that $\#\Delta_{lr}^n(x) \geqslant \#\Delta_{lr}(x)$ for all $n \geqslant 1$, from (8), we obtain that

$$p(\theta, 1) - \#\Delta_{lr}(x) \leqslant p(\theta, n) - n\#\Delta_{lr}(x) \leqslant p(\theta, 1) - \#\Delta + n(\#\Delta - \#\Delta_{lr}(x)) \quad \text{for all } n \geqslant 1,$$

and setting $n = m_0$, we obtain (14) from (15). □

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*Remark 2.11.* We note that estimation (14) of $\beta(x)$ in Theorem 2.10 is exact if $m_0 = 1$. If the integer $n_0$ of Lemma 2.9 is equal to one, then it is also possible to get an exact expression of $\beta(x)$ for $m_0 \geqslant 2$. This happens, for instance, if any atom of generation one contains at most one discontinuity point. In such a case, we can use (9) and (15) to obtain that

$$\beta(x) = p(\theta, 1) - \#\Delta + m_0(\#\Delta - \#\Delta_{lr}(x)) + \sum_{m=1}^{m_0-1} (\#\Delta_{lr}^m(x) - \#\Delta) \quad \text{for all } m_0 \geqslant 2.$$

We observe that the upper bound for $\beta(x)$ given by (14) is reached if and only if $\Delta_r^m(x) = \Delta$ for all $1 \leqslant m \leqslant m_0 - 1$. It follows that the upper bound is reached when $\#\Delta$ atoms of generation $m_0 - 1$ contain a discontinuity and a point of the orbit of $f^{m_0-1}(x)$ on each side of that discontinuity.

*Proof of part (1) of Theorem 1.2.* Let $x \in \widetilde{X}$ with itinerary $\theta$. Then equality (3) of Theorem 1.2, follows directly from equality (13) of Theorem 2.10 with $\alpha = \#\Delta_{lr}(x) \in \{0, 1, \ldots, N-1\}$. Also, for any $x \in \widetilde{X}$ with itinerary $\theta$,

$$p(\theta, 1) - \#\Delta \leqslant 1 \leqslant p(\theta, 1) - \#\Delta_{lr}(x), \tag{16}$$

which, together with (14), implies that $1 \leqslant \beta(x) \leqslant 1 + m_0(\#\Delta - \#\Delta_{lr}(x)) = 1 + m_0(N - 1 - \alpha)$. □

*Remark 2.12.* Now we give some direct consequences of Theorem 2.10 and we comment on their relationship with other results.

(1) There exists an itinerary with a complexity function which is not eventually constant if and only if there exists a discontinuity point $c$ which is lr-recurrently visited by an orbit of $\widetilde{X}$. In particular, if the limit set of $f$ does not contain any discontinuity point, then the complexity function of every itinerary is eventually constant (recall that if $c \in \Delta_{lr}(x)$, then $c$ belongs to the $\omega$-limit set of $x$). In this case, for any $x \in \widetilde{X}$ with itinerary $\theta$,

$$p(\theta, 1) \leqslant p(\theta, n) \leqslant p(\theta, 1) + (m_0 - 1)\#\Delta \quad \text{for all } n \geqslant 1,$$

and $p(\theta, n) = \beta(x)$ is constant for any $n \geqslant m_0$. Moreover, when the limit set of $f$ does not contain discontinuity points, there exists a smallest integer $m \geqslant 1$ such that no atom of generation $m$ contains discontinuities. This integer $m$ is an upper bound for $m_0$, which provides a uniform upper bound on $\beta(x)$ through inequality (14).

(2) If $\#\Delta_{lr}(x) = 1$ but $\Delta_{lr}(x) \neq \Delta$, then the itinerary $\theta$ of the orbit of $x$ satisfies $p(\theta, n) = n + \beta(x)$ for all $n$ large enough, where $\beta(x)$ may be larger than one. An example of a piecewise contracting map whose itineraries have such a complexity can be found in [**7**]. In [**8**], it is shown that, up to a prefix of finite length, a sequence of complexity $n + \beta$ is the image by a morphism of a Sturmian sequence. We conclude that if $\#\Delta_{lr}(x) = 1$ but $\Delta_{lr}(x) \neq \Delta$, the itinerary of any orbit is Sturmian up to a morphism. Hence, up to a morphism, it is the itinerary of an irrational rotation, with respect to a suitable partition of the circle.

(3) If $\#\Delta_{lr}(x) > 1$, the itinerary $\theta$ may be that of an irrational rotation: in fact, for some adequate values of $\alpha$ and $\beta$, the itineraries of an irrational rotation with respect to a suitable partition of the circle may have a complexity function of the form $\alpha n + \beta$ for all $n$ large enough. However, not every sequence with such a complexity is itself an itinerary of an irrational rotation [**1**].

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(4) If all the discontinuities are lr-recurrently visited by the orbit of a point $x \in \widetilde{X}$, i.e., $\Delta_{lr}(x) = \Delta$, then $m_0 = n_0$ (see the definition of $m_0$ in the proof of Theorem 2.10). In addition, from part (1) of Theorem 1.2, we know that $\beta(x) = 1$ in this case. So, equality (13) becomes

$$p(\theta, n) = (N-1)n + 1 \quad \text{for all } n \geqslant n_0. \tag{17}$$

In the particular case where the map has two contraction pieces ($N = 2$) and the (unique) discontinuity point is lr-recurrently visited by the orbit of $x$, then $n_0 = 1$ and $\theta(x)$ is a Sturmian sequence. Therefore, it is also an itinerary of an irrational rotation. In general, if the itinerary $\theta(x)$ satisfies (17) for some $N \geqslant 2$, then it has the complexity of an itinerary of an $N$-interval exchange transformation satisfying the so-called Keane's infinite distinct orbit condition [**17**, **18**]. In fact, the following result is proved in [**15**]: if an injective piecewise contracting map $f$ has no periodic orbit and is such that the image of any discontinuity and each lateral limit of $f$ belong to $\widetilde{X}$, then it is semi-conjugate to an interval exchange transformation. It is, however, not easy to exhibit examples satisfying these hypotheses since, generically, a piecewise contracting interval map has periodic points. In the next section, we will construct such examples for every $N \geqslant 2$, where, furthermore, equality (17) holds for any itinerary $\theta$.

## 3. *Existence of piecewise contracting interval maps of full complexity*

In the previous section, we proved Theorem 2.10, which implies that the complexity of the itinerary of any orbit of a piecewise contracting interval map satisfying the separation property is bounded from above by an affine function whose slope is equal to the number of discontinuities of the map. However, as far as we know, there is still no example of piecewise contracting interval maps with more than one lr-recurrently visited discontinuity. The purpose of this section is to construct such examples. We will do even more and construct examples for which all the discontinuities are lr-recurrently visited by all the orbits. These maps generate itineraries with the maximal complexity for the fixed number $N$ of contracting pieces. We say that they have 'full' complexity.

We are also interested in the asymptotic dynamics of such examples. It takes place in what we call the *attractor* $\Lambda$ of the piecewise contracting map $f : X \to X$. To define the attractor we first recall the definition of the atoms $A \in \mathcal{A}_n$ of generation $n$ for any natural number $n \geqslant 1$ (see Definition 2.1). We define the attractor $\Lambda \subset X$ as

$$\Lambda := \bigcap_{n=1}^{\infty} \Lambda_n \quad \text{where } \Lambda_n := \bigcup_{A \in \mathcal{A}_n} A \text{ for all } n \geqslant 1. \tag{18}$$

The sets $\Lambda_n$ can, equivalently, be defined recursively by $\Lambda_1 := \overline{f(X \backslash \Delta)}$ and $\Lambda_{n+1} := \overline{f(\Lambda_n \backslash \Delta)}$ for all $n \geqslant 1$.

Note that the attractor $\Lambda$ is non-empty and compact. In addition, $\Lambda$ contains all the non-wandering and $\omega$-limit points; see [**5**] for more details and examples.

In this section, we prove the following theorem.

THEOREM 3.1. *For every $N \geqslant 2$, there exists a piecewise affine contracting map which has $N$ contraction pieces and satisfies the separation property, whose attractor is a minimal Cantor set, and is such that each of its discontinuities is lr-recurrently visited by any orbit.*

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Theorem 3.1, together with (17), proves immediately equation (4) of Theorem 1.2 for any $n \geqslant n_0$. Later, we will prove that it is always possible to construct the maps in such a way that $n_0 = 1$ (see Lemma 3.8), to end the proof of part (2) of Theorem 1.2.

Observe that the attractor of the piecewise contracting map of Theorem 3.1 contains *all* the discontinuities of the map. In fact, any lr-recurrently visited discontinuity belongs to the $\omega$-limit set of some orbit, and the attractor contains all the $\omega$-limit sets.

To prove Theorem 3.1, we will prove the following stronger statement.

*Assertion A.* For every $N \geqslant 2$, there exist $N$ ordered disjoint open intervals $X_1 = [c_0, c_1)$, $X_2 = (c_1, c_2)$, $\ldots$, $X_N = (c_{N-1}, c_N]$ of $X := [c_0, c_N]$ and $f : X \to X$ with all the following properties.

(P1) The map $f$ is piecewise contracting with contraction pieces $X_1, \ldots, X_N$ and $f|_{X_i}$ is affine with slope $\lambda \in (0, 1)$.
(P2) The map $f$ satisfies the separation property.
(P3) The attractor $\Lambda$ of $f$ is a Cantor set.
(P4) The set $\bigcup_{i=1}^{N-1} \{f_i(c_i),\ f_{i+1}(c_i)\}$ is a subset of $\widetilde{X}$.
(P5) There exist $i \in \{1, \ldots, N-1\}$ such that $\{f^n(f_i(c_i))\}_{n\in\mathbb{N}}$ or $\{f^n(f_{i+1}(c_i))\}_{n\in\mathbb{N}}$ is dense in $\Lambda$.
(P6) For any $x \in \widetilde{X}$ and $i \in \{1, \ldots, N-1\}$ we have that $c_i \in \Delta_{lr}(x)$.

To prove Assertion A and Theorem 3.1, we will need the following lemma.

LEMMA 3.2. *Let $N \geqslant 2$ and let $c_0 < c_1 < \cdots < c_N$ in $\mathbb{R}$. Let $f : [c_0, c_N] \to [c_0, c_N]$ be a piecewise contracting map which satisfies the separation property and has contraction pieces $X_1 = [c_0, c_1)$, $X_2 = (c_1, c_2)$, $\ldots$, $X_N = (c_{N-1}, c_N]$. Suppose that there exists $i \in \{1, \ldots, N-1\}$ such that:*

(1) *$f_j(c_i) \in \widetilde{X}$ and $\{f^n(f_j(c_i))\}_{n\in\mathbb{N}}$ is dense in $\Lambda$ for some $j \in \{i, i+1\}$; and*
(2) *$c_i \in \Delta_{lr}(x_0)$ for some $x_0 \in \widetilde{X}$.*

*Then, for any $\epsilon > 0$ and $y \in \Lambda$ such that $\Lambda \cap (y, y+\nu) \neq \emptyset$ (respectively, $\Lambda \cap (y-\nu, y) \neq \emptyset$) for all $\nu > 0$, there exists $l \geqslant 0$ such that $f^l(x_0) \in (y, y+\epsilon)$ (respectively, $f^l(x_0) \in (y-\epsilon, y)$).*

*Proof.* We will make the proof for $y \in \Lambda$ such that $\Lambda \cap (y, y+\nu) \neq \emptyset$ for all $\nu > 0$ and, without loss of generality, we will suppose that $i = j = 1$. Let $\epsilon > 0$, $z \in \Lambda \cap (y, y+\epsilon)$ and $\delta = \frac{1}{2}\min\{z - y,\ y + \epsilon - z\}$. Since $\{f^n(f_1(c_1))\}_{n\in\mathbb{N}}$ is dense in $\Lambda$, there exists $n$ such that $|f^n(f_1(c_1)) - z| < \delta$. By injectivity of $f$ on $X\backslash\Delta$, the set $P := \bigcup_{l=0}^{n-1} f^{-l}(\Delta)$ is finite, and for $\rho := d(f_1(c_1), P) > 0$ the map $f^n$ is continuous on $(f_1(c_1) - \rho,\ f_1(c_1) + \rho)$. Using the continuity of $f_1$ on $[c_0, c_1]$, we obtain that there exists $\delta' > 0$ such that $|f^n(f_1(c_1)) - f^{n+1}(x)| < \delta$ for all $x \in (c_1 - \delta', c_1)$. As $c_1 \in \Delta_{lr}(x_0)$, there exists $m$ such that $f^m(x_0) \in (c_1 - \delta', c_1)$, and by the triangular inequality we deduce that $|f^l(x_0) - z| < 2\delta$ for $l = m + n + 1$, that is, $f^l(x_0) \in (y, y+\epsilon)$. □

Before proving Assertion A, we show that, together with Lemma 3.2, it implies Theorem 3.1.

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*Proof of Theorem 3.1 as a Corollary of Assertion A.* Suppose that Assertion A is true and let $f$ satisfy (P1–6)† for some $N \geqslant 2$. Then $f$ is a piecewise affine contracting interval map, it has the separation property, its attractor is a Cantor set and $\Delta = \Delta_{lr}(x)$ for any $x \in \widetilde{X}$. So, to prove Theorem 3.1, it remains to prove that $\Lambda$ is minimal and that $\Delta = \Delta_{lr}(x)$ for any $x \in X \backslash \widetilde{X}$. To this end, note that (P1–6) do not impose any condition on the definition of $f$ on $\Delta$, and therefore $f$ can be suitably defined on this set. So we can assume that $f(c_i) \in \{f_{i-1}(c_i), f_i(c_i)\}$ for any $i \in \{1, \ldots, N-1\}$, which implies, by (P4), that $f(\Delta) \subset \widetilde{X}$. Since $f$ satisfies (P1–6), it satisfies the hypotheses of Lemma 3.2. It follows that the orbit of any point $x_0 \in \Lambda \cap \widetilde{X}$ is dense in $\Lambda$. Now, since $f(c) \in \widetilde{X}$ for all $c \in \Delta$, the orbit of a point in $\Lambda \backslash \widetilde{X}$ is also dense in $\Lambda$. We deduce that $\Lambda$ is minimal. Finally, (P4), (P6) and $f(\Delta) \subset \widetilde{X}$ imply that, for any $x \in X \backslash \widetilde{X}$ and $c \in \Delta$, there exists $p \geqslant 1$ such that $c \in \Delta_{lr}(f^p(x))$. Since $\Delta_{lr}(f^p(x)) \subset \Delta_{lr}(x)$, we conclude that any discontinuity of $f$ is lr-recurrently visited by any orbit, which ends the proof of Theorem 3.1. □

In the following subsections, we will prove Assertion A. Let us describe briefly the route of the proof.

The proof goes by induction on the number $N \geqslant 2$ of contraction pieces of $f$. In §3.1, relying on a known example, we prove that there exists a map satisfying (P1–6) for $N = 2$. In §3.2, we construct a map $g$ satisfying (P1–6) with $N + 1$ contraction pieces, assuming the existence of a map $f$ satisfying (P1–6) with $N$ contraction pieces and with contracting constant $\lambda \in (0, 1)$. To construct $g$ from $f$, we first choose an adequate point $\xi_0 \in \widetilde{X}$, and its orbit $\{\xi_r\}_{r \in \mathbb{N}}$, where $\xi_r = f^r(\xi_0)$. Second, we 'cut' the interval $X$ at each point $\xi_r$ with $r \geqslant 1$ (but not at $\xi_0$) and insert an interval $G_r$ substituting the point $\xi_r$, such that, for all $r \geqslant 1$, the length of $G_r$ is $\lambda^r$. We define an affine map $g|_{G_r} : G_r \to G_{r+1}$ for all $r \geqslant 1$. In this way, we have added a new discontinuity point of $g$ at the point, say, $\xi_0$. For all $y \notin \bigcup_{r \geqslant 1} G_r$, we define the image $g(y)$ from the image $f(x)$ of the corresponding point $x \in X \backslash \{\xi_r\}_{r \geqslant 1}$. In particular, $g$ preserves the old $N - 1$ discontinuity points of $f$. So $g$ has $N$ discontinuity points and hence $N + 1$ continuity pieces. Finally, in Proposition 3.7, we show that there exists a good choice of the *cutting orbit* $\{\xi_r\}_{r \in \mathbb{N}}$, to make $g$ inherit the properties (P1–6) from $f$.

3.1. *Full complexity with a single discontinuity.* In this subsection, we prove that Assertion A holds for $N = 2$. We begin with a lemma about the sets $\Lambda_n$ (defined in (18)) for piecewise increasing maps with two contraction pieces.

LEMMA 3.3. *Let $X := [0, 1]$, $c \in (0, 1)$ and $f : X \to X$ be a piecewise contracting map with contraction pieces $X_1 = [0, c)$ and $X_2 = (c, 1]$. Suppose that the continuous extensions $f_1$ and $f_2$ of $f$ are increasing and such that $0 = f_2(c) < f_2(1) < f_1(0) < f_1(c) = 1$.*

*For every $k \in \mathbb{N}$, let $H_k := f^k((f(1), f(0)))$. If $c \notin \overline{H_k}$ for all $k \in \mathbb{N}$, then $H_k = (f^{k+1}(1), f^{k+1}(0))$ for all $k \in \mathbb{N}$ and*

$$\Lambda_n = [0, 1] \Bigg\backslash \bigcup_{k=0}^{n-1} H_k \quad \text{for all } n \geqslant 1. \tag{19}$$

† (P1–6) is a shorthand notation for 'the properties (P1) to (P6)'.

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*Moreover,* $\{0, 1\} \subset \widetilde{X}$ *and, for any* $n \geqslant 1$ *and* $A \in \mathcal{A}_n$*, there exists* $p$ *and* $q$ *in* $\mathbb{N}$ *such that* $A = [f^p(0),\ f^q(1)]$.

*Proof.* Assume that $H_k = (f^{k+1}(1),\ f^{k+1}(0))$ for some $k \in \mathbb{N}$. Then $H_k \subset [0, c)$ or $H_k \subset (c, 1]$ because $c \notin H_k$. Hence, $f$ is continuous and increasing on $H_k$, and $H_{k+1} = (f^{k+2}(1),\ f^{k+2}(0))$. As $H_0 = (f(1),\ f(0))$, we have proved by induction that $H_k = (f^{k+1}(1),\ f^{k+1}(0))$ for every $k \in \mathbb{N}$.

Now let us show (19) by induction. We have $\Lambda_1 := f_1([0, c]) \cup f_2([c, 1]) = [f(0), 1] \cup [0,\ f(1)] = [0, 1]\backslash H_0$ and hence (19) is true for $n = 1$. Now let $n \geqslant 1$, and assume that $\Lambda_n = [0, 1]\backslash \bigcup_{k=0}^{n-1} H_k$. We shall prove that $\Lambda_{n+1} = [0, 1]\backslash \bigcup_{k=0}^{n} H_k$. First, observe that

$$\Lambda_{n+1} = f_1(\overline{\Lambda_n \cap (0, c)}) \cup f_2(\overline{\Lambda_n \cap (c, 1)}) = f(\Lambda_n\backslash\{c\}) \cup \{f_1(c),\ f_2(c)\},$$

since $c \in [0, 1]\backslash \bigcup_{k=0}^{n-1} \overline{H}_k$, and therefore it belongs to the interior of $\Lambda_n$. It follows that

$$\Lambda_{n+1} = f\left(([0, 1]\backslash\{c\}) \setminus \bigcup_{k=0}^{n-1} H_k\right) \cup \{0, 1\}.$$

In addition, since $f$ is injective on $[0, 1]\backslash\{c\}$, we have

$$f\left(([0, 1]\backslash\{c\}) \setminus \bigcup_{k=0}^{n-1} H_k\right) = f([0, 1]\backslash\{c\}) \setminus f\left(\bigcup_{k=0}^{n-1} H_k\right).$$

On the other hand, we have $f(x) \notin \{0, 1\}$ for any $x \neq c$ and $c \notin H_k$ for all $k \in \mathbb{N}$, which implies that

$$\begin{aligned}\Lambda_{n+1} &= (f([0, 1]\backslash\{c\}) \cup \{0, 1\}) \setminus \bigcup_{k=0}^{n-1} f(H_k)\\ &= ([0, 1]\backslash H_0) \setminus \bigcup_{k=1}^{n} H_k = [0, 1] \setminus \bigcup_{k=0}^{n} H_k,\end{aligned}$$

as wanted.

Let us prove that $\{0, 1\} \subset \widetilde{X}$. Since $c \notin f^k([f(1),\ f(0)])$ for every $k \geqslant 0$, we have $c \notin \{f^k(0),\ f^k(1)\}$ for all $k \geqslant 1$. It follows that $\{0, 1\} \subset \widetilde{X}$, because $c \notin \{0, 1\}$.

To end the proof, first observe that $f_1([0, c]) = [f(0), 1]$ and $f_2([c, 1]) = [0,\ f(1)]$. Therefore, the atoms of $\mathcal{A}_1$ are of the form $[f^p(0),\ f^q(1)]$. Now, as an induction hypothesis, assume that, for some $n \geqslant 1$ and for any $A \in \mathcal{A}_n$, there exists $p$ and $q$ in $\mathbb{N}$ such that $A = [f^p(0),\ f^q(1)]$. If $B \in \mathcal{A}_{n+1}$, then, by definition of atoms, there exist $i \in \{1, 2\}$ and $A \in \mathcal{A}_n$ such that $B = f_i(\overline{A \cap X_i})$, where $X_1 := [0, c)$ and $X_2 := (c, 1]$. If $c \notin A$, then $A \subset X_i$ and $B = f(A) = [f^{p+1}(0),\ f^{q+1}(1)]$. If $c \in A$, then either $A \cap X_i = [f^p(0), c)$ or $A \cap X_i = (c,\ f^q(1)]$. In both cases, $A \cap X_i \neq \emptyset$, since $\{0, 1\} \subset \widetilde{X}$. It follows that either $B = [f^{p+1}(0), 1]$ or $B = [0,\ f^{q+1}(1)]$. □

PROPOSITION 3.4. *Let* $\lambda$ *and* $\mu \in (0, 1)$ *be such that* $\lambda + \mu > 1$ *and denote* $c := (1 - \mu)/\lambda$. *Let* $f_1 : [0, c] \to [0, 1]$ *and* $f_2 : [c, 1] \to [0, 1]$ *be defined by*

$$f_1(x) = \lambda x + \mu \quad \textit{for all } x \in [0, c] \quad \textit{and} \quad f_2(x) = \lambda x + \mu - 1 \quad \textit{for all } x \in [c, 1]. \tag{20}$$

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*Then any map* $f : [0, 1] \to [0, 1]$ *such that* $f|_{[0,c)} = f_1|_{[0,c)}$ *and* $f|_{(c,1]} = f_2|_{(c,1]}$, *and* $c \notin f^k([f(1), f(0)])$ *for all* $k \in \mathbb{N}$, *satisfies (P1–6) with* $X_1 = [0, c)$ *and* $X_2 = (c, 1]$.

*Proof.* Consider a map $f : [0, 1] \to [0, 1]$ which satisfies the hypotheses of Proposition 3.4. Then it is easy to check that $f$ satisfies (P1–2), with $c_0 = 0$, $c_1 = c$ and $c_2 = 1$. Also, $f$ satisfies all the hypotheses of Lemma 3.3, and it follows, in particular, that (P4) holds.

Let $\gamma \in \{0, 1\}$ and let us show that $\{f^n(\gamma)\}_{n\in\mathbb{N}}$ belongs to $\Lambda$. Since for any $k \in \mathbb{N}$ the set $H_k$ of Lemma 3.3 is an open set of $[0, 1]$, it cannot contain $\gamma$. Therefore, by (19), we have $\gamma \in \Lambda_n$ for all $n \in \mathbb{N}$, that is, $\gamma \in \Lambda$. Since $\gamma \in \widetilde{X}$ and $\Lambda \cap \widetilde{X}$ is forward invariant, we deduce that $\{f^n(\gamma)\}_{n\in\mathbb{N}} \subset \Lambda$.

Now, let us prove that $\Lambda$ has no isolated point. Let $x \in \Lambda$ and $\epsilon > 0$. Let $n \geqslant 1$ be such that $\operatorname{diam}(A) < \epsilon$ for every $A \in \mathcal{A}_n$. Let $A \in \mathcal{A}_n$ be such that $x \in A$, and let $p$ and $q \in \mathbb{N}$ be such that $A = [f^p(0), f^q(1)]$. If $x \in (f^p(0), f^q(1)]$, then $0 < |f^p(0) - x| < \epsilon$ and if $x \in [f^p(0), f^q(1))$, then $0 < |f^q(1) - x| < \epsilon$. Since both points $f^p(0)$ and $f^q(1) \in \Lambda$, we found a point in $\Lambda\backslash\{x\}$ which is at a distance less than $\epsilon$ of $x$. This shows that $\Lambda$ is a perfect set (recall that $\Lambda$ is compact) and it proves, at the same time, that $\{f^n(0)\}_{n\in\mathbb{N}}$ and $\{f^n(1)\}_{n\in\mathbb{N}}$ are dense in $\Lambda$, i.e., $f$ satisfy (P5). Now $\Lambda$ is totally disconnected because $f$ satisfies the separation property [**5**]. It follows that $\Lambda$ is a Cantor set and $f$ satisfies (P3).

Now we show that $f$ satisfies (P6). To this end, we prove that, for every $x \in \widetilde{X}$ and $\epsilon > 0$, there exists $l \in \mathbb{N}$ and $r \in \mathbb{N}$ such that $f^l(x) \in (c - \epsilon, c)$ and $f^r(x) \in (c, c + \epsilon)$. Let $x \in \widetilde{X}$ and $\epsilon > 0$. Let $n_0 \in \mathbb{N}$ be such that $\operatorname{diam}(A) < \epsilon/2$ for every $A \in \mathcal{A}_{n_0}$ and let $A \in \mathcal{A}_{n_0}$ be such that $f^{n_0}(x) \in A$. Let $p$ and $q$ be integers such that $A = [f^p(0), f^q(1)]$ and let $T = \{t \in \mathbb{N} : f^{n_0+t}(x) \in (c - \epsilon, c + \epsilon)\}$. Arguing by contradiction, assume that $T = \emptyset$ or that $f^{n_0+t}(x) \in (c, c + \epsilon)$ for all $t \in T$. Then, by induction on $t \in \mathbb{N}$, we deduce that

$$0 < (f^{q+t}(1) - c)(f^{n_0+t}(x) - c) \quad \text{and} \quad 0 \leqslant f^{q+t}(1) - f^{n_0+t}(x) < \epsilon/2 \quad \text{for all } t \in \mathbb{N}.$$

Therefore, for each $t \in T$, we have $f^{q+t}(1) \in (c, 1]$ and, for each $t \notin T$,

$$|f^{q+t}(1) - c| \geqslant |f^{n_0+t}(x) - c| - |f^{q+t}(1) - f^{n_0+t}(x)| > \epsilon/2.$$

We deduce that $f^{q+t}(1) \notin (c - \epsilon/2, c)$ for all $t \in \mathbb{N}$. Now let $\nu > 0$ be such that $f^k(1) \notin (c - \nu, c)$ for all $k \leqslant q$. Then $f^k(1) \notin (c - \epsilon_0, c)$ for all $k \in \mathbb{N}$, where $\epsilon_0 = \min\{\nu, \epsilon/2\}$. On the other hand, there exist $p'$ and $q'$ such that $[f^{p'}(0), f^{q'}(1)]$ is an atom of diameter strictly smaller than $\epsilon_0$ which contains $c$. Since $f^{p'}(0) \in \Lambda$ and $f^k(1) \notin (c - \epsilon_0, c)$ for all $k \in \mathbb{N}$, it follows that $\{f^k(1)\}_{k\in\mathbb{N}}$ is not dense in $\Lambda$, which is a contradiction. Therefore, $T \neq \emptyset$ and there exists $l \in \mathbb{N}$ such that $f^l(x) \in (c - \epsilon, c)$. Now, if we assume that $f^{n_0+t}(x) \in (c - \epsilon, c)$ for every $t \in T$, we deduce with an analogous proof that $\{f^k(0)\}_{k\in\mathbb{N}}$ is not dense in $\Lambda$. Therefore, there exists $r \in \mathbb{N}$ such that $f^r(x) \in (c, c + \epsilon)$. □

*Proof of Assertion A for* $N = 2$. Consider a map $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \lambda x + \mu \mod 1 \text{ for all } x \in [0, 1],$$

where $\lambda$ and $\mu \in (0, 1)$ are such that $\lambda + \mu > 1$. Then $f$ is a piecewise contracting map which satisfies hypothesis (20), with the particularity that $f(c) = 0$. Immediately, the 'gap' between the atoms of generation one is the interval $(f(1), f(0))$. It is standard to

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prove that if there exists a minimum natural number $k$ such that $c \in f^k([f(1), f(0)])$, then the attractor of $f$ contains only periodic points. On the other hand, it has been proved using a rotation number approach that there is an uncountable set of values of $(\lambda, \mu)$ such that $f$ has no periodic points [**3**, **4**, **6**, **7**, **16**]. It follows that, for such values of $(\lambda, \mu)$, $c \notin f^k([f(1), f(0)])$ for all $k \in \mathbb{N}$. Together with Proposition 3.4 this proves that Assertion A holds for $N = 2$. □

3.2. *Full complexity with any number of discontinuities.* In the previous subsection, we proved the existence of a map satisfying (P1–6) with $N = 2$ (a piecewise contracting interval map with a single discontinuity). In this subsection, we will complete the proof of Assertion A, by induction on $N$.

Let us assume that Assertion A holds for some $N \geqslant 2$. Then there exists $c_0 < c_1 < \cdots < c_N$ in $\mathbb{R}$ and $f : X \to X$, where $X = [c_0, c_N]$, which satisfies (P1–6) with the contraction pieces $X_1 = [c_0, c_1), \ldots, X_N = (c_{N-1}, c_N]$. In the following, $\Delta_f := \{c_i\}_{1 \leqslant i \leqslant N-1}$ denotes the set of the discontinuities of $f$, $\Lambda_f$ is the attractor of $f$ and $\widetilde{X}_f$ is the set defined by (2), where $\Delta = \Delta_f$.

Now we construct a new map $g$, with $N + 1$ contraction pieces and satisfying (P1–6), from the given map $f$ that satisfies (P1–6) and has $N$ contraction pieces. The construction involves what we call a *well-cutting* orbit of $f$, defined as follows.

*Definition 3.5.* We say that an orbit $\{\xi_r\}_{r \in \mathbb{N}}$ of $f$ is *well-cutting* if $\xi_0 \in \Lambda_f \cap \widetilde{X}_f$ and $\{\xi_r\}_{r \in \mathbb{N}}$ does not contain any point of the following sets:

(1) the boundaries of the gaps of the Cantor set $\Lambda_f$;
(2) the orbits of $c_0$ and $c_N$; or
(3) the orbits of $f_i(c_i)$ and $f_{i+1}(c_i)$ for all $i \in \{1, \ldots, N-1\}$.

Note that $f$ has an uncountable number of well-cutting orbits. Indeed, $\Delta_f$ and the sets of items (1)–(3) are countable. Therefore, the set $P$ of all their pre-images is also countable. Since $\Lambda_f$ is uncountable, the complement of $P$ in $\Lambda_f$ is uncountable and contains only a well-cutting orbit of $f$. Also, a well-cutting orbit is not eventually periodic. Indeed, by Theorem 2.10, property (P6) implies that the complexity function of the itinerary of $\xi_0$ is not eventually constant.

Let $\lambda \in (0, 1)$ be the slope of $f$ on any of its contraction pieces, and let $\{\xi_r\}_{r \in \mathbb{N}}$ be a well-cutting orbit of $f$. Consider the function $\phi : [c_0, c_N] \to \mathbb{R}$ defined for any $x \in [c_0, c_N]$ by

$$\phi(x) = x + \sum_{n \in \mathcal{N}(x)} \lambda^n \quad \text{where } \mathcal{N}(x) := \{n \geqslant 1 : \xi_n < x\}. \tag{21}$$

The following lemma gathers basic properties of $\phi$ that we will use in this section.

LEMMA 3.6. *The function $\phi$ is strictly increasing, left-continuous on $[c_0, c_N]$, continuous on $[c_0, c_N] \setminus \{\xi_r\}_{r \geqslant 1}$ (in particular at $\xi_0$), and $\lim_{x \searrow \xi_r} \phi(x) = \phi(\xi_r) + \lambda^r$ for all $r \geqslant 1$. Moreover,*

$$\phi([c_0, c_N]) = [\phi(c_0), \phi(c_N)] \setminus \bigcup_{r=1}^{\infty} G_r \quad \text{where } G_r := (\phi(\xi_r), \phi(\xi_r) + \lambda^r]$$

*for all $r \geqslant 1$,*

*and $\overline{G}_r \cap \overline{G}_l = \emptyset$ for all $r \neq l$.*

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*Proof.* Noting that $\mathcal{N}(x) \subset \mathcal{N}(x')$ for any $x < x'$, it is straightforward to show that $\phi$ is strictly increasing.

Now we show that the left-hand limit of $\phi$ at any point $x_0 \in (c_0, c_N]$ is equal to $\phi(x_0)$. Let $x_0 \in (c_0, c_N]$ and $\epsilon > 0$. Take $m_0$ such that $\sum_{n=m_0}^{\infty} \lambda^n < \epsilon/2$ and let $\rho > 0$ be such that $\xi_n \notin (x_0 - \rho, x_0)$ for all $n < m_0$. Now, if $\delta := \min\{\epsilon/2, \rho\}$, then, for any $x \in (x_0 - \delta, x_0)$,

$$|\phi(x_0) - \phi(x)| = x_0 - x + \sum_{n \in \mathcal{N}(x_0)} \lambda^n - \sum_{n \in \mathcal{N}(x)} \lambda^n < \frac{\epsilon}{2} + \sum_{n \in \mathcal{N}(x_0) \setminus \mathcal{N}(x)} \lambda^n < \epsilon,$$

since $\min \mathcal{N}(x_0) \setminus \mathcal{N}(x) \geqslant m_0$ if $\xi_n \notin (x_0 - \rho, x_0)$ for all $n < m_0$. With an analogous proof, we can show that the right-hand limit of $\phi$ at $x_0 \in [c_0, c_N)$ is equal to $\phi(x_0)$ if $x_0 \neq \xi_r$ for any $r \geqslant 1$, and equal to $\phi(\xi_r) + \lambda^r$ if $x_0 = \xi_r$ for some $r \geqslant 1$.

Taking into account that $\phi$ is strictly increasing, left-continuous and has a discontinuity jump of magnitude $\lambda^r$ at every point $\xi_r$ with $r \geqslant 1$, it is standard to check that

$$\phi([c_0, c_N]) = [\phi(c_0), \phi(c_N)] \setminus \bigcup_{r=1}^{\infty} G_r.$$

Finally we show that $\overline{G}_r \cap \overline{G}_l = \emptyset$ for all $r \neq l$. Let $l$ and $r \geqslant 1$ be such that $r \neq l$ and $\xi_r > \xi_l$. As $\phi(\xi_r) \in \phi([c_0, c_N])$, we have that $\phi(\xi_r) \notin G_l$, and, as $\phi$ is injective, we have that $\phi(\xi_r) \notin \overline{G}_l$. It follows that $\overline{G}_r \cap \overline{G}_l = \emptyset$, since $\phi(\xi_r) > \phi(\xi_l)$. □

The following proposition shows how a well-cutting orbit of $f$ and its associated function $\phi$ allow us to obtain a map which satisfies (P1–6) with $N + 1$ contraction pieces. Therefore, this ends the proof of Assertion A by induction on $N$.

PROPOSITION 3.7. *Let $\{\xi_r\}_{r \in \mathbb{N}}$ be a well-cutting orbit of $f$ and let $\phi$ be defined according to (21). Let $\Delta_g := \phi(\Delta_f \cup \{\xi_0\})$. Then any map $g : [\phi(c_0), \phi(c_N)] \to [\phi(c_0), \phi(c_N)]$ defined on $[\phi(c_0), \phi(c_N)] \setminus \Delta_g$ by*

$$g(y) = \begin{cases} \phi \circ f \circ \phi^{-1}(y) & \text{if } y \in \phi([c_0, c_N]) \setminus \Delta_g, \\ \lambda(y - \phi(\xi_r)) + \phi(\xi_{r+1}) & \text{if } y \in G_r \text{ and } r \geqslant 1 \end{cases} \tag{22}$$

*satisfies (P1–6) with $N + 1$ contraction pieces.*

*Proof.* Let $d_0 < d_1 < \cdots < d_{N+1}$ be such that $\{d_0, d_1, \ldots, d_{N+1}\} = \phi(\Delta_f \cup \{c_0, \xi_0, c_N\})$. We denote by $j_0$ the integer of $\{1, \ldots, N\}$ such that

$$d_{j_0} = \phi(\xi_0). \tag{23}$$

Let $Y_1 := [d_0, d_1)$, $Y_2 := (d_1, d_2)$, $\ldots$, $Y_{N+1} := (d_N, d_{N+1}]$ and $Y := [d_0, d_{N+1}]$. Since $\phi$ is strictly increasing, the sets $Y_j$ are all non-empty and pairwise disjoint. Let $g : Y \to Y$ be a map satisfying (22). We are going to show that $g$ satisfies (P1–6) with $Y_1, Y_2, \ldots, Y_{N+1}$.

(P1) We first show that, for any $j \in \{0, 1, \ldots, N\}$, the map $g$ is affine with slope $\lambda$ on the interval $(d_j, d_{j+1})$, that is,

$$g(y') - g(y) = \lambda(y' - y) \quad \text{for all } y, y' \in (d_j, d_{j+1}). \tag{24}$$

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To prove (24), we fix $j \in \{0, 1, \ldots, N\}$ and $y, y' \in (d_j, d_{j+1})$, and we consider three cases.

*Case 1.* Assume that $y, y' \in \phi([c_0, c_N])$ and denote $x := \phi^{-1}(y) > \phi^{-1}(d_j)$ and $x' := \phi^{-1}(y') < \phi^{-1}(d_{j+1})$. If we assume (with no loss of generality) that $y < y'$, then $x < x'$ and

$$g(y') - g(y) = f(x') - f(x) + \sum_{n \in \mathcal{N}_2} \lambda^n \quad \text{where } \mathcal{N}_2 := \{n \geqslant 1 : f(x) \leqslant \xi_n < f(x')\}.$$

Since $f$ is affine on $(\phi^{-1}(d_j), \phi^{-1}(d_{j+1}))$ and has slope $\lambda$, we have that $f(x') - f(x) = \lambda(x' - x)$. On the other hand, since $f$ is injective (separation property) and increasing on $(\phi^{-1}(d_j), \phi^{-1}(d_{j+1}))$, we have that $n \in \mathcal{N}_2$ if and only if $n \geqslant 2$ and $n - 1 \in \mathcal{N}_1$, where $\mathcal{N}_1 := \{n \geqslant 1 : x \leqslant \xi_n < x'\}$. It follows that

$$g(y') - g(y) = \lambda(x' - x) + \sum_{n \in \mathcal{N}_1} \lambda^{n+1} = \lambda(\phi(x') - \phi(x)) = \lambda(y' - y).$$

*Case 2.* Assume that $y \in G_r$ for some $r \geqslant 1$ and that $y' \in \phi([c_0, c_N])$. Since $G_r$ is an interval and $G_r \cap \phi([c_0, c_N]) = \emptyset$, we have $G_r \subset (d_j, d_{j+1})$. Therefore, $\phi(\xi_r) \in (d_j, d_{j+1})$, because $r \neq 0$ and $\{\xi_r\}_{r \in \mathbb{N}}$ is a well-cutting orbit. So we can use Case 1 to obtain

$$g(y') - g(\phi(\xi_r)) = \lambda(y' - \phi(\xi_r)).$$

On the other hand, by definition of $g$ on $G_r$

$$g(\phi(\xi_r)) - g(y) = \phi(\xi_{r+1}) - \lambda(y - \phi(\xi_r)) - \phi(\xi_{r+1}),$$

and the sum of these two equalities is (24).

*Case 3.* Assume that $y \in G_r$ and that $y' \in G'_r$ for some $r$ and $r' \geqslant 1$ and let $z \in \phi([c_0, c_N]) \cap (d_j, d_{j+1})$. Then, applying Case 2 twice, we obtain

$$g(y') - g(y) = g(y') - g(z) + g(z) - g(y) = \lambda(y' - y),$$

which ends the proof of (24).

According to (24), we know that $g$ is continuous on $\bigcup_{j=0}^{N}(d_j, d_{j+1})$. To study the continuity of $g$ on the border of these intervals, we compute the left-hand and right-hand limits of $g$ at the points of $\phi(\Delta_f \cup \{c_0, c_N\})$. Let $i \in \{1, 2, \ldots, N\}$ and consider an increasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $(c_{i-1}, c_i)$ which converges to $c_i$. Then the sequence $\{\phi(x_n)\}_{n \in \mathbb{N}}$ is increasing and converges to $\phi(c_i)$ because $\phi$ is continuous at any point that is not in the positive orbit of $\xi_0$ and $\{\xi_r\}_{r \in \mathbb{N}}$ is a well-cutting orbit. Using the definition of $g$ on $\phi([c_0, c_N]) \backslash \Delta_g$ and the continuous extension $f_i$ of $f|_{(c_{i-1}, c_i)}$ to $[c_{i-1}, c_i]$, we obtain that

$$\lim_{n \to \infty} g(\phi(x_n)) = \lim_{n \to \infty} \phi(f(x_n)) = \lim_{n \to \infty} \phi(f_i(x_n)) = \lim_{x \to f_i(c_i)} \phi(x).$$

Once again, since $\{\xi_r\}_{r \in \mathbb{N}}$ is a well-cutting orbit, $f_i(c_i)$ does not belong to $\{\xi_r\}_{r \in \mathbb{N}}$ and $\phi$ is continuous at $f_i(c_i)$. It follows that

$$\lim_{y \nearrow \phi(c_i)} g(y) = \phi(f_i(c_i)). \tag{25}$$

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Similarly, for any $i \in \{0, 1, \ldots, N-1\}$,

$$\lim_{y \searrow \phi(c_i)} g(y) = \phi(f_{i+1}(c_i)). \tag{26}$$

Since $f_{i+1}(c_i) \neq f_i(c_i)$ for any $i \in \{1, \ldots, N-1\}$ (separation property) and $\phi$ is injective, we deduce that $g$ is discontinuous on $\phi(\Delta_f)$. On the other hand, from (25) and (26), respectively, we obtain that $g$ is continuous at $\phi(c_N)$ and $\phi(c_0)$, respectively.

It remains to study $g$ at $\phi(\xi_0)$. As $\{\xi_r\}_{r\in\mathbb{N}}$ is a well-cutting orbit, there exists $i \in \{1, 2, \ldots, N\}$ such that $\xi_0 \in (c_{i-1}, c_i)$. Using the left-continuity of $\phi$, the monotonicity of $f$ on $(c_{i-1}, c_i)$ and the continuity of $f$ at $\xi_0$, we obtain that

$$\lim_{y \nearrow \phi(\xi_0)} g(y) = \phi(f(\xi_0)) = \phi(\xi_1). \tag{27}$$

On the other hand,

$$\lim_{y \searrow \phi(\xi_0)} g(y) = \lim_{x \searrow \xi_1} \phi(x) = \phi(\xi_1) + \lambda. \tag{28}$$

Equalities (27) and (28) prove that $g$ is discontinuous at $\phi(\xi_0)$. We conclude that the set of the discontinuity points of $g$ is $\Delta_g$, which, together with (24), proves that $g$ is a piecewise affine contracting map with contractions pieces $Y_1, Y_2, \ldots, Y_{N+1}$. In particular, the set of (2) writes for $g$ as $\widetilde{X}_g = \bigcap_{n=0}^{+\infty} g^{-n}(X \setminus \Delta_g)$.

(P2) Let us prove that $g$ satisfies the separation property. For any $j \in \{1, \ldots, N+1\}$, denote by $g_j : \overline{Y_j} \to [\phi(c_0), \phi(c_N)]$ the continuous extension of $g|_{Y_j}$ to $\overline{Y_j}$. Then

$$\overline{Y_{j_0}} = [\phi(c_{j_0-1}), \phi(\xi_0)], \ \overline{Y_{j_0+1}} = [\phi(\xi_0), \phi(c_{j_0})] \quad \text{and}$$
$$\overline{Y_j} = \begin{cases} [\phi(c_{j-1}), \phi(c_j)] & \text{if } 1 \leqslant j < j_0, \\ [\phi(c_{j-2}), \phi(c_{j-1})] & \text{if } j_0 + 1 < j \leqslant N+1, \end{cases}$$

where $j_0$ is defined by (23). For every $j \in \{1, \ldots, N+1\}$, the map $g_j$ is affine with slope $\lambda > 0$. Therefore, $g_j(\overline{Y_j})$ is an interval whose boundaries are obtained using (25)–(28). Indeed,

$$g_{j_0}(\overline{Y_{j_0}}) = [\phi(f_{j_0}(c_{j_0-1})), \phi(f_{j_0}(\xi_0))], \quad g_{j_0+1}(\overline{Y_{j_0+1}}) = [\phi(f_{j_0}(\xi_0)) + \lambda, \phi(f_{j_0}(c_{j_0}))],$$

and

$$g_j(\overline{Y_j}) = \begin{cases} [\phi(f_j(c_{j-1})), \phi(f_j(c_j))] & \text{if } 1 \leqslant j < j_0, \\ [\phi(f_{j-1}(c_{j-2})), \phi(f_{j-1}(c_{j-1}))] & \text{if } j_0 + 1 < j \leqslant N+1. \end{cases}$$

On the one hand, we have obtained that $g_{j_0}(\overline{Y}_{j_0}) \cap g_{j_0+1}(\overline{Y}_{j_0+1}) = \emptyset$. On the other hand, the monotonicity of $\phi$ and the separation property of $f$ imply that the sets

$$g_1(\overline{Y_1}), \ldots, g_{j_0-1}(\overline{Y_{j_0-1}}), g_{j_0}(\overline{Y_{j_0}}) \cup g_{j_0+1}(\overline{Y_{j_0+1}}), g_{j_0+2}(\overline{Y_{j_0+2}}), \ldots, g_{N+1}(\overline{Y_{N+1}})$$

are pairwise disjoint. It follows that $g$ satisfies the separation property.

(P3) Now we study the attractor of $g$, which can be written using the continuous extensions of $g$ as

$$\Lambda_g = \bigcap_{n=1}^{\infty} \Lambda_{g,n},$$

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where the sets $\Lambda_{g,n}$ are defined recursively as

$$\Lambda_{g,1} = \bigcup_{j=1}^{N+1} g_j(\overline{Y_j}) \quad \text{and} \quad \Lambda_{g,n+1} = \bigcup_{j=1}^{N+1} g_j(\overline{\Lambda_{g,n} \cap Y_j}) \quad \text{for all } n \geqslant 1.$$

We denote $G := \bigcup_{r=1}^{\infty} G_r$ and we recall that $\phi(X) = Y \setminus G$ by Lemma 3.6.

Let us show by induction that

$$\Lambda_{g,n} \cap \phi(X) = \phi(\Lambda_{f,n}) \quad \text{for all } n \geqslant 1. \tag{29}$$

First, note that

$$\begin{aligned} \Lambda_{g,1} \cap \phi(X) &= \bigcup_{j=1}^{N+1} (g_j(\overline{Y_j} \cap \phi(X)) \cup g_j(\overline{Y_j} \cap G)) \cap \phi(X) \\ &= \bigcup_{j=1}^{N+1} g_j(\overline{Y_j} \cap \phi(X)) \cap \phi(X), \end{aligned}$$

since, for any $j \in \{1, \ldots, N+1\}$, we have $\overline{Y_j} \cap G = Y_j \cap G$ and therefore $g_j(\overline{Y_j} \cap G) = g(Y_j \cap G) \subset G$. In addition, if $y \in \overline{Y_j} \cap \phi(X)$ and $y \neq d_{j_0}$, then

$$g_j(y) = \begin{cases} \phi \circ f_j \circ \phi^{-1}(y) & \text{if } 1 \leqslant j < j_0 + 1, \\ \phi \circ f_{j-1} \circ \phi^{-1}(y) & \text{if } j_0 + 1 \leqslant j \leqslant N+1, \end{cases} \tag{30}$$

and if $y \neq d_{j_0}$, then

$$g_{j_0}(y) = \phi \circ f_{j_0} \circ \phi^{-1}(y) \quad \text{and} \quad g_{j_0+1}(y) = \phi(f_{j_0}(\xi_0)) + \lambda = \phi(\xi_1) + \lambda \in G. \tag{31}$$

Denote by $Z_1, Z_2, \ldots, Z_N$ the sets defined by

$$Z_{j_0} = Y_{j_0} \cup Y_{j_0+1} \quad \text{and} \quad Z_j = \begin{cases} Y_j & \text{if } 1 \leqslant j < j_0, \\ Y_{j+1} & \text{if } j_0 < j \leqslant N. \end{cases}$$

Then

$$\Lambda_{g,1} \cap \phi(X) = \bigcup_{j=1}^{N} \phi \circ f_j \circ \phi^{-1}(\overline{Z_j} \cap \phi(X)).$$

As $\overline{Z_j} \cap \phi(X) = \phi(\overline{X_j})$ for any $j \in \{1, \ldots, N\}$, we deduce that

$$\Lambda_{g,1} \cap \phi(X) = \bigcup_{j=1}^{N} \phi(f_j(\overline{X_j})) = \phi(\Lambda_{f,1}).$$

Now assume that $\Lambda_{g,n} \cap \phi(X) = \phi(\Lambda_{f,n})$ for some $n \geqslant 1$. As before,

$$\begin{aligned} \Lambda_{g,n+1} \cap \phi(X) &= \bigcup_{j=1}^{N+1} g_j(\overline{\Lambda_{g,n} \cap Y_j} \cap \phi(X)) \cap \phi(X) \\ &= \bigcup_{j=1}^{N} \phi \circ f_j \circ \phi^{-1}(\overline{\Lambda_{g,n} \cap Z_j} \cap \phi(X)). \end{aligned}$$

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To obtain that $\Lambda_{g,n+1} \cap \phi(X) = \phi(\Lambda_{f,n+1})$ and complete the induction, it is enough to show that

$$\overline{\Lambda_{g,n} \cap Z_j} \cap \phi(X) = \phi(\overline{\Lambda_{f,n} \cap X_j}). \tag{32}$$

On the one hand, using the induction hypothesis, we obtain $\Lambda_{g,n} \cap \phi(X) = \phi(\Lambda_{f,n})$. On the other hand, we have $Z_j \cap \phi(X) = \phi(X_j)$. As $\phi$ is injective, it follows that

$$\Lambda_{g,n} \cap Z_j \cap \phi(X) = \phi(\Lambda_{f,n} \cap X_j) \quad \text{for all } j \in \{1, \dots, N\}.$$

Let $y \in \phi(\overline{\Lambda_{f,n} \cap X_j} \backslash (\Lambda_{f,n} \cap X_j))$ and $x \in \overline{\Lambda_{f,n} \cap X_j} \backslash (\Lambda_{f,n} \cap X_j)$ be such that $y = \phi(x)$. Then $x \in \{c_{j-1}, c_j\}$ and $\phi$ is continuous at $x$, since $x \notin \{\xi_r\}_{r \in \mathbb{N}}$. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of $\Lambda_{f,n} \cap X_j$ which converges to $x$. Then the sequence $\{\phi(x_n)\}_{n \in \mathbb{N}}$ belongs to $\Lambda_{g,n} \cap Z_j$ and converges to $y = \phi(x)$. It follows that $y \in \overline{\Lambda_{g,n} \cap Z_j} \cap \phi(X)$ and we have proved that $\phi(\overline{\Lambda_{f,n} \cap X_j}) \subset \overline{\Lambda_{g,n} \cap Z_j} \cap \phi(X)$. To show the converse inclusion, take $y \in \overline{\Lambda_{g,n} \cap Z_j} \cap \phi(X) \backslash (\Lambda_{g,n} \cap Z_j)$. Then $y \in \overline{Z_j} \backslash Z_j \subset \{\phi(c_{j-1}), \phi(c_j), \phi(\xi_0)\}$. If $y = \phi(\xi_0)$, then $j = j_0$ and $y \in \phi(\Lambda_{f,n} \cap X_{j_0})$, since $\xi_0 \in \Lambda_f \cap X_{j_0}$. As $c_{j-1}$ and $c_j$ are lr-recurrently visited, there exist two atoms $A_{j-1}$ and $A_j$ in the set of the atoms of generation $n$ of $f$ such that $c_{j-1}$ and $c_j$ belong to the interior of $A_{j-1}$ and $A_j$, respectively. As $\{c_{j-1}, c_j\} \subset \overline{X_j}$, it follows that $c_{j-1} \in \overline{A_{j-1} \cap X_j}$ and $c_j \in \overline{A_j \cap X_j}$. Recalling that $\Lambda_{f,n}$ is the union of all the atoms of generation $n$ of $f$, we deduce that $\{c_{j-1}, c_j\} \subset \overline{\Lambda_{f,n} \cap X_j}$ and $y \in \phi(\overline{\Lambda_{f,n} \cap X_j})$. This ends the proof of (32) and completes the proof by induction of (29).

Using the injectivity of $\phi$ and (29), we obtain that

$$\Lambda_g \cap \phi(X) = \phi(\Lambda_f). \tag{33}$$

Now we show that $\Lambda_g \cap G = \{\phi(\xi_r) + \lambda^r\}_{r \geqslant 1}$. To this end, we prove by induction that

$$\Lambda_{g,n} \cap G = \{\phi(\xi_r) + \lambda^r,\ 1 \leqslant r \leqslant n\} \cup \bigcup_{r=n+1}^{\infty} G_r \quad \text{for all } n \geqslant 1. \tag{34}$$

Using equations (30) and (31), it follows that

$$\begin{aligned} \Lambda_{g,1} \cap G &= \bigcup_{j=1}^{N+1} (g_j(\overline{Y_j} \cap \phi(X)) \cup g_j(\overline{Y_j} \cap G)) \cap G \\ &= \{g_{j_0+1}(d_{j_0})\} \cup \bigcup_{j=1}^{N+1} g_j(\overline{Y_j} \cap G) \cap G. \end{aligned}$$

Recalling the inclusion of $G$ in the union of the sets $Y_1, Y_2, \dots, Y_{N+1}$, and the definition of $g$ in $G$, we obtain

$$\Lambda_{g,1} \cap G = \{\phi(\xi_1) + \lambda\} \cup \bigcup_{r=2}^{N+1} G_r,$$

which proves (34) for $n = 1$.

Now, assume that (34) holds for some $n \geqslant 1$. As $d_{j_0} = \phi(\xi_0) \in \phi(\Lambda_f) = \Lambda_g \cap \phi(X)$, we have that $d_{j_0} \in \Lambda_g \subset \Lambda_{g,n}$. On the other hand, there exists a decreasing sequence in $\Lambda_f$ which converges to $\xi_0$, because $\Lambda_f$ is a Cantor set and $\xi_0$ is not a border

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of a gap of $\Lambda_f$. By (33), the image of this sequence by $\phi$ belongs to $\Lambda_g$ and it is decreasing by monotonicity of $\phi$. From the continuity of $\phi$ at $\xi_0$, it follows that $d_{j_0} \in \overline{\Lambda_g \cap Y_{j_0+1}} \subset \overline{\Lambda_{g,n} \cap Y_{j_0+1}}$. Now, using once again equations (30) and (31), we deduce that

$$\Lambda_{g,n+1} \cap G = \bigcup_{j=1}^{N+1} (g_j(\overline{\Lambda_{g,n} \cap Y_j} \cap \phi(X)) \cup g_j(\overline{\Lambda_{g,n} \cap Y_j} \cap G)) \cap G$$

$$= \{g_{j_0+1}(d_{j_0})\} \cup \bigcup_{j=1}^{N+1} g_j(\overline{\Lambda_{g,n} \cap Y_j} \cap G) \cap G.$$

As $\overline{\Lambda_{g,n} \cap Y_j} \setminus (\Lambda_{g,n} \cap Y_j) \subset \{d_{j-1}, d_j\}$, and $d_0, d_1, \ldots, d_{N+1}$ do not belong to $G$, we have that $g_j(\overline{\Lambda_{g,n} \cap Y_j} \cap G) = g(\Lambda_{g,n} \cap Y_j \cap G)$ for all $j \in \{1, 2, \ldots, N+1\}$. Therefore,

$$\Lambda_{g,n+1} \cap G = \{\phi(\xi_1) + \lambda\} \cup g(\Lambda_{g,n} \cap G).$$

Using the induction hypothesis, we obtain

$$\Lambda_{g,n+1} \cap G = \{\phi(\xi_r) + \lambda^r,\ 1 \leqslant r \leqslant n+1\} \cup \bigcup_{r=n+2}^{\infty} G_r,$$

which complete the proof of (34).

Note that (34) can also be written as

$$\Lambda_{g,n} \cap G = \{\phi(\xi_r) + \lambda^r\}_{r \geqslant 1} \cup \bigcup_{r=n+1}^{\infty} \mathrm{Int}(G_r) \quad \text{for all } n \geqslant 1,$$

and recall that the sets $G_r$ are pairwise disjoint. This implies that $\Lambda_g \cap G = \{\phi(\xi_r) + \lambda^r\}_{r \geqslant 1}$, which, together with (33), gives

$$\Lambda_g = \phi(\Lambda_f) \cup \{\phi(\xi_r) + \lambda^r\}_{r \geqslant 1}. \tag{35}$$

Now we show that $\Lambda_g$ is a Cantor set. By definition of attractor, $\Lambda_g$ is compact. In addition, it is totally disconnected because $g$ satisfies the separation property (see [**5**, Theorem 5.2]). It remains to show that $\Lambda_g$ has no isolated point. Let $y \in \phi(\Lambda_f)$ and $x \in \Lambda_f$ be such that $y = \phi(x)$. As $\Lambda_f$ is a Cantor set, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $\Lambda_f \setminus \{x\}$ which converges to $x$ and $\{\phi(x_n)\}_{n \in \mathbb{N}}$ belongs to $\Lambda_g \setminus \{y\}$. If $y \notin \{\phi(\xi_r)\}_{r \geqslant 1}$, then $\phi$ is continuous at $x$ and $\{\phi(x_n)\}_{n \in \mathbb{N}}$ converges to $y$. If $y = \phi(\xi_r)$ for some $r \geqslant 1$, then we can assume that $\{x_n\}_{n \in \mathbb{N}}$ is increasing, since $\xi_r$ does not belong to the boundaries of the gaps of the Cantor set $\Lambda_f$. Using the left-continuity of $\phi$, we obtain that $\{\phi(x_n)\}_{n \in \mathbb{N}}$ converges to $y$. Now let $y = \phi(\xi_r) + \lambda^r$ for some $r \geqslant 1$ and let $\{x_n\}_{n \in \mathbb{N}}$ be a decreasing sequence in $\Lambda_f$ which converges to $\xi_r$. Then $\{\phi(x_n)\}_{n \in \mathbb{N}}$ converges to $y$ and belongs to $\Lambda_g \setminus \{y\}$, since $y \notin \phi(X)$.

(P4) Let us prove that $g_j(d_j)$ and $g_{j+1}(d_j)$ belong to $\widetilde{X}_g$ for every $j \in \{1, 2, \ldots, N\}$. First, note that, for any $x$ such that $f^n(x) \notin \Delta_f \cup \{\xi_0\}$ for all $n \in \mathbb{N}$, we have

$$g^n \circ \phi(x) = \phi \circ f^n(x) \quad \text{for all } n \in \mathbb{N} \text{ and } \phi(x) \in \widetilde{X}_g. \tag{36}$$

Let $j \in \{1, 2, \ldots, N\}$ and $y \in \{g_j(d_j), g_{j+1}(d_j)\}$. If $y \neq g_{j_0+1}(d_{j_0})$, then, using (30) and (31), we obtain that $y = \phi(x)$ for some point $x \in \{f_j(c_j), f_{j+1}(c_j), f_{j-1}(c_{j-1}),$

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$f_j(c_{j-1})$, $f(\xi_0)\}$. Since $\{\xi_n\}_{n\in\mathbb{N}}$ is a well-cutting orbit and $f_i(c_i)$ and $f_{i+1}(c_i)$ belong to $\widetilde{X}_f$ for any $i \in \{1, 2, \ldots, N-1\}$, it follows from (36) that $y \in \widetilde{X}_g$. Now, if $y = g_{j_0+1}(d_{j_0})$, then, by (31), we have that $g^n(y) \in G$ for all $n \in \mathbb{N}$, and therefore $y \in \widetilde{X}_g$.

(P5) Let $i \in \{1, \ldots, N-1\}$ and $l \in \{i, i+1\}$ be such that $\{f^n(f_l(c_i))\}_{n\in\mathbb{N}}$ is dense in $\Lambda_f$. Let us denote $x_0 := f_l(c_i)$ and $y_0 := \phi(x_0)$. Using (30) we obtain that there exists $j \neq j_0$ such that $y_0 \in \{g_j(d_j), g_{j+1}(d_j)\}$. Therefore, to prove (P5) for $g$ it is enough to show that $\{g^n(y_0)\}_{n\in\mathbb{N}}$ is dense in $\Lambda_g$. Applying (36) to $x_0$, we obtain that $y_0 \in \widetilde{X}_g$ and that $g^n(y_0) = \phi(f^n(x_0))$ for all $n \in \mathbb{N}$. In addition, according to (33), we have $\{\phi(f^n(x_0))\}_{n\in\mathbb{N}} \subset \Lambda_g$, since $\{f^n(x_0)\}_{n\in\mathbb{N}} \subset \Lambda_f$.

First, let $y \in \Lambda_g \backslash \{\phi(\xi_r), \phi(\xi_r) + \lambda^r\}_{r\geqslant 1}$. Then $y \in \phi(\Lambda_f)$ and there exists $\{n_k\}_{k\in\mathbb{N}}$ such that $\{f^{n_k}(x_0)\}_{k\in\mathbb{N}}$ converges to $x := \phi^{-1}(y) \in \Lambda_f$. Since $x \notin \{\xi_r\}_{r\geqslant 1}$, it follows that $\phi$ is continuous at $x$ and $\{g^{n_k}(y_0)\}_{k\in\mathbb{N}}$ converges to $y$.

Now let $y = \phi(\xi_r)$ for some $r \geqslant 1$. Since $c_i$ is lr-recurrently visited by $\{f^k(x_0)\}_{k\in\mathbb{N}}$, and $\Lambda_f \cap (\xi_r - \nu, \xi_r) \neq \emptyset$ for all $\nu > 0$ (recall that $\xi_r$ is not border of gap) by Lemma 3.2 we have that the orbit of $x_0$ accumulates from the left on $\xi_r$. Using the left-continuity of $\phi$ we obtain that the orbit of $y_0$ accumulates on $y = \phi(\xi_r)$. Using now that $\Lambda_f \cap (\xi_r, \xi_r + \nu) \neq \emptyset$ for all $\nu > 0$, we obtain that there exists a subsequence of $\{f^n(x_0)\}_{n\in\mathbb{N}}$ which converges to $\xi_r$ from the right-hand side. The image by $\phi$ of this subsequence converges to $\phi(\xi_r) + \lambda^r$. It follows that $\phi(\xi_r) + \lambda^r$ is also a limit point of $\{g^n(y_0)\}_{n\in\mathbb{N}}$.

(P6) Now we prove that all the discontinuities of $g$ are lr-recurrently visited by the orbits of the points of $\widetilde{X}_g$. Let $j \in \{1, \ldots, N\}$ and $y \in \widetilde{X}_g$. We are going to show that there exist two sequences $\{n_k\}_{k\in\mathbb{N}}$ and $\{m_k\}_{k\in\mathbb{N}}$ going to infinity such that $g^{n_k}(y) < d_j < g^{m_k}(y)$ for all $k \in \mathbb{N}$ and $\{g^{n_k}(y)\}_{k\in\mathbb{N}}$ and $\{g^{m_k}(y)\}_{k\in\mathbb{N}}$ converge to $d_j$.

First, let us show the above assertion for $j \neq j_0$. We denote by $c$ the point of $\Delta_f$ such that $\phi(c) = d_j$. If $y \in \widetilde{X}_g \cap \phi(X)$, then

$$g^n(y) = \phi \circ f^n \circ \phi^{-1}(y) \quad \text{for all } n \in \mathbb{N}, \tag{37}$$

since $\widetilde{X}_g \cap \phi(X)$ is forward invariant by $g$. It follows that $x := \phi^{-1}(y)$ belongs to $\widetilde{X}_f$ and $c$ is lr-recurrently visited by $\{f^k(x)\}_{k\in\mathbb{N}}$ by property (P6) of $f$. Therefore, there exist two sequences $\{n_k\}_{k\in\mathbb{N}}$ and $\{m_k\}_{k\in\mathbb{N}}$ going to infinity such that $f^{n_k}(x) < c < f^{m_k}(x)$ for all $k \in \mathbb{N}$ and $\{f^{n_k}(x)\}_{k\in\mathbb{N}}$ and $\{f^{m_k}(x)\}_{k\in\mathbb{N}}$ converge to $c$. Using (37), the monotonicity of $\phi$ and its continuity at $c$, we deduce that $\{g^{n_k}(y)\}_{k\in\mathbb{N}}$ and $\{g^{m_k}(y)\}_{k\in\mathbb{N}}$ satisfy the required properties.

Now assume that $y \in G \subset \widetilde{X_g}$ and let $r \geqslant 1$ be such that $y \in G_r$. Then $g^n(y) \in G_{r+n}$ for all $n \in \mathbb{N}$, that is,

$$\phi(f^n(\xi_r)) < g^n(y) \leqslant \phi(f^n(\xi_r)) + \lambda^{n+r} \quad \text{for all } n \in \mathbb{N}. \tag{38}$$

As $\xi_r \in \widetilde{X}_f$, we have that $c$ is lr-recurrently visited by $\{f^k(\xi_r)\}_{k\in\mathbb{N}}$. Therefore, there exist two sequences $\{n_k\}_{k\in\mathbb{N}}$ and $\{m_k\}_{k\in\mathbb{N}}$ going to infinity such that $f^{n_k}(\xi_r) < c < f^{m_k}(\xi_r)$ for all $k \in \mathbb{N}$ and $\{f^{n_k}(\xi_r)\}_{k\in\mathbb{N}}$ and $\{f^{m_k}(\xi_r)\}_{k\in\mathbb{N}}$ converge to $c$. Using (38) and the continuity of $\phi$ at $c$, we have that both $\{g^{n_k}(y)\}_{k\in\mathbb{N}}$ and $\{g^{m_k}(y)\}_{k\in\mathbb{N}}$ converge to $\phi(c) = d_j$. On the other hand, by monotonicity of $\phi$ and the left-hand side of (38), we have that $d_j < g^{m_k}(y)$ for all $k \in \mathbb{N}$. Again by monotonicity of $\phi$, we have $\phi(f^{n_k}(\xi_r)) < d_j$, which implies that $\phi(f^{n_k}(\xi_r)) + \lambda^{n_k+r} < d_j$ because $d_j \notin G$. We deduce from (38) that $g^{n_k}(y) < d_j$.

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Finally, we show that $d_{j_0} = \phi(\xi_0)$ is lr-recurrently visited by the orbit of any point of $\widetilde{X}_g$. Let $d_j \in \Delta_g \backslash \{d_{j_0}\}$ be a discontinuity of $g$ such that $\{g^n(g_l(d_j))\}_{n\in\mathbb{N}}$ is dense in $\Lambda_g$ for some $l \in \{j, j+1\}$ (we proved in (P6) that it exists). Since $d_j \neq d_{j_0}$, we already know that $d_j$ is lr-recurrently visited by $\{g^k(y)\}_{k\in\mathbb{N}}$ for any $y \in \widetilde{X}_g$. Now, since $\phi(\xi_0)$ is not a border of gap of $\Lambda_g$ (because $\xi_0$ is not a border of gap of $\Lambda_f$ and $\phi$ is continuous at this point), we can apply Lemma 3.2 to deduce that $d_{j_0}$ is lr-recurrently visited by $\{g^k(y)\}_{k\in\mathbb{N}}$ for any $y \in \widetilde{X}_g$. □

LEMMA 3.8. *If any atom of generation one of $f$ contains at most one point of $\Delta_f$, then there exists a well-cutting orbit of $f$ such that any atom of generation one of $g$ contains at most one point of $\Delta_g$.*

*Proof.* As $f$ satisfies (P6), any discontinuity of $f$ is contained in an atom of generation one of $f$ and this atom is unique because of (P2). It follows that one of the $N$ atoms of generation one of $f$ does not contain any discontinuity. Let us denote this atom by $A_{i_1}$, where $i_1 \in \{1, \ldots, N\}$ is such that $A_{i_1} = \overline{f((c_{i_1-1}, c_{i_1}))}$. Then $A_{i_1} \cap \Lambda_{f,n} \neq \emptyset$ for all $n \geqslant 1$, since by (P6), for any $n \geqslant 2$, there exists an atom of generation $n-1$ which contains $c_{i_1}$ in its interior. It follows that $A_{i_1} \cap \Lambda_f \neq \emptyset$. Moreover, $A_{i_1} \cap \Lambda_f$ is compact, totally disconnected and any point of $\mathrm{Int}(A_{i_1}) \cap \Lambda_f$ is not isolated, because $\Lambda_f$ is a Cantor set. Now, as the atoms are compact and disjoint, if $x \in (A_{i_1} \setminus \mathrm{Int}(A_{i_1})) \cap \Lambda_f$, then $x$ is a border of gap of $\Lambda_f$ and there exists a sequence in $\mathrm{Int}(A_{i_1}) \cap \Lambda_f$ which converges to $x$. We deduce that $A_{i_1} \cap \Lambda_f$ is a Cantor set. Therefore, there exists $\xi_0 \in A_{i_1}$ such that $\{f^k(\xi_0)\}_{k\in\mathbb{N}}$ is a well-cutting orbit of $f$.

For every $i \in \{1, \ldots, N\}$, let $B_i = \overline{g((\phi(c_i-1), \phi(c_i)))}$, where $\phi$ is the function defined in (21) with a well-cutting orbit of $f$ such that $\xi_0 \in A_{i_1}$. Then, for any atom $B$ of generation one of $g$, there exists $i \in \{1, \ldots, N\}$ such that $B \subseteq B_i$. Moreover, we can show that

$$B_i \cap \phi(X) = \phi(A_i) \quad \text{for all } i \in \{1, \ldots, N\},$$

where $A_i := \overline{f((c_{i-1}, c_i))}$ is an atom of generation one of $f$. Now let $d \neq d' \in \Delta_g$ and assume by contradiction that there exists an atom of generation one of $g$ which contains $d$ and $d'$. Then there exits an atom of generation one of $f$ which contains two elements of $\Delta_f \cup \{\xi_0\}$, which is a contradiction. □

*Proof of part (2) of Theorem 1.2.* From Theorem 3.1, we deduce that equality (4) of Theorem 1.2 holds for any $n \geqslant n_0$, where $n_0$ is defined in Lemma 2.9 and is not necessarily equal to one. For any piecewise contracting map $f$, the integer $n_0$ is bounded above by $n_{0,f} := \min\{n \geqslant 1 : \#A \cap \Delta_f \leqslant 1 \text{ for all } A \in \mathcal{A}_{f,n}\}$, where $\mathcal{A}_{f,n}$ is the set of the atoms of generation $n$ of $f$. Obviously, if $f$ has only two contraction pieces, then $n_{0,f} = 1$. Lemma 3.8 proves that if $f$ satisfies (P1–6) and $n_{0,f} \neq 1$, then, for a suitable choice of the well-cutting orbit of $f$, we have $n_{0,g} = 1$ for the map $g$ of Proposition 3.7. □

## 4. *Proof of Corollary 1.3*

Let $N \geqslant 2$ and $\alpha \in \{1, \ldots, N-1\}$. According to part (2) of Theorem 1.2, there exist a compact interval $[a, b] \subset \mathbb{R}$ and a piecewise affine contracting map $g : [a, b] \to [a, b]$

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which has $\alpha + 1$ contraction pieces $X_1, \ldots, X_{\alpha+1}$ and satisfies the separation property, such that

$$p(\theta, n) = \alpha n + 1 \quad \text{for all } n \geqslant 1 \tag{39}$$

for every itinerary $\theta$ of $g$. In particular, if $\alpha = N - 1$, then the corollary is true with $f = g$.

Now suppose that $\alpha < N - 1$. Let $c > b$ and $X_{\alpha+2}, \ldots, X_N$ be pairwise disjoint open intervals such that $[b, c] = \bigcup_{i=\alpha+2}^{N} \overline{X_i}$. Consider a map $h : [b, c] \to [a, c] \setminus \bigcup_{i=1}^{\alpha+1} \overline{g(X_i)}$ which is affine, injective and contracting in each piece $X_{\alpha+2}, \ldots, X_N$ and satisfies $\overline{h(X_i)} \cap \overline{h(X_j)} = \emptyset$ for all $i \neq j \in \{\alpha + 2, \ldots, N\}$. As $g$ satisfies the separation property, there exists a non-empty open set $J \subset [a, b] \setminus \bigcup_{i=1}^{\alpha+1} \overline{g(X_i)}$. It is easy to construct $h$ in such a way that, for any $x \in [b, c]$, there exists $j \geqslant 1$ such that $h^j(x) \in J$.

Finally, let $X := [a, c]$ and consider the map $f : X \to X$ defined for all $x \in [a, c]$ by $f(x) := g(x)$ if $x \in [a, b]$ and $f(x) := h(x)$ if $x \in (b, c]$. Then $f$ is a piecewise affine contracting interval map, which has $N$ contraction pieces $X_1, \ldots X_N$ and satisfies the separation property (note also that $[a, b]$ is an invariant set for $f$). Let $x \in \widetilde{X}$ and let $\theta$ be its itinerary by $f$. If $x \in [a, b]$, then $\theta$ is an itinerary of $g$, and by (39) we have $p(\theta, n) = \alpha n + 1$ for all $n \geqslant 1$. Now suppose that $x \in (b, c]$ and let $j$ be such that $f^j(x) \in J$. Then, from (39) and equality (13) of Theorem 2.10, we deduce that $\#\Delta_{lr}(f^j(x)) = \alpha$. As $\Delta_{lr}(x) = \Delta_{lr}(f^j(x))$, by applying Theorem 2.10 to $x$, we obtain that there exist $m_0 \geqslant 1$ and $\beta$ such that $p(\theta, n) = \alpha n + \beta$ for all $n \geqslant m_0$. Using (14), or by applying directly part (1) of Theorem 1.2, we obtain that $\beta \in \{1, \ldots, 1 + m_0(N - 1 - \alpha)\}$.

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